Journal of Lie Theory
Volume **29** (2019) 997–1005

Some Upper Bounds on the Dimension of the Higher Multiplier of a Pair of Lie Algebras

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Communicated by M. Schlichenmaier

Abstract. We investigate the concept of higher Schur multiplier of a pair of Lie algebras and discuss relevant exact sequences. We will also derive some inequalities between the dimensions of higher Schur multipliers and pairs of their special factor in case of nilpotent Lie algebras. Furthermore, we show new upper bounds for the dimensions.

Mathematics Subject Classification: 17B30, 17B56, 17B99.

Key Words: Nilpotent Lie algebras, pair of Lie algebras, c-nilpotent multiplier.

1. Introduction and preliminaries

Throughout this article, we will use the term Lie algebra to mean a Lie algebra over a fixed field $\mathbb{F}$ and $[\ ,\ ]$ denotes the Lie bracket. By a pair of Lie algebras (L, N), we mean a Lie algebra L together an ideal N. For a pair (L, N) of Lie algebras we define a series of ideals of L contained in N as follows:

$$N = [N,_0 L] \supseteq [N, L] \supseteq [N, L, L] \supseteq \cdots \supseteq [N,_n L] \supseteq ..., \text{ with } [N,_n L] = [N, \underbrace{L, ..., L}_{n-times}]$$

for all $n > 0$. We call such a series, the *lower central series* of N in L. We say that a pair (L, N) of Lie algebras is *nilpotent* if it has a finite lower central series i.e. $[N,_n L] = 0$, for some positive integer n. The shortest length of such series is called the *nilpotency* class of the pair (L, N). Note that $[N,_n L]$ sometimes is shown by $\gamma_{n+1}(N, L)$, for $n \geq 0$. Similarly we may define the *upper central series* of N in L as follows:

$$0 = Z_0(N, L) \subseteq Z_1(N, L) \subseteq \cdots \subseteq Z_m(N, L) \subseteq \cdots,$$

where $Z_m(N, L) = \{n \in N \mid [n, l_1, \ldots, l_m] = 0, \ for\ all\ l_1, \ldots, l_m \in L\}$. It can be easily checked that a pair (L, N) of Lie algebras is nilpotent of class at most c if and only if $Z_c(N, L) = N$.

Let $0 \longrightarrow R \longrightarrow F \stackrel{\pi}{\longrightarrow} L \longrightarrow 0$ be a free presentation of a Lie algebra L, where F is a free Lie algebra. Then the *Schur multiplier* of L, denoted by $\mathcal{M}(L)$, is defined to be the factor Lie algebra $(R \cap F^2)/[R, F]$. Furthermore, if L is finite dimensional,

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ISSN 0949–5932 / \$2.50

then $\mathcal{M}(L)$ is isomorphic to $H^2(L,\mathbb{F})$, where L acts trivially on $\mathbb{F}$ (see [1, 2, 3, 7, 8] for more information on the *Schur multiplier* of a finite dimensional Lie algebra). Let (L,N) be a pair of Lie algebras, then we have the *Schur multiplier* of the *pair* (L,N) to be the abelian Lie algebra $\mathcal{M}(L,N)$ appearing in the following natural exact sequence of Lie algebras

$$H_3(L) \longrightarrow H_3(L,N) \longrightarrow \mathcal{M}(L,N) \longrightarrow \mathcal{M}(L) \longrightarrow \mathcal{M}(L/N)$$
$$\longrightarrow L/[N,L] \longrightarrow L/L^2 \longrightarrow L/(L^2+N) \longrightarrow 0,$$

where $\mathcal{M}(X)$ and $H_3(X)$ denote the Schur multiplier and the third homology of a Lie algebra X, respectively. This is analogous to the definition of the Schur multiplier of a pair of groups given by G. Ellis in [5]. Now, if the ideal N possesses a complement in L, then $\mathcal{M}(L) \cong \mathcal{M}(L,N) \oplus \mathcal{M}(L/N)$. In this case,

$$\mathcal{M}(L,N) = \ker(\mu : \mathcal{M}(L) \longrightarrow \mathcal{M}(L/N)),$$

in which μ is the natural homomorphism. So if S is an ideal of F such that $N \cong S/R$, then $\mathcal{M}(L,N) = (R \cap [S,F])/[R,F]$ (for more information see [12, 15]). Now we define the c-*nilpotent multiplier* of a pair (L,N) of Lie algebras for $c \geq 1$.

Definition 1.1. Let (L,N) be a pair of Lie algebras, $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$ be a free presentation of a Lie algebra L, where F is a free Lie algebra and $N \cong \frac{S}{R}$ for an ideal S of F. Then the *higher Schur multiplier* or the c-*nilpotent multiplier* $(c \geq 1)$ of the pair (L,N), is defined by

$$\mathcal{M}^{(c)}(L,N) = \frac{R \cap \gamma_{c+1}(S,F)}{\gamma_{c+1}(R,F)}.$$

Rismanchain and Araskhan in [12] investigated the notion of the Schur multiplier $\mathcal{M}(L,N)$ of an arbitrary pair (L,N) of Lie algebras, and derived some inequalities about dimension of it in the case of finite dimensional. Also some inequalities and upper bounds for the dimension of $\mathcal{M}^{(c)}(L)$ and the dimension of $\mathcal{M}(L,N)$ are given by more authors, for instance Salemkar et al. in [16], Edalatzadeh in [4] and Rismanchian et al. in [15]. In the next section, we will derive some inequalities for the dimensions of the c-nilpotent multiplier of pair of finite dimensional nilpotent Lie algebras and their special factor Lie algebras. So, in Theorems 2.9 and 2.11, we obtain some upper bounds for the dimensions and give an example for comparing the upper bounds. These our main results specially generalize some works of Yankosky [18], Niroomand and Russo [10, 11].

2. Main results

This section starts with the following remark which explains Definition 1.1.

Remark 2.1. (i) $\mathcal{M}^{(c)}(L,N) = ker(\mu^c : \mathcal{M}^{(c)}(L) \to \mathcal{M}^{(c)}(\frac{L}{N}))$, where μ^c is the natural homomorphism;

(ii) $\mathcal{M}^{(c)}(L,N)$ is abelian and independent of the choice of the free presentation of L;

(iii) If we put $c = 1$ in 1.1, then clearly

$$\mathcal{M}^{(1)}(L,N) = \frac{R \cap \gamma_2(S,F)}{\gamma_2(R,F)}$$

which is equal to $\mathcal{M}(L,N)$, the well-known Schur multiplier of a pair of Lie algebras (see also [12, 15]);

(iv) If one puts $N = L$, then clearly $\mathcal{M}^{(c)}(L,N)$ is equal to $\mathcal{M}^{(c)}(L)$ (see more information in [14]).

An exact sequence E : $0 \longrightarrow M \longrightarrow K \longrightarrow L \longrightarrow 0$ of Lie algebras is a *central extension* of L if M is a central subalgebra of K. In 1994, Moneyhun [9] proved that in the extension E, if L is finite dimensional then so is $\mathcal{M}(L)$. In the following lemma, by considering another technique, we have a similar result.

Lemma 2.2. *Let L be a finite dimensional Lie algebra and N be an ideal of L. Then there exists a Lie algebra H with an ideal K such that*

$$\dim \gamma_{c+1}(N,L) + \dim \mathcal{M}^{(c)}(L,N) = \dim \gamma_{c+1}(K,H) < \infty.$$

In particular, $\dim \gamma_{c+1}(L) + \dim \mathcal{M}^{(c)}(L) = \dim \gamma_{c+1}(H) < \infty.$

Proof. Let $L = F/R$ be a free presentation of Lie algebra L and S be an ideal of the free Lie algebra F such that $N \cong S/R$. Since $R/\gamma_{c+1}(R,F) \subseteq Z_c(F/\gamma_{c+1}(R,F))$ and L is finite dimensional, then so is $\frac{F/\gamma_{c+1}(R,F)}{Z_c(F/\gamma_{c+1}(R,F))}$. Therefore, by [16, Proposition 2.7], the Lie algebra $\gamma_{c+1}(F)/\gamma_{c+1}(R,F)$ is finite dimensional. Now, suppose that $H = F/\gamma_{c+1}(R,F)$ and $K = S/\gamma_{c+1}(R,F)$, then

$$\dim \gamma_{c+1}(K,H) = \dim(\frac{\gamma_{c+1}(S,F)}{\gamma_{c+1}(R,F)}) = \dim(\frac{\gamma_{c+1}(S,F)}{R \cap \gamma_{c+1}(S,F)}) + \dim(\frac{R \cap \gamma_{c+1}(S,F)}{\gamma_{c+1}(R,F)}).$$

Also, $\gamma_{c+1}(N,L) = \dfrac{\gamma_{c+1}(S,F) + R}{R} \cong \dfrac{\gamma_{c+1}(S,F)}{R \cap \gamma_{c+1}(S,F)}$. Thus the result holds. ■

In the following lemma we show that there is a close relationship between the c-nilpotent multipliers $\mathcal{M}^{(c)}(L,N)$ and $\mathcal{M}^{(c)}(L/K,N/K)$, for ideals K and N in L such that $K \subseteq N$.

Lemma 2.3. *Let L be a Lie algebra with a free presentation*

$$0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0.$$

In addition suppose that S and T are ideals of the Lie algebra F such that $T \subseteq S, N \cong S/R$ and $K \cong T/R$, then the following sequences are exact:

(i)
$$0 \longrightarrow \mathcal{M}^{(c)}(L,K) \longrightarrow \mathcal{M}^{(c)}(L,N) \longrightarrow \mathcal{M}^{(c)}(L/K,N/K) \longrightarrow \frac{K \cap \gamma_{c+1}(N,L)}{\gamma_{c+1}(K,L)} \longrightarrow 0;$$

(ii) *If* $\gamma_{m+1}(N,L)=0$ *and* $K=\gamma_m(N,L)=\dfrac{\gamma_m(S,F)+R}{R}$, *then*

$$0\longrightarrow \frac{\gamma_{c+m}(S,F)}{\gamma_{c+1}(R,F)\cap\gamma_{c+m}(S,F)}\longrightarrow \mathcal{M}^{(c)}(L,N)$$

$$\longrightarrow \mathcal{M}^{(c)}(L/K,N/K)\longrightarrow \gamma_m(N,L)\cap\gamma_{c+1}(N,L)\longrightarrow 0.$$

Proof. We prove only part (ii). Using the assumption, then we have $\gamma_{c+m}(S,F)\subseteq R$. One can easily check that the following sequence is exact:

$$0\longrightarrow \frac{\gamma_{c+m}(S,F)}{\gamma_{c+1}(R,F)\cap\gamma_{c+m}(S,F)}\longrightarrow \frac{R\cap\gamma_{c+m}(S,F)}{\gamma_{c+1}(R,F)}$$

$$\longrightarrow \frac{(\gamma_m(S,F)+R)\cap\gamma_{c+1}(S,F)}{\gamma_{c+1}(\gamma_m(S,F)+R,F)}\longrightarrow \frac{\big(\gamma_m(S,F)\cap\gamma_{c+1}(S,F)\big)+R}{R}\longrightarrow 0. \qquad \blacksquare$$

By the above lemma, we have the following corollary.

Corollary 2.4. *Let* (L,N) *be a pair of finite dimensional nilpotent Lie algebras. Then*

(i) $\dim\big(K\cap\gamma_{c+1}(N,L)\big)+\dim\mathcal{M}^{(c)}(L,N)$

$$=\dim\Big(\frac{R\cap\gamma_{c+1}(T,F)}{\gamma_{c+1}(R,F)}\Big)+\dim\big(\mathcal{M}^{(c)}(L/K,N/K)\big)+\dim\big(\gamma_{c+1}(K,L)\big);$$

(ii) $\dim(\mathcal{M}^{(c)}(L,N))\leq\dim\Big(\dfrac{\gamma_{c+m}(S,F)}{\gamma_{c+1}(R,F)\cap\gamma_{c+m}(S,F)}\Big)+\dim(\mathcal{M}^{(c)}(L/K,N/K))$.

Now, we need the following two technical lemmas.

Lemma 2.5. ([13, Lemma 2.5]) *Let* H,N *be ideals of Lie algebra* L *and*

$$N=N_0\supseteq N_1\supseteq\cdots,$$

be a chain of ideals of N *such that* $[N_i,L]\subseteq N_{i+1}$ *for all* $i=1,2,\ldots$. *Then* $[N_i,[H,_j L]]\subseteq N_{i+j+1}$ *for all* i,j.

Lemma 2.6. *Let* L *be a finite dimensional nilpotent Lie algebra of class* $m\geq 2$,

$$0\longrightarrow R\longrightarrow F\longrightarrow L\longrightarrow 0$$

be a free presentation of L *and* $N\cong S/R$ *for some ideal* S *of the free Lie algebra* F. *Then* $\frac{\gamma_{c+1}([S,_{m-1}F]+R,F)}{\gamma_{c+1}(R,F)}$ *is a homomorphic image of*

$$[N,_{m-1}L]\otimes\underbrace{\frac{L}{Z_{m-1}(N,L)}\otimes\cdots\otimes\frac{L}{Z_{m-1}(N,L)}}_{c-times}.$$

Proof. Put $Z_k(N,L)=T_k/R$ for $0\leq k\leq m$. Now consider the following chain

$$S=T_m\supseteq\cdots\supseteq T_k\supseteq T_{k-1}\supseteq\cdots\supseteq T_1\supseteq T_0=R.$$

Since $[T_k,F]\subseteq T_{k-1}$, by Lemma 2.5, $[T_{m-1},[S,_{m-2}F]]\subseteq T_{m-1-(m-2+1)}=T_0=R$.

Therefore,

$$[[S,_{m-1} F]+R, \underbrace{T_{m-1},\dots,T_{m-1}}_{c-times}] \subseteq [[S,_{m-1} F], \underbrace{T_{m-1},\dots,T_{m-1}}_{c-times}] + [R, \underbrace{T_{m-1},\dots,T_{m-1}}_{c-times}]$$
$$\subseteq [[S,_{m-1} F], \underbrace{T_{m-1},\dots,T_{m-1}}_{c-times}] + \gamma_{c+1}(R,F)$$
$$\subseteq [[[T_{m-1},F],[S,_{m-2} F]], \underbrace{T_{m-1},\dots,T_{m-1}}_{(c-1)-times}]$$
$$+[[[T_{m-1},[S,_{m-2} F]],F], \underbrace{T_{m-1},\dots,T_{m-1}}_{(c-1)-times}] + \gamma_{c+1}(R,F)$$
$$\subseteq [[T_{m-2},[S,_{m-2} F]], \underbrace{T_{m-1},\dots,T_{m-1}}_{(c-1)-times}] + \gamma_{c+1}(R,F)$$
$$\vdots$$
$$\subseteq [[T_0,[S,_0 F]], \underbrace{T_{m-1},\dots,T_{m-1}}_{(c-1)-times}] + \gamma_{c+1}(R,F) \subseteq \gamma_{c+1}(R,F).$$

The latter inclusion gives the following epimorphism

$$\frac{[S,_{m-1} F]+R}{R} \times \underbrace{\frac{F}{T_{m-1}} \times \cdots \frac{F}{T_{m-1}}}_{c-times} \longrightarrow \frac{\gamma_{c+1}([S,_{m-1} F]+R,F)}{\gamma_{c+1}(R,F)},$$

$$(x+R, f_1+T_{m-1},\dots,f_c+T_{m-1}) \longmapsto [x,f_1,\dots,f_c]+\gamma_{c+1}(R,F). \qquad \blacksquare$$

Corollary 2.7. *Under the assumptions and notations of the above lemma, we have*

$$\dim \mathcal{M}^{(c)}(L,N) + \dim\big([N,_{m-1} L]\cap[N,_c L]\big) \le$$
$$\dim \mathcal{M}^{(c)}\big(L/[N,_{m-1} L], N/[N,_{m-1} L]\big) + \dim[N,_{m-1} L]\Big[\dim\big(\frac{L}{Z_{m-1}(N,L)}\big)\Big]^c.$$

Proof. In Corollary 2.4(i), we take $K=[N,_{m-1} L]=\frac{[S,_{m-1}F]+R}{R}$.
By Lemma 2.6, we have

$$\dim \mathcal{M}^{(c)}(L,N) + \dim\big([N,_{m-1} L]\cap[N,_c L]\big)$$
$$= \dim \mathcal{M}^{(c)}\big(L/[N,_{m-1} L], N/[N,_{m-1} L]\big) + \dim\big(\frac{\gamma_{c+1}([S,_{m-1} F]+R,F)}{\gamma_{c+1}(R,F)}\big)$$
$$\le \dim \mathcal{M}^{(c)}\big(L/[N,_{m-1} L], N/[N,_{m-1} L]\big)$$
$$+\dim\Bigg([N,_{m-1} L]\otimes \underbrace{\frac{L}{Z_{m-1}(N,L)}\otimes\cdots\otimes\frac{L}{Z_{m-1}(N,L)}}_{c-times}\Bigg)$$
$$\le \dim \mathcal{M}^{(c)}\Big(L/[N,_{m-1} L], N/[N,_{m-1} L]\Big) + \dim[N,_{m-1} L]\Big[\dim\big(\frac{L}{Z_{m-1}(N,L)}\big)\Big]^c. \qquad \blacksquare$$

Remark 2.8. (i) In the sequel $d(X)$ is the minimum number of generators of the Lie algebra X;

(ii) Let F be a free Lie algebra on the set $X=\{x_1,x_2,...,x_d\}$. Then the basic commutators in F are inductively defined as follows. The generators $x_1,x_2,...,x_d$ are basic commutators of length one and ordered by setting $x_i<x_j$ if $i<j$. If

all the basic commutators c_i of length less than k, where $k > 1$ is integer, have been defined and ordered, then we define the basic commutators of length k to be all commutators of the form $[c_i, c_j]$ such that the sum of length of c_i and c_j is k, $c_i > c_j$ and if $c_i = [c_s, c_t]$, then $c_j \geq c_t$. The basic commutators of length k follow those of length less than k in any order with respect to each other. Basic commutators will be numbered so that they will be ordered by their subscripts.

A. I. Shirshov [17] indicated that the set of all basic commutators on X is a basis of F. It also is shown in [6] that the number of basic commutators on X of length n, denoted by $l_d(n)$, is obtained by the following formula

$$l_d(n) = \frac{1}{n} \sum_{m|n} \mu(m) d^{\frac{n}{m}},$$

where $\mu(m)$ is Möbius function, defined $\mu(1) = 1$, $\mu(k) = 0$ if divisible by a square, and $\mu(p_1 ... p_s) = (-1)^s$ if $p_1, ..., p_s$ are distinct prime numbers.

The following theorem generalizes [4, Theorem A] by different technique and under the weaker conditions.

Theorem 2.9. *Let (L, N) be a pair of finite dimensional nilpotent Lie algebras. Then*

$$\dim \mathcal{M}^{(c)}(L, N) \leq \dim \mathcal{M}^{(c)}(L/[N, L], N/[N, L]) + \dim[N, L] d(\frac{L}{Z(N, L)})^c - \dim[N,_c L].$$

Proof. Let m be the nilpotency class of the pair (L, N). We use induction on m. If (L, N) is of class 1, then $[N, L] = 0$ and the result holds. Suppose that the result holds for a pair nilpotent Lie algebras of class to be less than m and let (L, N) has nilpotency class $m - 1$. Note that $[N,_{m-1} L] \subseteq Z(N, L)$, $[N, L] \subseteq Z_{m-1}(N, L)$, $\big[N/[N,_{m-1} L], L/[N,_{m-1} L]\big] = [N, L]/[N,_{m-1} L]$ and $Z(N, L)/[N,_{m-1} L] \subseteq Z(N/[N,_{m-1} L], L/[N,_{m-1} L])$. For convenience, let

$$A = \frac{(L/[N,_{m-1} L])}{Z(N/[N,_{m-1} L], L/[N,_{m-1} L])}, \quad B = \frac{L}{Z(N, L)} = \frac{(L/[N,_{m-1} L])}{(Z(N, L)/[N,_{m-1} L])}.$$

Since A is a homomorphic image of B, we get $\dim A/A^2 \leq \dim B/B^2$. By induction,

$$\begin{aligned} \dim \mathcal{M}^{(c)}(L/[N,_{m-1} L], N/[N,_{m-1} L]) &\leq \dim \mathcal{M}^{(c)}\big(L/[N, L], N/[N, L]\big) \\ &+ \dim([N, L]/[N,_{m-1} L])(d(A))^c - \dim \gamma_{c+1}\big(N/[N,_{m-1} L], L/[N,_{m-1} L]\big) \\ &\leq \dim \mathcal{M}^{(c)}\big(L/[N, L], N/[N, L]\big) \\ &+ \dim([N, L]/[N,_{m-1} L])(d(B))^c - \dim \Big(\frac{[N,_c L] + [N,_{m-1} L]}{[N,_{m-1} L]}\Big). \end{aligned}$$

By Corollary 2.7,

$$\begin{aligned} \dim \mathcal{M}^{(c)}(L, N) &\leq \dim \mathcal{M}^{(c)}(L/[N,_{m-1} L], N/[N,_{m-1} L]) \\ &+ \dim([N,_{m-1} L])[\dim(L/Z_{m-1}(N, L))]^c - \dim\big([N,_{m-1} L] \cap [N,_c L]\big). \end{aligned}$$

Also $\quad \dim(L/Z_{m-1}(N, L)) \leq \dim\Big(L/(L^2 + Z(N, L))\Big) = \dim(B/B^2).$

Therefore,

$$\dim \mathcal{M}^{(c)}(L,N) \le \dim \mathcal{M}^{(c)}\big(L/[N,L], N/[N,L]\big) + \dim([N,L]/[N,_{m-1}L])(d(B))^c$$
$$+ \dim([N,_{m-1}L])(d(B))^c - \dim\Big(\frac{[N,_c L]+[N,_{m-1}L]}{[N,_{m-1}L]}\Big) - \dim\big([N,_{m-1}L]\cap[N,_c L]\big)$$
$$\le \dim \mathcal{M}^{(c)}\big(L/[N,L], N/[N,L]\big) + \dim([N,L])(d(B))^c - \dim[N,_c L]. \qquad \blacksquare$$

We immediately obtain the following corollary.

Corollary 2.10. *Let (L,N) be a nilpotent pair of Lie algebras. Then*

$$\dim \mathcal{M}^{(c)}(L,N) \le \dim \mathcal{M}^{(c)}(L/[N,L], N/[N,L]) + \dim[N,L]d(L)^c - \dim[N,_c L].$$

Proof. In the proof of the above theorem, since

$$d(B) = \dim(B/B^2) \le \dim(L/L^2) = d(L),$$

the result holds. ■

Theorem 2.11. *Let (L,N) be a nilpotent pair of finite dimentional Lie algebras of class m. Then*

$$\dim \mathcal{M}^{(c)}(L,N) \le \sum_{i=1}^{m} \dim\Big(\frac{\gamma_{c+i}(S,F)}{\gamma_{c+i+1}(S,F)}\Big).$$

Proof. We use induction on m, the nilpotency class of the pair (L,N). If $m=1$, then $[S,F]\subseteq R$, so

$$\dim(M^{(c)}(L,N)) = \dim\big(\gamma_{c+1}(S,F)/\gamma_{c+1}(R,F)\big) \le \dim\big(\gamma_{c+1}(S,F)/\gamma_{c+2}(S,F)\big).$$

By induction

$$\dim\big(\mathcal{M}^{(c)}(\frac{L}{[N,_{m-1}L]}, \frac{N}{[N,_{m-1}L]})\big) \le \sum_{i=1}^{m-1} \dim\Big(\frac{\gamma_{c+i}(S,F)}{\gamma_{c+i+1}(S,F)}\Big).$$

The hypothesis implies that $\gamma_{c+m+1}(S,F)$ is contained in $\gamma_{c+1}(R,F)\cap\gamma_{c+m}(S,F)$. By Corollary 2.4 (ii), we have

$$\dim \mathcal{M}^{(c)}(L,N) \le \dim\big(\mathcal{M}^{(c)}(\frac{L}{[N,_{m-1}L]}, \frac{N}{[N,_{m-1}L]})\big)$$
$$+ \dim\Big(\frac{\gamma_{c+m}(S,F)}{\gamma_{c+1}(R,F)\cap\gamma_{c+m}(S,F)}\Big)$$
$$\le \dim\big(\mathcal{M}^{(c)}(\frac{L}{[N,_{m-1}L]}, \frac{N}{[N,_{m-1}L]})\big) + \dim\Big(\frac{\gamma_{c+m}(S,F)}{\gamma_{c+m+1}(S,F)}\Big)$$
$$\le \sum_{i=1}^{m} \dim\Big(\frac{\gamma_{c+i}(S,F)}{\gamma_{c+i+1}(S,F)}\Big). \qquad \blacksquare$$

Niroomand and Russo [11, Theorem 1.2] obtained an upper bound for $\dim \mathcal{M}(L,N)$. In the following result, we present another upper bounds for this dimension. It will be interesting to compare these upper bounds.

Corollary 2.12. *Let (L,N) be a nilpotent pair of finite dimensional Lie algebras of class m. Then*

(i) $\dim \mathcal{M}(L,N) + \dim[N,L] \leq \dim \mathcal{M}(L/[N,L], N/[N,L]) + \dim[N,L] d(\frac{L}{Z(N,L)})$.

(ii) $\dim \mathcal{M}(L,N) \leq \sum_{i=1}^{m} \dim \Big(\frac{\gamma_{i+1}(S,F)}{\gamma_{i+2}(S,F)}\Big)$.

Proof. The results hold, if one puts $c=1$ in Theorems 2.9 and 2.11. ∎

Now, we compare the upper bounds given in Corollary 2.10 and Theorem 2.11.

Example 2.13. Let F be a free Lie algebra on 2 generators, $L = F/F^4$ and $N = F^2/F^4$. Then L is a Lie algebra of 2 generators, $\dim L = 5$, $\dim N = 3$ and $[N,_2 L] = 1$. By Corollary 2.10, for c=1,

$$\begin{aligned}\dim M(L,N) &\leq \dim M(L/[N,L], N/[N,L]) + \dim[N,L]d(L) - \dim[N,L]\\ &= \dim M(F/F^3, F^2/F^3) + \dim F^3/F^4 \dim F/F^2 - \dim F^3/F^4\\ &= \dim F^3/F^4 + l_2(3).l_2(1) - \dim F^3/F^4 = 4.\end{aligned}$$

Also, by Theorem 2.11, for c=1,

$$\dim M(L,N) \leq \sum_{j=1}^{2} \dim \Big(\frac{[S,_i F]}{[S,_{i+1} F]}\Big) = \dim F^3/F^4 + \dim F^4/F^5 = l_2(3) + l_2(4) = 5.$$

Thus, the upper bound given in Corollary 2.10 is better than the upper bound obtained in Theorem 2.11.
For c=2, using Corollary 2.10, we have

$$\begin{aligned}\dim M^{(2)}(L,N) &\leq \dim M^{(2)}(L/[N,L], N/[N,L]) + \dim[N,L]d(L)^2 - \dim[N,N,L]\\ &= \dim M^{(2)}(F/F^3, F^2/F^3) + \dim F^3/F^4 \big(\dim F/F^2\big)^2 - \dim F^4/F^4\\ &= \dim F^4/F^5 + l_2(3).(l_2(1))^2 - 0 = 3 + 2\cdot 4 = 11.\end{aligned}$$

Also, using Theorem 2.11,

$$\begin{aligned}\dim M^{(2)}(L,N) &\leq \sum_{i=1}^{2} \dim \Big(\frac{[S,_{i+1} F]}{[S,_{i+2} F]}\Big) = \dim F^4/F^5 + \dim F^5/F^6\\ &= l_2(4) + l_2(5) = 3 + 6 = 9.\end{aligned}$$

Therefore, the upper bound given in Theorem 2.11 is better than the upper bound obtained in Corollary 2.10.

Thus, according to the above example, we can say that two obtained upper bounds in Corollary 2.10 and Theorem 2.11 are not frequently comparable.

Acknowledgments. The authors would like to thank the referee for useful comments that helped to improve the presentation of the article.

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Received January 3, 2019
and in final form July 4, 2019