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Left Invariant Ricci Solitons on Three-Dimensional Lie Groups

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Abstract. We give a necessary and sufficient condition for an arbitrary real Lie group, to admit an algebraic Ricci soliton. As an application, we classify all algebraic Ricci solitons on three-dimensional real Lie groups, up to automorphism. This classification shows that, in dimension three, there exist a solvable Lie group and a simple Lie group such that they do not admit any algebraic Ricci soliton. Also it is shown that there exist three-dimensional unimodular and non-unimodular Lie groups with left invariant Ricci solitons. Finally, for a unimodular solvable Lie group, the solution of the Ricci soliton equation is given, explicitly.

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1. Introduction

Let M be a smooth manifold equipped with a complete Riemannian metric g. The Riemannian metric g is called a Ricci soliton if

$$\mathsf{ric}_g = cg + \mathsf{L}_X g, \tag{1}$$

for a real number c and a complete vector field X on M, where ric_g and $\mathsf{L}_X g$ denote the Ricci tensor and Lie derivative of the Riemannian metric g, respectively. If X is the gradient vector field of a differentiable function $\phi\colon M \longrightarrow \mathbb{R}$ then g is named a gradient Ricci soliton. In the equation (1), if $c < 0$, $c = 0$ or $c > 0$, then the Ricci soliton g is called expanding, steady, or shrinking. We can easily see that the concept of Ricci soliton is a natural generalization of the concept of Einstein metric. But the more important reason for studying Ricci solitons is that g is a Ricci soliton if and only if the one-parameter family of Riemannian metrics

$$g_t = (-2ct + 1)\phi_t^* g, \tag{2}$$

is a solution to the Ricci flow equation

$$\frac{\partial}{\partial t} g_t = -2\mathsf{ric}_{g_t}, \tag{3}$$

where ϕ_t is a one-parameter group of diffeomorphism of M (for more details see [10] and [11]). In this article, our interest is in algebraic Ricci solitons. Let G be a

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real Lie group equipped with a left invariant Riemannian metric g, and $\mathfrak{g}$ be its Lie algebra. The Riemannian metric g is named an algebraic Ricci soliton if there exists a derivation $D \in \mathsf{Der}(\mathfrak{g})$ such that

$$\mathsf{Ric}_g = c.Id + D, \tag{4}$$

where $c \in \mathbb{R}$ and Ric_g denotes the $(1,1)$-Ricci tensor of g. We can see any algebraic Ricci soliton is a Ricci soliton (see [7] and [9]).
In the case of nilpotent Lie groups with left invariant Riemannian metrics, Lauret showed that the equations (1) and (4) are equivalent. In fact left invariant Riemannian metrics on nilpotent Lie groups are Ricci solitons if and only if they are algebraic Ricci solitons ([9] and [6]). The definition of algebraic Ricci soliton can generalize to homogeneous spaces. Jablonski proved that all homogeneous Ricci solitons are algebraic [6]. So the classification of algebraic Ricci solitons on Lie groups (homogeneous spaces) is very important. This problem was studied in many recent works (for more details see [2], [3], [5], [6], [7], [8], [9], [10], [11] and for a non-Riemannian case see [14]).
In this paper, for an arbitrary real Lie group, we give a necessary and sufficient condition based on structural constants, to admit an algebraic Ricci soliton. By using this condition, we give all algebraic Ricci solitons on three-dimensional real Lie groups, up to automorphism. We mention that in this classification, unlike [1] and [3] which have considered Ricci solitons only on the set of left invariant vector fields, we do not limit the problem to this set. Then, we show that there exist a solvable Lie group and a simple Lie group, in dimension three, such that they do not admit any algebraic Ricci soliton. Also we prove that there exist three-dimensional unimodular and non-unimodular Lie groups such that they admit Ricci solitons. Finally, for a special unimodular solvable Lie group, the solution of the Ricci soliton equation is given.

2. Algebraic Ricci solitons

During recent years, left invariant Ricci solitons (a left invariant metric on a Lie group which is a Ricci soliton) on some Lie groups have been studied (see [1] and [3]). But, for simplicity, the authors only considered left invariant vector fields to solve the Ricci soliton equation (1). For example, in [1], the authors showed that the Ricci soliton equation (1) on a special solvable Lie group G^n equipped with a special left invariant Riemannian metric, in the space of left invariant vector fields, does not admit a solution. We can find another example in [3], where the author has proven the same previous result for unimodular Lie groups.
As we mentioned in the introduction in [6] it is proven that homogeneous Ricci solitons are algebraic. So we can characterize the left invariant Ricci solitons, by using equation (4) without the restriction of considering only left invariant vector fields.
In this section, using structural constants, we give a necessary and sufficient condition for any left invariant Riemannian metric on an arbitrary Lie group to be a Ricci soliton (or equivalently to be an algebraic Ricci soliton).

Lemma 2.1. *Let G be a n-dimensional Lie group with a left invariant Riemannian metric g. Suppose that $\{E_1, \cdots, E_n\}$ is an orthonormal basis for the Lie algebra $\mathfrak{g}$ of G, with respect to the Riemannian metric g.*

If α_{ijk} *denote the structural constants defined by*

$$[E_i, E_j] = \sum_{k=1}^{n} \alpha_{ijk} E_k, \tag{5}$$

then the $(1,1)$*-Ricci tensor of* g *can be computed as follows,*

$$\begin{aligned} \mathsf{Ric}_g(E_i) &= \frac{1}{4}\sum_{l=1}^{n}\Big\{\sum_{j=1}^{n}\sum_{r=1}^{n} 2\alpha_{rjj}(\alpha_{irl}+\alpha_{lir}-\alpha_{rli}) - 2\alpha_{ijr}(\alpha_{rjl}+\alpha_{lrj}-\alpha_{jlr}) \\ &\qquad -(\alpha_{ijr}+\alpha_{rij}-\alpha_{jri})(\alpha_{jrl}+\alpha_{ljr}-\alpha_{rlj})\Big\}E_l. \end{aligned} \tag{6}$$

Proof. Considering the curvature tensor formula of Theorem 2.9 in [12], we have

$$\begin{aligned} \mathsf{Ric}_g(E_i) &= \sum_{j=1}^{n} R(E_i, E_j)E_j \\ &= \frac{1}{4}\sum_{j=1}^{n}\sum_{l=1}^{n}\Big\{\sum_{r=1}^{n}(\alpha_{jjr}+\alpha_{rjj}-\alpha_{jrj})(\alpha_{irl}+\alpha_{lir}-\alpha_{rli}) \\ &\quad -(\alpha_{ijr}+\alpha_{rij}-\alpha_{jri})(\alpha_{jrl}+\alpha_{ljr}-\alpha_{rlj}) - 2\alpha_{ijr}(\alpha_{rjl}+\alpha_{lrj}-\alpha_{jlr})\Big\}E_l. \end{aligned} \tag{7}$$

Now the equations $\alpha_{jjr} = 0$ and $\alpha_{rjj} = -\alpha_{jrj}$ complete the proof. ∎

Now, by using structural constants of the Lie algebra of a Lie group, we give a necessary and sufficient condition for a left invariant Riemannian metric on an arbitrary Lie group G to be a Ricci soliton. We mention that, this condition is given in general case and we do not restrict ourself to the set of left invariant vector fields to solve the Ricci soliton equation (1).

Theorem 2.2. *Suppose that* G *is a* n*-dimensional Lie group equipped with a left invariant Riemannian metric* g*. Assume that* $\{E_1, \cdots, E_n\}$ *is an orthonormal basis for* $\mathfrak{g}$*, the Lie algebra of* G*, and* α_{ijk} *are the structural constants with respect to this basis. The Riemannian metric* g *is a Ricci soliton if and only if there exists a constant* $c \in \mathbb{R}$ *such that, for any* $t, p, q = 1, \cdots, n$ *we have,*

$$\begin{aligned} &c\alpha_{qpt} + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{n} 2\alpha_{rjj}\Big(\alpha_{iqt}(\alpha_{pri}+\alpha_{ipr}-\alpha_{rip}) - \alpha_{ipt}(\alpha_{qri}+\alpha_{iqr}-\alpha_{riq})\Big) \\ &+ 2(\alpha_{rji}+\alpha_{irj}-\alpha_{jir})(\alpha_{ipt}\alpha_{qjr}-\alpha_{iqt}\alpha_{pjr}) + (\alpha_{jri}+\alpha_{ijr}-\alpha_{rij}) \\ &\times\Big(\alpha_{ipt}(\alpha_{qjr}+\alpha_{rqj}-\alpha_{jrq}) - \alpha_{iqt}(\alpha_{pjr}+\alpha_{rpj}-\alpha_{jrp})\Big) \\ &- 2\alpha_{pqi}\alpha_{rjj}(\alpha_{irt}+\alpha_{tir}-\alpha_{rti}) + 2\alpha_{pqi}\alpha_{ijr}(\alpha_{rjt}+\alpha_{trj}-\alpha_{jtr}) \\ &+ \alpha_{pqi}(\alpha_{ijr}+\alpha_{rij}-\alpha_{jri})(\alpha_{jrt}+\alpha_{tjr}-\alpha_{rtj}) = 0. \end{aligned} \tag{8}$$

Proof. We know that a left invariant Riemannian metric on a Lie group is a Ricci soliton if and only if it is an algebraic Ricci soliton (see [6]). So we replace the equation (1) with the equation (4). Now, by using equation (6) of Lemma 1, we see that the Riemannian metric g satisfies the algebraic Ricci soliton equation (4) if and only if, for any $i = 1, \cdots, n$,

$$D(E_i) = -\,cE_i + \frac{1}{4}\sum_{l=1}^{n}\Big\{\sum_{j=1}^{n}\sum_{r=1}^{n} 2\alpha_{rjj}(\alpha_{irl} + \alpha_{lir} - \alpha_{rli}) \\ - 2\alpha_{ijr}(\alpha_{ril} + \alpha_{lrj} - \alpha_{jlr}) - (\alpha_{ijr} + \alpha_{rij} - \alpha_{jri})(\alpha_{jrl} + \alpha_{ljr} - \alpha_{rlj})\Big\}E_l. \tag{9}$$

On the other hand we know that a linear map $D\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ is a derivation if and only if, for any $p, q = 1, \cdots, n$,

$$D([E_p, E_q]) = [D(E_p), E_q] + [E_p, D(E_q)]. \tag{10}$$

Now substituting the equations (9) in the equations (10) completes the proof. ∎

Unlike the Ricci soliton equation (1) and the algebraic Ricci soliton equation (4), computing with equation (8) is very simple. In the next section we will use this equation to classify all left invariant Ricci solitons on three-dimensional Lie groups.

3. Classification of algebraic Ricci solitons on 3-dimensional Lie groups

Ha and Lee [4] classified all left invariant Riemannian metrics on simply connected three-dimensional Lie groups, up to automorphism (see also [13]), to fifteen cases. Here we use Theorem 2.2 and this classification to classify all algebraic Ricci solitons on simply connected three-dimensional Lie groups.

Let G be an arbitrary simply connected three-dimensional real Lie group equipped with any left invariant Riemannian metric g, and $\mathfrak{g}$ denotes its Lie algebra. Then, up to automorphism, (G, g) is one of the following fifteen cases. In any case the set $\{E_1, E_2, E_3\}$ is an orthonormal basis for $\mathfrak{g}$ with respect to g.

3.1. Case 1. The first case is the trivial case $G = \mathbb{R}^3$. In this case all structural constants α_{ijk} are zero, so $\mathsf{Ric}_g = 0$ and in equation (4) for any $c \in \mathbb{R}$, $D = -c.Id$.

3.2. Case 2. Let G be the Heisenberg Lie group. It is shown that, up to automorphism, G admits only one family of left invariant metrics g such that for the orthonormal basis $\{E_1, E_2, E_3\}$, the structural constants are of the following form,

$$\alpha_{123} = -\alpha_{213} = \frac{1}{\lambda},$$

where λ is a positive real number. We can see g satisfies equation (8) if and only if

$$\frac{c}{\lambda} + \frac{3}{2\lambda^3} = 0,$$

or equivalently, if and only if $c = -\frac{3}{2\lambda^2}$. So, up to automorphism, any algebraic Ricci soliton g on the Heisenberg Lie group satisfies the equation

$$\mathsf{Ric}_g = -\frac{3}{2\lambda^2}Id + D,$$

where, by using (9), the representation of D in the basis $\{E_1, E_2, E_3\}$ is as follows:

$$D = \begin{pmatrix} \frac{1}{\lambda^2} & 0 & 0 \\ 0 & \frac{1}{\lambda^2} & 0 \\ 0 & 0 & \frac{2}{\lambda^2} \end{pmatrix}.$$

Remark 3.1. In the cases 3 and 4, we consider the Lie group $G = \mathbb{R}^2 \rtimes_\varphi \mathbb{R}$, where for any $t \in \mathbb{R}$,

$$\varphi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}.$$

3.3. Case 3. G is a solvable Lie group such that the nonzero commutators of its Lie algebra are as follows,

$$[E_1, E_3] = \frac{1}{\sqrt{\nu}} E_2, \qquad [E_2, E_3] = \frac{1}{\sqrt{\nu}} E_1,$$

where $\nu > 0$. In fact we have

$$\alpha_{132} = -\alpha_{312} = \alpha_{231} = -\alpha_{321} = \frac{1}{\sqrt{\nu}}.$$

Now Theorem 2.2 shows that g is an algebraic Ricci soliton if and only if $c = -\frac{2}{\nu}$. Therefore we have

$$\mathsf{Ric}_g = -\frac{2}{\nu} Id + D,$$

where, by using equation (9),

$$D = \begin{pmatrix} \frac{2}{\nu} & 0 & 0 \\ 0 & \frac{2}{\nu} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3.4. Case 4. The previous Lie group, up to automorphism, admits another left invariant Riemannian metric such that the structural constants with respect to the orthonormal basis $\{E_1, E_2, E_3\}$ are as follows,

$$\alpha_{231} = -\alpha_{321} = \frac{\sqrt{\mu + \sqrt{\mu}}}{\sqrt{\nu}\sqrt{\mu - \sqrt{\mu}}}, \quad \alpha_{312} = -\alpha_{132} = \frac{-\sqrt{\mu - \sqrt{\mu}}}{\sqrt{\nu}\sqrt{\mu + \sqrt{\mu}}},$$

where $\nu > 0$ and $\mu > 1$. If g satisfies equation (8), then $\mu = 1$, which is a contradiction. So g is not a Ricci soliton.

3.5. Case 5. This case specialized to the solvable Lie group $G = \tilde{E}_0(2) = \mathbb{C} \rtimes \mathbb{R}$ where, for any (c, r) and $(d, s) \in G$, we have $(c, r).(d, s) = (c + e^{2\pi i r} d, r + s)$. Up to automorphism, G admits a left invariant Riemannian metric g. The Lie algebra $\mathfrak{g}$ of G has an orthonormal basis $\{E_1, E_2, E_3\}$ such that

$$\alpha_{132} = -\alpha_{312} = \frac{\sqrt{\mu}}{\sqrt{\nu}}, \quad \alpha_{321} = -\alpha_{231} = \frac{1}{\sqrt{\mu\nu}},$$

where $\nu > 0$ and $0 < \mu \leq 1$. Using Theorem 2.2, a direct computation shows that g is a Ricci soliton if and only if $c = 0$ and $\mu = 1$. But in this case we have $D = 0$, which means that (G, g) is an Einstein manifold. So the Lie group $\tilde{E}_0(2)$ does not admit any non-Einstein algebraic Ricci soliton.

3.6. Case 6. Suppose that G is the simple Lie group $\widetilde{PSL}(2, \mathbb{R})$, the universal covering group of $SL(2, \mathbb{R})$. G admits only one family of left invariant Riemannian metrics g, up to automorphism. Then the Lie algebra $\mathfrak{g}$ of G admits an orthonormal basis $\{E_1, E_2, E_3\}$ such that

$$\alpha_{123} = -\alpha_{213} = \frac{2\nu}{\sqrt{\lambda\mu\nu}}, \quad \alpha_{312} = -\alpha_{132} = \frac{2\mu}{\sqrt{\lambda\mu\nu}}, \quad \alpha_{321} = -\alpha_{231} = \frac{2\lambda}{\sqrt{\lambda\mu\nu}},$$

where $\lambda > 0$ and $0 < \nu \leq \mu$. We can easily see that, for positive constants λ, ν and μ, (G, g) does not satisfy the equation (8). So there is no algebraic Ricci soliton on the simple Lie group $G = \widetilde{PSL}(2, \mathbb{R})$.

3.7. Case 7. Let G be the Lie group $SU(2)$, the simply connected Lie group corresponding to $\mathfrak{so}(3)$. The only left invariant Riemannian metric, up to automorphism, on G is the metric induced by the orthonormal basis $\{E_1, E_2, E_3\}$ such that

$$\alpha_{123} = -\alpha_{213} = \frac{\nu}{\sqrt{\lambda\mu\nu}}, \quad \alpha_{312} = -\alpha_{132} = \frac{\mu}{\sqrt{\lambda\mu\nu}}, \quad \alpha_{231} = -\alpha_{321} = \frac{\lambda}{\sqrt{\lambda\mu\nu}},$$

where, $0 < \nu \leq \mu \leq \lambda$. A direct computation together with Theorem 2.2 show that the left invariant Riemannian metric g is an algebraic Ricci soliton if and only if $0 < \lambda = \mu = \nu$. The assumption $\lambda = \mu = \nu$ implies that $c = \frac{1}{2\nu}$. In this case we have $D = 0$ which shows that for $\lambda = \mu = \nu$, the Riemannian metric g is Einstein and the Lie group $G = SU(2)$ does not admit non-Einstein algebraic Ricci soliton.

Remark 3.2. The cases 8 to 15 are specialized to non-unimodular solvable Lie groups.

3.8. Case 8. In this case we consider the non-unimodular simply connected Lie group $G_I \cong \mathbb{R}^2 \rtimes_{\varphi_I} \mathbb{R}$, where $\varphi_I(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}$. For any left invariant Riemannian metric g on G_I, the Lie algebra $\mathfrak{g}_I$ admits an orthonormal basis $\{E_1, E_2, E_3\}$ with

$$\alpha_{122} = -\alpha_{212} = \alpha_{133} = -\alpha_{313} = \frac{1}{\sqrt{\nu}},$$

where $\nu > 0$. The Riemannian metric g is a Ricci soliton if and only if $c = \frac{-2}{\nu}$. Then we have $D = 0$. Therefore (G, g) is an Einstein manifold and there is not any non-Einstein algebraic Ricci soliton on G_I.

Remark 3.3. In the cases 9 to 15, we have considered the Lie group $G_h \cong \mathbb{R}^2 \rtimes_{\varphi_h} \mathbb{R}$ where for $h = 1$,

$$\varphi_h(t) = e^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + te^t \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix},$$

and for $h \neq 1$,

$$\varphi_h(t) = e^t \frac{e^{zt} + e^{-zt}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^t \frac{e^{zt} - e^{-zt}}{2z} \begin{pmatrix} -1 & -h \\ 1 & 1 \end{pmatrix},$$

where $z = \sqrt{1-h} \neq 0$. Similar to G_I, G_h is a non-unimodular solvable Lie group. For the Lie group G_h, there are 7 family of left invariant Riemannian metrics, up to automorphism. We will study the existence of algebraic Ricci solitons on G_h in the cases 9 to 15. Again, we mention that the set $\{E_1, E_2, E_3\}$ is an orthonormal basis with respect to the left invariant Riemannian metric of any case.

3.9. Case 9. Let g be the left invariant Riemannian metric induced by $\{E_1, E_2, E_3\}$. Then for the structural constants we have

$$\alpha_{123} = -\alpha_{213} = \frac{\sqrt{\mu}}{\sqrt{\nu}}, \quad \alpha_{312} = -\alpha_{132} = \frac{h}{\sqrt{\mu\nu}}, \quad \alpha_{133} = -\alpha_{313} = \frac{2}{\sqrt{\nu}},$$

where $\nu > 0$ and $0 < \mu < |h|$, and $h < 0$. We can see g satisfies in the equation (8) if and only if $h = -\mu$ and $c = \frac{-2\mu-4}{\nu}$. Now, by using equation (9), for D we have

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2\mu+4}{\nu} & -\frac{2\sqrt{\mu}}{\nu} \\ 0 & -\frac{2\sqrt{\mu}}{\nu} & \frac{2\mu}{\nu} \end{pmatrix}.$$

3.10. Case 10. In this case, the Riemannian metric and the structural constants are the same as the previous case, but with condition $h = 0$ and $0 < \mu, \nu$. A similar computation shows that if g is an algebraic Ricci soliton then $\mu = 0$ which is a contradiction. So g is not a Ricci soliton.

3.11. Case 11. In this case, we consider the left invariant Riemannian metric g such that

$$\alpha_{122} = -\alpha_{212} = \frac{1}{2\sqrt{\nu}}, \quad \alpha_{213} = -\alpha_{123} = \alpha_{312} = -\alpha_{132} = \frac{\sqrt{3}}{2\sqrt{\nu}}, \quad \alpha_{133} = -\alpha_{313} = \frac{3}{2\sqrt{\nu}},$$

where $h = 0$ and $\nu > 0$. By using equation (8) of Theorem 2.2 we can see g is an algebraic Ricci soliton if and only if $c = -\frac{4}{\nu}$. Now, the equation (9) shows that the derivation D is as follows

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{\nu} & \frac{\sqrt{3}}{\nu} \\ 0 & \frac{\sqrt{3}}{\nu} & \frac{1}{\nu} \end{pmatrix}.$$

3.12. Case 12. In this case for $\nu > 0$, $h = 1$ and $0 < \mu \leq 1$ we have,

$$\alpha_{123} = -\alpha_{213} = \frac{\sqrt{\mu}}{\sqrt{\nu}}, \quad \alpha_{312} = -\alpha_{132} = \frac{1}{\sqrt{\mu\nu}}, \quad \alpha_{133} = -\alpha_{313} = \frac{2}{\sqrt{\nu}}.$$

A direct computation shows that if g is an algebraic Ricci soliton then $2\frac{\sqrt{\mu}}{\sqrt{\nu}}(1+\mu) = 0$ which is a contradiction. Therefore, in this case, g is not a Ricci soliton.

3.13. Case 13. With respect to the left invariant Riemannian metric g considered in this case, the structural constants are as follows

$$\alpha_{122} = -\alpha_{212} = \frac{\lambda}{\sqrt{\nu}}, \quad \alpha_{123} = -\alpha_{213} = \frac{\sqrt{1-\lambda^2}}{\sqrt{\nu}},$$
$$\alpha_{312} = -\alpha_{132} = \frac{1-\lambda^2}{\sqrt{(1-\lambda^2)\nu}}, \quad \alpha_{133} = -\alpha_{313} = \frac{2-\lambda}{\sqrt{\nu}},$$

for $\nu > 0$, $h = 1$ and $0 < \lambda < 1$. Equation (8) shows that there is not any $c \in \mathbb{R}$ and $D \in \mathsf{Der}(\mathfrak{g}_h)$ such that the Riemannian metric g satisfies the equation (8). So g is not a Ricci soliton.

3.14. Case 14. In this case the left invariant Riemannian metric g is defined by the orthonormal basis $\{E_1, E_2, E_3\}$ where we have

$$\alpha_{132} = -\alpha_{312} = -\frac{h-1}{\sqrt{(\mu-1)\nu}}, \quad \alpha_{123} = -\alpha_{213} = \frac{\sqrt{\mu-1}}{\sqrt{\nu}},$$
$$\alpha_{122} = -\alpha_{212} = \alpha_{133} = -\alpha_{313} = \frac{1}{\sqrt{\nu}},$$

where $\nu > 0$, $h > 1$ and $1 < \mu \leq h$. The Riemannian metric g satisfies the equation (8) if and only if $h = \mu$ and $c = -\frac{2}{\nu}$. But in this case equation (9) shows that $D = 0$. So (G, g) is an Einstein manifold.

3.15. Case 15. In the last case for the structural constants, with respect to the the orthonormal basis $\{E_1, E_2, E_3\}$ we have,

$$\alpha_{122} = -\alpha_{212} = \frac{1-\sqrt{1-h}}{\sqrt{\nu}}, \quad \alpha_{312} = -\alpha_{132} = \frac{2\mu\sqrt{1-h}}{\sqrt{\nu(1-\mu^2)}},$$
$$\alpha_{133} = -\alpha_{313} = \frac{1+\sqrt{1-h}}{\sqrt{\nu}},$$

where $\nu > 0$, $0 \leq \mu < 1$ and $0 < h < 1$. Now we can see that the Riemannian metric g satisfies in the equation (8) if and only if $\mu = 0$ and $c = \frac{2h-4}{\nu}$. So by using equation (9) we have

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{2(h-1-\sqrt{1-h})}{\nu} & 0 \\ 0 & 0 & -\frac{2(h-1+\sqrt{1-h})}{\nu} \end{pmatrix}.$$

We summarize the above results in Table 1.

Corollary 3.4. *There exists a solvable Lie group which does not admit any left invariant Ricci soliton. In fact, Table* 1*, cases* 12 *and* 13*, show that, for* $h = 1$*, the solvable non-unimodular Lie group* $G_h \cong \mathbb{R}^2 \rtimes_{\varphi_h} \mathbb{R}$ *does not admit any algebraic Ricci soliton (neither Einstein nor non-Einstein).*

Corollary 3.5. *There exists a simple Lie group which does not admit any left invariant Ricci soliton. More precisely, the case* 6 *of the table 1 proves that there is not any algebraic Ricci soliton (neither Einstein nor non-Einstein) Riemannian metric on the simple Lie group* $\widetilde{PSL}(2, \mathbb{R})$.

Remark 3.6. Although it is shown that the Ricci soliton equation (1) for a three-dimensional non-unimodular Lie group has not any solution in the set of left invariant vector fields (see [3], Proposition 3.5), but cases 11 and 15 show that there are three-dimensional non-unimodular Lie groups that admit algebraic Ricci solitons in the set of all vector fields.

Remark 3.7. In [3], it is proven that there is not any left invariant Ricci soliton on a unimodular Lie group such that the vector field X, in the Ricci soliton equation (1),

Table 1: The classification of algebraic Ricci solitons on simply connected 3-dimensional real Lie groups

case	The simply connected Lie group	Structural constants with respect to the orthonormal basis $\{E_1, E_2, E_3\}$	Conditions for parameters	Einstein Ricci soliton	Non-Einstein Ricci soliton
case 1	$\mathbb{R}^3$	0	$-$	+ (if $c = 0$)	+ (if $c \neq 0$)
case 2	The Heisenberg group	$\alpha_{123} = -\alpha_{213} = \frac{1}{\lambda}$	$\lambda > 0$	$-$	+(if $c = -\frac{3}{2\lambda^2}$)
case 3	The solvable Lie group $G = \mathbb{R}^2 \rtimes_\varphi \mathbb{R}$	$\alpha_{132} = -\alpha_{312} = \alpha_{231} = -\alpha_{321} = \frac{1}{\sqrt{\nu}}$	$\nu > 0$	$-$	+ (if $c = -\frac{2}{\nu}$)
case 4	"	$\alpha_{231} = -\alpha_{321} = \frac{\sqrt{\mu+\sqrt{\mu}}}{\sqrt{\nu}\sqrt{\mu-\sqrt{\mu}}}, \alpha_{312} = -\alpha_{132} = \frac{-\sqrt{\mu-\sqrt{\mu}}}{\sqrt{\nu}\sqrt{\mu+\sqrt{\mu}}}$	$\mu > 1, \nu > 0$	$-$	$-$
case 5	The solvable Lie group $G = \tilde{E}_0(2) = \mathbb{C} \rtimes \mathbb{R}$	$\alpha_{132} = -\alpha_{312} = \frac{\sqrt{\mu}}{\sqrt{\nu}}, \alpha_{321} = -\alpha_{231} = \frac{1}{\sqrt{\mu\nu}}$	$0 < \mu \leq 1$, $\nu > 0$	+ (if $c = 0$ and $\mu = 1$)	$-$
case 6	The simple Lie group $\widetilde{PSL}(2, \mathbb{R})$	$\alpha_{123} = -\alpha_{213} = \frac{2\nu}{\sqrt{\lambda\mu\nu}}, \alpha_{312} = -\alpha_{132} = \frac{2\mu}{\sqrt{\lambda\mu\nu}}, \alpha_{321} = -\alpha_{231} = \frac{2\lambda}{\sqrt{\lambda\mu\nu}}$	$\mu \geq \nu > 0$, $\lambda > 0$	$-$	$-$
case 7	The simple Lie group $SU(2)$	$\alpha_{123} = -\alpha_{213} = \frac{\nu}{\sqrt{\lambda\mu\nu}}, \alpha_{312} = -\alpha_{132} = \frac{\mu}{\sqrt{\lambda\mu\nu}}, \alpha_{231} = -\alpha_{321} = \frac{\lambda}{\sqrt{\lambda\mu\nu}}$	$\lambda \geq \mu \geq \nu > 0$	+ (if $\lambda = \nu = \mu$)	$-$
case 8	The non-unimodular Lie group $G_I \cong \mathbb{R}^2 \rtimes_{\varphi_I} \mathbb{R}$	$\alpha_{122} = -\alpha_{212} = \alpha_{133} = -\alpha_{313} = \frac{1}{\sqrt{\nu}}$	$\nu > 0$	+ (if $c = \frac{-2}{\nu}$)	$-$
case 9	The non-unimodular Lie group $G_h \cong \mathbb{R}^2 \rtimes_{\varphi_h} \mathbb{R}$	$\alpha_{123} = -\alpha_{213} = \frac{\sqrt{\mu}}{\sqrt{\nu}}, \alpha_{312} = -\alpha_{132} = \frac{h}{\sqrt{\mu\nu}}, \alpha_{133} = -\alpha_{313} = \frac{2}{\sqrt{\nu}}$	$0 < \mu \leq \|h\|$, $\nu > 0$, $h < 0$	$-$	+ (if $h = -\mu$ and $c = \frac{-2\mu-4}{v}$)
case 10	"	$\alpha_{123} = -\alpha_{213} = \frac{\sqrt{\mu}}{\sqrt{\nu}}, \alpha_{133} = -\alpha_{313} = \frac{2}{\sqrt{\nu}}$	$\mu, \nu > 0$, $h = 0$	$-$	$-$
case 11	"	$\alpha_{122} = -\alpha_{212} = \frac{1}{2\sqrt{\nu}}, \alpha_{213} = -\alpha_{123} = \alpha_{312} = -\alpha_{132} = \frac{\sqrt{3}}{2\sqrt{\nu}}, \quad \alpha_{133} = -\alpha_{313} = \frac{3}{2\sqrt{\nu}}$	$\nu > 0$, $h = 0$	$-$	+ (if $c = -\frac{4}{\nu}$)
case 12	"	$\alpha_{123} = -\alpha_{213} = \frac{\sqrt{\mu}}{\sqrt{\nu}}, \alpha_{312} = -\alpha_{132} = \frac{1}{\sqrt{\mu\nu}}, \quad \alpha_{133} = -\alpha_{313} = \frac{2}{\sqrt{\nu}}$	$\nu > 0$, $h = 1$, $0 < \mu \leq 1$	$-$	$-$
case 13	"	$\alpha_{122} = -\alpha_{212} = \frac{\lambda}{\sqrt{\nu}}, \quad \alpha_{123} = -\alpha_{213} = \frac{\sqrt{1-\lambda^2}}{\sqrt{\nu}}, \alpha_{312} = -\alpha_{132} = \frac{1-\lambda^2}{\sqrt{(1-\lambda^2)\nu}}, \alpha_{133} = -\alpha_{313} = \frac{2-\lambda}{\sqrt{\nu}}$	$\nu > 0$, $h = 1$, $0 < \lambda < 1$	$-$	$-$
case 14	"	$\alpha_{132} = -\alpha_{312} = -\frac{h-1}{\sqrt{(\mu-1)\nu}}, \alpha_{123} = -\alpha_{213} = \frac{\sqrt{\mu-1}}{\sqrt{\nu}}, \alpha_{122} = -\alpha_{212} = \alpha_{133} = -\alpha_{313} = \frac{1}{\sqrt{\nu}}$	$\nu > 0$, $h > 1$, $1 < \mu \leq h$	+ (if $h = \mu$ and $c = -\frac{2}{\nu}$)	$-$
case 15	"	$\alpha_{122} = -\alpha_{212} = \frac{1-\sqrt{1-h}}{\sqrt{\nu}}, \alpha_{312} = -\alpha_{132} = \frac{2\mu\sqrt{1-h}}{\sqrt{\nu(1-\mu^2)}}, \alpha_{133} = -\alpha_{313} = \frac{1+\sqrt{1-h}}{\sqrt{\nu}}$	$0 \leq \mu < 1$, $\nu > 0$, $0 < h < 1$	$-$	+ (if $\mu = 0$ and $c = \frac{2h-4}{\nu}$)

is a left invariant vector field. But the case 3 shows that there is a three-dimensional unimodular Lie group that admits algebraic Ricci solitons. Certainly, in this case, the vector field X in the equation (1) is not a left invariant vector field. In the following example we compute such a vector field, explicitly.

Example 3.8. Similar to [1], let G^n be the $2n+1$-dimensional real Lie group which is defined as follows:

$$G^n = \left\{ \left(\begin{array}{ccccc} e^{u_0} & 0 & \cdots & 0 & x_0 \\ 0 & e^{u_1} & \cdots & 0 & x_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{u_n} & x_n \\ 0 & 0 & \cdots & 0 & 1 \end{array} \right) \middle| \begin{array}{c} (x_0, x_1, ..., x_n, u_1, ..., u_n) \in \mathbb{R}^{2n+1}, \\ u_0 = -\sum_{i=1}^{n} u_i \end{array} \right\}. \tag{11}$$

Suppose that $X_i = e^{u_i}\frac{\partial}{\partial x_i}$ and $U_\alpha = \frac{\partial}{\partial u_\alpha}$, where $i = 0, \cdots, n$ and $\alpha = 1, \cdots, n$. Easily we can see the set $\{X_1, \cdots, X_{n+1}, U_1, \cdots, U_n\}$ is a basis for the Lie algebra $\mathfrak{g}^n$ of G^n. Now, let $n = 1$. We can see the Lie algebra $\mathfrak{g}^1$ is the same Lie algebra of the case 3. The non-zero brackets are as follows

$$[X_0, U_1] = X_0, \qquad [X_1, U_1] = -X_1.$$

Assume that g is the left invariant Riemannian metric on G^1 such that the set $\{X_0 = e^{-u_1}\frac{\partial}{\partial x_0}, X_1 = e^{u_1}\frac{\partial}{\partial x_1}, U_1 = \frac{\partial}{\partial u_1}\}$ is an orthonormal set. In fact with respect to the coordinates (x_0, x_1, u_1) we have

$$g = \left(\begin{array}{ccc} e^{2u_1} & 0 & 0 \\ 0 & e^{-2u_1} & 0 \\ 0 & 0 & 1 \end{array} \right). \tag{12}$$

If we want to compare with the case 3, it is sufficient to put $\nu = 1$, $E_1 = \frac{1}{\sqrt{2}}(X_0 + X_1)$, $E_2 = \frac{1}{\sqrt{2}}(X_0 - X_1)$ and $E_3 = \frac{1}{\sqrt{\nu}}U_1$. Similar to [1], we use the basis $\{X_0, X_1, U_1\}$. In this basis for the Levi-Civita connection ∇ we have

$$\begin{array}{c|ccc} \nabla & X_0 & X_1 & U_1 \\ \hline X_0 & -U_1 & 0 & X_0 \\ X_1 & 0 & U_1 & -X_1 \\ U_1 & 0 & 0 & 0 \end{array}. \tag{13}$$

Let $V = \alpha X_0 + \beta X_1 + \gamma U_1$ be an arbitrary element of the Lie algebra $\mathfrak{g}^1$, where $\alpha, \beta, \gamma \in \mathbb{R}$. Easily for $(1,1)$-Ricci tensor of g we have

$$\mathsf{Ric}_g(V) = -2\gamma U_1. \tag{14}$$

So the Riemannian metric g satisfies the algebraic Ricci soliton equation (4), where $c = -2$ and, with respect to the the basis $\{X_0, X_1, U_1\}$, D is as follows:

$$D = \left(\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right). \tag{15}$$

Now we compute the vector field X such that satisfies the Ricci soliton equation (1). For the Riemannian manifold (G^1, g), it is shown that there is not any solution for the Ricci soliton equation (1) in the set of left invariant vector fields (see [1]). But in above, and also in case 3, we proved that there is a solution for the equation (1) if we consider the equation on the set of all vector fields.

Suppose that $X = \theta X_0 + \eta X_1 + \zeta U_1$ is an arbitrary vector field on G^1, where θ, η, and ζ are smooth real functions on G^1. A direct computation shows that the Ricci soliton equation (1) reduces to the following system

$$\begin{cases} 2(\zeta + X_0(\theta)) = 2c \\ X_0(\eta) + X_1(\theta) = 0 \\ X_0(\zeta) + U_1(\theta) - \theta = 0 \\ 2(X_1(\eta) - \zeta) = 2c \\ \eta + U_1(\eta) + X_1(\zeta) = 0 \\ U_1(\zeta) = 2(c+2). \end{cases} \tag{16}$$

On the other hand we see that $X = \mathsf{grad}\Phi$, for some smooth functions Φ, if and only if

$$\begin{cases} \frac{\partial \Phi}{\partial x_0} = e^{u_1}\theta \\ \frac{\partial \Phi}{\partial x_1} = e^{-u_1}\eta \\ \frac{\partial \Phi}{\partial u_1} = \zeta. \end{cases} \tag{17}$$

We see that, for $c = -2$, there exist at least three solutions to the Ricci soliton equation (1).

Solution 1: If $\theta = x_0 e^{u_1}$, $\eta = -5x_1 e^{-u_1}$ and $\zeta = -3$, then
$X = \theta X_0 + \eta X_1 + \zeta U_1 = x_0 \frac{\partial}{\partial x_0} - 5x_1 \frac{\partial}{\partial x_1} - 3\frac{\partial}{\partial u_1}$
satisfies equation (1) or, equivalently, X satisfies system (16).

Solution 2: If $\theta = -2x_0 e^{u_1}$, $\eta = -2x_1 e^{-u_1}$ and $\zeta = 0$, then
$X = \theta X_0 + \eta X_1 + \zeta U_1 = -2x_0 \frac{\partial}{\partial x_0} - 2x_1 \frac{\partial}{\partial x_1}$
is a solution to (16).

Solution 3: If $\theta = -5x_0 e^{u_1}$, $\eta = x_1 e^{-u_1}$ and $\zeta = 3$, then
$X = \theta X_0 + \eta X_1 + \zeta U_1 = -5x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1} + 3\frac{\partial}{\partial u_1}$
is a solution to (16).

So the left invariant Riemannian metric g is a left invariant expanding Ricci soliton on G^1. The system (17) shows that the left invariant Ricci soliton g is a non-gradient expanding Ricci soliton.

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