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Homogeneous Principal Bundles over Manifolds with Trivial Logarithmic Tangent Bundle

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Abstract. Winkelmann considered compact complex manifolds X equipped with a reduced effective normal crossing divisor $D \subset X$ such that the logarithmic tangent bundle $TX(-\log D)$ is holomorphically trivial. He characterized them as pairs (X, D) admitting a holomorphic action of a complex Lie group $\mathbb{G}$ satisfying certain conditions (see J. Winkelmann, *On manifolds with trivial logarithmic tangent bundle*, Osaka J. Math. 41 (2004) 473–484; and *On manifolds with trivial logarithmic tangent bundle: the non-Kähler case*, Transform. Groups 13 (2008) 195–209); this $\mathbb{G}$ is the connected component, containing the identity element, of the group of holomorphic automorphisms of X that preserve D. We characterize the homogeneous holomorphic principal H-bundles over X, where H is a connected complex Lie group. Our characterization says that the following three are equivalent:

(1) E_H is homogeneous.

(2) E_H admits a logarithmic connection singular over D.

(3) The family of principal H-bundles $\{g^*E_H\}_{g\in\mathbb{G}}$ is infinitesimally rigid at the identity element of the group $\mathbb{G}$.

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1. Introduction

The present work was motivated by a work of Winkelmann [8]. We begin by very briefly recalling from [8]. Let X be a compact connected complex manifold and $D \subset X$ a reduced effective normal crossing divisor (definition will be recalled in Section 2.1), such that the corresponding logarithmic tangent bundle $TX(-\log D)$ is holomorphically trivial. Let $\mathbb{G}$ be the connected component of the group of holomorphic automorphisms of X that preserve D. This $\mathbb{G}$ is a complex Lie group that acts transitively on the complement $X \setminus D$. The tautological action of $\mathbb{G}$ on X has certain properties (they are recalled in Section 3.1); these properties actually characterize pairs (X, D) of the above type [8, p. 196, Theorem 1].

Let H be a connected complex Lie group and E_H a holomorphic principal H-bundle over X. It is called *homogeneous* if for every $g \in \mathbb{G}$ the pulled back principal H-bundle g^*E_H is holomorphically isomorphic to E_H (Definition 4.3 and Proposition 4.4).

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Let $\varpi \colon \mathbb{G} \times X \longrightarrow X$ be the evaluation map defined by $(g, x) \longmapsto g(x)$. Let

$$\mathcal{E}_H := \varpi^* E_H \longrightarrow \mathbb{G} \times X$$

be the holomorphic principal H-bundle over $\mathbb{G} \times X$ obtained by pulling back E_H using the above holomorphic map ϖ. Consider $\mathcal{E}_H$ as a holomorphic family of holomorphic principal H-bundles over X parameterized by $\mathbb{G}$. Let

$$f \colon \mathrm{Lie}(\mathbb{G}) = T_e\mathbb{G} \longrightarrow H^1(X, \mathrm{ad}(E_H))$$

be the infinitesimal deformation map at the identity element $e \in \mathbb{G}$ for this family of holomorphic principal H-bundles over X. Let E_H be any holomorphic principal H-bundle over X. We prove that the following three statements are equivalent:

1. *E_H is homogeneous.*
2. *E_H admits a logarithmic connection singular over D.*
3. *The above homomorphism f vanishes identically (meaning the above family of principal H-bundles is infinitesimally rigid at $e \in \mathbb{G}$).*

Corollary 4.5 says that the first two statements are equivalent. Proposition 5.1 says that the first and the third statements are equivalent.

2. Logarithmic connections on a holomorphic principal bundle

2.1. Logarithmic differential forms

Let X be a complex manifold of complex dimension d. A reduced effective divisor $D \subset X$ is said to be a *normal crossing divisor* if for point $x \in D$ there are holomorphic coordinate functions $z_1, \ldots, z_d$ defined on an open neighborhood $U \subset X$ of x with $z_1(x) = \cdots = z_d(x) = 0$, and there is an integer $1 \leq k \leq d$, such that

$$D \cap U = \{y \in U \mid z_1(y) = \cdots = z_k(y) = 0\}. \tag{1}$$

Note that it is not assumed that the irreducible components of D are smooth. In [7] and [8], the terminology "locally simple normal crossing divisor" is used; however, it seems that "normal crossing divisor" is used more often in the literature; see [3].

The holomorphic cotangent and tangent bundles of X will be denoted by Ω^1_X and TX respectively. Take a normal crossing divisor D on X. Let

$$TX(-\log D) \subset TX \tag{2}$$

be the coherent analytic subsheaf generated by all locally defined holomorphic vector fields v on X such that $v(\mathcal{O}_X(-D)) \subset \mathcal{O}_X(-D)$. In other words, if v is a holomorphic vector field defined over $U \subset X$, then v is a section of $TX(-\log D)|_U$ if and only if $v(f)|_{U\cap D} = 0$ for all holomorphic functions f on U that vanish on $U \cap D$. It is straightforward to check that the stalk of sections of $TX(-\log D)$ at the point x in (1) is generated by

$$z_1\frac{\partial}{\partial z_1}, \cdots, z_k\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_{k+1}}, \cdots, \frac{\partial}{\partial z_d}.$$

The condition that D is a normal crossing divisor implies that the coherent analytic sheaf $TX(-\log D)$ is in fact locally free. Clearly, we have

$$TX(-\log D)|_{X\setminus D} = TX|_{X\setminus D}.$$

This vector bundle $TX(-\log D)$ is called the *logarithmic tangent bundle* for the pair (X, D). Restricting the natural homomorphism $TX(-\log D) \longrightarrow TX$ to the divisor D, we get a homomorphism

$$\psi\colon TX(-\log D)|_D \longrightarrow TX|_D .$$

Let
$$\mathbb{L} := \text{kernel}(\psi) \subset TX(-\log D)|_D \tag{3}$$

be the kernel. To describe $\mathbb{L}$, let $\nu\colon \widetilde{D} \longrightarrow D$ be the normalization; the given condition on D implies that $\widetilde{D}$ is smooth. Then $\mathbb{L}$ is identified with the direct image

$$\mathbb{L} = \nu_* \mathcal{O}_{\widetilde{D}} . \tag{4}$$

The key point in the construction of this isomorphism is the following: Let Y be a Riemann surface and $y_0 \in Y$ a point; then for any holomorphic coordinate function z around y_0, with $z(y_0) = 0$, the evaluation of the local section $z\frac{\partial}{\partial z}$ of $TY \otimes \mathcal{O}_Y(-y_0)$ at the point y_0 does not depend on the choice of the coordinate function z.

Consider the Lie bracket operation on the locally defined holomorphic vector fields on X. The holomorphic sections of $TX(-\log D)$ are closed under this Lie bracket. Indeed, if v_1, v_2 are holomorphic sections of $TX(-\log D)$ over $U \subset X$, and f is a holomorphic function on U that vanishes on $U \cap D$, then from the identity

$$[v_1, v_2](f) = v_1(v_2(f)) - v_2(v_1(f))$$

we conclude that the function $[v_1, v_2](f)$ vanishes on $U \cap D$.

The dual vector bundle $TX(-\log D)^*$ is denoted by $\Omega^1_X(\log D)$. From (2) we have

$$(TX)^* = \Omega^1_X \subset \Omega^1_X(\log D) .$$

The stalk of sections of $\Omega^1_X(\log D)$ at the point x in (1) is generated by

$$\frac{1}{z_1}dz_1, \cdots, \frac{1}{z_k}dz_k, dz_{k+1}, \cdots, dz_d .$$

For every integer $i \geq 0$, define $\Omega^i_X(\log D) := \bigwedge^i \Omega^1_X(\log D)$. Let

$$\eta\colon D \longrightarrow X$$

be the inclusion map. Taking dual of the homomorphism ψ (see (3)), and using (4), we get the following short exact sequence of coherent analytic sheaves on X

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log D) \xrightarrow{\mathcal{R}} (\eta \circ \nu)_* \mathcal{O}_{\widetilde{D}} \longrightarrow 0 ,$$

where ν is the map in (4); the above homomorphism $\mathcal{R}$ is known as the *residue map.* We refer the reader to [6] for more details on logarithmic forms and vector fields.

2.2. Atiyah bundle and logarithmic connection

Let H be a complex Lie group. The Lie algebra of H will be denoted by $\mathfrak{h}$. Let

$$p\colon E_H \longrightarrow X \tag{5}$$

be a holomorphic principal H-bundle; we recall that this means that E_H is a holomorphic fiber bundle over X equipped with a holomorphic right-action of the group H

$$q' \colon E_H \times H \longrightarrow E_H \tag{6}$$

such that $p(q'(z, h)) = p(z)$ for all $(z, h) \in E_H \times H$, where p is the projection in (5) and, furthermore, the resulting map to the fiber product

$$E_H \times H \longrightarrow E_H \times_X E_H\,, \quad (z, h) \longrightarrow (z, q'(z, h))$$

is a biholomorphism. For notational convenience, the point $q'(z, h) \in E_H$, where $(z, h) \in E_H \times H$, will be denoted by zh.

As before, let $D \subset X$ be a normal crossing divisor. Since p in (5) is a holomorphic submersion, the inverse image

$$\widehat{D} := p^{-1}(D) \subset E_H \tag{7}$$

is also a normal crossing divisor. Consider the action of H on the tangent bundle TE_H given by the action of H on E_H in (6). This action of H on TE_H clearly preserves the subsheaf $TE_H(-\log \widehat{D}) \subset TE_H$. The corresponding quotient

$$\mathrm{At}(E_H)(-\log D) := TE_H(-\log \widehat{D})/H \longrightarrow E_H/H = X \tag{8}$$

is evidently a holomorphic vector bundle over X; it is called the *logarithmic Atiyah bundle* (see [1] for the case where D is the zero divisor).

Let $dp \colon TE_H \longrightarrow p^*TX$ be the differential of the projection p in (5). Let

$$\mathcal{K} := \mathrm{kernel}(dp) \subset TE_H$$

be the kernel of dp. So we have the following short exact sequence of holomorphic vector bundles on E_H:

$$0 \longrightarrow \mathcal{K} \longrightarrow TE_H \stackrel{dp}{\longrightarrow} p^*TX \longrightarrow 0\,. \tag{9}$$

Note that we have $\mathcal{K} \subset TE_H(-\log \widehat{D})$, and $dp(TE_H(-\log \widehat{D})) = p^*(TX(-\log D))$. Therefore, the short exact sequence in (9) gives the following short exact sequence of holomorphic vector bundles over E_H

$$0 \longrightarrow \mathcal{K} \longrightarrow TE_H(-\log \widehat{D}) \stackrel{dp}{\longrightarrow} p^*(TX(-\log D)) \longrightarrow 0 \tag{10}$$

(the restriction of dp to $TE_H(-\log \widehat{D})$ is also denoted by dp). The above action of H on TE_H clearly preserves the subbundle $\mathcal{K}$. The quotient

$$\mathrm{ad}(E_H) := \mathcal{K}/H \longrightarrow E_H/H = X$$

is called the *adjoint vector bundle* for E_H. We note that $\mathrm{ad}(E_H)$ is identified with the holomorphic vector bundle $E_H \times^H \mathfrak{h} \longrightarrow X$ associated to the principal H-bundle E_H for the adjoint action of H on the Lie algebra $\mathfrak{h}$. This isomorphism between $\mathcal{K}/H$ and $E_H \times^H \mathfrak{h}$ is obtained from the fact that the action of H on E_H identifies $\mathcal{K}$ with the trivial holomorphic vector bundle $E_H \times \mathfrak{h}$ over E_H with fiber $\mathfrak{h}$.

Take quotient of the vector bundles in (10) by the actions of H. From (10) we get a short exact sequence of holomorphic vector bundles over X

$$0 \longrightarrow \mathrm{ad}(E_H) := \mathcal{K}/H \xrightarrow{\iota_0} (TE_H(-\log \widehat{D}))/H =: \mathrm{At}(E_H)(-\log D) \tag{11}$$

$$\xrightarrow{\beta} (p^*(TX(-\log D)))/H = TX(-\log D) \longrightarrow 0\,;$$

it is called the *logarithmic Atiyah exact sequence* for E_H. The homomorphism dp in (10) descents to the homomorphism β in (11).
A *logarithmic connection* on E_H singular over D is a holomorphic homomorphism of vector bundles

$$\varphi\colon TX(-\log D) \longrightarrow \mathrm{At}(E_H)(-\log D)$$

such that

$$\beta \circ \varphi = \mathrm{Id}_{TX(-\log D)}\,, \tag{12}$$

where β is the projection in (11). In other words, giving a logarithmic connection on E_H singular over D is equivalent to giving a holomorphic splitting of the short exact sequence in (11). See [4] for logarithmic connections (see also [2]).

As noted before, the locally defined holomorphic sections of the logarithmic tangent bundles $TX(-\log D)$ and $TE_H(-\log \widehat{D})$ are closed under the Lie bracket operation of vector fields. The locally defined holomorphic sections of the subbundle $\mathcal{K}$ in (9) are clearly closed under the Lie bracket operation. The homomorphisms in the exact sequence (9) are all compatible with the Lie bracket operation. Since the Lie bracket operation commutes with diffeomorphisms, for any two H-invariant holomorphic vector fields v, w defined on an H-invariant open subset of E_H, their Lie bracket $[v, w]$ is again holomorphic and H-invariant. Therefore, the sheaves of sections of the three vector bundles in (11) are all equipped with a Lie bracket operation. Moreover, all the homomorphisms in (11) commute with these operations.
Take a homomorphism

$$\varphi\colon TX(-\log D) \longrightarrow \mathrm{At}(E_H)(-\log D)$$

satisfying the condition in (12). Then for any two holomorphic sections v_1, v_2 of $TX(-\log D)$ over $U \subset X$, consider

$$\mathbb{K}(v_1, v_2) := [\varphi(v_1), \varphi(v_2)] - \varphi([v_1, v_2])\,.$$

The projection β in (11) intertwines the Lie bracket operations on the sheaves of sections of $\mathrm{At}(E_H)(-\log D)$ and $TX(-\log D)$, and hence we have $\beta(\mathbb{K}(v_1, v_2)) = 0$. Consequently, $\mathbb{K}(v_1, v_2)$ is a holomorphic section of $\mathrm{ad}(E_H)$ over U. From the identity $[fv, w] = f[v, w] - w(f)\cdot v$, where f is a holomorphic function while v and w are holomorphic vector fields, it follows that

$$\mathbb{K}(fv_1, v_2) = f\mathbb{K}(v_1, v_2)\,.$$

Also, we have $\mathbb{K}(v_1, v_2) = -\mathbb{K}(v_2, v_1)$. Therefore, the mapping $(v_1, v_2) \mapsto \mathbb{K}(fv_1, v_2)$ defines a holomorphic section

$$\mathbb{K}(\varphi) \in H^0(X, \Omega^2_X(\log D) \otimes \mathrm{ad}(E_H))\,. \tag{13}$$

The section $\mathbb{K}(\varphi)$ in (13) is called the *curvature* of the logarithmic connection φ.

2.3. Residue of a logarithmic connection

The quotient $(TE_H)/H$ is a holomorphic vector bundle over $E_H/H = X$. It is the Atiyah bundle for E_H; let $\mathrm{At}(E_H)$ denote this Atiyah bundle (see [1]). Taking quotient of the vector bundles in (9) by the actions of H, from (9) we get a short exact sequence of holomorphic vector bundles over X

$$0 \longrightarrow \mathrm{ad}(E_H) := \mathcal{K}/H \longrightarrow (TE_H)/H =: \mathrm{At}(E_H) \stackrel{\beta'}{\longrightarrow} (p^*TX)/H = TX \longrightarrow 0\,, \tag{14}$$

which is known as the Atiyah exact sequence for E_H (see [1]); note that β in (11) is the restriction of β' in (14). Restricting to D the exact sequences in (11) and (14), we get the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ad}(E_H)|_D & \stackrel{\widehat{\iota_0}}{\longrightarrow} & \mathrm{At}(E_H)(-\log D)|_D & \stackrel{\widehat{\beta}}{\longrightarrow} & TX(-\log D)|_D & \longrightarrow & 0 \\ & & \| & & \downarrow \mu & & \downarrow \psi & & \\ 0 & \longrightarrow & \mathrm{ad}(E_H)|_D & \stackrel{\iota_1}{\longrightarrow} & \mathrm{At}(E_H)|_D & \stackrel{\widehat{\beta'}}{\longrightarrow} & TX|_D & \longrightarrow & 0 \end{array} \tag{15}$$

whose rows are exact; the map ψ in the one in (3) and μ is the homomorphism given by the natural homomorphism $\mathrm{At}(E_H)(-\log D) \longrightarrow \mathrm{At}(E_H)$. In (15) the following convention is employed: the restriction to D of a map on X is denoted by the same symbol after adding a hat. From (3) we know that the kernel of ψ is $\mathbb{L} = \nu_*\mathcal{O}_{\widetilde{D}}$ (see (4)). Let

$$\iota_{\mathbb{L}} : \mathbb{L} \longrightarrow TX(-\log D)|_D$$

be the inclusion map. Let $\varphi : TX(-\log D) \longrightarrow \mathrm{At}(E_H)(-\log D)$ be a logarithmic connection on E_H singular over D. Consider the composition

$$\widehat{\varphi} \circ \iota_{\mathbb{L}} : \mathbb{L} \longrightarrow \mathrm{At}(E_H)(-\log D)|_D$$

(the restriction of φ to D is denoted by $\widehat{\varphi}$). From the commutativity of the diagram in (15) it follows that

$$\widehat{\beta'} \circ \mu \circ \widehat{\varphi} \circ \iota_{\mathbb{L}} = \psi \circ \widehat{\beta} \circ \widehat{\varphi} \circ \iota_{\mathbb{L}}\,. \tag{16}$$

But $\widehat{\beta} \circ \widehat{\varphi} = \mathrm{Id}_{TX(-\log D)|_D}$ by (12), while $\psi \circ \iota_{\mathbb{L}} = 0$ by (3), so that

$$\psi \circ \widehat{\beta} \circ \widehat{\varphi} \circ \iota_{\mathbb{L}} = 0\,.$$

Hence from (16) we conclude that $\widehat{\beta'} \circ \mu \circ \widehat{\varphi} \circ \iota_{\mathbb{L}} = 0$.

Now from the exactness of the bottom row in (15) it follows that the image of $\mu \circ \widehat{\varphi} \circ \iota_{\mathbb{L}}$ is contained in the image of the injective map ι_1 in (15). Therefore, $\mu \circ \widehat{\varphi} \circ \iota_{\mathbb{L}}$ defines a map

$$\mathcal{R}_\varphi : \mathbb{L} \longrightarrow \mathrm{ad}(E_H)|_D\,. \tag{17}$$

The homomorphism $\mathcal{R}_\varphi$ in (17) is called the *residue* of the logarithmic connection φ [4].

3. Manifolds with trivial logarithmic tangent bundle

3.1. Trivialization of the logarithmic tangent bundle

We now assume that

1. X is compact, and
2. the logarithmic tangent bundle $TX(-\log D)$ is holomorphically trivial.

Such pairs $(X,\, D)$ were classified in [8]; this was done earlier in [7] under an extra assumption that X lies in class $\mathcal{C}$. Below we briefly recall from [8].

Let $X_0 := X \setminus D$ be the complement. Denote by $\mathbb{G}$ the connected component of the group of holomorphic automorphisms of X that preserve D. This $\mathbb{G}$ is a complex Lie group and it acts transitively on X_0. For each point of X_0, the isotropy subgroup of $\mathbb{G}$ is discrete. Let C denote the connected component of the center of $\mathbb{G}$ containing the identity element. It is a semi-torus, meaning C is a quotient of the additive group $(\mathbb{C}^{\dim C},\, +)$ by a discrete subgroup that generates the vector space $\mathbb{C}^{\dim C}$. There is a locally holomorphically trivial fibration $\pi\colon X \longrightarrow Y$, such that

- Y is a compact parallelizable manifold, more precisely, the quotient group $\mathbb{G}/C$ acts transitively on Y with discrete cocompact isotropies,
- the projection π is $\mathbb{G}$–equivariant and it admits a holomorphic connection preserved by the action of $\mathbb{G}$,
- the typical fiber of the fiber bundle π is an equivariant compactification of C, such that all isotropy subgroups are semi-tori, and
- C is the structure group of the fiber bundle.

(See [8, p. 196, Theorem 1].)

Let $\mathfrak{g}$ denote the Lie algebra of the above defined group $\mathbb{G}$. Let $\mathcal{G} := X \times \mathfrak{g} \longrightarrow X$ be the trivial holomorphic vector bundle over X with fiber $\mathfrak{g}$.

The tautological action of $\mathbb{G}$ on X (recall that $\mathbb{G}$ is a subgroup of the group of holomorphic automorphisms of X) produces a homomorphism

$$\gamma_0\colon \mathfrak{g} \longrightarrow H^0(X,\, TX)\,.$$

This homomorphism γ_0 preserves the Lie algebra structures of $\mathfrak{g}$ and $H^0(X,\, TX)$ (its Lie algebra structure is given by Lie bracket of vector fields). The action of $\mathbb{G}$ on X, by definition, preserves D, and from this it follows that

$$\gamma_0(\mathfrak{g}) \subset H^0(X,\, TX(-\log D))\,. \tag{18}$$

Let

$$\gamma\colon \mathcal{G} \,:=\, X \times \mathfrak{g} \longrightarrow TX(-\log D) \tag{19}$$

be the $\mathcal{O}_X$–linear homomorphism defined by

$$\gamma(x)(v) \,=\, \gamma_0(v)(x) \in TX(-\log D)_x\,, \quad \forall\, x \in X\,,\ v \in \mathfrak{g}\,.$$

Lemma 3.1. *The homomorphism γ in* (19) *is an isomorphism.*

Proof. Since $\mathbb{G}$ acts transitively on $X \setminus D$, we have $\dim \mathfrak{g} \geq \dim X = d$. Take any $x \in X_0 = X \setminus D$.

The isotropy subgroup of $\mathbb{G}$ for x is discrete, and hence the homomorphism

$$\gamma(x)\colon \mathfrak{g} \longrightarrow T_x X$$

is injective. Now from the above inequality $\dim \mathfrak{g} \geq \dim X$ it follows immediately that $\gamma(x)$ is an isomorphism if $x \in X_0$. On the other hand, both the vector bundles $\mathcal{G}$ and $TX(-\log D)$ in (19) are holomorphically trivial, and X is compact. From these it follows that γ is an isomorphism. To see this, consider the homomorphism of top exterior products

$$\bigwedge^d \gamma \,:\, \mathcal{O}_X \,=\, \bigwedge^d \mathcal{G} \longrightarrow \bigwedge^d TX(-\log D) \,=\, \mathcal{O}_X$$

induced by γ in (19), where $d = \dim_{\mathbb{C}} X$. Any homomorphism $\mathcal{O}_X \longrightarrow \mathcal{O}_X$ is given by a globally defined holomorphic function on X. From the above observation that γ is an isomorphism over X_0 it follows immediately that $\bigwedge^d \gamma$ is an isomorphism over $X \setminus D$. Therefore, we conclude that $\bigwedge^d \gamma$ corresponds to a nowhere zero holomorphic function on X. This implies that $\bigwedge^d \gamma$ is an isomorphism over entire X. From this it follows immediately that γ is an isomorphism over X. ■

Corollary 3.2. *The homomorphism*

$$\gamma_0\colon \mathfrak{g} \longrightarrow H^0(X,\, TX(-\log D))$$

in (18) *is an isomorphism.*

Proof. The global holomorphic sections of the two holomorphic vector bundles $\mathcal{G}$ and $TX(-\log D)$ are $\mathfrak{g}$ and $H^0(X,\, TX(-\log D))$ respectively. The homomorphism γ_0 evidently coincides with the homomorphism of global sections

$$H^0(X,\, \mathcal{G}) \longrightarrow H^0(X,\, TX(-\log D))$$

given by γ in (19). But γ is an isomorphism by Lemma 3.1. Consequently, γ_0 is an isomorphism. ■

3.2. Equivariant bundles

Let M be a connected complex Lie group and $\rho\colon M \longrightarrow \mathbb{G}$ a surjective holomorphic homomorphism, where $\mathbb{G}$ is the group in Section 3.1. Let H be a *connected* complex Lie group.

Definition 3.3. A ρ*–equivariant* principal H-bundle is a pair $(E_H,\, \delta)$, where E_H as in (5) is a holomorphic principal H-bundle over X, and

$$\delta\colon M \times E_H \longrightarrow E_H$$

is a left–action of the group M on E_H, such that the following conditions hold:

1. the action of M is holomorphic, meaning the map δ is holomorphic,
2. the actions of M and H on E_H commute,
3. for any $(y,\, z) \in M \times E_H$,

$$p(\delta(y, z)) \,=\, \rho(y)(p(z))\,, \tag{20}$$

where p is the projection in (5) (recall that $\mathbb{G} \subset \mathrm{Aut}(X)$).

4. A criterion for equivariance

Theorem 4.1. (1) *Let (E_H, δ) be a ρ-equivariant holomorphic principal H-bundle, where $\rho\colon M \longrightarrow \mathbb{G}$ is a surjective holomorphic homomorphism of connected complex Lie groups. Then E_H admits a logarithmic connection singular over D.*

(2) *Let E_H be a holomorphic principal H-bundle over X admitting a logarithmic connection singular over D. Then there is connected complex Lie group M and a surjective holomorphic homomorphism $\rho\colon M \longrightarrow \mathbb{G}$, such that there is a left-action δ of M on E_H with the property that (E_H, δ) is a ρ–equivariant holomorphic principal H-bundle.*

Proof. To prove (1), let (E_H, δ) be a ρ–equivariant holomorphic principal H-bundle, where $\rho\colon M \longrightarrow \mathbb{G}$ is a surjective holomorphic homomorphism of connected complex Lie groups. The Lie algebra of M will be denoted by $\mathfrak{m}$. Let

$$\phi\colon \mathfrak{m} \longrightarrow H^0(E_H, TE_H)$$

be the homomorphism given by the action δ of M on E_H. From the given condition, that the actions of M and H on E_H commute, it follows immediately that

$$\phi(\mathfrak{m}) \subset H^0(E_H, TE_H)^H\,,$$

where $H^0(E_H, TE_H)^H \subset H^0(E_H, TE_H)$ is the space of H-invariant holomorphic vector fields on E_H for the natural action of H on E_H. From the definition $\mathrm{At}(E_H) := (TE_H)/H$ it evidently follows that

$$H^0(E_H, TE_H)^H \,=\, H^0(X, \mathrm{At}(E_H))\,,$$

so we have $\phi(\mathfrak{m}) \subset H^0(X, \mathrm{At}(E_H))$. Since the action of $\mathbb{G}$ on X, by definition, preserves D, from (20) it follows that the action of M on E_H preserves the inverse image $\widehat{D} = p^{-1}(D)$ in (7). Hence we have

$$\phi\colon \mathfrak{m} \longrightarrow H^0(E_H, TE_H(-\log \widehat{D}))^H \subset H^0(E_H, TE_H)^H\,.$$

From (8) it now follows that

$$\phi(\mathfrak{m}) \subset H^0(X, \mathrm{At}(E_H)(-\log D)) \subset H^0(X, \mathrm{At}(E_H))\,. \tag{21}$$

Let

$$\rho'\colon \mathfrak{m} \longrightarrow \mathfrak{g} \tag{22}$$

be the homomorphism of Lie algebras for the homomorphism ρ of Lie groups. From (20) it follows that for all $w \in \mathfrak{m}$, we have

$$\beta \circ \phi(w) \,=\, \gamma_0 \circ \rho'(w) \in H^0(X, TX(-\log D))\,,$$

where β and γ_0 are the homomorphisms in (11) and Corollary 3.2 respectively. The homomorphism ρ' is surjective because ρ is so. Combining this with Corollary 3.2 it follows that the homomorphism

$$\gamma_0 \circ \rho'\colon \mathfrak{m} \longrightarrow H^0(X, TX(-\log D)) \tag{23}$$

is surjective. Fix a $\mathbb{C}$–linear map

$$\xi\colon H^0(X,\, TX(-\log D)) \longrightarrow \mathfrak{m} \tag{24}$$

such that

$$(\gamma_0 \circ \rho') \circ \xi \,=\, \mathrm{Id}_{H^0(X, TX(-\log D))}\,. \tag{25}$$

Now consider the composition

$$\phi \circ \xi\colon H^0(X,\, TX(-\log D)) \longrightarrow H^0(X,\, \mathrm{At}(E_H)(-\log D))\,, \tag{26}$$

where ϕ and ξ are the maps in (21) and (24) respectively. From (25) it follows that

$$\beta_* \circ (\phi \circ \xi) \,=\, \mathrm{Id}_{H^0(X, TX(-\log D))}\,, \tag{27}$$

where

$$\beta_*\colon H^0(X,\, \mathrm{At}(E_H)(-\log D)) \,\longrightarrow\, H^0(X,\, TX(-\log D)) \tag{28}$$

is the homomorphism of global sections induced by the map β of vector bundles in (11). Since the vector bundle $TX(-\log D)$ is holomorphically trivial, the map $\phi \circ \xi$ in (26) defines a homomorphism

$$\widetilde{\xi}\colon TX(-\log D) \longrightarrow \mathrm{At}(E_H)(-\log D)\,. \tag{29}$$

To construct this map $\widetilde{\xi}$, for any $x \,\in\, X$ and $v \,\in\, TX(-\log D)_x$, let

$$\widetilde{v} \,\in\, H^0(X,\, TX(-\log D))$$

be the unique holomorphic section such that $\widetilde{v}(x) \,=\, v$. The map $\widetilde{\xi}$ sends v to $\phi \circ \xi(\widetilde{v})(x) \,\in\, \mathrm{At}(E_H)(-\log D)_x$. From (27) it follows immediately that

$$\beta \circ \widetilde{\xi} \,=\, \mathrm{Id}_{TX(-\log D)}\,.$$

In other words, $\widetilde{\xi}$ is a logarithmic connection on E_H singular over D. This proves (1) in the theorem.

We shall now prove (2) in the theorem. Let $\mathbb{A}$ denote the space of all pairs of the form $(\tau,\, f)$, where $\tau\colon X \longrightarrow X$ is a biholomorphism and $f\colon E_H \longrightarrow E_H$ is a biholomorphism such that

- $\tau \circ p \,=\, p \circ f$, where p is the projection in (5), and
- $f(q'(z,h)) \,=\, q'(f(z),h)$, for all $(z,\, h) \in E_H \times H$, where q' is the action in (6) (in other words, f is H–equivariant).

We note that $\mathbb{A}$ is a group, with group operation map and inverse map respectively given by

$$(\tau_1,\, f_1)\cdot(\tau_2,\, f_2) \,=\, (\tau_1 \circ \tau_2,\, f_1 \circ f_2) \quad \text{and} \quad (\tau,\, f)^{-1} \,=\, (\tau^{-1},\, f^{-1})\,.$$

This $\mathbb{A}$ is a complex Lie group with Lie algebra $H^0(X,\, \mathrm{At}(E_H))$ (the Lie algebra structure on $H^0(X,\, \mathrm{At}(E_H))$ is given by the Lie bracket of vector fields). Let $\mathbb{A}_D \subset \mathbb{A}$ denote the subgroup consisting of all $(\tau,\, f)$ of the above type such that $\tau(D) \,=\, D$. It is a complex Lie subgroup with Lie algebra $H^0(X,\, \mathrm{At}(E_H)(-\log D))$.

Let $\mathbb{A}_D^0 \subset \mathbb{A}_D$ be the connected component containing the identity element. Define a homomorphism

$$\theta\colon \mathbb{A}_D^0 \longrightarrow \mathbb{G}\,, \quad (\tau,\, f) \longmapsto \tau\,, \tag{30}$$

where $\mathbb{G}$ is the group defined in Section 3.1. Now assume that E_H admits a logarithmic connection singular over D. We shall show that the homomorphism θ in (30) is surjective.

To prove that θ is surjective, fix a logarithmic connection

$$\varphi\colon TX(-\log D) \longrightarrow \mathrm{At}(E_H)(-\log D)$$

on E_H singular over D. Let

$$\varphi_*\colon H^0(X,\, TX(-\log D)) \longrightarrow H^0(X,\, \mathrm{At}(E_H)(-\log D)) \tag{31}$$

be the homomorphism of global sections induced by φ. Let

$$\theta'\colon \mathrm{Lie}(\mathbb{A}_D^0) \;=\; H^0(X,\, \mathrm{At}(E_H)(-\log D)) \longrightarrow \mathfrak{g} \tag{32}$$

be the homomorphism of Lie algebras associated to the homomorphism θ in (30). To prove that θ is surjective it suffices to show that θ' is surjective, because $\mathbb{G}$ is connected. Now consider the composition

$$\theta' \circ \varphi_*\colon H^0(X,\, TX(-\log D)) \longrightarrow \mathfrak{g}\,,$$

where φ_* and θ' are constructed in (31) and (32) respectively. From the constructions of $\theta' \circ \varphi_*$ and the map γ_0 (see Corollary 3.2) it follow immediately that

$$(\theta' \circ \varphi_*) \circ \gamma_0 \;=\; \mathrm{Id}_{\mathfrak{g}}\,.$$

On the other hand, from Corollary 3.2 we know that the homomorphism γ_0 is an isomorphism. Hence $\theta' \circ \varphi_*$ is also an isomorphism. This implies that θ' is surjective. As noted before, the surjectivity of the homomorphism θ in (30) follows from the surjectivity of θ'.

The group $\mathbb{A}_D^0$ in (30) has a tautological holomorphic action on E_H (recall that it is a subgroup of the automorphism group of the complex manifold E_H); let

$$\mathbb{T}_D\colon \mathbb{A}_D^0 \times E_H \longrightarrow E_H \tag{33}$$

be this tautological action of $\mathbb{A}_D^0$ on E_H. The pair $(E_H,\, \mathbb{T}_D)$ is evidently a θ–equivariant holomorphic principal H-bundle, where θ is the homomorphism in (30). This completes the proof of (2). ∎

As in Theorem 4.1(1), let $(E_H,\, \delta)$ be a ρ–equivariant holomorphic principal H-bundle, where $\rho\colon M \longrightarrow \mathbb{G}$ is a surjective holomorphic homomorphism of connected complex Lie groups. Fix a homomorphism ξ as in (24) satisfying the condition in (25). Construct the homomorphism $\widetilde{\xi}$ in (29) using ξ. It was shown in the proof of Theorem 4.1(1) that $\widetilde{\xi}$ is a logarithmic connection on E_H singular on D. Let

$$\mathbb{K}(\widetilde{\xi}) \in H^0(X,\, \Omega_X^2(\log D) \otimes \mathrm{ad}(E_H)) \tag{34}$$

be the curvature of the logarithmic connection on E_H defined by $\widetilde{\xi}$; curvature was defined in (13). We shall compute $\mathbb{K}(\widetilde{\xi})$ in (34). For that, consider the linear map

$$K_0\colon \bigwedge^2 H^0(X,\, TX(-\log D)) \longrightarrow \mathfrak{m}\,, \quad v\wedge w \longmapsto [\xi(v),\, \xi(w)] - \xi([v,\, w])\,; \quad (35)$$

so K_0 measures how the $\mathbb{C}$–linear homomorphism ξ fails to be a homomorphism of Lie algebras. We have the composition homomorphism

$$(\beta_*\circ\phi)\circ K_0\colon \bigwedge^2 H^0(X,\, TX(-\log D)) \longrightarrow H^0(X,\, TX(-\log D))\,,$$

where β_* is the homomorphism in (28) of global sections given by β in (11), and ϕ is the homomorphism in (21). From (25), and the fact that both ϕ and β are Lie algebra structure preserving, it follows immediately that

$$(\beta_*\circ\phi)\circ K_0 \;=\; 0\,.$$

Hence from the short exact sequence in (11) it now follows that

$$\phi\circ K_0(\bigwedge^2 H^0(X,\, TX(-\log D))) \subset H^0(X,\, \mathrm{ad}(E_H))$$
$$\subset\; H^0(X,\, \mathrm{At}(E_H)(-\log D))\,.$$

Recall that the holomorphic vector bundle $TX(-\log D)$ is holomorphically trivial. So the holomorphic vector bundle $\bigwedge^2(TX(-\log D))$ is also holomorphically trivial, and, moreover, we have

$$\bigwedge^2 H^0(X,\, TX(-\log D)) \;=\; H^0(X,\, \bigwedge^2(TX(-\log D)))\,.$$

So the above homomorphism $\phi\circ K_0$ can be considered as a homomorphism

$$\phi\circ K_0\colon H^0(X,\, \bigwedge^2(TX(-\log D))) \longrightarrow H^0(X,\, \mathrm{ad}(E_H))\,. \quad (36)$$

Since $\bigwedge^2(TX(-\log D))$ is holomorphically trivial, the homomorphism $\phi\circ K_0$ in (36) produces a homomorphism of coherent analytic sheaves

$$(\phi\circ K_0)'\colon \bigwedge^2(TX(-\log D)) \longrightarrow \mathrm{ad}(E_H)$$

as follows: for any $x\in X$ and $w\in\bigwedge^2(TX(-\log D))_x$, let $\widetilde{w}$ be the unique element of $H^0(X,\, \bigwedge^2(TX(-\log D)))$ satisfying the condition that $\widetilde{w}(x) \,=\, w$. Now set

$$(\phi\circ K_0)'(x)(w) \;=\; (\phi\circ K_0)(\widetilde{w})(x) \in \mathrm{ad}(E_H)_x\,.$$

The above homomorphism $(\phi\circ K_0)'$ defines an element

$$\sigma\in H^0(X,\mathrm{Hom}(\bigwedge^2(TX(-\log D)),\, \mathrm{ad}(E_H))) = H^0(X,\Omega^2_X(\log D)\otimes\mathrm{ad}(E_H)). \quad (37)$$

From the definition of $\mathbb{K}(\widetilde{\xi})$ in (34) (see (13)) and the construction of σ in (37) it is straightforward to check that

$$\sigma \;=\; \mathbb{K}(\widetilde{\xi})\,. \quad (38)$$

The residue of the logarithmic connection $\widetilde{\xi}$ (constructed in (17)) can also be described in terms of ξ.

Lemma 4.2. *Let (E_H, δ) be a ρ–equivariant holomorphic principal H-bundle, where $\rho\colon M \longrightarrow \mathbb{G}$ is a surjective holomorphic homomorphism of connected complex Lie groups, such that the homomorphism $\gamma_0 \circ \rho'$ in (23) is an isomorphism. Then E_H admits a flat holomorphic connection singular over D which is preserved by the action δ of M on E_H.*

Proof. Set ξ in (24) to be the inverse of the isomorphism $\gamma_0 \circ \rho'$. Note that as $\gamma_0 \circ \rho'$ is an isomorphism, the condition in (25) forces ξ to be the inverse of $\gamma_0 \circ \rho'$. Therefore, ξ is an isomorphism of Lie algebras. Hence K_0 in (35) vanishes. Now from (38) it follows that connection $\widetilde{\xi}$ in (29) is flat. Since the image of ξ in $\mathfrak{m}$ is preserved by the adjoint action of M on $\mathfrak{m}$ (in fact the image of ξ is entire $\mathfrak{m}$), it is straightforward to deduce that the action of M on E_H preserves the connection $\widetilde{\xi}$. ∎

Definition 4.3. A holomorphic principal H-bundle E_H over X is called *homogeneous* if there is a connected complex Lie group M and a surjective holomorphic homomorphism $\rho\colon M \longrightarrow \mathbb{G}$, such that there is a holomorphic action δ of M on E_H satisfying the condition that (E_H, δ) is a ρ–equivariant holomorphic principal H-bundle.

Proposition 4.4. *A holomorphic principal H-bundle E_H over X is homogeneous if and only if the holomorphic principal H-bundle g^*E_H is holomorphically isomorphic to E_H for every $g \in \mathbb{G}$.*

Proof. First assume that there is a connected complex Lie group M and a surjective holomorphic homomorphism $\rho\colon M \longrightarrow \mathbb{G}$, such that there is a holomorphic action δ of M on E_H satisfying the condition that (E_H, δ) is a ρ–equivariant holomorphic principal H-bundle. For any $g \in \mathbb{G}$, take any $\widetilde{g} \in \rho^{-1}(g)$. Then the action of $\widetilde{g}$ on E_H (given by δ) is a holomorphic isomorphism of the holomorphic principal H-bundle g^*E_H with E_H.
To prove the converse, assume that g^*E_H is holomorphically isomorphic to E_H for every $g \in \mathbb{G}$. Consider the homomorphism θ in (30). Since g^*E_H is holomorphically isomorphic to E_H for every $g \in \mathbb{G}$, we conclude that the homomorphism θ is surjective. Now in Definition 4.3, set $M = \mathbb{A}^0_D$, $\rho = \theta$ and δ to be the tautological action $\mathbb{T}_D$ of $\mathbb{A}^0_D$ on E_H (see (33)). Since θ is surjective, it follows that E_H is homogeneous. ∎

The following is an immediate consequence of Theorem 4.1 and Proposition 4.4.

Corollary 4.5. *A holomorphic principal H-bundle E_H over X admits a logarithmic connection singular over D if and only if for every $g \in \mathbb{G}$ the holomorphic principal H-bundle g^*E_H is holomorphically isomorphic to E_H.*

5. Infinitesimal deformations of principal bundles

The space of all infinitesimal deformations of a holomorphic principal H-bundle $E_H \stackrel{p}{\longrightarrow} X$ are parameterized by $H^1(X, \mathrm{ad}(E_H))$. This in particular means the following. Let T be a complex manifold with a base point $t_0 \in T$. Let $\mathcal{E}_H \longrightarrow T \times X$ be a holomorphic principal H-bundle. For any $t \in T$, the holomorphic principal

H-bundle $\mathcal{E}_H|_{\{t\}\times X}$ on X will be denoted by $\mathcal{E}_H^t$. Assume that we are given a holomorphic isomorphism of E_H with the principal H-bundle $\mathcal{E}_H^{t_0}$; here X is identified with $\{t_0\}\times X$ using the map $x \longmapsto (t_0, x)$. So $\{\mathcal{E}_H^t\}_{t\in T}$ is a holomorphic family of holomorphic principal H-bundles over X such that the holomorphic principal H-bundle over X for $t_0 \in T$ is the given holomorphic principal H-bundle E_H. Then there is a $\mathbb{C}$–linear homomorphism

$$T_{t_0}T \longrightarrow H^1(X,\, \mathrm{ad}(E_H)) \tag{39}$$

which is compatible with the pullback operation of families of holomorphic principal H-bundles on X (see [5] for more details). The homomorphism in (39) is called the *infinitesimal deformation map* at t_0 for the family $\mathcal{E}_H$ of holomorphic principal H-bundles over X. Consider the Lie group $\mathbb{G}$ in Section 3.1. Let

$$\varpi\colon \mathbb{G}\times X \longrightarrow X, \quad (g,\, x) \longmapsto g(x) \tag{40}$$

be the evaluation map (recall that $\mathbb{G}$ is a subgroup of the automorphism group of X). Note that this map ϖ is holomorphic. As in (5), consider a holomorphic principal H-bundle $E_H \xrightarrow{p} X$. Let

$$\mathcal{E}_H := \varpi^* E_H \longrightarrow \mathbb{G}\times X \tag{41}$$

be the pulled back holomorphic principal H-bundle over $\mathbb{G}\times X$. Note that the restriction $\mathcal{E}_H^e = (\varpi^* E_H)|_{\{e\}\times X}$, where e is the identity element of $\mathbb{G}$, is identified with the holomorphic principal H-bundle E_H. Therefore, as in (39), we have the infinitesimal deformation map

$$f\colon T_e\mathbb{G} = \mathfrak{g} \longrightarrow H^1(X,\, \mathrm{ad}(E_H))\,. \tag{42}$$

Proposition 5.1. *A holomorphic principal H-bundle E_H over X is homogeneous (see Definition* 4.3) *if and only if the homomorphism f in* (42) *is the zero map.*

Proof. First assume that E_H is homogeneous. So there is a connected complex Lie group M and a surjective holomorphic homomorphism $\rho\colon M \longrightarrow \mathbb{G}$, such that there is a holomorphic action

$$\delta\colon M\times E_H \longrightarrow E_H \tag{43}$$

as in Definition 3.3 satisfying the condition that $(E_H,\, \delta)$ is a ρ–equivariant holomorphic principal H-bundle. We have the principal H-bundle

$$\mathcal{F}_H := (\rho\times \mathrm{Id}_X)^*\mathcal{E}_H \longrightarrow M\times E_H\,, \tag{44}$$

where $\mathcal{E}_H$ is the holomorphic principal H-bundle in (41). Let $e' \in M$ be the identity element. For the family of principal H-bundles $\mathcal{F}_H$ parameterized by M, as in (39), we have the infinitesimal deformation map

$$f_1\colon T_{e'}M = \mathfrak{m} \longrightarrow H^1(X,\, \mathrm{ad}(E_H))\,, \tag{45}$$

where $\mathfrak{m}$ as before is the Lie algebra of M. Since the homomorphism in (39) is compatible with the pullback operation of families of holomorphic principal H-bundles over X, we have

$$f_1 = f\circ \rho'\,, \tag{46}$$

where f, f_1 and ρ' are the homomorphisms in (42), (45) and (22) respectively.
It can be shown that the action δ in (43) identifies the family of holomorphic principal H-bundles $\mathcal{F}_H$ in (44) with the trivial family

$$p_2^* E_H \longrightarrow M \times X\,, \tag{47}$$

where $p_2 \colon M \times X \longrightarrow X$ is the projection $(z,\, x) \longmapsto x$. To prove this, consider the map

$$\widetilde{\delta} \colon M \times E_H \longrightarrow M \times E_H\,, \quad (z,\, y) \longmapsto (z,\, \delta(z, y))\,.$$

From the equality in (20) it follows that this map $\widetilde{\delta}$ is a holomorphic isomorphism of the holomorphic principal H-bundle $\mathcal{F}_H$ in (44) with the holomorphic principal H-bundle $p_2^* E_H$ in (47). Since $p_2^* E_H$ is a constant family, the infinitesimal deformation map

$$T_{e'} M \;=\; \mathfrak{m} \longrightarrow H^1(X,\, \mathrm{ad}(E_H))$$

for $p_2^* E_H$ is the zero homomorphism. Consequently, the infinitesimal deformation map f_1 in (45) is the zero homomorphism. Since ρ' in (22) is surjective (as ρ is surjective) from (46) it now follows that $f \;=\; 0$.
To prove the converse, assume that $f \;=\; 0$. Consider the short exact sequence in (11). Let

$$H^0(X,\, \mathrm{At}(E_H)(-\log D)) \stackrel{\beta_*}{\longrightarrow} H^0(X,\, TX(-\log D)) \stackrel{\lambda}{\longrightarrow} H^1(X,\, \mathrm{ad}(E_H)) \tag{48}$$

be the long exact sequence of cohomologies associated to it, where β_* is the homomorphism in (28) and λ is the connecting homomorphism. Consider the isomorphism

$$\gamma_0 \colon \mathfrak{g} \longrightarrow H^0(X,\, TX(-\log D))$$

in Corollary 3.2. We have

$$\lambda \circ \gamma_0 \;=\; f\,,$$

where λ is the homomorphism in (48). Since $f \;=\; 0$, and γ_0 is an isomorphism, we conclude that $\lambda \;=\; 0$.
The homomorphism β_* in (48) is surjective, because $\lambda \;=\; 0$. Fix a homomorphism

$$b \colon H^0(X,\, TX(-\log D)) \longrightarrow H^0(X,\, \mathrm{At}(E_H)(-\log D))$$

such that

$$\beta_* \circ b \;=\; \mathrm{Id}_{H^0(X, TX(-\log D))}\,. \tag{49}$$

As the holomorphic vector bundle $TX(-\log D)$ is holomorphically trivial, the homomorphism b in (49) produces a homomorphism

$$b' \colon TX(-\log D) \longrightarrow \mathrm{At}(E_H)(-\log D)$$

that sends any $v \in TX(-\log D)_x$ to

$$b(\widetilde{v})(x) \;\in\; \mathrm{At}(E_H)(-\log D)_x\,,$$

where $\widetilde{v} \in H^0(X,\, TX(-\log D))$ is the unique holomorphic section such that $\widetilde{v}(x) \;=\; v$. From (49) it follows that

$$\beta \circ b' \;=\; \mathrm{Id}_{TX(-\log D)}\,,$$

where β is the projection in (11). Hence b' defines a logarithmic connection on E_H singular over D. Now from Theorem 4.1(2) we conclude that E_H is homogeneous. ∎

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References

[1] M. F. Atiyah: *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. 85 (1957) 181–207.

[2] I. Biswas, V. Heu, J. Hurtubise: *Isomonodromic deformations of logarithmic connections and stability*, Math. Ann. 366 (2016) 121–140.

[3] B. Conrad: *From normal crossings to strict normal crossings*, http://math.stanford.edu/∼conrad/249BW17Page/handouts/crossings.pdf.

[4] P. Deligne: *Équations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Mathematics 163, Springer, Berlin et al. (1970).

[5] I. F. Donin: *Construction of a versal family of deformations for holomorphic bundles over a compact complex space*, Math. USSR Sb. 23 (1974) 405–416.

[6] K. Saito: *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo, Section 1A (Math.) 27 (1980) 265–291.

[7] J. Winkelmann: *On manifolds with trivial logarithmic tangent bundle*, Osaka J. Math. 41 (2004) 473–484.

[8] J. Winkelmann: *On manifolds with trivial logarithmic tangent bundle: the non-Kähler case*, Transform. Groups 13 (2008) 195–209.

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