

A Note on The Spectral Transfer Morphism for Affine Hecke Algebras

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Communicated by B. Krötz

Abstract. Opdam introduced the notion of spectral transfer morphisms of affine Hecke algebras to study the formal degree of a unipotent discrete series representation. Based on the uniqueness property of supercuspidal unipotent representations established by Opdam and the author, Opdam proved that unipotent discrete series representations of classical groups can be classified by the associated formal degrees, in the same spirit as Reeder’s result for split exceptional adjoint groups. The present paper aims at verifying that three specific families of finite maps of algebraic tori are spectral transfer morphisms. These spectral transfer morphisms are used in the proof of Opdam’s result mentioned above.

Mathematics Subject Classification: 20G25, 22E50.

Key Words: Affine Hecke algebra, unipotent representation, discrete series representation, formal degree, spectral transfer morphism.

1. Introduction

Let k be a non-archimedean local field with a finite residue field \mathfrak{f} . Fix a separable closure k_s of k , and let $k_{nr} \subset k_s$ be the maximal unramified extension of k . Throughout this paper, assume that \mathbf{G} is a connected absolutely simple algebraic group of adjoint type defined and quasi-split over k such that \mathbf{G} splits over k_{nr} . Lusztig [11] defined unipotent representations for $\mathbf{G}(k)$ (the group of k -rational points of \mathbf{G}). We will recall this definition in more detail in Section 4. The irreducible unipotent representations of $\mathbf{G}(k)$ are the irreducible subquotients of representations of the form $V := \text{c-Ind}_{\mathcal{P}_k}^{\mathbf{G}(k)} E$ where (σ, E) is a cuspidal unipotent representation of a parahoric subgroup $\mathcal{P}_k \subset \mathbf{G}(k)$. In the case where \mathcal{P}_k is a maximal parahoric subgroup of $\mathbf{G}(k)$ then V is actually a finite direct sum of supercuspidal unipotent representations.

A conjecture of Hiraga, Ichino and Ikeda [9] states that for a discrete series representation of a connected reductive group defined over a local field, the formal degree $\text{fdeg}(\pi)$ equals to $|\gamma(\varphi_\pi)|$ times a rational factor, where $\gamma(\varphi_\pi)$ is the adjoint gamma factor associated with the Langlands parameter φ_π corresponding to π . An earlier result of Reeder [21] on unipotent discrete series representations of split exceptional adjoint groups is compatible with this conjecture (see [9, §3.4]).

*This work was supported by European Research Council (ERC) Advanced Grant No. 268105 and by Netherlands Organisation for Scientific Research (NWO) Vidi Grant No. 639.032.528.

Recently, Opdam [18, Thm. 4.11] proved the Hiraga–Ichino–Ikeda (HII) conjecture for unipotent discrete series representations of $\mathbf{G}(k)$, for classical groups \mathbf{G} satisfying our assumption. The main tool Opdam used is called *Spectral Transfer Morphisms* for affine Hecke algebras which he developed in [17].

It is known that there is a bijection between the collection $\text{Irr}_{\text{upt}}(\mathbf{G}(k), \mathcal{P}_k, \sigma)$ of representations of $\mathbf{G}(k)$ which are irreducible subquotients of the compactly induced representation V and simple modules over the algebra $\text{End}_{\mathbf{G}(k)}(V)$. This endomorphism algebra is an extension of an affine Hecke algebra $\mathcal{H}(\mathbf{G}(k), \mathcal{P}_k, \sigma)$. Parameters of $\mathcal{H}(\mathbf{G}(k), \mathcal{P}_k, \sigma)$ are explicitly determined by \mathbf{G} , the parahoric subgroup \mathcal{P}_k and the cuspidal unipotent character σ . (An explicit expression of these parameters can be found in [11, §5.12].)

If the unipotent discrete series representation is supercuspidal, the spectral transfer morphism is particularly simple to describe (using the main result of [18]). We illustrate it briefly. Suppose now \mathcal{P}_k is a *maximal* parahoric subgroup and σ a cuspidal unipotent character of the reductive quotient $\overline{\mathcal{P}_k}$ of \mathcal{P}_k (we lift it to a representation of \mathcal{P} via the natural projection $\mathcal{P}_k \twoheadrightarrow \overline{\mathcal{P}_k}$). If π belongs to $\text{Irr}_{\text{upt}}(\mathbf{G}(k), \mathcal{P}_k, \sigma)$, then π is supercuspidal, and $\mathcal{H}(\mathbf{G}(k), \mathcal{P}_k, \sigma)$ is a direct product of copies of \mathbf{C} , whose simple direct factors are in canonical bijection with the irreducible summands of V . Let $\Lambda = \mathbf{C}[v, v^{-1}]$ be the ring of complex polynomials in variables v, v^{-1} , and let $\mathcal{H}^{\text{IM}}(\mathbf{G}(k))$ be the generic Iwahori–Matsumoto Hecke algebra (defined over Λ) of \mathbf{G} . In the case at hand, a spectral transfer morphism from $\mathcal{H}(\mathbf{G}(k), \mathcal{P}_k, \sigma)$ to $\mathcal{H}^{\text{IM}}(\mathbf{G}(k))$ is simply the co-pairing of spectral transfer morphisms of the individual factors. For each individual factor, the spectral transfer morphism is defined by a morphism of the form:

$$\mathcal{S}: \text{Spec } \Lambda \rightarrow \text{Spec } Z(\mathcal{H}^{\text{IM}}(\mathbf{G}(k))),$$

where $Z(\mathcal{H}^{\text{IM}}(\mathbf{G}(k)))$ is the centre of $\mathcal{H}^{\text{IM}} := \mathcal{H}^{\text{IM}}(\mathbf{G}(k))$. Here $v \in \mathcal{H}^{\text{IM}}$ acts on V as multiplication by the positive square root of $|f|$. The main result of [18] states that the morphism \mathcal{S} defining a spectral transfer map from $\mathcal{H}(\mathbf{G}(k), \mathcal{P}_k, \sigma)$ to $\mathcal{H}^{\text{IM}}(\mathbf{G}(k))$ is closely related to the (discrete, unramified) Langlands parameter of π , in the way we will now explain.

We shall relate $\text{Spec } Z(\mathcal{H}^{\text{IM}})$ to the Langlands L -group ${}^L G$ of \mathbf{G} via a theorem of Borel (cf. [1, Prop. 6.7]). This theorem can be formulated as: $\text{Spec } Z(\mathcal{H}^{\text{IM}})$ is bijectively corresponding to the set of G^\vee -conjugacy classes of semi-simple elements in ${}^L G$, which in turn can be identified with the quotient $\mathcal{W}_0 \backslash T^\vee$ with $T^\vee = S^\vee / (1 - \theta)S^\vee$. Here S^\vee is a maximal torus in the complex dual group G^\vee , and \mathcal{W}_0 is the associated finite Weyl group. In other words, \mathcal{S} corresponds to, in this special case, a \mathcal{W}_0 -orbit of a Λ -valued point of T^\vee . This Λ -valued point of T^\vee is obtained from a Langlands parameter for π as follows. Let

$$\varphi_\pi: \text{Frob}^{\mathbf{Z}} \times \mathbf{SL}_2(\mathbf{C}) \rightarrow {}^L G = G^\vee \rtimes \langle \vartheta \rangle$$

be a discrete unramified Langlands parameter associated with π , where ϑ denotes the automorphism on G^\vee induced by Frob. We denote $u := \varphi_\pi(\text{id}, (\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}))$, and let $r \in {}^L G$ be a semi-simple element such that

$$rur^{-1} = u^{|f|}. \tag{1}$$

It is known that (cf. [10]) there is a semi-simple element c in G^\vee such that $r = cs$, where $s := \varphi_\pi(\text{Frob}, I_2)$, and u must be a distinguished unipotent element in the connected centraliser $Z_{G^\vee}(s)^\circ$. Existence of distinguished unipotent elements in $Z_{G^\vee}(s)^\circ$ forces that s must be an isolated element in ${}^L G$ (see [22, §3.7]). The G^\vee -conjugacy class of r can be identified with the \mathcal{W}_0 -orbit of a Λ -valued point $\mathbf{r} \in T^\vee$. Then [18, Thm. 4.11] implies that the morphism \mathcal{S} defined by this \mathcal{W}_0 -orbit $\mathcal{W}_0\mathbf{r}$ defines a spectral transfer map from $\mathcal{H}(\mathbf{G}(k), \mathcal{P}_k, \sigma)$ to $\mathcal{H}^{\text{IM}}(\mathbf{G}(k))$.

For the spectral transfer morphisms of supercuspidal unipotent representations of non-split but quasi-split groups, we shall discuss in Section 4.4.

The affine Hecke algebra $\mathcal{H}(\mathbf{G}(k), \mathcal{P}_k, \sigma)$ attached to some $\pi \in \text{Irr}_{\text{cpt}}(\mathbf{G}(k), \mathcal{P}_k, \sigma)$ can have positive rank in general. Let \mathcal{H} be the generic affine Hecke algebra defined over Λ which specialises to $\mathcal{H}(\mathbf{G}(k), \mathcal{P}_k, \sigma)$ if we substitute the square root of $|\mathfrak{f}|$ for v . A spectral transfer morphism from $\mathcal{H} := \mathcal{H}(\mathbf{G}(k), \mathcal{P}_k, \sigma)$ to $\mathcal{H}^{\text{IM}} := \mathcal{H}^{\text{IM}}(\mathbf{G}(k))$ is given by a finite morphism from $\text{Spec } Z(\mathcal{H})$ to $\text{Spec } Z(\mathcal{H}^{\text{IM}})$ satisfying some extra conditions (see Def. 3.1). Among these defining conditions for spectral transfer morphism, the most important one is that a spectral transfer morphism should preserve the Plancherel measure in the following sense. We define a trace functional τ' on an affine Hecke algebra \mathcal{H} . The spectral measure of this trace functional is called the *Plancherel measure*. The density of this Plancherel measure is expressed by a function, called μ -function, which is a certain rational function defined on $\text{Spec } Z(\mathcal{H}) = W \backslash T$, where T is a certain algebraic torus associated with \mathcal{H} , and W denotes the underlying finite Weyl group. A spectral transfer morphism between two pairs $(\mathcal{H}_i, \tau'_i), i = 1, 2$, is given by a finite morphism of torsors of algebraic tori $\mathcal{S}: T_1 \rightarrow T_2$ which, among other conditions, preserves the associated Plancherel measure. Roughly speaking, this means the pull-back under \mathcal{S} of the regularization of μ_2 along the image of \mathcal{S} , up to a non-zero rational constant, is equal to μ_1 . (See the condition (T3) in Def. 3.1.)

This paper aims at verifying that three specific maps of algebraic tori are spectral transfer morphisms. In order to do this, we first recall the structure of the affine Hecke algebra $\mathcal{H}(\mathbf{G}(k), \mathcal{P}_k, \sigma)$ and define the μ -function in Sections 2.1 and 2.2 respectively. We will also recall residual cosets, which is necessary to give the definition of spectral transfer morphisms in Section 3. Then we turn to unipotent representations in Section 4. The materials here are borrowed from [18], except for Section 4.4. We will recall the definition of unipotent representation and the associated affine Hecke algebra. In particular, we will give the parameters of the relevant affine Hecke algebras for classical groups in Section 4. After these preparation, we give expressions of three morphisms in Section 4, and will prove in Section 5 that these morphisms are spectral transfer morphisms.

2. Affine Hecke algebras and μ -functions

2.1. Generic affine Hecke algebras

Let $\mathcal{R} = (X, R_0, F_0, Y, R_0^\vee, F_0^\vee)$ be a reduced irreducible and semi-simple root datum, with a perfect pairing $\langle \cdot, \cdot \rangle: X \times Y \rightarrow \mathbf{Z}$. Here F_0 is a base of R_0 . Let $\mathbf{Z}R_0$ be the root lattice spanned by the root system R_0 . Then $\Omega_X := X/\mathbf{Z}R_0$ is a finite abelian group. Let $\mathcal{W}_0 = \mathcal{W}_0(R_0)$ be the finite Weyl group of the root system R_0 , with a

set S_0 of distinguished generators, in bijection with F_0 . We equip the vector space $E^* := \mathbf{R} \otimes_{\mathbf{Z}} X$ with a Euclidean structure which is invariant under the action of \mathcal{W}_0 . Following the convention in [17], we call elements of the set $R := R_0^\vee \times \mathbf{Z}$ *affine roots*. If $a = \alpha^\vee + n \in R$, the reflection r_a induced by a is defined by $r_a(x) = x - a(x)\alpha$, where $a(x) = \langle x, \alpha^\vee \rangle + n$ for all $x \in E^*$. Such reflections are viewed as affine linear transformations on E^* .

The semi-direct product $\mathcal{W} := X \rtimes \mathcal{W}_0$ is called the *extended affine Weyl group*. Inside \mathcal{W} sits a normal subgroup $\mathcal{W}_a \simeq \mathbf{Z}R_0 \rtimes \mathcal{W}_0$ (the unextended affine Weyl group). \mathcal{W}_a is an affine Coxeter group. Let S_a be a set of distinguished generators of \mathcal{W}_a . Let $C \subset E^*$ be the fundamental alcove for the action of \mathcal{W}_a on E^* . The closure \overline{C} of C is a fundamental domain for the action of \mathcal{W}_a on E^* . The stabiliser $\text{Stab}_{\mathcal{W}}(C) := \{w \in \mathcal{W} : w(C) = C\}$ of C in \mathcal{W} is isomorphic to the quotient $\mathcal{W}/\mathcal{W}_a$, and hence isomorphic to Ω_X .

On the affine Coxeter system (\mathcal{W}_a, S_a) we have a canonical length function $l: \mathcal{W}_a \rightarrow \mathbf{Z}_+$ such that every distinguished generator $s \in S_a$ has length one. We extend l to \mathcal{W} by defining that $l(w) = 0, \forall w \in \Omega_X$. Let $t_x \in \mathcal{W}$ denote the translation corresponding to $x \in X$. Then l satisfies

$$l(wt_x) = l(w) + \sum_{\alpha \in R_0^+} \langle x, \alpha^\vee \rangle \tag{2}$$

for all $w \in \mathcal{W}_0$ and all $x \in X^+ := \{x \in X : \langle x, \alpha^\vee \rangle \geq 0, \forall \alpha \in F_0\}$.

Let $\Lambda := \mathbf{C}[v^{\pm 1}]$ be the Laurent polynomial ring in variables v, v^{-1} . We define an associative algebra $\mathcal{H}(\mathcal{R}, m)$ over Λ , with distinguished Λ -basis $\{N_w : w \in \mathcal{W}\}$ parametrized by $w \in \mathcal{W}$, satisfying the following relations:

- (i) If $l(ww') = l(w) + l(w')$, then $N_w N_{w'} = N_{ww'}$.
- (ii) $(N_s - v^{m(s)})(N_s + v^{-m(s)}) = 0$ for all $s \in S_a$.

Here the parameters $m(s) \in \mathbf{Z}$ are given by a function $m: \mathcal{W} \setminus S_a \rightarrow \mathbf{Z}$, which is defined on the \mathcal{W} -conjugacy classes of S_a . The algebra $\mathcal{H}(\mathcal{R}, m)$ is called the *generic affine Hecke algebra* (with parameter m). If there is no danger of ambiguity, we shall simply write $\mathcal{H}(\mathcal{R})$ for $\mathcal{H}(\mathcal{R}, m)$.

Remark 2.1. (i) Using the *Bernstein presentation* (see e.g. [17, Thm. 2.6]) for $\mathcal{H}(\mathcal{R}, m)$, one can show that the centre of $\mathcal{H}(\mathcal{R}, m)$ is $(\Lambda[X])^{\mathcal{W}_0}$.

- (ii) From the definition, we note that $\mathcal{H}(\mathcal{R}, m)$ is not sensitive to the orientation of the double or triple edges in the Dynkin diagram of R_0 in case that the roots in R_0 have different lengths.

Define $\tau: \mathcal{H}(\mathcal{R}) \rightarrow \Lambda$ by $\tau(\sum_w a_w N_w) = a_1$. Let $\mathcal{H}_{\mathbf{v}}(\mathcal{R})$ be the specialisation of $\mathcal{H}(\mathcal{R})$ at $\mathbf{v} \in \mathbf{R}_{>1}$. Then τ determines a family of \mathbf{C} -valued linear functional $\{\tau_{\mathbf{v}} \mid \mathbf{v} \in \mathbf{R}_{>1}\}$ on the family of the algebras $\{\mathcal{H}_{\mathbf{v}}(\mathcal{R}) \mid \mathbf{v} \in \mathbf{R}_{>1}\}$. Define an anti-involution $*$ on $\mathcal{H}_{\mathbf{v}}(\mathcal{R})$ by $(\sum_w a_w N_w)^* = \sum_w \overline{a_w} N_{w^{-1}}$, then the functional $\tau_{\mathbf{v}}$ is positive with respect to this anti-involution. Moreover, for all $\mathbf{v} \in \mathbf{R}_{>1}$, the triple $(\mathcal{H}_{\mathbf{v}}(\mathcal{R}), \tau_{\mathbf{v}}, *)$ form a type I Hilbert algebra in the sense of Dixmier [7].

The spectral measure associated with $\tau_{\mathbf{v}}$ on $\mathcal{H}_{\mathbf{v}}$ is called the *Plancherel measure*. Its density is given by the so-called μ -function which we will review in next section.

2.2. The μ -function

Recall the root datum $\mathcal{R} = (X, R_0, F_0, Y, R_0^\vee, F_0^\vee)$ underlying $\mathcal{H}(\mathcal{R})$. Let \mathcal{T} be the diagonalisable group scheme with character lattice $X \times \mathbf{Z}$. We view \mathcal{T} as a group scheme over $\text{Spec } \Lambda = \mathbf{C}^\times$ via the homomorphism $\Lambda \rightarrow \mathbf{C}[X \times \mathbf{Z}]$ defined by $v^n \mapsto (0, n)$. The fibre at $\mathbf{v} \in \mathbf{C}^\times$ will be denoted by $\mathcal{T}_{\mathbf{v}}$. The function $\mu(R_0, v)$ associated with $\mathcal{H}(\mathcal{R}, m)$ is a rational function on \mathcal{T} with rational coefficient.

To define $\mu(R_0, v)$ we need to introduce some notations. Let $R := R_0^\vee \times \mathbf{Z}$. We define a function $\mathfrak{m}_R: R \rightarrow \mathbf{Z}$ by the rules that (i) \mathfrak{m}_R is \mathcal{W} -invariant and (ii) $\mathfrak{m}_R(a) = m(s)$ if the simple reflection s is conjugate to the reflection r_a induced by $a \in R$. This function \mathfrak{m}_R gives rise to the functions $\mathfrak{m}_\pm: \mathcal{W}_0 \setminus R_0 \rightarrow \frac{1}{2}\mathbf{Z}$ defined as

$$\mathfrak{m}_+(\alpha) = \frac{\mathfrak{m}_R(\alpha^\vee) + \mathfrak{m}_R(1 + \alpha^\vee)}{2}, \quad \mathfrak{m}_-(\alpha) = \frac{\mathfrak{m}_R(\alpha^\vee) - \mathfrak{m}_R(1 + \alpha^\vee)}{2} \tag{3}$$

One observes that $\mathfrak{m}_-(\cdot) = 0$ if the two reflections r_{α^\vee} and $r_{1+\alpha^\vee}$ are \mathcal{W} -conjugate. It is well-known that if $\mathfrak{m}_-(\cdot) \neq 0$, then R_0 contains a component of type B.

For each $\alpha \in R_0$, define

$$c_{\mathfrak{m}}(\alpha, v) := \frac{(1 - v^{-2\mathfrak{m}_+(\alpha)}\alpha^{-1})(1 + v^{-2\mathfrak{m}_-(\alpha)}\alpha^{-1})}{1 - \alpha^{-2}}. \tag{4}$$

We put $c_{\mathfrak{m}} := \prod_{\alpha \in R_0^+} c_{\mathfrak{m}}(\alpha, v)$. The Weyl group \mathcal{W}_0 acts on $c_{\mathfrak{m}}(\alpha)$, and hence on $c_{\mathfrak{m}}$, via its action on the roots. The μ -function of $\mathcal{H}(\mathcal{R}, m)$ is defined as

$$\mu(R_0, v) = v^{-2\mathfrak{m}_{\mathcal{W}}(w_0)} \frac{d(v)}{(w_0 \cdot c_{\mathfrak{m}})c_{\mathfrak{m}}} = \frac{d(v)}{v^{2\mathfrak{m}_{\mathcal{W}}(w_0)}} \prod_{\alpha \in R_0} \frac{1}{c_{\mathfrak{m}}(\alpha, v)} \tag{5}$$

where $w_0 \in \mathcal{W}_0$ is the longest element, $\mathfrak{m}_{\mathcal{W}}: \mathcal{W} \rightarrow \mathbf{Z}$ is defined by

$$\mathfrak{m}_{\mathcal{W}}(w) = \sum_{a \in R^+ \cap w^{-1}(R^-)} \mathfrak{m}_R(a),$$

and the factor $d(v)$ is a rational function in variable v . Clearly, the μ -function is \mathcal{W}_0 -invariant.

The explicit expression of $d(v)$ will be given in Section 3. Note that when the root system R_0 is empty, we have $\mu(\emptyset, v) = d(v)$. Here we point out that we shall regard $d(v)$ as a normalisation factor in the following sense: let $1 \in \mathcal{W}$ be the identity element of the extended affine Weyl group, then the functional $N_w \mapsto \delta_{w,1}d(v)$ defines a trace $\tau'(\cdot) := d(v)\tau(\cdot)$ on $\mathcal{H}(\mathcal{R}, m)$.

By abuse of language, we call $\mu_{\mathfrak{m},d}(R_0, \mathbf{v}) := \mu(R_0, v)|_{v=\mathbf{v}}$ the μ -function associated with $(\mathcal{H}_{\mathbf{v}}(\mathcal{R}, m), d)$, where $d := d(\mathbf{v})|_{v=\mathbf{v}}$.

2.3. Residual cosets and residual points

Let \mathcal{T} be the complex torus defined as in Section 2, so that the character lattice of \mathcal{T} is $X \times \mathbf{Z}$. Let $L \subset \mathcal{T}$ be a coset of a subtorus of \mathcal{T} . We define (where $\epsilon = \pm$ is a sign)

$$p_\epsilon(L) = \{\alpha \in R_0 : \epsilon\alpha|_L = v^{-2\mathfrak{m}_\epsilon(\alpha)}\}, \quad z_\epsilon(L) = \{\alpha \in R_0 : \epsilon\alpha|_L = 1\} \tag{6}$$

The coset $L \subset \mathcal{T}$ is called *residual* if

$$|p_+(L)| + |p_-(L)| - |z_+(L)| - |z_-(L)| = \text{codim}(L). \tag{7}$$

Suppose L is a residual coset for $\mu_{m,d}(R_0, v)$ with $p_\epsilon(L)$ and $z_\epsilon(L)$ as defined above, by the *regularisation of μ along L* we mean the following expression

$$\mu_{m,d}^L(R_0, v) := \frac{(d/v^{2m\nu(w_0)}) \prod_{\alpha \in R_0 \setminus z_-(L)} (1 + \alpha^{-1}) \prod_{\alpha \in R_0 \setminus z_+(L)} (1 - \alpha^{-1})}{\prod_{\alpha \in R_0 \setminus p_-(L)} (1 + v^{-2m_-(\alpha)} \alpha^{-1}) \prod_{\alpha \in R_0 \setminus p_+(L)} (1 - v^{-2m_+(\alpha)} \alpha^{-1})}. \tag{8}$$

The regularisation $\mu_{m,d}^L(R_0, v)$ restricting onto L defines a nonzero rational function $\mu_{m,d}^{(L)}(R_0, v)$ in v over the residual coset L . In particular, if $L = \{\mathbf{r}(v)\}$ is just a residual point with values in Λ , we call the expression $\mu_{m,d}^{\{\mathbf{r}\}}(R_0, v)$ the ‘‘residue’’ of the μ -function at the residual point \mathbf{r} , and we shall use the notation $\text{Res}(\mu_{m,d}(R_0, v), \mathbf{r})$ for $\mu_{m,d}^{\{\mathbf{r}\}}(R_0, v)$, to distinguish the role of the residual point \mathbf{r} . (Compare with the computation of residue of a one-variable meromorphic function at a simple pole.)

Recall that $T := \mathcal{T}_{\mathbf{v}}$ is the fibre of \mathcal{T} at $\mathbf{v} \in \mathbf{C}^\times$. Since T is a complex torus, it has a polar decomposition $T = T_u T_+$, where T_u is unitary and T_+ is the real vector group of T . For $\mathbf{r}(v) \in \mathcal{T}$, we usually write $s = s(\mathbf{v})$ for the unitary part of $\mathbf{r}(\mathbf{v})$.

3. The spectral transfer morphisms for affine Hecke algebras

Given two normalised affine Hecke algebras $(\mathcal{H}(\mathcal{R}_1, m_1), d_1)$ and $(\mathcal{H}(\mathcal{R}_2, m_2), d_2)$, let T_1, T_2 be the associated tori. Let $L = \mathbf{r}T^L \subset T_2$ be a residual coset, with $\mathbf{r} \in T_L$ a residual point. Here T^L is a subtorus of T and we denote the quotient T/T^L by T_L . Let $\mathcal{W}_{2,0}$ be the Weyl group associated with the root datum of \mathcal{R}_2 . Denote by $\mathcal{W}_{2,0}(L)$ the quotient group of the stabiliser of L in $\mathcal{W}_{2,0}$ by the pointwise stabiliser.

Put
$$K_L^n := T_L \cap T^L \cap \mathcal{W}_{2,0}(L), \quad L_n := L/K_L^n. \tag{9}$$

Definition 3.1. [17, Def. 5.1] Let $L \subset T_2$ be a residual coset and let L_n be given as in (9). A map $\phi_T: T_1 \rightarrow L_n$ is called a *spectral transfer map* if it satisfies the following conditions:

- (T1) ϕ_T is a finite morphism (in the sense of algebraic geometry);
- (T2) ϕ_T maps the identity e of T_1 into $(L \cap T_L)/K_L^n$, and if we declare $\phi_T(e)$ to be the base point of L (regarded as a T^L/K_L^n -torsor), then ϕ_T is a homomorphism of algebraic tori.
- (T3) There exists a constant $c \in \mathbf{Q}^\times$ such that $\phi_T^*(\mu_{m_2, d_2}^{(L)}(\mathcal{R}_2)) = c\mu_{m_1, d_1}(\mathcal{R}_1)$.

Remark 3.2. There is an extra defining condition (T4). However, to deal with unipotent representations we do not need it and hence we do not state it. In practice, the conditions (T1), (T2) and (T4) are usually easily verified. So the key point is the verification of (T3). (See also [17, Def. 5.1] and [19, Def. 4.3.1, Remark 4.3.2].)

Remark 3.3. In the present paper, the quotient L_n will in fact always be L itself, so the composition of these spectral transfer maps is simply given by the naive composition.

Here we give one elementary example of spectral transfer morphisms. For more examples of spectral transfer morphisms, the reader can consult [17, §7].

Example 3.4. Let \mathcal{W}_0 be the finite Weyl group associated to the root datum \mathcal{R} . Every $w \in \mathcal{W}_0$ acts on the associate complex torus as automorphism. Since the μ -function is \mathcal{W}_0 -invariant, this morphism clearly satisfies the condition (T3). These spectral transfer maps induced by elements in \mathcal{W}_0 form one equivalence class, which defines the identity spectral transfer morphism.

A surjective spectral transfer morphism is called a *spectral covering*. Examples of spectral coverings can be found in [17, 7.1.3]. However, the main examples in this paper all satisfy $\dim T_1 < \dim T_2$, and hence are not surjective.

4. Unipotent representations and affine Hecke algebras

4.1. Unipotent representations

We will define unipotent representations for $\mathbf{G}(k)$ and discuss the affine Hecke algebras derived from unipotent representations. Firstly, let us give the assumptions on the ground field k and on the linear algebraic group \mathbf{G} .

Let k be a non-archimedean local field with finite residue field \mathfrak{f} . Fix a separable closure k_s of k . Let $k_{nr} \subset k_s$ be the maximal unramified extension of k , with residual field $\overline{\mathfrak{F}}$, an algebraic closure of \mathfrak{f} . Recall that there exist isomorphisms of Galois groups: $\text{Gal}(k_{nr}/k) \simeq \text{Gal}(\overline{\mathfrak{F}}/\mathfrak{f}) \simeq \hat{\mathbf{Z}}$. The *geometric Frobenius element* Frob , whose *inverse* induces the automorphism $x \mapsto x^{|\mathfrak{f}|}$ for any $x \in \overline{\mathfrak{F}}$, is a topological generator of $\text{Gal}(k_{nr}/k)$.

We assume that \mathbf{G} is a connected absolutely simple algebraic group of adjoint type defined and quasi-split over k such that \mathbf{G} splits over k_{nr} . The Galois group $\text{Gal}(k_s/k)$ acts on $\mathbf{G}(k_s)$. Since \mathbf{G} is k_{nr} -split, this action factors through an action of $\text{Gal}(k_{nr}/k)$ on $G := \mathbf{G}(k_{nr})$. It turns out that the action of $\text{Gal}(k_{nr}/k)$ on G is completely determined by the action of Frob on it. We shall denote $F \in \text{Aut}(G)$ the automorphism induced by Frob .

Recall that the equivalence classes of inner forms of \mathbf{G} are parameterized by the Galois cohomology set $H^1(k, \mathbf{G}_{ad}) = H^1(k, \mathbf{G})$. By a theorem of Steinberg (cf. [24, Thm. 1.9]), we know that $H^1(\text{Gal}(k_s/k_{nr}), \mathbf{G})$ is trivial. We thus have an isomorphism $H^1(k, \mathbf{G}) \cong H^1(F, G) := H^1(\text{Gal}(k_{nr}/k), G)$. Let $z \in Z^1(F, G)$ be a cocycle and denote $u := z(\text{Frob}) \in G$. Let $F_u := \text{Ad}(u) \circ F$ be an inner twist of F by u . Then F_u -action on G defines a k -structure on G . If \mathbf{G}^u is the inner form of \mathbf{G} corresponding to u , then $\mathbf{G}^u(k) = G^{F_u}$. In particular, when $[z] = 1 \in H^1(F, G)$ is the privileged element, we have $\mathbf{G}(k) = G^F$.

We can find a parahoric subgroup $\mathcal{P} \subset G$ such that $F_u(\mathcal{P}) = \mathcal{P}$. Let \mathcal{P}_+ be the pro-unipotent radical of \mathcal{P} . The quotient $\mathcal{P}/\mathcal{P}_+$ is reductive. Since \mathcal{P}_+ is connected, we can apply the pro-algebraic version of Lang's theorem and obtain $(\mathcal{P}/\mathcal{P}_+)^{F_u} = \mathcal{P}^{F_u}/\mathcal{P}_+^{F_u}$, which is a finite reductive group. We shall denote this finite group by $\overline{\mathcal{P}}^{F_u}$.

A smooth representation π of G^{F_u} is called *unipotent*, if there exists an F_u -stable parahoric subgroup \mathcal{P} of G , with a cuspidal unipotent representation σ of $\overline{\mathcal{P}}^{F_u}$, such that the $\mathcal{P}_+^{F_u}$ -invariants of π contain σ . (Here we lift σ to a representation of \mathcal{P}^{F_u} via the natural projection $\mathcal{P}^{F_u} \rightarrow \overline{\mathcal{P}}^{F_u}$). The pair $\mathfrak{s} := (\mathcal{P}^{F_u}, \sigma)$ is a *type* of G^{F_u} in the sense of Bushnell–Kutzko [3].

Definition 4.1. Let $\mathcal{P} \subset G$ be a F_u -stable parabolic subgroup, and let σ be a cuspidal unipotent representation of $\overline{\mathcal{P}}^{F_u}$. We call the pair $\mathfrak{s} = (\mathcal{P}^{F_u}, \sigma)$ a *unipotent type* of G^{F_u} , and denote by $\text{Irr}(G^{F_u}, \mathfrak{s})$ the totality of isomorphism classes of irreducible unipotent representations of G^{F_u} attached to a unipotent type $\mathfrak{s} = (\mathcal{P}^{F_u}, \sigma)$ of G^{F_u} .

4.2. Unipotent affine Hecke algebras

We fix a unipotent type $\mathfrak{s} = (\mathcal{P}^{F_u}, \sigma)$ of G^{F_u} in this section. Define a representation $\text{c-Ind}_{\mathcal{P}^{F_u}}^{G^{F_u}} \sigma$ of G^{F_u} . Let V be the vector space of this representation. The endomorphism algebra $\text{End}(V)$ is isomorphic to the $(\mathcal{P}^{F_u}, \sigma)$ -spherical Hecke algebra of G^{F_u} . We denote this $(\mathcal{P}^{F_u}, \sigma)$ -spherical Hecke algebra by $\mathcal{H}_{\mathfrak{s}}$. Let $\text{Irr}(\mathcal{H}_{\mathfrak{s}})$ be the set of isomorphism classes of (finite dimensional) simple $\mathcal{H}_{\mathfrak{s}}$ -modules. From the work of Moy–Prasad [15], Morris [13] and Lusztig [11], we know that

- Theorem 4.2.** (i) *There is a natural bijection between the sets $\text{Irr}(G^{F_u}, \mathfrak{s})$ and $\text{Irr}(\mathcal{H}_{\mathfrak{s}})$.*
 (ii) *Two sets $\text{Irr}(G^{F_u}, \mathfrak{s})$ and $\text{Irr}(G^{F_u}, \mathfrak{s}')$ are either disjoint or identical. They are equal if and only if there exists an element $g \in G^{F_u}$ which conjugates \mathcal{P} to \mathcal{P}' , and σ to a representation isomorphic to σ' .*

The structure of $\mathcal{H}_{\mathfrak{s}}$ has been determined by Lusztig [11]. Briefly, $\mathcal{H}_{\mathfrak{s}}$ is the tensor product of an affine Hecke algebra with the group algebra of a finite abelian group. This finite abelian group can be described in terms of the group of diagram automorphisms of the affine Dynkin diagram $\text{Dyn}(\Sigma_0^{(1)})$ of G . Here $\Sigma_0^{(1)}$ is the affine root system attached to \mathbf{G} .

Let $\mathcal{I} \subset \mathcal{P}$ be an Iwahori subgroup of G . Define a group $\Omega := N_G(\mathcal{I})/\mathcal{I}$. It is known that Ω is finite and abelian (cf. [6]). After conjugation (if necessary), we may and will assume that \mathcal{I} is standard. Thus, the group Ω acts on $\text{Dyn}(\Sigma_0^{(1)})$ of \mathbf{G} as diagram automorphisms.

Let $\tilde{\mathcal{I}}$ denote the set of all nodes of $\text{Dyn}(\Sigma_0^{(1)})$. Suppose that \mathcal{P} in the type $\mathfrak{s} = (\mathcal{P}^{F_u}, \sigma)$ corresponds to a subset $\mathcal{J} \subset \tilde{\mathcal{I}}$. The Galois group $\text{Gal}(k_s/k)$ acts on $\text{Dyn}(\Sigma_0^{(1)})$. This action factors through the action of $\text{Gal}(k_{nr}/k)$, because \mathbf{G} is k_{nr} -split. Denote by θ the diagram automorphism of $\text{Dyn}(\Sigma_0^{(1)})$ induced by F (the automorphism of G induced by Frob). We also have the twisted group θ_u , induced by F_u . Since \mathcal{P} is F_u -stable, we see that θ_u preserves \mathcal{J} .

Furthermore, let Ω^{θ_u} denotes the elements in Ω commuting with θ_u . Since Ω is abelian, we thus have $\Omega^{\theta_u} = \Omega^{\theta}$. The action of Ω on $\text{Dyn}(\Sigma_0^{(1)})$ induces an action of Ω^{θ} on the θ_u -orbits of this Dynkin diagram. Denote by $\Omega^{\theta}(\mathcal{P})$ the isotropy group of \mathcal{P} . Then $\Omega^{\theta}(\mathcal{P})$ acts on the set $(\tilde{\mathcal{I}} - \mathcal{J})/\theta_u$ of θ_u -orbits as permutations. Let $\Omega_1^{\theta}(\mathcal{P})$ be the pointwise stabiliser of $(\tilde{\mathcal{I}} - \mathcal{J})/\theta_u$, and let $\Omega_2^{\theta}(\mathcal{P}) := \Omega^{\theta}(\mathcal{P})/\Omega_1^{\theta}(\mathcal{P})$.

Lusztig [11, 1.18, 1.19, 1.20] described the structure of $\mathcal{H}_{\mathfrak{s}}$. It decomposes into a direct sum decomposition of two-sided ideals \mathcal{H}^{ψ} such that every ideal \mathcal{H}^{ψ} is isomorphic to the other, and they are parameterised by irreducible characters ψ of $\Omega_1^{\theta}(\mathcal{P})$. Moreover, \mathcal{H}^{ψ} is isomorphic to an extended affine Hecke algebra which is the smash product of an unextended affine Hecke algebra \mathcal{H}^a and $\Omega_2^{\theta}(\mathcal{P})$. More precisely, Lusztig constructed an isomorphism between \mathcal{H}^{ψ} and an affine Hecke algebra obtained as a generic affine Hecke algebra specialised at $v = |\mathfrak{f}|^{1/2}$, and thus gave the Iwahori-Matsumoto presentation of \mathcal{H}^{ψ} for all types of simple algebraic

groups which are inner forms of split groups. Associated to \mathcal{H}^ψ is an affine Weyl group $W_{\mathfrak{s}}$ with a set $S_{\mathfrak{s}}$ of distinguished generators. The structure of $W_{\mathfrak{s}}$ and the parameter function $m_{\mathfrak{s}}$ can be entirely determined by the data of \mathbf{G} and \mathfrak{s} (cf. [14]). However, there is no canonical way to choose the set $S_{\mathfrak{s}}$. But different choices of $S_{\mathfrak{s}}$ give us algebras which are essentially the same, in the sense that there exist admissible isomorphisms (cf. [17, 2.1.7] and [18, text around Eq. (19)]) among them.

The following proposition is a consequence of the theory of *types* due to [3]. It is an enhancement of the bijection (i) in Proposition 4.2. The claim of bijection is proved at the end of [15] as an equivalence of two categories of representations. Using the main theorem in [17] we obtain the other claim.

Theorem 4.3. *Let $\mathfrak{s} = (\mathcal{P}^{F_u}, \sigma)$ be a unipotent type of G^{F_u} . The bijection between $\text{Irr}(G^{F_u}, \mathfrak{s})$ and $\text{Irr}(\mathcal{H}_{\mathfrak{s}})$ respects the notion of temperedness, and preserves the Plancherel measure on the level of irreducible tempered representations.*

We call \mathcal{H}^ψ a unipotent affine Hecke algebra, and denote it by \mathcal{H}_{upt} . In particular, we can define discrete series representations for unipotent affine Hecke algebras, and study their formal degrees in place of the formal degrees of unipotent discrete series representations of G^F .

In Section 4, we will discuss the parameters for \mathcal{H}_{upt} when \mathbf{G} is isogenous to a classical group (except for the special linear group).

4.3. Cuspidal unipotent characters and the normalisation factor d

From [4, §13.7] we know that cuspidal unipotent characters are rare for classical finite groups of Lie type. The root system type must verify selective conditions. For instance, an odd orthogonal group has one cuspidal unipotent character only if the rank is of the form $n(n + 1)$ for some positive integer n , and has no such character otherwise. Therefore, we can determine the subset $J \subset \tilde{I}$ corresponding to \mathcal{P} such that $\overline{\mathcal{P}^{F_u}}$ has a cuspidal unipotent character. Consequently, we are able to explicitly define the normalisation factor $d(\mathbf{v})$ of the μ -function of \mathcal{H}_{upt} . We proceed as follows.

We normalise the Haar measure on $\mathbf{G}(k)$ as [6], then the volume of a parahoric subgroup \mathcal{P}^{F_u} equals to

$$\text{Vol}(\mathcal{P}^{F_u}) = \mathbf{v}^{-a} |\overline{\mathcal{P}^{F_u}}|, \tag{10}$$

where \mathbf{v} is the positive square root of $|f|$, a is the rank of \mathbf{G} regarded as an algebraic group defined over a separable closure k_s of k , and $\overline{\mathcal{P}_k}$ is the reductive quotient of \mathcal{P}_k by its pro-unipotent radical $\mathcal{P}_{k,+}$. So $\overline{\mathcal{P}_k}$ is a finite group of Lie type. Using the results in [4, §2.9] we can determine $|\overline{\mathcal{P}_k}|$, the order of $\overline{\mathcal{P}_k}$.

For a unipotent discrete series representation belongs to $\text{Irr}_{\text{upt}}(G^{F_u}, \mathfrak{s})$ with $\mathfrak{s} = (\mathcal{P}^{F_u}, \sigma)$, via the analysis in [18, §2.4.1], the normalisation factor $d = d(\mathbf{v})$ is given by [18, Eq. (24)]

$$d(\mathbf{v}) = |\Omega_1^{\theta}(\mathcal{P})|^{-1} \text{Vol}(\mathcal{P}^{F_u})^{-1} \text{deg}(\sigma). \tag{11}$$

One can check case-by-case that $d(\mathbf{v})$ satisfies that $d(\mathbf{v}) = d(\mathbf{v}^{-1})$.

Definition 4.4. We define $d(v)$ to be the rational function with rational coefficients such that the value of $d(v)$ at $v = \mathbf{v}$ is equal to $d(\mathbf{v})$ given by Eq. (11). This rational function $d(v)$ will be regarded as the normalisation factor of the μ -function $\mu(R_0, v)$.

4.4. The case of non-split but quasi-split exceptional groups

Let \mathbf{G} be an adjoint k -group of type ${}^3\mathbf{D}_4$ or ${}^2\mathbf{E}_6$. In this section, we show that for a given supercuspidal unipotent representation π of $\mathbf{G}(k)$, there is only one Weyl group orbit $\mathcal{W}_0\mathbf{r}$ of residual point verifying the following equation

$$\text{fdeg}(\pi, q) = C_\pi \text{Res}(\mu^{\text{IM}}, \mathbf{r}) \tag{12}$$

for some constant $C_\pi \in \mathbf{Q}^\times$ independent of q . (Here $q = v^2$ is a variable, and Eq. (12) is understood as an equation between rational functions in variable q .) Thus Definition 3.1 (T3) implies that $\mathcal{W}_0\mathbf{r}$ is the only orbit of Λ -valued points eligible to define a spectral transfer morphism from \mathcal{H}_{upt} to \mathcal{H}^{IM} . It can be easily checked that $\mathcal{W}_0\mathbf{r}$ indeed defines such a spectral transfer morphism.

The group of type ${}^3\mathbf{D}_4$

Let \mathbf{G} be an adjoint k -group of type ${}^3\mathbf{D}_4$. Under the Frobenius action, the relative root system is \mathbf{G}_2 . So the underlying root system of the unipotent affine Hecke algebra is $R_0 = \mathbf{G}_2$. Let us denote by $F_0 = \{\alpha_1, \alpha_2\}$ the base of R_0 , where α_1 is a short root and α_2 is a long root. In terms of [17, (4) and (8)], the parameters of the associated μ -function are given as follows: $m_+(\alpha_1) = 1, m_+(\alpha_2) = 3$. The associated Iwahori-Hecke algebra is denoted as $\mathbf{G}_2(3, 1)[q]$ in notations of [18, 3.2.1], and the normalisation factor $d^{\text{IM}}(v)$ in the μ^{IM} -function is given explicitly as

$$d^{\text{IM}}(v) = \frac{v - v^{-1}}{v^3 - v^{-3}} \frac{1}{(v - v^{-1})^2} = \frac{1}{(v - v^{-1})(v^3 - v^{-3})} = \frac{q^2}{(q - 1)(q^3 - 1)}.$$

The only conjugacy class of maximal parahoric subgroups \mathcal{P}_k of $\mathbf{G}(k)$ which support a cuspidal unipotent representation has root system type ${}^3\mathbf{D}_4$. From [4, §13.7] we know that there are in total two cuspidal unipotent characters of the finite reductive group of type ${}^3\mathbf{D}_4$. They are denoted by ${}^3\mathbf{D}_4[1]$ and ${}^3\mathbf{D}_4[-1]$ (see the right-most column $\mathcal{U}_{\text{cusp}}$ at Table 1). The algebra $\mathbf{G}_2(3, 1)[q]$ has four orbits of residual orbits. Among them, we find two residual orbits with representatives $\mathbf{r}_1 = (1, q)$ and $\mathbf{r}_2 = (-q^{-3}, q)$ verify Eq. (12) for these two cuspidal unipotent characters respectively. These are the only residual orbits of $\mathbf{G}_2(3, 1)[q]$ supporting supercuspidal unipotent representations. In terms of spectral transfer morphism, for irreducible supercuspidal unipotent representations of $\mathbf{G} = {}^3\mathbf{D}_4$ occurring as subquotient of the representation induced by the cuspidal unipotent character ${}^3\mathbf{D}_4[1]$ (resp. by ${}^3\mathbf{D}_4[-1]$), the associated spectral transfer morphism is determined uniquely by the condition that the image is the \mathcal{W}_0 -orbit of $\mathbf{r}_1 = (1, q)$ (resp. $\mathbf{r}_2 = (-q^{-3}, q)$).

Table 1: Cuspidal unipotent characters of ${}^3\mathbf{D}_4$.

s	\mathbf{r}	$\text{Res}(\mu_{q'}^{\text{IM}}, \mathbf{r})$	$\mathcal{U}_{\text{cusp}}$
\mathbf{G}_2	(q^3, q)	$(\Phi_5\Phi_9)/(\Phi_2^2\Phi_3^2\Phi_6^2\Phi_{12})$	
$\mathbf{G}_2(a_1)$	$(1, q)$	$(\Phi_2^2\Phi_3^2\Phi_6^2)^{-1}$	${}^3\mathbf{D}_4[1]$
\mathbf{A}_2	$(q^3, \vartheta q^{-3})$	$\Phi_3^{-2}\Phi_6^{-2}\Phi_9\Phi_{12}^{-1}$	
$\mathbf{A}_1\mathbf{A}_1$	$(-q^{-3}, q)$	$(\Phi_2^2\Phi_6^2\Phi_{12})^{-1}$	${}^3\mathbf{D}_4[-1]$

The group of type 2E_6

Now let \mathbf{G} be an adjoint k -group of type 2E_6 . Under the Frobenius action we have a relative root system of type F_4 . So the root system of the unipotent affine Hecke algebra is $R_0 = F_4$. The parameter function m_+ takes value 2 (resp. 1) at long (resp. short) roots in $R_0 = F_4$. The Iwahori–Hecke algebra is denoted as $F_4(2, 1)[q]$ in notations of [18, 3.2.1]. The normalisation factor for the μ^{IM} -function is

$$d = (v - v^{-1})^{-2}(v^2 - v^{-2})^{-2} = q^{-3}(q - 1)^{-2}(q^2 - 1)^{-2},$$

and there are eight residual orbits. The only conjugacy class of maximal parahoric subgroups \mathcal{P}_k of $\mathbf{G}(k)$ which support a cuspidal unipotent representation has root system type 2E_6 . From [4, §13.7] we know that there are in total three cuspidal unipotent characters of the finite reductive group of type 2E_6 . They are denoted as ${}^2E_6[1]$, ${}^2E_6[\theta]$ and ${}^2E_6[\theta^2]$ where θ is a primitive third root of unity (see the right-most column at Table 2). Note that two cuspidal unipotent characters ${}^2E_6[\theta]$ and ${}^2E_6[\theta^2]$ have the same degree.

Table 2: Cuspidal unipotent characters of 2E_6 .

s	\mathbf{r}	$\text{Res}(\mu_{q'}^{\text{IM}}, \mathbf{r})$	$\mathcal{U}_{\text{cusp}}$
F_4	(q^2, q^2, q, q)	$\frac{\Phi_5\Phi_7\Phi_{11}\Phi_{16}}{\Phi_2^6\Phi_3^2\Phi_4^2\Phi_6^3\Phi_{10}\Phi_{12}\Phi_{18}}$	
$F_4(a_1)$	$(q^2, 1, q, q)$	$\Phi_2^{-6}\Phi_3^{-2}\Phi_4^{-2}\Phi_5^2\Phi_6^{-3}\Phi_7\Phi_{12}^{-1}$	
$F_4(a_2)$	$(q^2, 1, q, 1)$	$\Phi_2^{-6}\Phi_3^{-2}\Phi_4^2\Phi_5\Phi_6^{-3}\Phi_{10}^{-1}$	
$F_4(a_3)$	$(1, 1, q, 1)$	$(\Phi_2^6\Phi_3^2\Phi_4^2\Phi_6^3)^{-1}$	${}^2E_6[1]$
A_1C_3	$(-q^{-7}, q^2, q, q)$	$\Phi_2^{-6}\Phi_5\Phi_6^{-3}\Phi_{18}^{-1}$	
A_2A_2	$(q^2, \theta q^{-4}, q, q)$	$(\Phi_3^2\Phi_6^3\Phi_{12}\Phi_{18})^{-1}$	${}^2E_6[\theta], {}^2E_6[\theta^2]$
A_3A_1	$(q^2, q^2, \sqrt{-1}q^{7/2}, q)$	$\Phi_2^{-6}\Phi_4\Phi_6^{-3}\Phi_8\Phi_{10}^{-1}\Phi_{18}^{-1}$	
$B_4(1)$	$(q^2, q^2, q, -q^{-8})$	$\Phi_2^{-6}\Phi_4^{-2}\Phi_5\Phi_6^{-3}\Phi_7\Phi_{12}^{-1}\Phi_{16}\Phi_{18}^{-1}$	
$B_4(2)$	$(q^2, 1, q, -q^{-5})$	$\Phi_2^{-6}\Phi_4^{-2}\Phi_5\Phi_6^{-3}\Phi_8\Phi_{10}^{-1}\Phi_{12}^{-1}$	

There are 24 positive roots in F_4 . With the help of Maple, we obtain Table 2. We see that two orbits with representatives $\mathbf{r}_1 = (1, 1, q, 1)$ and $\mathbf{r}_2 = (q^2, \theta q^{-4}, q, q)$ verify Eq. (12) for these three cuspidal unipotent characters respectively. These are the only residual orbits of $F_4(2, 1)[q]$ supporting supercuspidal unipotent representations. In terms of spectral transfer morphism, for each irreducible supercuspidal unipotent representation of $\mathbf{G} = {}^2E_6$ occurring as subquotient of the representation induced by the cuspidal unipotent character ${}^2E_6[1]$ (resp. by ${}^2E_6[\theta]$ or ${}^2E_6[\theta^2]$ equivalently), the associated spectral transfer morphism is determined uniquely by the condition that the image is the \mathcal{W}_0 -orbit of $\mathbf{r}_1 = (1, 1, q, 1)$ (resp. $\mathbf{r}_2 = (q^2, \theta q^{-4}, q, q)$).

Remark 4.5. The term s at the left-most column of Tables 1 and 2 has two-fold meaning. Firstly, it is the unitary part of the residual point $\mathbf{r} \in T$, so s must be a torsion element in G^\vee (see text around Eq. (1)). Secondly, from the classification of torsion elements in complex algebraic groups, we know that s corresponds to a node in the Kac diagram (cf. [22, §3.8 and Table 1]). We thus indicate s by the

root system type of the complement of this node. The bijection mentioned in [16, Prop. B.1] shows that these two interpretations of s are compatible.

4.5. The case of classical groups

We still need to determine the parameters $\mathfrak{m}_\pm(\alpha)$ for each \mathcal{W}_0 -conjugacy classes of roots $\alpha \in R_0$. For classical groups, these parameters can be given via another two closely related parameters m_\pm such that $4m_\pm \in \mathbf{Z}$ and $2(m_- \pm m_+) \in \mathbf{Z}$. These parameters m_\pm are determined from two natural numbers a and b coming from the condition that $\overline{\mathcal{P}}$ has a cuspidal unipotent character. The relationship between $\{a, b\}$ and $\{m_-, m_+\}$ (here we understand them as multi-sets, i.e. the elements can be equal) are explicitly given by [18, 3.2.1]. An important fact is, the data $\{a, b\}$ come from the representation (or arithmetic) side, while the data $\{m_-, m_+\}$ come from the spectral (or geometry) side, and the correspondence of them should be viewed as a correspondence satisfying Langlands’ philosophy.

Let \mathcal{V} be the parameter space of the pair $\{m_-, m_+\}$. Recall that we have $4m_\pm \in \mathbf{Z}$. We partition \mathcal{V} into six disjoint subsets as follows.

$$\begin{aligned}
 \{m_-, m_+\} \in \mathcal{V}^{\text{I}} & \quad \text{iff } m_\pm \in \mathbf{Z}/2 \text{ and } m_- - m_+ \notin \mathbf{Z}, \\
 \{m_-, m_+\} \in \mathcal{V}^{\text{II}} & \quad \text{iff } m_\pm \in \mathbf{Z} + 1/2 \text{ and } m_- - m_+ \in \mathbf{Z}, \\
 \{m_-, m_+\} \in \mathcal{V}^{\text{III}} & \quad \text{iff } m_\pm \in \mathbf{Z} \text{ and } m_- - m_+ \notin 2\mathbf{Z}, \\
 \{m_-, m_+\} \in \mathcal{V}^{\text{IV}} & \quad \text{iff } m_\pm \in \mathbf{Z} \text{ and } m_- - m_+ \in 2\mathbf{Z}, \\
 \{m_-, m_+\} \in \mathcal{V}^{\text{V}} & \quad \text{iff } m_\pm \in \mathbf{Z} \pm 1/4 \text{ and } \delta_- \neq \delta_+, \\
 \{m_-, m_+\} \in \mathcal{V}^{\text{VI}} & \quad \text{iff } m_\pm \in \mathbf{Z} \pm 1/4 \text{ and } \delta_- = \delta_+.
 \end{aligned}
 \tag{13}$$

The numbers δ_\pm appearing in cases V and VI are determined by the following method. If $m \in \mathbf{Z} \pm 1/4$, we write

$$|m| = \kappa + \frac{2\epsilon - 1}{4},
 \tag{14}$$

where $\epsilon \in \{0, 1\}$ and $\kappa \in \mathbf{Z}_{\geq 0}$. The integer κ determines a number $\delta \in \{0, 1\}$ by the rule that $\kappa - \delta \in 2\mathbf{Z}$. In other words, δ indicates the parity of κ .

Remark 4.6. The isogeny classes of the corresponding classical groups are: unitary groups in case I; odd special orthogonal groups in case II; symplectic groups in cases III and V; even special orthogonal groups in cases IV and VI. Cases V and VI correspond to the Hecke algebras of unipotent types of an inner twists of \mathbf{G} by elements of \mathbf{G}_{ad} which are not in the image of any nontrivial central isogeny. Their irreducible representations correspond to enhancements of Langlands parameters on which the centre of the spin group acts nontrivially. Following [18], we call them the *extra-special* cases.

Finally, the parameters $\mathfrak{m}_\pm(\alpha)$ are obtained from m_\pm by the following rules: Observe that from Lusztig’s list [11, §7] we can assume that the relevant root system R_0 of the Iwahori–Hecke algebra always have type B. Let $(t_i)_{1 \leq i \leq l}$ be the standard basis of the character lattice which contains B_l . We write the roots multiplicatively. Then:

$$\mathfrak{m}_-(t_i t_j^\pm) = 0, \mathfrak{m}_+(t_i t_j^\pm) = \mathfrak{b}; \mathfrak{m}_-(t_i) = \mathfrak{b}m_-, \mathfrak{m}_+(t_i) = \mathfrak{b}m_+,
 \tag{15}$$

where $\mathfrak{b} = 1$ if both $m_+ \pm m_- \in \mathbf{Z}$, and $\mathfrak{b} = 2$ else.

4.6. Three morphisms

We now look at three morphisms which are from unipotent affine Hecke algebras associated with *classical* groups. Opdam defined them in [18, 3.2.6] and claimed there that these morphisms are spectral transfer morphisms. We now write out their definitions and prove that they are indeed spectral transfer morphisms. In what follows, we will write T^n to indicate that the rank of the algebraic torus T is n .

- (1) For $m_{\pm} \in \mathbf{Z} + 1/2$, define a homomorphism $\phi_{T,m_-,m_+} : T^n \rightarrow T^{n+m_+-1/2}$ of algebraic tori over \mathbf{C} by

$$\phi_{T,m_-,m_+}(t_1, \dots, t_n) = (t_1, \dots, t_n, v^b, v^{3b}, \dots, v^{2b(m_+-1)}). \tag{16}$$

- (2) For $m_+ \in \mathbf{Z}_{>0}$, define a homomorphism $\psi_{T,m_-,m_+} : T^n \rightarrow T^{n+2m_+-2}$ of algebraic tori over \mathbf{C} by $\psi_{T,m_-,m_+}(t_1, \dots, t_n) =$

$$= (t_1, \dots, t_n, 1, q^b, q^b, q^{2b}, q^{2b}, \dots, q^{b(m_+-2)}, q^{b(m_+-2)}, q^{b(m_+-1)}). \tag{17}$$

- (3) (The extra-special cases \mathcal{V}^V and \mathcal{V}^{VI}). For $m_{\pm} > 0$, set

$$l := \begin{cases} 2n + (a/2)(a + 1) + 2b(b + 1) & \text{if } \mathcal{X} = V, \\ 2n + (a/2)(a + 1) + 2b^2 - \delta_+ & \text{if } \mathcal{X} = VI. \end{cases} \tag{18}$$

Let $\kappa_{\pm}, \epsilon_{\pm}$ be defined as in (14). If we let $l_{\pm} := \kappa_{\pm}(\kappa_{\pm} + \epsilon_{\pm} - 1/2)$, then $l = 2n + [l_-] + [l_+]$ in both parameter types V, VI. For $m \in \mathbf{Z} \pm \frac{1}{4}$ and $m > 1$, let

$$\sigma_e(m) = (q^{\delta}, q^{\delta+1}, \dots, q^{2m-\frac{3}{2}}).$$

Define the residual points $r_e(m)$ recursively by putting

$$r_e\left(\frac{1}{4}\right) = r_e\left(\frac{3}{4}\right) := \emptyset; \quad r_e(m) = (\sigma_e(m); r_e(m - 1)) \text{ if } m > 1.$$

Finally we define the representing morphism $\xi_{T,m_-,m_+} : T^n \rightarrow T^l$ of the extra-special STM by

$$\xi_{T,m_-,m_+}(t_1, \dots, t_n) = (-r_e(m_-), v^{-1}t_1, vt_1, \dots, v^{-1}t_n, vt_n, r_e(m_+)). \tag{19}$$

It is easy to verify the defining conditions (T1)–(T3) for the first two families of morphisms. However, for the extra-special cases the verification of condition (T3) is much more complicated. Our main result is the following theorem.

Theorem 4.7. *The morphisms ϕ_{T,m_-,m_+} (16), ψ_{T,m_-,m_+} (17) and ξ_{T,m_-,m_+} (19) are spectral transfer morphisms.*

5. The proofs

We will prove Theorem 4.7 in this section. The conditions (T1), (T2) are easy to verify, and we shall only verify condition (T3). A key idea to the proof is the factorisation of the μ -function by a parabolic root sub-system (cf. Section 5). We shall interpret this parabolic factorisation in terms of Lusztig’s classification of unipotent representations.

5.1. Lusztig’s classification and the parabolic factorisation of the μ -function

Recall that Lusztig’s classification [11] is achieved by matching the Iwahori-Matsumoto presentations of two affine Hecke algebras. One of these affine Hecke algebras is $\mathcal{H}_s = \mathcal{H}_{\text{upt}}(\mathcal{P}^{F_u}, \sigma)$, derived from the irreducible unipotent representation $\pi \in \text{Irr}(\mathbf{G}(k), \mathcal{P}^{F_u}, \sigma)$ (as we have seen in §2.1). The other one, here we denote it by $\mathbf{H} := \mathbf{H}(s, \mathcal{N}, \mathcal{F})$, is obtained by considering cuspidal local systems \mathcal{F} on unipotent orbits \mathcal{N} in a connected complex reductive group $M := Z_{G^\vee}(s)^\circ$ where $s \in {}^L G$ is semisimple and G^\vee is the complex dual group of G^{F_u} . The conclusion is, π is corresponding to the data $(s, \mathcal{N}, \mathcal{F})$ if and only if the two affine Hecke algebras $\mathbf{H}(s, \mathcal{N}, \mathcal{F})$ and $\mathcal{H}_{\text{upt}}(\mathcal{P}^{F_u}, \sigma)$ have the same Iwahori–Matsumoto presentation.

Let I denote a set of simple roots of G^\vee , and let $J \subset I$ be the subset corresponding to the reductive subgroup M . One can establish a bijection between $K = I - J$ and the simple roots in the underlying root system of \mathbf{H} . (The case that $K = \emptyset$ is very simple, and we shall not consider this case here.) More precisely, starting from the root systems of types I and J , one can determine a unique reduced root system whose base is in bijection with K . This reduced root system is the underlying root system of \mathbf{H} . Then, the Iwahori-Matsumoto presentation of \mathbf{H} is determined, with the explicit expression of the parameters given in [11, 5.12]. The same procedure applies to $(\tilde{I} - J)/\theta_u$ and hence the Iwahori-Matsumoto presentation of $\mathcal{H}_{\text{upt}}(\mathcal{P}^{F_u}, \sigma)$ follows. (See [11, 1.17–1.20].) In particular, via Lusztig’s classification, we obtain a bijection between K and the set $(\tilde{I} - J)/\theta_u$ of θ_u -orbits of $(\tilde{I} - J)$.

Now we turn to spectral transfer morphism. We consider the following morphism:

$$\text{Spec } Z(\mathcal{H}_{\text{upt}}(\mathcal{P}^{F_u}, \sigma)) \rightarrow \text{Spec } Z(\mathbf{H}_I),$$

where \mathbf{H}_I is an affine Hecke algebra attached to the root system I (and given by the same recipe as the derivation of \mathbf{H}). By Lusztig’s classification, $\mathcal{H}_{\text{upt}}(\mathcal{P}^{F_u}, \sigma)$ can be replaced by \mathbf{H} , so we can turn to consider a morphism between complex tori

$$\mathcal{S}: T^K \rightarrow T^I,$$

where T_K (resp. T^I) is the complex torus associated with \mathbf{H} (resp. \mathbf{H}_I). Let T^J be a subtorus of T^I determined by the subset $J \subset I$ (that is, if a root $\alpha \in \mathbf{R}J \cap I$, then $\alpha|_{T^J}$ is constant). We now interpret the image of \mathcal{S} as a coset L of T^J . In order that \mathcal{S} to be a spectral transfer morphism, we must choose an appropriate base point $\mathbf{r} \in T^J \cap (T^I/T^J)$ of L to verify the condition (T3). To this end, we are led to the conclusion that the unitary part of \mathbf{r} and the semisimple element s in the triple $(s, \mathcal{N}, \mathcal{F})$ must be \mathcal{W}_0 -conjugate (cf. [16, Appendix B], [21, 18, 8] and also [9, Conj. 1.5]).

In this regard, the three morphisms (16), (17) and (19) explicitly give the coordinates of the point \mathbf{r} , as a residual point of the μ -function attached to \mathbf{H}_I . Further, those coordinates in \mathbf{r} not involving with the original coordinates in T_K , give rise to a residual point of the root system of the type of J (in the sense of [20]).

In general, suppose $L \subset T$ is a coset of a subtorus $T^L \subset T$. The natural projection $T \rightarrow T_L := T/T^L$ corresponds to a sublattice $X_L \subset X$. Then L is a principal homogeneous space of T^L . Suppose \mathbf{r} is the base point of L , then we can write $L = \mathbf{r}T^L$.

The intersection $R_L := R_0 \cap X_L$ is a parabolic root sub-system of R_0 . If necessary, we use an appropriate element in \mathcal{W}_0 to map R_L to a standard position, then, we can choose a base J of R_L which is a subset of the base F_0 of R_0 . We have a root datum $\mathcal{R}_L := (X^*(T_L), R_L, X_*(T_L), R_L^\vee)$ and a parabolic subgroup \mathcal{W}_0^L of \mathcal{W}_0 . (In fact, the root datum \mathcal{R}_L and the Weyl group \mathcal{W}_0^L only depend on the subset J of F_0 .) Consequently, we can decompose the μ -function as

$$\mu_{\mathfrak{m},d}^{(L)}(\mathcal{R}) = \mu_{\mathfrak{m},d}^{\{\mathfrak{r}\}}(\mathcal{R}_L) \frac{v^{-2\mathfrak{m}_{\mathcal{W}_0}(w^L)}}{\prod_{\alpha \in R_{0,+} \setminus R_{L,+}} (c_{\mathfrak{m}}(\alpha)c_{\mathfrak{m}}(\alpha^{w^L}))}, \tag{20}$$

where $w^L := w_0 w_L^{-1} \in \mathcal{W}^L \subset \mathcal{W}$ is the longest element.

5.2. Conventions and the proofs

Before the proof, we have some conventions. Given a nonzero rational number x we denote by $\epsilon(x) = x/|x| \in \{\pm 1\}$ its sign. Recall that the normalisation factor $d(v)$, regarded as a rational function in v , satisfies $d(v) = d(v^{-1})$, and hence so does the μ -function. Thus, we do not need to care about the minus signs. Moreover, we will neglect the powers of q , because we are only interested in the q' -part. Consequently, we can replace every expression $1 - q^{-a}$ by $1 - q^a$. Accordingly, we can rewrite the μ -function as

$$\mu_{m_{\pm},d}(R) = \prod_{\alpha \in R_+} \frac{d(1 - \alpha(t))^2}{(1+v^{2m_- - \alpha(t)})(1+v^{-2m_- - \alpha(t)})(1-v^{2m_+ + \alpha(t)})(1-v^{-2m_+ + \alpha(t)})} \tag{21}$$

To simplify the notations, in the proofs below, we write $\text{Res}(\mu, \mathfrak{r})$ for $\mu^{\{\mathfrak{r}\}}(\mathfrak{r})$.

The involutions

$$\eta_+ : m_- \mapsto -m_+, m_+ \mapsto m_-, \quad \eta : \text{interchanging } m_+ \text{ and } m_-$$

act on the μ -functions, inducing isomorphisms of normalised affine Hecke algebras. The group $\mathbf{Iso} = \langle \eta_+, \eta \rangle$ is isomorphic to the dihedral group of order eight, and it is called the spectral isomorphism group (cf. [17, §7]). By virtue of \mathbf{Iso} , it is enough to verify the condition (T3) $m_+ \geq m_- \geq 0$. If $m_+ = 1/2$ then $\phi_{T,m_-,1/2} = \eta_+$. Hence we may and will assume that $m_+ > 1/2$ from now on.

The morphism $\phi_{T,m_-,m_+}^r : T^r \rightarrow T^{r+m_+-1/2}$ (16).

We first check that the image L of ϕ_{T,m_-,m_+}^r in $T^{r+m_+-1/2}$ is a residual coset. The subset $R_L \subset R_0^{(1)}$ (the affine extension of R_0) of roots which are constant on the image is the set

$$R_L = \{t_i^{\pm 1} : r < i \leq r + m_+ - 1/2\} \cup \{t_i t_j^{-1} : r < i \neq j \leq r + m_+ - 1/2\},$$

a root system of type $B_{m_+-1/2}$. Consider the factorisation $T^{r+m_+-1/2} = T^r \times T^{m_+-1/2}$. By [17, Thm. 3.7(ii)] we need to show that the point \mathfrak{r}_{m_-,m_+-1} in the torus $\{\text{id}\} \times T^{m_+-1/2}$ defined by $t_i(\mathfrak{r}_{m_-,m_+-1}) = v^{2(i-r)-1}$ for $i = r + 1, \dots, r + m_+ - 1/2$ is a residual point with respect to the μ -function $\mu_L := \mu_{m_-,m_+-1}^{m_+-1/2}$ associated to the roots of R_L . By definition of the residual points we need to check that there are precisely $m_+ - 1/2$ factors in the denominator of $\mu_{m_-,m_+-1}^{m_+-1/2}$ which are zero in this point. Indeed, these are the factors corresponding to the roots $t_{i+1} t_i^{-1}$ (for $i = r + 1, \dots, r + m_+ - 3/2$) and the factor corresponding to $t_{r+m_+-1/2}$.

It is a straightforward computation to determine the factor

$$\text{Res}(\mu_{m_L,d}(\mathcal{R}_L), \mathbf{r}_{m_-,m_+-1}),$$

which is equal to $d \cdot \text{Res}(\mu_{m_L,1}(\mathcal{R}_L), \mathbf{r}_{m_-,m_+-1})$ (in the notation of (20), with $d = d_{m_-,m_+-1}$). We obtain

$$\text{Res}(\mu_{m_L,1}(\mathcal{R}_L), \mathbf{r}_{m_-,m_+-1}) = -q^{(m_- - 1/2)(m_+ - 1/2)} \frac{(q - 1)^{m_+ - 1/2}}{\prod_{k=m_- - m_+ + 1}^{m_- + m_+ - 1} (q^k + 1)} \tag{22}$$

In this computation it is not necessary to compute the exact power of q , since *a priori* we know that $\text{Res}(\mu_{m_L,d}(\mathcal{R}_L), \mathbf{r}_{m_-,m_+-1})$ is invariant with respect to the substitution $v \mapsto v^{-1}$. This remark simplifies the computation considerably.

In order to establish (T3) for ϕ_T we need to check firstly that

$$d_{m_-,m_+} = \lambda \cdot [d_{m_-,m_+-1} \cdot \text{Res}(\mu_{m_L,1}(\mathcal{R}_L), \mathbf{r}_{m_-,m_+-1})] \tag{23}$$

for some $\lambda \in \mathbf{Q}^\times$. This is easy using (22). (Again, it is not necessary to compute the power of q).

Next, we need to establish (T3) for the non-constant part of μ^L , by considering the restriction to L of the roots in $B_{r+m_+-1/2}$ which are not constant on $\{\text{id}\} \times T^{m_+ - 1/2}$. For the long roots this is trivial, and by the Weyl group symmetry of μ we only need to consider the roots that restrict to one fixed short root and its opposite, let us say t_{r+1} and t_{r+1}^{-1} . It is straightforward to collect all such factors and check that, after cancellation of equal terms in numerator and denominator, we will obtain

$$\frac{(1 - t_{r+1}^{-2})(1 - t_{r+1}^2)}{(1 + v^{-2m_-} t_{r+1}^{-1})(1 + v^{-2m_-} t_{r+1})(1 - v^{-2m_+} t_{r+1}^{-1})(1 - v^{-2m_+} t_{r+1})}. \tag{24}$$

This is a factor of $\text{Res}(\mu_{m_L,1}(\mathcal{R}_L), \mathbf{r}_{m_-,m_+-1})$ with $m_L = (m_-, m_+)$. In short, the pull-back of the STM ϕ_{T,m_-,m_+} (16) on the μ -function $\mu_{m_-,m_+-1,d_{m_-,m_+-1}}$, is equal to the μ -function $\mu_{m_-,m_+,d_{m_-,m_+}}$ (up to a non-zero rational constant). As desired, finishing the proof. ■

Next, we discuss the case that $\mathbf{m}_\pm \in \mathbb{Z}$ and $m_+ \neq 0$.

The morphism $\psi_{T,m_-,m_+}^r : T^r \rightarrow T^{r+2m_+-2}$ (17).

We sketch a proof similar to the proof of ϕ_{T,m_-,m_+} . It is enough to consider the case $m_+ > 0$. Let us check that the image of ψ_{T,m_-,m_+}^r is a residual coset. The parabolic subset R_L of roots which are constant on the image is the set

$$R_L = \{t_i^{\pm 1} : r < i \leq r + 2(m_+ - 1)\} \cup \{t_i t_j^{-1} : r < i \neq j \leq r + 2(m_+ - 1)\},$$

a root system of type $B_{2(m_+-1)}$. Consider $T^{r+2(m_+-1)} = T^r \times T^{2(m_+-1)}$. Again by [17, Theorem 3.7(ii)] we need to show that the point \mathbf{r}_{m_-,m_+-2} in the torus $\{\text{id}\} \times T^{2(m_+-1)}$ defined by $t_i(\mathbf{r}_{m_-,m_+-2}) = v^{2\lfloor(i-r)/2\rfloor}$ for $i = r + 1, \dots, r + 2(m_+ - 1)$ is a residual point with respect to the μ -function $\mu_L := \mu_{m_-,m_+-1}^{r+2(m_+-1)}$ associated to the roots of R_L . Precisely $2(m_+ - 1)$ roots are equal to 1 on this point, so by the definition of residual points we need to check that there are precisely $2(m_+ - 1) + 2(m_+ - 1) = 4(m_+ - 1)$ factors in the denominator of $\mu_{m_-,m_+-1}^{2(m_+-1)}$ which are zero in this point. Indeed, there

are $4(m_+ - 2)$ such factors corresponding to roots of the form $t_i t_j^{-1}$, 2 such factors from the roots t_i with $i = r + 2m_+ - 4$ and $i = r + 2m_+ - 3$, and finally 2 such factors from the roots $t_{r+1} t_{r+2}$ and $t_{r+1} t_{r+3}$.

This time we have (in the notation of (20), with $d = d_{m_-, m_+ - 2}$):

$$\begin{aligned} \text{Res}(\mu_{m_L, 1}(\mathcal{R}_L), \mathbf{r}_{m_-, m_+ - 2}) = \\ \frac{q^{m_-(m_+ - 1)}(q - 1)^{2(m_+ - 1)}}{\prod_{k=m_- - m_+ + 2}^{m_- + m_+ - 2} (q^k + 1)^2 (q^{(m_+ - m_-) - 1} + 1) (q^{(m_+ + m_-) - 1} + 1)} \end{aligned} \tag{25}$$

Again, in this computation it is not necessary to compute the exact power of q . In view of the factorisation (20) of μ^L , in order for (T3) to hold we need to check firstly that

$$d_{m_-, m_+} = \lambda \cdot [d_{m_-, m_+ - 2} \cdot \text{Res}(\mu_{m_L, 1}(\mathcal{R}_L), \mathbf{r}_{m_-, m_+ - 2})] \tag{26}$$

for some $\lambda \in \mathbf{Q}^\times$. This is easy using (22). (Again, it is not necessary to compute the exact power of q). The rest of the argument is an elementary computation as in the proof for ϕ_{T, m_-, m_+} , finishing the proof. ■

Now we turn to the most complicated case.

Proposition 5.1. *The morphism ξ_{T, m_-, m_+}^r defines a spectral transfer morphism*

$$\xi_{T, m_-, m_+}^r : T^r \rightarrow T^l \tag{27}$$

with l as defined in (18), which is called *extra-special*.

Remark 5.2. Note that the unipotent affine Hecke algebra \mathcal{H}_{m_-, m_+}^r associated with T^r has parameter q^2 , while the unipotent affine Hecke algebra $\mathcal{H}_{\delta_-, \delta_+}^l$ associated with T^l has parameter q .

We will use induction, and the following lemma will serve as the induction basis.

Lemma 5.3. *Proposition 5.1 is true for $(m_-, m_+) = (1/4, 1/4), (3/4, 1/4), (1/4, 3/4)$ and $(3/4, 3/4)$.*

Proof of Lemma 5.3.

Suppose $m_\pm = 1/4$. Then $\delta_\pm = 0$, and $a = b = 0$, and $r_e(1/4) = \emptyset$. If moreover the rank $r = 0$ then we have nothing to verify because $d_{1/4, 1/4}$ reduces to 1, as does $d_{0, 0}$, and there is no root contributing the μ -functions so that both μ -functions reduce to 1 as well.

Now assume $r > 0$. Since $m_\pm = 1/4$, we note that this is of parameter type VI with $a = b = 0$. Hence $l = 2r$, so that $d_{1/4, 1/4} = (v^2 - v^{-2})^{-r}$ (recall that we have parameter q^2 here) and $d_{0, 0} = (v - v^{-1})^{-2r}$. The μ -function $\mu_{0, 0}^{2r}$ has only contributions from the type D roots:

$$\mu_{0, 0}^{2r} = (v - v^{-1})^{-2r} \prod_{1 \leq i < j \leq 2r} \frac{(1 - t_i t_j^{-1})^2 (1 - t_i t_j)^2}{(1 - q t_i t_j^{-1})(1 - q^{-1} t_i t_j^{-1})(1 - q t_i t_j)(1 - q^{-1} t_i t_j)}$$

We compute the pull-back $\xi^*(\mu_{0,0}^{2r})^{(L)}$, so we need to make substitution

$$(t_1, t_2, \dots, t_{2r-1}, t_{2r}) = (vs_1, v^{-1}s_1, vs_2, v^{-1}s_2, \dots, vs_r, v^{-1}s_r)$$

in $\mu_{0,0}^{2r}$, after regularising the expression along the image L of ξ . Concerning the regularisation, observe that the parabolic subsystem of roots which is constant on the image is

$$\{(t_1 t_2^{-1})^{\pm 1}, (t_3 t_4^{-1})^{\pm 1}, \dots, (t_{2r-1} t_{2r}^{-1})^{\pm 1}\},$$

of type A_1^r . After dropping the singular factors, the remaining ‘‘constant factors’’ yield (including the normalisation factor $(v - v^{-1})^{-2r}$), up to a power of v which is irrelevant for us (as explained before): $(v^2 - v^{-2})^{-r}$, which is indeed the normalisation factor $d_{1/4,1/4}$.

For the non-constant part, by Weyl group invariance, it is enough to check the contribution of one type B root in each \mathcal{W}_0 -orbit. First consider $s_1 s_2$. The roots whose pull-back along ξ equals a nonzero power of $s_1 s_2$ times a power of v are $t_1 t_3, t_2, t_3, t_1 t_4, t_2 t_4$ (together with their opposites). These yield $q s_1 s_2, s_1 s_2, s_1 s_2$ and $q^{-1} s_1 s_2$ respectively, so these root give us the factor (after some cancellations, and up to powers of v and other characters of T^r , which are irrelevant anyhow):

$$\frac{(1 - s_1 s_2)^2}{(1 - q^{-2} s_1 s_2)(1 - q^2 s_1 s_2)} \tag{28}$$

which shows that the pull back of $\mu_{0,0}^{2r}$ yields a factor which is a type D μ function with base q^2 , as desired for $\mu_{1/4,1/4}^r$.

The type D roots which pull back to a power of v times a nonzero power of s_1 (a type A_1 root) are only $t_1 t_2$ (and its opposite). Its pull back is s_1^2 . This gives a factor in $\xi^*(\mu_{0,0}^{2r})^{(L)}$ of the form:

$$\frac{(1 - s_1)^2(1 + s_1)^2}{(1 + v^{-1}s_1)(1 + vs_1)(1 - v^{-1}s_1)(1 - vs_1)} \tag{29}$$

This second fraction is exactly the factor in $\mu_{1/4,1/4}^r$ (with parameter q^2) for this last remaining type of root in the type B root system. In other words, the condition (T3) is satisfied. Hence we have verified that $\xi_{T,1/4,1/4}^r$ represents an extra-special STM. ■

Now we turn to the proof of Proposition 5.1 itself.

Proof of Proposition 5.1.

Recall that $m_{\pm} \in \mathbf{Z} \pm \frac{1}{4}$. Using spectral isomorphisms from $\text{Iso} = \langle \eta_+, \eta \rangle$, we can assume that $m_{\pm} > 0$. So we can write $m_- = \kappa_- + (2\epsilon_- - 1)/4$ with $\kappa_- \in \mathbf{Z}_{\geq 0}$ and $\epsilon_- \in \{0, 1\}$. Let δ_- be defined by $\kappa_- - \delta_- \in 2\mathbf{Z}$. In other words, $\delta_- = 0$ (resp. 1) if κ_- is even (resp. odd). Similarly we have $m_+ = \kappa_+ + (2\epsilon_+ - 1)/4$ and δ_+ .

We will apply an inductive argument on $m_- + m_+$, where the induction base is provided by Lemma 5.3. Let us first assume that $m_+ \geq m_- > 0$ and that $m_+ > 1$. By induction we may now assume that ξ_{T,m_-,m_+}^r represents an STM.

Note that $\delta_{\pm}, \epsilon_{\pm} \in \{0, 1\}$. We define $\delta_{\pm}^c, \epsilon_{\pm}^c$ by the rules that $\delta_{\pm} + \delta_{\pm}^c = 1, \epsilon_{\pm} + \epsilon_{\pm}^c = 1$. Also we define $A(m) = 2m - \frac{3}{2}$ and write $A_{\pm} = A(m_{\pm}) = 2(\kappa_{\pm} - 1) + \epsilon_{\pm} \in \mathbf{Z}_{\geq -1}$. Observe that $A_+ \geq 1$ by our assumptions.

To proceed, we need more notations. The μ -function associated to the source normalised Hecke algebra $(\mathcal{H}_{m_-,m_+}^r, \tau^d)$ of the alleged STM $\xi = \xi_{m_-,m_+}^r$ with parameters

m_{\pm} will be denoted by $\mu_{m_-,m_+,d}^r$, where we often omit the rank r if there should be no confusion. When $d = d_{m_-,m_+}^r$ we will simply write μ_{m_-,m_+}^r . The μ -function of the target is denoted by $\mu_{\delta_-, \delta_+}^l$, with l given as in (18). Recall that we have, up to irrelevant factors, $\mu_{\delta_-, \delta_+}^l = d_{\delta_-, \delta_+}^l \mu_{\delta_-, \delta_+}^{D,l} \mu_{\delta_-, \delta_+}^{A,l}$ with

$$d_{\delta_-, \delta_+}^l = (v - v^{-1})^{-l} (v + v^{-1})^{-\delta_- \delta_+}. \tag{30}$$

The second factor arises in [18, Prop. 2.5] from the fact that the reductive quotient $\overline{\mathbb{P}}$ of a minimal F -stable parahoric in the case $(\delta_-, \delta_+) = (1, 1)$ is an \mathbb{F}_q -torus of split rank l whose maximal \mathbb{F}_q -anisotropic subtorus has $q + 1$ rational points over \mathbb{F}_q . Furthermore,

$$\begin{aligned} \mu^{D,l} &= \prod_{1 \leq i < j \leq l} \frac{(1 - t_i t_j^{-1})^2}{(1 - q t_i t_j^{-1})(1 - q^{-1} t_i t_j^{-1})} \prod_{1 \leq i < j \leq l} \frac{(1 - t_i t_j)^2}{(1 - q t_i t_j)(1 - q^{-1} t_i t_j)}, \\ \mu_{\delta_-, \delta_+}^{A,l} &= \prod_{i=1}^l \frac{(1 - t_i^2)^2}{(1 + q^{\delta_-} t_i)(1 + q^{-\delta_-} t_i)(1 - q^{\delta_+} t_i)(1 - q^{-\delta_+} t_i)}. \end{aligned} \tag{31}$$

On the other hand $\mu_{m_-,m_+}^r = d_{m_-,m_+}^r \mu_{m_-,m_+}^{D,r}(q^2) \mu_{m_-,m_+}^{A,r}$ with

$$d_{m_-,m_+}^r = (v - v^{-1})^{-r} d_{m_-,m_+}^0,$$

and the normalisation factor d_{m_-,m_+}^0 is given by

$$d_{m_-,m_+}^0 = \prod_{i=1}^{\lfloor m_- - m_+ \rfloor} \left(\frac{v^{2|m_- - m_+| - 2i}}{1 + q^{2|m_- - m_+| - 2i}} \right)^i \prod_{j=1}^{\lfloor m_- + m_+ \rfloor} \left(\frac{v^{2|m_- + m_+| - 2j}}{1 + q^{2|m_- + m_+| - 2j}} \right)^j$$

as in [18, Eq. (36)]. Here $\mu_{m_-,m_+}^{D,r}(q^2)$ is similar to $\mu_{m_-,m_+}^{D,l}$, only with rank r instead of l and with parameter q^2 instead of q . Finally:

$$\mu_{m_-,m_+}^{A,r} = \prod_{i=1}^r \frac{(1 - s_i^2)^2}{(1 + v^{4m_-} s_i)(1 + v^{-4m_-} s_i)(1 - v^{4m_+} s_i)(1 - v^{-4m_+} s_i)} \tag{32}$$

Recall that the morphism $\xi := \xi_{T,m_-,m_+}^r : T_r \rightarrow T_l$ is defined by

$$\xi_{T,m_-,m_+}^r(s_1, \dots, s_r) = (-r_e(m_-), v^{-1}s_1, vs_1, \dots, v^{-1}s_r, vs_r, r_e(m_+)) \tag{33}$$

where $r_e(\frac{1}{4}) = r_e(\frac{3}{4}) = \emptyset, r_e(m) = (\sigma_e(m), r_e(m - 1))$ and

$$\sigma_e(m) = (q^\delta, q^{\delta+1}, \dots, q^{2m - \frac{3}{2}}).$$

We denote $\mathbf{r}_0 := (-r_e(m_-), r_e(m_+))$. Note that if we re-order the coordinates of $\xi(s_1, \dots, s_r)$ (or of \mathbf{r}_0), or invert them, then the result lies in the same $W_{2,0}$ -orbit. The μ -function is invariant under the $W_{2,0}$ -action, hence also invariant under such operation.

We now consider the rank 0 case, i.e. $r = 0$. This is the main challenge, as we will see.

We need to verify the condition (T3), assuming that $m_+ \geq m_- > 0$ and $m_+ > 1$, and (by the induction hypothesis) that ξ_{T,m_-,m_+-1}^0 represents an STM.

Write $l_0 = \lfloor l_- \rfloor + \lfloor l_+ \rfloor$ for the rank of the target Hecke algebra if we use the parameters (m_-, m_+) , and $l'_0 = \lfloor l_- \rfloor + \lfloor l'_+ \rfloor$ if we use the parameters $(m'_-, m'_+) :=$

$(m_-, m_+ - 1)$. Observe that $\delta'_- = \delta_-$ and $\delta'_+ = \delta_+^c$. By our normalisations of the Hecke algebras it suffices to show the following identity for the ratio of residues:

$$C_{m_-, m_+ - 1} := \frac{\text{Res}(\mu^{D, l_0}, \mathbf{r}_0) \cdot \text{Res}(\mu_{\delta_-, \delta_+}^{A, l_0}, \mathbf{r}_0)}{\text{Res}(\mu^{D, l'_0}, \mathbf{r}'_0) \cdot \text{Res}(\mu_{\delta_-, \delta_+^c}^{A, l'_0}, \mathbf{r}'_0)} = \frac{d_{\delta_-, \delta_+^c}^{l'_0} \cdot \text{Res}(\mu_{\delta_-, \delta_+}^{l_0}, \mathbf{r}_0)}{d_{\delta_-, \delta_+}^{l_0} \cdot \text{Res}(\mu_{\delta_-, \delta_+^c}^{l'_0}, \mathbf{r}'_0)}. \tag{34}$$

The last expression equals, up to powers of v and rational constants,

$$\mathbf{A}(m_+) := \frac{d_{\delta_-, \delta_+^c}^{l'_0} \cdot d_{m_-, m_+}^0}{d_{\delta_-, \delta_+}^{l_0} \cdot d_{m_-, m_+ - 1}^0} = (v - v^{-1})^{A_+ + \delta_+^c} (v + v^{-1})^{\delta_- (\delta_+ - \delta_+^c)} \frac{d_{m_-, m_+}^0}{d_{m_-, m_+ - 1}^0}.$$

The second equality is easy to check, using the relations $l_0 - l'_0 = A_+ + \delta_+^c = 2m_+ - \frac{1}{2} - \delta_+$.

Notation. In the equations below we will simplify notations by omitting the references to the rank (if the arguments are given, the rank equals the number of coordinates of the argument so the explicit references to the ranks are superfluous), and we will simply write “reg” to indicate that we are using the regularisation of μ -functions, i.e. omitting the factors that are identically 0 after evaluation at the argument. Finally, an expression like $\mu^{D, \text{reg}}(\mathbf{r}_1; \mathbf{r}_2)$ means that we only consider the product in the numerator and the denominator of those type D-roots $t_i^{\pm 1} t_j^{\pm 1}$ for which t_i is a coordinate of \mathbf{r}_1 and t_j is a coordinate of \mathbf{r}_2 , and only those factors which are not identically 0.

Since $\mathbf{r}_0 = (\mathbf{r}'_0, \sigma_e(m_+))$, we see that $C_{m_-, m_+ - 1}$ is equal to

$$\begin{aligned} & \frac{\mu_{\delta_-, \delta_+}^{A, \text{reg}}(\mathbf{r}'_0)}{\mu_{\delta_-, \delta_+^c}^{A, \text{reg}}(\mathbf{r}'_0)} \mu_{\delta_-, \delta_+}^{A, \text{reg}}(\sigma_e(m_+)) \\ & \times \mu^{D, \text{reg}}(\sigma_e(m_+)) \mu^{D, \text{reg}}(-r_e(m_-); \sigma_e(m_+)) \mu^{D, \text{reg}}(\sigma_e(m_+); r_e(m_+ - 1)) \end{aligned} \tag{35}$$

It is easy to see that

$$\frac{\mu_{\delta_-, \delta_+}^{A, \text{reg}}(\mathbf{r}'_0)}{\mu_{\delta_-, \delta_+^c}^{A, \text{reg}}(\mathbf{r}'_0)} = \prod_{u_i \in r_e(m_-)} \frac{(1 + q^{\delta_+^c} u_i)(1 + q^{-\delta_+^c} u_i)}{(1 + q^{\delta_+} u_i)(1 + q^{-\delta_+} u_i)} \prod_{w_j \in r_e(m_+ - 1)} \frac{(1 - q^{\delta_+^c} w_j)(1 - q^{-\delta_+^c} w_j)}{(1 - q^{\delta_+} w_j)(1 - q^{-\delta_+} w_j)} \tag{36}$$

while

$$\begin{aligned} \mu_{\delta_-, \delta_+}^{A, \text{reg}}(\sigma_e(m_+)) &= \prod_{t_k \in \sigma_e(m_+)} \frac{(1 - t_k^2)^2}{(1 + q^{\delta_-} t_k)(1 + q^{-\delta_-} t_k)(1 - q^{\delta_+} t_k)(1 - q^{-\delta_+} t_k)} \\ &= \begin{cases} 1 & (\delta_-, \delta_+) = (0, 0) \\ \frac{1 + q^{2m_+ - 3/2}}{(1 + q)(1 + q^{2m_+ - 1/2})} & (\delta_-, \delta_+) = (1, 0) \\ \frac{(1 - q^{2m_+ - 3/2})(1 - q)}{1 - q^{2m_+ - 1/2}} & (\delta_-, \delta_+) = (0, 1) \\ \frac{(1 - q^{4m_+ - 3})(1 - q^2)}{1 + q^{2m_+ - 1/2}} & (\delta_-, \delta_+) = (1, 1) \end{cases} \tag{37} \\ &= \left(\frac{1 + q^{A_+}}{1 + q^{A_+ + 1}} \right)^{\delta_-} \left(\frac{1 - q^{A_+}}{1 - q^{A_+ + 1}} \right)^{\delta_+} (1 + q)^{(-1)^{\delta_+^c} \delta_-} (1 - q)^{\delta_+}. \end{aligned}$$

We denote this last expression by $(P1) = \mu_{\delta_-, \delta_+}^{A, \text{reg}}(\sigma_e(m_+))$.

Next we consider $\mu^{D, \text{reg}}(-r_e(m_-); \sigma_e(m_+))$ and $\mu^{D, \text{reg}}(\sigma_e(m_+); r_e(m_+ - 1))$.

$$\begin{aligned} & \mu^{D, \text{reg}}(-r_e(m_-); \sigma_e(m_+)) \\ & := \prod_{t_i \in \sigma_e(m_+), t_j \in -r_e(m_-)} \frac{(1 - t_i t_j^{-1})^2 (1 - t_i t_j)^2}{(1 - q t_i t_j^{-1})(1 - q^{-1} t_i t_j^{-1})(1 - q t_i t_j)(1 - q^{-1} t_i t_j)} \\ & = \prod_{t_j \in r_e(m_-)} \frac{(1 + q^{\delta_+} t_j^{-1})(1 + q^{A_+} t_j^{-1})(1 + q^{\delta_+} t_j)(1 + q^{A_+} t_j)}{(1 + q^{\delta_+ - 1} t_j^{-1})(1 + q^{A_+ + 1} t_j^{-1})(1 + q^{\delta_+ - 1} t_j)(1 + q^{A_+ + 1} t_j)} \end{aligned} \tag{38}$$

where $A_+ = 2m_+ - 3/2$. We can likewise obtain

$$\begin{aligned} & \mu^{D, \text{reg}}(\sigma_e(m_+), r_e(m_+ - 1)) \\ & = \prod_{t_i \in \sigma_e(m_+), t_j \in r_e(m_+ - 1)} \frac{(1 - t_i t_j^{-1})^2 (1 - t_i t_j)^2}{(1 - q t_i t_j^{-1})(1 - q^{-1} t_i t_j^{-1})(1 - q t_i t_j)(1 - q^{-1} t_i t_j)} \\ & = \prod_{t_j \in r_e(m_+ - 1)} \frac{(1 - q^{\delta_+} t_j^{-1})(1 - q^{A_+} t_j^{-1})(1 - q^{\delta_+} t_j)(1 - q^{A_+} t_j)}{(1 - q^{\delta_+ - 1} t_j^{-1})(1 - q^{A_+ + 1} t_j^{-1})(1 - q^{\delta_+ - 1} t_j)(1 - q^{A_+ + 1} t_j)} \end{aligned} \tag{39}$$

Observe that up to some power of v , we can cancel some factors from (36), (38) and (39) and obtain

$$\begin{aligned} & \frac{\mu_{\delta_-, \delta_+}(\mathbf{r}'_0)}{\mu_{\delta_-, \delta_+^c}(\mathbf{r}'_0)} \times \mu(D, q, -r_e(m_-), \sigma_e(m_+)) \times \mu(D, q, \sigma_e(m_+), r_e(m_+ - 1)) \\ & = \prod_{t_j \in r_e(m_-)} \frac{(1 + q^{A_+} t_j^{-1})(1 + q^{A_+} t_j)}{(1 + q^{A_+ + 1} t_j^{-1})(1 + q^{A_+ + 1} t_j)} \prod_{t_j \in r_e(m_+ - 1)} \frac{(1 - q^{A_+} t_j^{-1})(1 - q^{A_+} t_j)}{(1 - q^{A_+ + 1} t_j^{-1})(1 - q^{A_+ + 1} t_j)}. \end{aligned} \tag{40}$$

Denote $(P2) = \prod_{t_j \in r_e(m_-)} \frac{(1 + q^{A_+} t_j^{-1})(1 + q^{A_+} t_j)}{(1 + q^{A_+ + 1} t_j^{-1})(1 + q^{A_+ + 1} t_j)}$

and $(P3) = \prod_{t_j \in r_e(m_+ - 1)} \frac{(1 - q^{A_+} t_j^{-1})(1 - q^{A_+} t_j)}{(1 - q^{A_+ + 1} t_j^{-1})(1 - q^{A_+ + 1} t_j)}$.

They can be simplified further. Observe that

$$r_e(m_-) = \left(\sigma_e(m_-), r_e(m_- - 1) \right) = \left(\sigma_e(m_-), \sigma_e(m_- - 1), \dots, \sigma_e\left(\frac{7 - 2\epsilon_-}{4}\right) \right)$$

The number of σ_e 's in $r_e(m_-)$ is κ_- . Recall that for $g \in \mathbf{Z}_{>0}$ we defined:

$$\sigma_e\left(g + \frac{2\epsilon_- - 1}{4}\right) = (q^{\bar{g}}, q^{\bar{g}+1}, \dots, q^{2(g-1)+\epsilon_-})$$

where $\bar{g} = 0$ if g is even and $\bar{g} = 1$ if g is odd. Therefore we can write (P2) as

$$(P2) = \prod_{g=2-\epsilon_-}^{\kappa_-} \prod_{t_j \in \sigma_e\left(g + \frac{2\epsilon_- - 1}{4}\right)} \frac{(1 + q^{A_+} t_j^{-1})(1 + q^{A_+} t_j)}{(1 + q^{A_+ + 1} t_j^{-1})(1 + q^{A_+ + 1} t_j)}$$

$$\begin{aligned}
 &= \prod_{g=2-\epsilon_-}^{\kappa_-} \frac{(1 + q^{A_+-(2g-2+\epsilon_-)})(1 + q^{A_++\bar{g}})}{(1 + q^{A_++1-\bar{g}})(1 + q^{A_++(2g-1+\epsilon_-)})} \\
 &= \prod_{g=2-\epsilon_-}^{\kappa_-} \frac{(1 + q^{A_+-(2g-2+\epsilon_-)})}{(1 + q^{A_++(2g-1+\epsilon_-)})} \times \frac{(1 + q^{A_++\bar{g}})}{(1 + q^{A_++1-\bar{g}})}. \tag{41}
 \end{aligned}$$

Notice that δ_- indicates the parity of κ_- , so

$$\prod_{g=2-\epsilon_-}^{\kappa_-} \frac{(1 + q^{A_++\bar{g}})}{(1 + q^{A_++1-\bar{g}})} = \begin{cases} 1 & \text{if } \epsilon_- \neq \delta_- \\ \frac{1+q^{A_+}}{1+q^{A_++1}} & \text{if } \epsilon_- = \delta_- = 0 \\ \frac{1+q^{A_++1}}{1+q^{A_+}} & \text{if } \epsilon_- = \delta_- = 1. \end{cases}$$

So (P2) is equal to

$$\left(\frac{1 + q^{A_++\epsilon_-}}{1 + q^{A_++1-\epsilon_-}} \right)^{\epsilon_- \delta_- + \epsilon_-^c \delta_-^c} \frac{(1 + q^{A_+-2+\epsilon_-})(1 + q^{A_+-4+\epsilon_-}) \dots (1 + q^{A_+-A_-})}{(1 + q^{A_++3-\epsilon_-})(1 + q^{A_++5-\epsilon_-}) \dots (1 + q^{A_++A_+-1})}, \tag{42}$$

where $A_- = 2m_- - 3/2$. (Recall that $m_+ \geq m_- > 0$ and $m_+ > 1$, hence $A_+ \geq A_- \geq -1$ and $A_+ \geq 1$. Observe that (P2) = 1 if $0 < m_- < 1$.) Similarly we can compute that (P3) is equal to

$$\left(\frac{1 - q^{A_++\epsilon_+}}{1 - q^{A_++1-\epsilon_+}} \right)^{\epsilon_+ \delta_+^c + \epsilon_+^c \delta_+} \frac{(1 - q^2)(1 - q^4) \dots (1 - q^{A_+-2+\epsilon_+})}{(1 - q^{2A_+-1})(1 - q^{2A_+-3}) \dots (1 - q^{A_++3-\epsilon_-})}. \tag{43}$$

The final term to compute is $\mu^{D, \text{reg}}(\sigma_e(m_+)) = \mu^{D_0, \text{reg}}(\sigma_e(m_+))\mu^{D_{\neq 0}, \text{reg}}(\sigma_e(m_+))$. Here D_0 denotes the type D-roots whose coordinates sum up to zero (this is a maximal proper parabolic root subsystem, irreducible of type A), and $D_{\neq 0}$ denotes the remaining roots of type D (this is not a root subsystem).

Recall that $\sigma_e(m_+) = (q^{\delta_+}, q^{\delta_++1}, \dots, q^{2m_+-3/2})$. We can easily compute that

$$\mu^{D_0, \text{reg}}(\sigma_e(m_+)) = \prod_{1 \leq i < j \leq A_+-\delta_+} \frac{(1 - q^{j-i})^2}{(1 - q^{j-i+1})(1 - q^{j-i-1})} = \frac{(1 - q)^{A_++1-\delta_+}}{(1 - q^{A_++1-\delta_+})} \tag{44}$$

by considering the multiplicities of the range of $j - i \in \{1, 2, \dots, A_+ - \delta_+ - 1\}$. The same idea applies to computing $\mu^{D_{\neq 0}, \text{reg}}(\sigma_e(m_+))$:

$$\begin{aligned}
 \mu^{D_{\neq 0}, \text{reg}}(\sigma_e(m_+)) &= \prod_{1 \leq i < j \leq A_+-\delta_+} \frac{(1 - q^{i+j+2(\delta_+-1)})}{(1 - q^{i+j+2\delta_+-1})(1 - q^{i+j+2\delta_+-3})} \tag{45} \\
 &= \begin{cases} \frac{(1 - q^{1+2\delta_+})(1 - q^{3+2\delta_+}) \dots (1 - q^{A_++\delta_+-1})(1 - q^{A_++\delta_++1}) \dots (1 - q^{2A_+-1})}{(1 - q^{2\delta_+})(1 - q^{2+2\delta_+}) \dots (1 - q^{A_++\delta_+-2})(1 - q^{A_++\delta_++2}) \dots (1 - q^{2A_+})} & \text{if } \epsilon_+ = \delta_+ \\ \frac{(1 - q^{1+2\delta_+})(1 - q^{3+2\delta_+}) \dots (1 - q^{A_++\delta_+})(1 - q^{A_++\delta_++1}) \dots (1 - q^{2A_+-1})}{(1 - q^{2\delta_+})(1 - q^{2+2\delta_+}) \dots (1 - q^{A_++\delta_+-1})(1 - q^{A_++\delta_++1}) \dots (1 - q^{2A_+})} & \text{if } \epsilon_+ \neq \delta_+. \end{cases}
 \end{aligned}$$

Here, if $\delta_+ = 0$ then the denominator starts with $(1 - q^2)$.

We denote $\mu^{D, \text{reg}}(\sigma_e(m_+))$, $\mu^{D_0, \text{reg}}(\sigma_e(m_+))$, $\mu^{D_{\neq 0}, \text{reg}}(\sigma_e(m_+))$ respectively by (P4), (P4a) and (P4d).

Now we multiply (P1), (P2), (P3) and (P4). The 4 parameters $\epsilon_+, \delta_+, \epsilon_-, \delta_-$ take values in $\{0, 1\}$ independently. So basically we need to consider 16 cases (The parity of A_+ is the same as ϵ_+). But we spot a simplification when taking the product of (P3) and (P4d). We see that:

$$\begin{aligned} & \text{(P3)} \times \text{(P4d)} \\ &= \left(\frac{1 - q^{A_+ + \epsilon_+}}{1 - q^{A_+ + 1 - \epsilon_+}} \right)^{\epsilon_+ \delta_+^c + \epsilon_+^c \delta_+} \frac{(1 - q^2)(1 - q^4) \cdots (1 - q^{A_+ - 2 + \epsilon_+})}{(1 - q^{2A_+ - 1})(1 - q^{2A_+ - 3}) \cdots (1 - q^{A_+ + 3 - \epsilon_-})} \times \text{(P4d)} \\ &= \frac{(1 - q^{1 + 2\delta_+})(1 - q^{3 + 2\delta_+}) \cdots (1 - q^{A_+ + 1 - \epsilon_+})}{(1 - q^{A_+ + 2 + \epsilon_+}) \cdots (1 - q^{2A_+})} \\ &= \frac{(1 - q)[1 - q^2][1 - q^3][1 - q^4] \cdots (1 - q^{A_+ + 1 - \epsilon_+}) (1 - q^{A_+ + 1})^{\epsilon_+}}{[1 - q^2][1 - q^4][1 - q^6][1 - q^8] \cdots (1 - q^{2A_+}) (1 - q)^{\delta_+}} \\ &= \frac{(1 - q)[1 - q^2](1 - q^3)[1 - q^4] \cdots (1 - q^{A_+ + 1 - \epsilon_+}) (1 - q^{2(A_+ + 1)})^{\epsilon_+} (1 - q^{A_+ + 1})^{\epsilon_+}}{[1 - q^2][1 - q^4][1 - q^6][1 - q^8] \cdots (1 - q^{2(A_+ + 1 - \epsilon_+)}) (1 - q)^{\delta_+}} \\ &= \frac{(1 + q^{A_+ + 1})^{\epsilon_+} (1 - q^{A_+ + 1})}{(1 + q)(1 + q^2) \cdots (1 + q^{A_+ + 1 - \epsilon_+}) (1 - q)^{\delta_+}} \\ &= \frac{1}{(1 + q)(1 + q^2) \cdots (1 + q^{A_+})} \frac{1 - q^{A_+ + 1}}{(1 - q)^{\delta_+}}, \end{aligned}$$

no matter if ϵ_+ equals to δ_+ or not. Here in the third and the fourth equations we insert in both the numerators and denominators the factors in square brackets to produce the factors $1 + q^*$ ($\star = 1, \dots, A_+$) in the denominator.

We proceed to combine with (P4a) and (P1) to obtain

$$\begin{aligned} & \frac{(1 - q)^{A_+ + 1 - \delta_+}}{(1 - q^{A_+ + 1 - \delta_+})} \times \left(\frac{1 - q^{A_+}}{1 - q^{A_+ + 1}} \right)^{\delta_+} \times \left(\frac{1 + q^{A_+}}{1 + q^{A_+ + 1}} \right)^{\delta_-} (1 + q)^{\delta_- (-1)^{\delta_+^c}} (1 - q)^{\delta_+} \\ & \times \frac{1}{(1 + q)(1 + q^2) \cdots (1 + q^{A_+})} \frac{1 - q^{A_+ + 1}}{(1 - q)^{\delta_+}} \\ &= \frac{(1 + q)^{\delta_- (-1)^{\delta_+^c}} (1 - q)^{A_+ + 1 - \delta_+}}{(1 + q)(1 + q^2) \cdots (1 + q^{A_+})} \left(\frac{1 + q^{A_+}}{1 + q^{A_+ + 1}} \right)^{\delta_-} \\ &= \text{(P1)} \times \text{(P3)} \times \text{(P4)}, \end{aligned}$$

no matter the value of δ_+ . Finally we multiply with (P2). Note first that

$$\left(\frac{1 + q^{A_+ + \epsilon_-}}{1 + q^{A_+ + 1 - \epsilon_-}} \right)^{\epsilon_- \delta_- + \epsilon_-^c \delta_-^c} \left(\frac{1 + q^{A_+}}{1 + q^{A_+ + 1}} \right)^{\delta_-} = \frac{1 + q^{A_+ + \epsilon_-}}{1 + q^{A_+ + 1}}.$$

Hence no matter the values of ϵ_- and δ_- , the total product $C_{m_-, m_+ - 1}$ is equal to

$$\begin{aligned} & \frac{(1 + q)^{\delta_- (-1)^{\delta_+^c}} (1 - q)^{A_+ + 1 - \delta_+}}{(1 + q) \cdots (1 + q^{A_+}) (1 + q^{A_+ + 1})} \frac{(1 + q^{A_+ + \epsilon_-})(1 + q^{A_+ - 2 + \epsilon_-}) \cdots (1 + q^{A_+ - A_-})}{(1 + q^{A_+ + 3 - \epsilon_-})(1 + q^{A_+ + 5 - \epsilon_-}) \cdots (1 + q^{A_+ + A_- + 1})} \\ &= (1 - q)^{A_+ + 1 - \delta_+} (1 + q)^{\delta_- (-1)^{\delta_+^c}} \times \frac{1}{(1 + q)(1 + q^2) \cdots (1 + q^{A_+ - A_- - 1})} \times \\ & \times \frac{1}{(1 + q^{A_+ - A_- + 1})(1 + q^{A_+ - A_- + 3}) \cdots (1 + q^{A_+ + A_- + 1})}. \end{aligned} \tag{46}$$

(The second and third factor of this product may be equal to 1, if $A_+ \leq A_- + 1$ or if $A_- = -1$ respectively). In view of the expression of d_{m_-,m_+}^0 , we now verify easily that $C_{m_-,m_+-1} = \mathbf{A}(m_+)$. Furthermore, we remark that if $m_- > m_+$ then we should compute $C_{m_- - 1, m_+}$ which can be obtained by changing the subscripts $+$ in (P1) to (P4) to $-$ and similarly to obtain $\mathbf{A}(m_-)$. One can likewise verify that $C_{m_- - 1, m_+} = \mathbf{A}(m_-)$. To sum up, we have verified by induction on $m_- + m_+$ that in the rank 0 case of ξ_T indeed represents a spectral transfer morphism.

Now we consider the *positive rank* cases. Again, assume that $m_+ \geq m_-$. Still we need to verify the condition (T3) for the μ -functions. We put $s = (s_1, \dots, s_r)$. Let $\xi(s) := \xi_{T, m_-, m_+}^r(s)$ and $\xi'(s) := \xi_{T, m_-, m_+-1}^r(s)$ be given as in (33). Observe that $\xi(s) = (\xi'(s), \sigma_e(m_+))$. Let L be the image of ξ , and let L' be the image of ξ' . Observe that the parabolic root system R_L of roots of $R_0 = B_l$ which restrict to constant functions on L is isomorphic to $A_1^r \times B_{l_0}$, while the roots of $R'_0 = B_{l'}$ restricting to constant roots on L' form a parabolic subsystem of type $A_1^r \times B_{l'_0}$. Using (34) it suffices to prove that the ratio

$$\frac{\mu_{\delta_-, \delta_+}^{l, (L)}(\xi(s))}{\mu_{\delta_-, \delta_+^c}^{l', (L')}(\xi'(s))} = \frac{(v + v^{-1})^{-r} \text{Res}(\mu_{\delta_-, \delta_+}^{l_0}, \mathbf{r}_0) \cdot \mu_{\delta_-, \delta_+, R_0 \setminus R_L}(\xi(s))}{(v + v^{-1})^{-r} \text{Res}(\mu_{\delta_-, \delta_+^c}^{l'_0}, \mathbf{r}'_0) \cdot \mu_{\delta_-, \delta_+^c, R'_0 \setminus R_{L'}}(\xi'(s))} \tag{47}$$

which we denote as $C_{m_-, m_+-1}^r(s)$, is equal to $C_{m_-, m_+-1} \times \frac{\mu_{m_-, m_+}^r(s)}{\mu_{m_-, m_+-1}^r(s)}$, where C_{m_-, m_+-1} is defined in (34), and

$$\frac{\mu_{m_-, m_+}^r(s)}{\mu_{m_-, m_+-1}^r(s)} = \prod_{i=1}^r \frac{(1 - q^{-2(m_+-1)} s_i)(1 - q^{2(m_+-1)} s_i)}{(1 - q^{-2m_+} s_i)(1 - q^{2m_+} s_i)}. \tag{48}$$

Based on the result of rank 0 case, to prove that

$$C_{m_-, m_+-1}^r(s) = C_{m_-, m_+-1} \times \frac{\mu_{m_-, m_+}^r(s)}{\mu_{m_-, m_+-1}^r(s)},$$

we just need to verify that

$$\frac{\mu_{\delta_-, \delta_+, R_0 \setminus R_L}(\xi'(s), \sigma_e(m_+))}{\mu_{\delta_-, \delta_+^c, R'_0 \setminus R_{L'}}(\xi'(s))} = \prod_{i=1}^r \frac{(1 - q^{-2(m_+-1)} s_i)(1 - q^{2(m_+-1)} s_i)}{(1 - q^{-2m_+} s_i)(1 - q^{2m_+} s_i)}. \tag{49}$$

It is enough to consider the contribution to the factors of the right hand side involving s_1 . For this we need to consider the contribution in the numerator of the left hand side of the set of type D-roots $t_i^{\pm 1} t_j^{\pm 1}$ such that $t_i = v^{-1} s_1$ or vs_1 , and t_j is a coordinate of $\sigma_e(m_+)$. In addition, we need to consider the contribution in the numerator and denominator of the left hand side of the roots $t_1 = v^{-1} s_1$, $t_2 = vs_1$ and their opposites. Therefore, in this computation we may assume that $r = 1$.

The contribution from the type A_1^r roots $t_1 = v^{-1} s_1$ and $t_2 = vs_1$ is:

$$\begin{aligned} \frac{\mu_{\delta_-, \delta_+}^2(v^{-1} s_1, vs_1)}{\mu_{\delta_-, \delta_+^c}^2(v^{-1} s_1, vs_1)} &= \frac{(1 - q^{\delta_+^c - 1/2} s_1)(1 - q^{\delta_+^c + 1/2} s_1)(1 - q^{-\delta_+^c - 1/2} s_1)(1 - q^{-\delta_+^c + 1/2} s_1)}{(1 - q^{\delta_+ - 1/2} s_1)(1 - q^{\delta_+ + 1/2} s_1)(1 - q^{-\delta_+ - 1/2} s_1)(1 - q^{-\delta_+ + 1/2} s_1)} \\ &= \left(\frac{(1 - q^{3/2} s_1)(1 - q^{-3/2} s_1)}{(1 - q^{1/2} s_1)(1 - q^{-1/2} s_1)} \right)^{(-1)^{\delta_+}}. \end{aligned}$$

Next, consider the type D-roots $t_1 t_j^{\pm 1}$ and $t_2 t_j^{\pm 1}$ in the numerator, with $t_1 = v^{-1} s_1$, $t_2 = v s_1$, and $t_j \in \sigma_e(m_+)$. These yield:

$$\begin{aligned} & \prod_{s=\delta_+}^{2m_+-3/2} \frac{(1 - q^{-s+1/2} s_1)(1 - q^{-s-1/2} s_1)(1 - q^{s+1/2} s_1)(1 - q^{s-1/2} s_1)}{(1 - q^{-s+3/2} s_1)(1 - q^{-s-3/2} s_1)(1 - q^{s+3/2} s_1)(1 - q^{s-3/2} s_1)} \\ &= \frac{(1 - q^{-2m_++2} s_1)(1 - q^{2m_+-2} s_1)(1 - q^{-\delta_+-1/2} s_1)(1 - q^{-\delta_++1/2} s_1)}{(1 - q^{-\delta_++3/2} s_1)(1 - q^{\delta_+-3/2} s_1)(1 - q^{-2m_+} s_1)(1 - q^{2m_+} s_1)}. \end{aligned}$$

Multiplying the contributions from the type A and type D roots, we are quickly led to (49). Using induction on $m_- + m_+$ (as in the rank zero case), and the induction basis Lemma 5.3, we can finish the proof that ξ_{T,m_-,m_+}^r represents an extra-special spectral transfer morphism. ■

We conclude this paper by the following remark.

Remark 5.4. Compositions of ϕ_{T,m_-,m_+} and ψ_{T,m_-,m_+} are obviously defined. By applying (the compositions of) ϕ_T, ψ_T and ξ_T , we obtain a spectral transfer morphism from a rank zero unipotent affine Hecke algebras to the one of maximal rank with parameters attain the minimum in the same parameter type. These three spectral transfer morphisms are necessary in the proof of the main result, Theorem 3.4 in [18], which is called the *essential uniqueness* of spectral transfer morphism.

Acknowledgement. The author wants to express his gratitude to Eric Opdam and Maarten Solleveld, for their patient guidance and helpful discussion.

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Received September 17, 2018
and in final form March 19, 2019