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The Rosenfeld Planes

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Abstract. We construct the Lie triples of the exceptional symmetric spaces with dimensions 16, 32, 64, 128 and isometry groups of Dynkin types F_4, E_6, E_7, E_8. We start with a certain representation for the spin group $Spin_{k+l}$ by 2×2-matrices over $\mathbb{K} \otimes \mathbb{L}$ where $\mathbb{K}, \mathbb{L}$ are normed real division algebras with dimensions $k, l \in \{1, 2, 4, 8\}$. The Lie triple is $(\mathbb{K} \otimes \mathbb{L})^2$ with this representation of $Spin_{k+l}$ as isotropy representation, extended by certain scalars in $\mathbb{K} \otimes \mathbb{L}$. Six of these ten representations are shown to be isotropy representations of classical symmetric spaces. This observation greatly simplifies the check of Jacobi identity and the computation of the root decomposition. We call the corresponding symmetric spaces (generalized) Rosenfeld planes. They contain half dimensional subspaces with Lie triple $(\mathbb{K} \otimes \mathbb{L})^1$, so called Rosenfeld lines, which are shown to be certain real Grassmannians.

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1. Introduction

The irreducible compact symmetric spaces are classified [7]. There are two types: classical symmetric spaces which come in infinite series, and finitely many exceptions, called exceptional symmetric spaces. Both types fall into three subclasses: Lie groups, Grassmannians and spaces of "structures". Among the classical spaces, the Grassmannians are the spaces of linear subspaces in $\mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n$ while the "structures" are real and quaternionic structures on $\mathbb{C}^n$ and complex structures on $\mathbb{R}^n$ and $\mathbb{H}^n$. Among the exceptional spaces we have the exceptional Lie groups, and the role of the Grassmannians is played by the so called *Rosenfeld planes* with dimensions 16, 32, 64, 128 whose isometry groups are of Dynkin type F_4, E_6, E_7, E_8, respectively, and there are also certain spaces of "structures", cf. [2, p. 313f]. The 16-dimensional space is the octonionic projective plane. While Freudenthal [6] developed a separate geometry for each of these four spaces, Rosenfeld [8] tried to describe them as projective planes over the algebra $\mathbb{K} \otimes \mathbb{O}$ where $\mathbb{O}$ denotes the octonion algebra and $\mathbb{K}$ one of the four normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Though Rosenfeld's attempt failed, these spaces have in fact a strong connection to $\mathbb{K} \otimes \mathbb{O}$, and they show also similarities to projective planes: e.g. they contain "projective lines", and the space of projective lines in a Rosenfeld plane is isomorphic to the Rosenfeld plane itself ("duality"), cf. [3]. Extending work of Witt [9] and Adams [1], we show the relationship

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to $\mathbb{K}\otimes\mathbb{O}$ using a certain spin representation on $(\mathbb{K}\otimes\mathbb{L})^2$. Our work is based on the thesis of the first author [4].

Recall that the *spin group* $Spin_n$ is contained in the Clifford algebra over $\mathbb{R}^{n-1}$, denoted by Cl_{n-1}. It is a subgroup of the multiplicative group $Cl_{n-1}^{\times}$, generated by the unit sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^{n-1}\oplus\mathbb{R}\cdot 1\subset Cl_{n-1}$. Clearly, any matrix representation of the algebra Cl_{n-1} restricts to a representation of $Spin_n$. Given two normed division algebras $\mathbb{K},\mathbb{L}\in\{\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}\}$, the vector space $V=(\mathbb{K}\otimes\mathbb{L})^2$ is a Cl_{n-1}-module for $n=k+l$ with $k=\dim\mathbb{K}\le l=\dim\mathbb{L}$ where $\mathbb{K},\mathbb{L}\subset\mathbb{R}^{k+l}=\mathbb{K}\oplus\mathbb{L}$ act on V by diagonal and anti-diagonal matrices

$$\begin{pmatrix} e\otimes 1 & \\ & \bar e\otimes 1\end{pmatrix},\ e\in\mathbb{K},\quad\text{and}\quad \begin{pmatrix} & -1\otimes\bar f\\ 1\otimes f & \end{pmatrix},\ f\in\mathbb{L}.$$

We will show that this representation is essentially (up to complex or quaternionic unit scalars in both tensor factors) the isotropy representation of a symmetric space whose infinitesimal structure (“Lie triple structure”) on V is determined by this representation. From the ten algebras $\mathbb{K}\otimes\mathbb{L}$ with $\mathbb{K}\subset\mathbb{L}$ we obtain ten symmetric spaces called *generalized Rosenfeld planes*. Among them are the projective planes ($\mathbb{K}=\mathbb{R}$) and the Rosenfeld planes ($\mathbb{L}=\mathbb{O}$). For $\mathbb{L}\neq\mathbb{O}$ these are classical symmetric spaces, certain real Grassmannians as we shall show. Since the ten spaces are nested into each other, we can use the classical subspaces to establish the Lie triple structure and the root systems for the Rosenfeld planes without lengthy calculations. This gives also another construction for the exceptional simple Lie algebras.

2. Clifford and spin representations

Recall that the *Clifford algebra* Cl_{n-1} is the algebra (with 1) generated by euclidean $(n-1)$-space and subject to the relations

$$vw+wv=-2\langle v,w\rangle\cdot 1 \tag{1}$$

for all $v,w\in\mathbb{R}^{n-1}$, or equivalently,

$$e_ie_j+e_je_i=-2\delta_{ij} \tag{2}$$

for any orthonormal basis $e_1,\dots,e_{n-1}$ of $\mathbb{R}^{n-1}$. Cl_{n-1} contains euclidean n-space $\mathbb{R}^n=\mathbb{R}\cdot 1\oplus\mathbb{R}^{n-1}$. Its unit sphere $\mathbb{S}^{n-1}=\{(t,v)=t\cdot 1+v : t^2+|v|^2=1\}$ consists of invertible elements of Cl_{n-1} since $(t+v)(t-v)=t^2+|v|^2=1$. The group generated by $\mathbb{S}^{n-1}$ using the algebra multiplication in Cl_{n-1} is the *spin group* $Spin_n$.

A *representation* of Cl_{n-1} is an algebra homomorphism ϕ into some matrix algebra, $\phi\colon Cl_{n-1}\to\mathbb{R}^{p\times p}$, making the vector space $\mathbb{R}^p$ a Cl_{n-1}-module. By (2), a representation is given by any system of anticommuting complex structures $J_1,\dots,J_{n-1}$ on $\mathbb{R}^p$ (that is $J_i^2=-I$ and $J_iJ_k=-J_kJ_i$ for $k\neq i$).

Any representation of the algebra Cl_{n-1} induces a representation of the group $Spin_n\subset Cl_{n-1}$. The Lie algebra $\mathfrak{spin}_n$ is simple unless $n=4$ where $Spin_4=\mathbb{S}^3\times\mathbb{S}^3$. In all other cases, the kernel of $\phi|_{Spin_n}$ is a finite normal subgroup, thus it is contained in the center $C\subset Spin_n$. If n is odd, $C=\{\pm 1\}$. If n is even, then $C=\{\pm 1,\pm\omega\}$ where ω is the *volume element*, $\omega=e_0e_1\dots e_{n-1}$ with $e_0=1$ and $e_1,\dots,e_{n-1}$ being an orthonormal basis of $\mathbb{R}^{n-1}$. The volume element has order 2 (it squares to 1) if n is divisible by 4, otherwise it has order 4 and generates C. We consider two types of examples of (not necessarily irreducible) representations of Cl_{n-1}. They will be used in the sequel.

Example 2.1. $n = k \in \{1,2,4,8\}$. Then $\mathbb{R}^n$ is a normed division algebra, $\mathbb{R}^n = \mathbb{K} \in \{\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}\}$. We have $p = n$, and $\mathbb{K} = \mathbb{R} \oplus \mathbb{R}^{n-1}$ (which generates the algebra Cl_{n-1}) acts on $\mathbb{K}$ by left multiplication $\phi_o(a) = L(a)$, $a \in \mathbb{K}$. This extends to an algebra homomorphism $\phi_o\colon Cl_{n-1} \to \mathrm{End}(\mathbb{K})$. Then $\phi_o(Spin_k)$ acts irreducibly on $\mathbb{K}$ as $\{\pm I\}$, $\mathbb{S}^1$, $\mathbb{S}^3$, SO_8 for $k = 1,2,4,8$, respectively.
In fact, the Lie algebra $\mathfrak{spin}_n$ is generated by e_i and their products e_ie_j for distinct $i,j \in \{1,\dots,n-1\}$.[1] In the case $n=4$ we have $\phi_o(e_1e_2) = L(i)L(j) = L(k) = \phi_o(e_3)$ etc., thus $\phi_o|_{Spin_4}$ has a three-dimensional kernel whose Lie algebra is spanned by $e_ie_j - e_k$ for $ijk = 123, 231, 312$, and $Spin_4 = \mathbb{S}^3 \times \mathbb{S}^3$ is mapped onto $\mathbb{S}^3$.
In the case $n = 8$, for the volume element $\omega = e_1 \dots e_8 \in Spin_8 \subset Cl_7$ we have $\phi_o(\omega) = -I$ since $L(e_1)\dots L(e_8) = -I$, and $\ker \phi_o = \{1, -\omega\}$.

Example 2.2. $n = k+l$ with $k,l \in \{1,2,4,8\}$ Then $\mathbb{R}^n = \mathbb{K}\oplus\mathbb{L}$ with $k = \dim\mathbb{K}$ and $l = \dim\mathbb{L}$. Further, $p = 2kl$, and the Cl_{n-1}-module is $(\mathbb{K}\otimes\mathbb{L})^2$ where $\mathbb{K}\otimes\mathbb{L}$ denotes the tensor product algebra. The representation ϕ is determined by the following embedding of $\mathbb{R}^n = \mathbb{K}\oplus\mathbb{L}$ into $(\mathbb{K}\otimes\mathbb{L})^{2\times 2}$, the space of 2×2-matrices with coefficients in $\mathbb{K}\otimes\mathbb{L}$, which is a subspace of all real $(2kl \times 2kl)$-matrices,

$$\mathbb{K}\oplus\mathbb{L} \ni (p,q) \xrightarrow{\phi} \begin{pmatrix} \bar p & -\bar q' \\ q' & p \end{pmatrix}. \tag{3}$$

Here we put $p = p\otimes 1$ and $q' = 1 \otimes q \in \mathbb{K}\otimes\mathbb{L}$. Thus $\phi(\mathbb{K}\oplus\mathbb{L})$ is spanned by the identity matrix I and $k+l-1$ anticommuting complex structures

$$\begin{pmatrix} -e & \\ & e \end{pmatrix},\ \begin{pmatrix} & -1 \\ 1 & \end{pmatrix},\ \begin{pmatrix} & f' \\ f' & \end{pmatrix} \tag{4}$$

where e and f run through standard orthonormal bases of the imaginary parts of $\mathbb{K}$ and $\mathbb{L}$. The latter matrices span the tangent space of $\mathbb{S}^{n-1} \subset Spin_n$, and together with their Lie products they generate the whole Lie algebra $\mathfrak{spin}_n$. Commutators of these anticommuting matrices are twice their products which are again either diagonal or anti-diagonal. There are no double commutators (cf. footnote 1). Thus the full Lie algebra $\phi_*(\mathfrak{spin}_{k+l})$ is spanned by the matrices in (4) together with their products,

$$\begin{pmatrix} & e \\ e & \end{pmatrix},\ \begin{pmatrix} -f' & \\ & f' \end{pmatrix},\ \begin{pmatrix} ef & \\ & ef \end{pmatrix},\ \begin{pmatrix} e'f' & \\ & e'f' \end{pmatrix},\ \begin{pmatrix} & -ef' \\ ef' & \end{pmatrix} \tag{5}$$

where ef means $L(e)L(f)$ (with $e < f$ with respect to the natural ordering of the orthonormal basis). We may assume $\mathbb{K} \subset \mathbb{L}$, and we assume that the orthonormal bases are chosen such that the basis of Im $\mathbb{K}$ is contained in the basis of Im $\mathbb{L}$.

[1] Note that the $e_i \in Cl_{n-1}$ anticommute with each other, therefore double commutators $[[e_i,e_j],e_k]$ must vanish for i,j,k distinct since e_k commutes with e_ie_j.

Allowing now e, f to run through the full basis of $\mathbb{K}$ and $\mathbb{L}$, including the unit element 1, but with $(e,f) \neq (1,1)$, we can combine (4) and (5) to the following basis of $\phi_*(\mathfrak{spin}_{k+l})$,

$$\begin{pmatrix} \overline{ef} & \\ & ef \end{pmatrix}, \begin{pmatrix} \overline{e'f'} & \\ & e'f' \end{pmatrix}, \begin{pmatrix} & -\overline{ef'} \\ ef' & \end{pmatrix}. \tag{6}$$

Remark 2.3. If n is a multiple of 4, the volume $\omega = e_1 \dots e_n$ is a center element of order 2. If $n = 4 = 2+2$, this is not in the kernel of ϕ: it is mapped to $ii'I \neq I$. In the remaining cases $n = 4+4, 4+8, 8+8$, the volume ω is mapped to the identity matrix since $L(e_1)\dots L(e_k) = -I$ on $\mathbb{K} = \mathbb{H}, \mathbb{O}$. Since -1 is never in the kernel (it is mapped to $-I$), we have $\phi(Spin_n) = Spin_n/\{1,\omega\} = SO'_n$. In the case $n = 8$, the triality automorphism of $Spin_8$ maps ω to -1, thus $SO'_8 \cong SO_8$. But in the other cases $n = 12, 16$, the groups SO_n and SO'_n are not isomorphic.

Remark 2.4. Clearly, $Spin_{k+l}$ with $k, l \leq 8$ is contained in $Spin_{16}$. Left translations with elements of the standard basis of $\mathbb{O} \otimes \mathbb{O}$ permute this basis up to sign. Therefore their squared norm is $\dim \mathbb{O} \otimes \mathbb{O} = 64$, and consequently all the generators of the Lie algebra in (6) have squared norm 128 in the trace metric of $Spin_{16}$.

3. Extensions by unit scalars

The connected group S_k of *unit scalars* in a normed division algebra $\mathbb{K}$ is $\mathbb{S}^1$ for $\mathbb{K} = \mathbb{C}$ and $\mathbb{S}^3$ for $\mathbb{K} = \mathbb{H}$ (acting by right multiplication) and $\{1\}$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{O}\}$.[2] The actions of S_k and $Spin_k$ on $\mathbb{K}$ (Example 1, section 2) commute. Together they define a linear action of $Spin_k \times S_k$ on $\mathbb{K}$ which is the natural action of $SO_1 = \{1\}$, $SO_2 = \mathbb{S}^1$, $SO_4 = L(\mathbb{S}^3) \cdot R(\mathbb{S}^3)$, $SO_8 = \langle L(\mathbb{S}^7) R(\mathbb{S}^7) \rangle$ on $\mathbb{K} = \mathbb{R}, \mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^8$, respectively. We call these actions of SO_k on $\mathbb{K}$ the *extended actions* of Example 1. Further, the natural action of $SS' := S_k \times S_l$ on $(\mathbb{K} \otimes \mathbb{L})^2$ commutes with the action of $Spin_{k+l}$ (Example 2, section 2). Together they define a group

$$K = \phi(Spin_{k+l}) \cdot S_k \cdot S_l, \tag{7}$$

whose action on the real vector space $(\mathbb{K} \otimes \mathbb{L})^2$ is the *extended action* of Example 2.

Lemma 3.1. *K is contained in $\phi(Spin_{16})$.*

Proof. As mentioned before, $\phi(Spin_{k+l}) \subset \phi(Spin_{16})$. We have to show that $S_k \cdot S_l \subset Spin_{16}$. It suffices to prove this for the largest associative case, $\mathbb{K} \otimes \mathbb{L} = \mathbb{H} \otimes \mathbb{H}$ ($k = l = 4$). We observe that for any $q \in \mathbb{H}$, the right translation $R(q)$ on $\mathbb{H}$ can be replaced by a composition of octonionic left translations:

$$R(q) = -L(l)L(\bar{q}l) \tag{8}$$

where $\mathbb{O} = \mathbb{H} \oplus l\,\mathbb{H}$ with $l^2 = -1$. In fact, if $q \in \mathbb{R}$, this follows since $L(l)^2 = -I$. If $q \perp \mathbb{R}$ (that is $\bar{q} = -q$) and $|q| = 1$, then $l \perp lq$, and for any $x \in \mathbb{H}$ we have

$$L(l)L(ql)x = l((ql)x) = \left\{ \begin{array}{rcrcr} -l^2 q & = & q & \text{if} & x = 1 \\ -l^2 q^2 & = & -1 & \text{if} & x = q \\ l^2 qx & = & -qx & \text{if} & x \perp 1, q \end{array} \right\} = xq$$

[2]The non-real unit elements of $\mathbb{O}$ do not commute with all left or right translations.

where we have used anti-associativity[3] $l((lq)x) = -(l(lq))x$ in the case $x \perp 1, q$. A similar statement holds for l' and q' instead of l and q. By (5), the matrices $ef \cdot \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ and $e'f' \cdot \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ with $e = -l$ and $f = lq$ are contained in $\phi_*(\mathfrak{spin}_{16})$ and hence[4] the group $\mathbb{S}^3 \cdot \mathbb{S}^3$ of unit $(\mathbb{H} \otimes \mathbb{H})$-scalar multiplications is contained in $\phi(Spin_{16})$. ∎

Remark 3.2. The unit generators for the Lie algebra of SS' are matrices of type $\left(\begin{smallmatrix} L & \\ & L \end{smallmatrix}\right)$ with $L = L(e)L(f)$ or $L(e')L(f')$ where $e, f \in \mathrm{Im}\,\mathbb{O}$ are certain elements of the standard basis $B_{\mathbb{O}}$. Thus L is just a permutation of the standard basis of $\mathbb{O}\otimes\mathbb{O}$, up to sign, whose squared norm is 64, and in $Spin_{16}$ the unit generators for the Lie algebra of SS' have the same squared norm 128 as the unit generators for $\rho_*\mathfrak{spin}_{k+l}$, see Remark 2.4.

4. $(\mathbb{K} \otimes \mathbb{L})^2$ is a Lie triple

A main goal of the present paper is the proof of the following

Theorem 4.1. *The representation of K on $V = (\mathbb{K}\otimes\mathbb{L})^2$ is an s-representation, that is the isotropy representation of a symmetric space $X = G/K$.*

In order to prove the theorem, we have to show that the vector space $\mathfrak{g} := \mathfrak{k} \oplus V$ carries a Lie bracket extending the one on $\mathfrak{k}$, and $\mathfrak{k}$ and V are the (± 1)-eigenspaces of an involution on $\mathfrak{g}$ (a Lie algebra automorphism σ with $\sigma^2 = \mathrm{id}$). The idea is simple: for all $A \in \mathfrak{k}$ and $v, w \in V$ we let $[A, v] = -[v, A] = Av$ and define $[v, w] \in \mathfrak{k}$ by its inner product with any element in $\mathfrak{k}$:

$$\langle A, [v, w]\rangle_{\mathfrak{k}} := \langle Av, w\rangle\,. \tag{9}$$

The inner product $\langle\ ,\ \rangle_{\mathfrak{k}}$ on $\mathfrak{k}$ must be invariant under the adjoint action of K. Then this “Lie bracket” is invariant under K,

$$[kv, kw] = k[v, w]k^{-1} \tag{10}$$

for all $k \in K$ and $v, w \in V$. In fact, $\langle A, [kv, kw]\rangle = \langle Akv, kw\rangle = \langle k^TAkv, w\rangle = \langle k^TAk, [v, w]\rangle = \langle A, k[v, w]k^T\rangle$.

The $\mathrm{Ad}(K)$-invariant inner product $\langle\ ,\ \rangle_{\mathfrak{k}}$ on $\mathfrak{k}$ is not quite arbitrary: eventually it must be the restriction of an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. There is always such a metric, e.g. the trace metric for the representation of K on $\mathfrak{g}$ (the restriction of the Killing form metric of $\mathfrak{g}$ when $\mathfrak{g}$ is a Lie algebra). Of course, if $\mathfrak{k}$ is simple (like $\mathfrak{k} = \mathfrak{spin}_{16}$), it carries only one $\mathrm{Ad}(\mathfrak{k})$-invariant metric up to a factor. In our case we will embed $\mathfrak{k}$ into $\mathfrak{spin}_{16}$ and use the induced metric on $\mathfrak{k}$, see Section 7.

So far, this construction works for every representation. We will call $(V, [\ ,\ ])$ a *fake Lie triple*, and a subspace $V_1 \subset V$ a *subtriple* if $[u, v]w \in V_1$ for all $u, v, w \in V_1$. A fake Lie triple V is a (true) Lie triple if the Jacobi identity holds:

$$[u, v]w + [v, w]u + [w, u]v = 0 \tag{J}$$

for all $u, v, w \in V$. In our case, (J) will follow easily using subtriples which are already known to be Lie triples, see section 7.

[3]When $a, b \in \mathrm{Im}\,\mathbb{O}$ are perpendicular and $c \perp 1, a, b, ab$, then $a(bc) = -(ab)c$.

[4]If a matrix Lie algebra contains a complex structure J (that is $J^2 = -I$), then J is also contained in the corresponding matrix group since $e^{(\pi/2)J} = J$, like $e^{(\pi/2)i} = i$.

5. Rosenfeld lines

First we restrict our attention to the subspace

$$V_1 = \mathbb{K}\otimes\mathbb{L} = \{v = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in V : y = 0\} \subset V = (\mathbb{K}\otimes\mathbb{L})^2.$$

Theorem 5.1. *The subspace $V_1 \subset V$ is a subtriple which is a true Lie triple, corresponding to the Grassmannian $G_k(\mathbb{R}^{k+l})$.*

Proof. We have to show first $[v,w] \in \mathfrak{k}_1$ for all $v, w \in V_1$ where $\mathfrak{k}_1$ is the Lie algebra of $K_1 = \{k \in K : k(V_1) = V_1\}$, the stabilizer of V_1. This is the subgroup of block diagonal matrices corresponding to the decomposition $V = V_1 \oplus V_2$ where $V_i = (\mathbb{K}\otimes\mathbb{L})\mathbf{e}_i$ with $\mathbf{e}_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\mathbf{e}_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. But as we see from (6), the Lie algebra $\mathfrak{k}$ also decomposes as $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ where $\mathfrak{k}_2$ contains the anti-diagonal matrices which interchange V_1 and V_2; note that the Lie algebra of the scalars $S_k S_l$ belongs to $\mathfrak{k}_1$. For any $v, w \in V_1$ and $A \in \mathfrak{k}_2$ we have

$$\langle A, [v,w]\rangle = \langle Av, w\rangle \in \langle V_2, V_1\rangle = 0, \quad \text{thus} \quad [v,w] \in \mathfrak{k}_2^\perp = \mathfrak{k}_1.$$

The Lie algebra $\mathfrak{k}_1$ consists of the diagonal matrices in $\mathfrak{k}$, but for the action on V_1 only the upper left entry matters. According to (6), the Lie algebra for this representation is spanned by $L(\bar e)L(\bar f)$ and $L(\bar e')L(\bar f')$ and possibly $R(ef')$, due to the factor $S_k \cdot S_l$, see (7). Thus K_1 acts on $\mathbb{K}\otimes\mathbb{L}$ through left and right multiplications (L, R) by (possibly several) elements of $\mathbb{K}\otimes 1$ and $1\otimes\mathbb{L}$. This is a tensor representation: the left and right multiplications by $\mathbb{K} = \mathbb{K}\otimes 1$ act only on the first tensor factor $\mathbb{K}$ while those by $\mathbb{L}' = 1\otimes\mathbb{L}$ act only on the second tensor factor $\mathbb{L}$. The action on each of the tensor factors $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ is the extended action of Example 1, see the first paragraph of section 3. It is the natural action of SO_k on $\mathbb{R}^k$ for $k = 1, 2, 4, 8$, respectively. Thus the tensor representation on $\mathbb{K}\otimes\mathbb{L}$ is the isotropy representation of the Grassmannian $G_k(\mathbb{R}^{k+l})$ of k-dimensional linear subspaces in $\mathbb{R}^{k+l}$.[5] ∎

6. Associative Rosenfeld planes

When both $\mathbb{K}$ and $\mathbb{L}$ are associative, $(\mathbb{K}\otimes\mathbb{L})^2$ is a module over the ring of "scalars" $\mathbb{K}\otimes\mathbb{L}$ acting by right translations, and $\phi(Spin_{k+l})$ is $(\mathbb{K}\otimes\mathbb{L})$-linear: since it acts by matrices of left translations, it commutes with $R(p\otimes q)$ for all $p \in \mathbb{K}$ and $q \in \mathbb{L}$, see also [5].

We will show that K acts on V as the isotropy representation of some classical symmetric space. Clearly, if $\mathbb{K} = \mathbb{R}$, this is the projective plane over $\mathbb{L}$. The cases $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{H}$ remain.

[5] Recall that the k-planes in $\mathbb{R}^n$ close to $\mathbb{R}^k = \{x \in \mathbb{R}^n : x_{k+1} = \cdots = x_n = 0\}$ are just the graphs of linear maps $F\colon \mathbb{R}^k \to (\mathbb{R}^k)^\perp = \mathbb{R}^{n-k} = \mathbb{R}^l$. Therefore the tangent space of $G_k(\mathbb{R}^{k+l})$ at $\mathbb{R}^k$ is $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^l)$. The group SO_n acts transitively on $G_k(\mathbb{R}^n)$, and the isotropy group of the (oriented) k-plane $\mathbb{R}^k \in G_k(\mathbb{R}^n)$ is $SO_k \times SO_l$ acting on the tangent space $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^l)$ by $((A,B),F) \mapsto BFA^{-1}$. Using the identification $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^l) = \mathbb{R}^k \otimes \mathbb{R}^l$, this is the tensor representation

$$((A,B), v\otimes w) \mapsto Av \otimes Bw.$$

Case 1: $\mathbb{K} = \mathbb{C}$.
If $\mathbb{K} = \mathbb{C}$ and $\mathbb{L} \in \{\mathbb{C}, \mathbb{H}\}$, we consider the right translation $R(ii')$ which commutes with the action of $Spin_{k+l}$, hence the eigenspaces of $R(ii')$ are invariant under $Spin_{k+l}$. Since $(ii')^2 = 1$, the only eigenvalues are ± 1, and we have $V = E_+ \oplus E_-$ where $R(i) = \mp R(i')$ on $E_\pm$. Moreover, matrices in $Spin_{k+l}$ commute with the complex structure $R(i) = L(i)$ which leaves E_+ and E_- invariant, hence $Spin_{k+l}$ acts by complex linear maps, and $E_\pm$ are invariant complex subspaces. Complex conjugation in $\mathbb{K}$ anticommutes with $R(ii')$, hence it interchanges E_+ and E_- which shows $\dim E_+ = \dim E_- = \frac{1}{2} \dim V$. The group $Spin_{k+l} \cdot S_k$ acts complex-linearly on both subspaces E_- and E_+ (where $S_k = \mathbb{S}^1 \subset \mathbb{C} = \mathbb{K}$ acts by complex scalars).

Subcase 1a: $\mathbb{L} = \mathbb{C}$.
Then $k+l = 4$ and $E_\pm \cong \mathbb{C}^2$, and on $E_\pm$ we have $R(i') = \mp R(i)$, hence $R(q') = R(\bar{q})$ on E_+ and $R(q') = R(q)$ on E_-. Thus on $E_\pm$, the matrices in (3) reduce to the SU_2-matrix $\begin{pmatrix} p & -\bar{q} \\ q & \bar{p} \end{pmatrix}$ for $p, q \in \mathbb{C}$ with $|p|^2 + |q|^2 = 1$. Therefore $K' = \phi(Spin_4)$ acts on $E_\pm$ as SU_2. To see the kernels of these actions we pass to the Lie algebra $\mathfrak{k}' = \phi_*(\mathfrak{spin}_4)$, see (6). This is a module over the commutative ring $\mathbb{C} \otimes \mathbb{C}$, like $V = (\mathbb{C} \otimes \mathbb{C})^2$ on which it acts. Any such module M decomposes into submodules, the (± 1)-eigenspaces $M_\pm$ for the scalar multiplication with ii', and $M_\pm = (1 \pm ii')M$. In particular, $\mathfrak{k}'_\pm = (1 \pm ii')\mathfrak{k}' = (1 \pm ii')\mathfrak{su}_2$, and these are Lie subalgebras since $(1 \pm ii')^2 = 2(1 \pm ii')$. Moreover, from $(1+ii')(1-ii') = 0$ we have $\mathfrak{k}'_+ E_- = \mathfrak{k}'_- E_+ = 0$. Thus $K'_\pm = \exp \mathfrak{k}'_\pm \subset K'$ acts as SU_2 on $E_\pm \cong \mathbb{C}^2$ and trivially on $E_\mp$, and since $K' \cong SU_2 \times SU_2$, we obtain $K' = K'_+ \times K'_-$ (direct product of groups).

The other group $SS' \subset \mathbb{C} \otimes \mathbb{C}$ acts on V by scalar multiplication, and $SS' \cong (\mathbb{S}^1 \times \mathbb{S}^1)/\pm$. This is a square torus: $SS' = S_+ \times S_-$ with factors $S_+ = \{zz' : z \in \mathbb{S}^1\}$ and $S_- = \{z\bar{z}' : z \in \mathbb{S}^1\}$. On E_+ we have $i' = -i$, hence $z' = \bar{z}$, and $zz' = z\bar{z} = 1$ while $z' = z$ and $zz' = z^2$ on E_-. Thus S_+ acts trivially on E_+ and "with double speed" on E_-, and vice versa for S_-. Therefore we obtain $K = K_+ \times K_-$ with $K_+ = K'_+ \cdot S_+ \cong U_2$ and $K_- = K'_- \cdot S_- \cong U_2$, and the representation is the isotropy representation for $\mathbb{CP}^2 \times \mathbb{CP}^2$ on $\mathbb{C}^2 \oplus \mathbb{C}^2$.

Subcase 1b: $\mathbb{L} = \mathbb{H}$.
Then $R(j')$ anticommutes with $R(ii')$ and interchanges the eigenspaces E_+ and E_-. We consider the $\mathbb{C}$-linear map

$$F : E_+ \otimes_{\mathbb{C}} \mathbb{H} \to V = (\mathbb{C} \otimes \mathbb{H})^2, \quad v \otimes q \mapsto vq' = R(q')v.$$

Here E_+ and $\mathbb{H}$ are considered as complex vector spaces with complex structures $R(i) = -R(i')$ on E_+ and $-L(i)$ on $\mathbb{H}$. Then

$$\begin{aligned} F(vi \otimes q) &= viq' = vq'i, \qquad (11) \\ F(v \otimes (-iq)) &= -vi'q' = viq' = vq'i. \end{aligned}$$

Hence F is well defined and complex linear where the complex structure on the target space $(\mathbb{C} \otimes \mathbb{H})^2$ is $R(i)$. Moreover, F is onto: obviously, $E_+ = F(E_+ \otimes 1)$ and $E_- = R(j')E_+ = F(E_+ \otimes j)$. Thus F is a linear isomorphism by equality of dimensions: both the domain and the range have real dimension 16.

Further, F is K-equivariant. In fact, $\phi(Spin_6)$ keeps E_+ invariant since each element of $Spin_6$ is a matrix of left translations and thus it commutes with $R(ii')$.

The complex structure $R(i)$ on E_+ is also invariant, hence $\phi(Spin_6)$ is contained in the unitary group of E_+. Since E_+ has real dimension $\frac{1}{2}\dim(\mathbb{C}\otimes\mathbb{H})^2 = 8$, it is complex isomorphic to $\mathbb{C}^4$. Therefore $\phi(Spin_6)$ is a 15-dimensional subgroup of U_4 which can only be SU_4.

The other group $SS' = \mathbb{S}^1\mathbb{S}^3$ acts by right translations on $V = (\mathbb{C}\otimes\mathbb{H})^2$. Pulling back this action to $E_+ \otimes_{\mathbb{C}} \mathbb{H}$ via F, it changes only the second tensor factor $\mathbb{H}$: For $(z,p) \in \mathbb{S}^1 \times \mathbb{S}^3$ and $v \in E_+$, the "scalar" $z \otimes p = zp' \in \mathbb{C}\otimes\mathbb{H}$ acts on V by right translation $R(\bar{z}\bar{p}')$:

$$(z,p).vq' = vq'\bar{z}\bar{p}' = v\bar{z}q'\bar{p}' = vz'q'\bar{p}' = v(zq\bar{p})',$$

using $v\bar{z} = vz'$ for $v \in E_+$. Thus the action of $\mathbb{S}^1 \times \mathbb{S}^3$ on the second factor $\mathbb{H}$ is $((z,p),q) \mapsto zqp^{-1}$. It commutes with the complex structure $-L(i)$ on $\mathbb{H}$ and is precisely the action of U_2 on $\mathbb{H} = \bar{\mathbb{C}}^2$ (recall that the complex structure on $\mathbb{H}$ was $-L(i)$, not $L(i)$). Therefore the action of K on V is equivalent to the tensor representation of $SU_4 \times U_2$ on $\mathbb{C}^4 \otimes_{\mathbb{C}} \bar{\mathbb{C}}^2 = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^2, \mathbb{C}^4)$ which is the isotropy representation of the complex Grassmannian $G_2(\mathbb{C}^6)$ (see also [5]).

Case 2: $\mathbb{K} = \mathbb{L} = \mathbb{H}$.

Here we consider the two commuting involutions $R(ii')$ and $R(jj')$. They have a common eigenspace decomposition

$$V = E_{++} + E_{+-} + E_{-+} + E_{--}$$

where E_{++} is the intersection of the $(+1)$-eigenspaces of $R(ii')$ and $R(jj')$ etc. As before, $\phi(Spin_8)$ commutes with all right translations, thus it keeps these four subspaces invariant. We put $E = E_{++}$ and consider the linear map

$$F\colon E\otimes\mathbb{H} \to (\mathbb{H}\otimes\mathbb{H})^2 : v\otimes p \mapsto vp = v\bar{p}' \tag{12}$$

(recall that $R(p) = -R(p')$ on E for all imaginary $p \in \mathbb{H}$). The action of $Spin_8$ (pulled back to $E\otimes\mathbb{H}$ via F) lives only on E, while the action of $S = \mathbb{S}^3\mathbb{S}^3$ by right translations transforms $p \in \mathbb{H}$: for any $ab' \in S$ and $v \in E$ we have

$$\begin{aligned} a.vp &= vp\bar{a}, \\ b'.vp &= vp\bar{b}' = v\bar{b}'p = vbp. \end{aligned}$$

In other words, $ab' \in S$ acts only on the second tensor factor of $E\otimes\mathbb{H}$, and it acts by $p \mapsto bp\bar{a}$, which is the standard action of

$$SO_4 = \{L(b)R(\bar{a}) : b, a \in \mathbb{S}^3\}$$

on $\mathbb{H} = \mathbb{R}^4$. The linear map F in (12) is an isomorphism (onto and the same dimensions for domain and range) and moreover it is K-equivariant with respect to the representation of $K = \phi(Spin_8)\cdot S$ as tensor representation of $SO_8 \times SO_4$ on $E\otimes\mathbb{H}$. This is the isotropy representation of the Grassmannian $G_4(\mathbb{R}^{12})$.

Remark 6.1. The generators of $\mathfrak{spin}_8 = \mathfrak{so}_8$ corresponding to the matrices (6) have squared trace norm 8 on $\mathbb{R}^8$ and $8\cdot 4$ on $\mathbb{R}^8\otimes\mathbb{R}^4$. Likewise, the generators of $\mathfrak{so}_4$ corresponding to $R(e)$ and $R(e')$ have squared norm 4 on $\mathbb{R}^4$ and $4\cdot 8$ on

$\mathbb{R}^4 \otimes \mathbb{R}^8$. Thus the two sorts of generators have the same length with respect to the isotropy representation of $G_4(\mathbb{R}^{12})$ and also as a Lie subalgebra of $\mathfrak{spin}_{16}$, see Remark 3.2. Thus the two metrics on $\mathfrak{k}$ are proportional which is important in view of (9).

Remark 6.2. Since the "Lie bracket" (9) on $V = (\mathbb{O} \otimes \mathbb{O})^2$ (the "Lie triple") depends only on the representation of K and the invariant metric on $\mathfrak{k}$, its restriction to $V' = (\mathbb{H} \otimes \mathbb{H})^2$ is a true Lie triple, corresponding the Grassmannian $G_4(\mathbb{R}^{12})$. In particular, the Jacobi identity holds on $(\mathbb{H} \otimes \mathbb{H})^2$.

Remark 6.3. Using (9), we have an uncommon way to determine the Lie triple (curvature tensor) for these Grassmannians using $(\mathbb{O}\otimes\mathbb{O})^2$. E.g. to compute $[\mathbf{e}_1, \mathbf{e}_2]$ we look for all matrices A in the basis (6) of $\mathfrak{spin}_{16}$ with $\langle A, [\mathbf{e}_1, \mathbf{e}_2]\rangle = \langle A\mathbf{e}_1, \mathbf{e}_2\rangle = 1$, see section 8 for a systematic treatment. The only one with $\langle A\mathbf{e}_1, \mathbf{e}_2\rangle \neq 0$ is $\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$, hence $[\mathbf{e}_1, \mathbf{e}_2] = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$ and $[\mathbf{e}_1, [\mathbf{e}_1, \mathbf{e}_2]] = -\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)\mathbf{e}_1 = -\mathbf{e}_2$.

Next we compute $[\mathbf{e}_1, i\mathbf{e}_1]$. This time, there are four matrices A in the basis (6) of $\mathfrak{spin}_{16}$ with $\langle Ai\mathbf{e}_1, \mathbf{e}_1\rangle \neq 0$, namely $\left(\begin{smallmatrix} i & \\ & -i \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} L_s & \\ & L_s \end{smallmatrix}\right)$, $s = 1, 2, 3$, where $L_1 = L(j)L(k)$ and $L_2 = L(l)L(il)$ and $L_3 = L(kl)L(jl)$ since $i = jk = l(il) = (kl)(jl)$. Since $A\mathbf{e}_1 = i\mathbf{e}_1$ for all four matrices, $[\mathbf{e}_1, i\mathbf{e}_1]$ is the sum of these matrices by (9), and we obtain $[\mathbf{e}_1, [\mathbf{e}_1, i\mathbf{e}_1]] = -4i\mathbf{e}_1$. Thus the operator $-\operatorname{ad}(\mathbf{e}_1)^2$ measuring the curvature has eigenvalues 1 and 4 as we know already in $\mathbb{CP}^2$.

It is amazing that the the well known pinching factor $4 = \frac{\max \mathrm{K}}{\min \mathrm{K}}$ for the sectional curvature K of $\mathbb{CP}^2$, $\mathbb{HP}^2$, $\mathbb{OP}^2$ is related to the fact that any imaginary basis octonion has precisely four different representations as a product of two basis octonions, e.g. $i = 1 \cdot i = jk = l(il) = (kl)(jl)$.

7. The Jacobi identity

Now we can show the Jacobi identity for the "Lie triple" defined by (9) on $V = (\mathbb{K} \otimes \mathbb{O})^2$. It suffices to consider the case $\mathbb{K} = \mathbb{O}$ since $(\mathbb{K} \otimes \mathbb{O})^2 \subset (\mathbb{O} \otimes \mathbb{O})^2$. Since $\mathfrak{spin}_{16}$ is a simple Lie algebra, there is just one Ad-invariant inner product on it, up to a scaling factor, thus we may use in (9) the trace metric of $\phi_*(\mathfrak{spin}_{16})$, divided by 128 (then the identity matrix has norm 1). We consider the orthonormal basis $\hat{B} = B\mathbf{e}_1 \cup B\mathbf{e}_2$ of $V = (\mathbb{O} \otimes \mathbb{O})^2$ where B is the basis of $\mathbb{O} \otimes \mathbb{O}$ given by the tensor products of the standard basis of $\mathbb{O}$

$$B_{\mathbb{O}} = \{1, i, j, k, l, il, jl, kl\} \tag{13}$$

in each tensor factor. It is enough to show the Jacobi identity on $\hat{B}$,

$$[[\mathbf{a}, \mathbf{b}], \mathbf{c}] + [[\mathbf{b}, \mathbf{c}], \mathbf{a}] + [[\mathbf{c}, \mathbf{a}], \mathbf{b}] = 0 \tag{14}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \hat{B}$. Using the equivariance for $K = \phi(Spin_{16})$ (see (10)), we may assume that $\mathbf{c} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. In fact, after applying one of the anti-diagonal matrices in $\mathfrak{k} \cap K$ in (6) if necessary, we may assume that $\mathbf{c} = \gamma\mathbf{e}_1$ for some $\gamma = e \otimes f = ef'$. Applying the inverse of $\begin{pmatrix} e & \\ & -e \end{pmatrix} \cdot \begin{pmatrix} f' & \\ & -f' \end{pmatrix} \in K$ we change $\mathbf{c}$ to $\mathbf{e}_1$, and $\pm\hat{B}$ is kept invariant by this transformation.

The remaining basis elements are $\mathbf{a} = \alpha\mathbf{e}_i$ and $\mathbf{b} = \beta\mathbf{e}_j$ (with $i, j \in \{1, 2\}$) with $\alpha = ab'$ and $\beta = cd'$ in B. But the two octonions a, c lie in a common quaternionic subalgebra $\mathbb{H} \subset \mathbb{O}$, and the same holds for b, d. Thus we are in a "classical" subspace of the form $(\mathbb{H}\otimes\mathbb{H})^2 \subset (\mathbb{O}\otimes\mathbb{O})^2$ where the Jacobi identity already holds (section 6, Case 2). Thus we have established a Lie algebra structure on $\mathfrak{g} = \mathfrak{k} \oplus V$ in all cases.

Remark 7.1. We may use the group $K = \phi(Spin_{16})$ to carry $\alpha\mathbf{e}_i$ and $\beta\mathbf{e}_j$ into the standard subspace $(\mathbb{H} \otimes \mathbb{H})^2$.[6] According to (6), the group K contains the matrix $\begin{pmatrix} N & \\ & -N \end{pmatrix}$ where $N = L(e)L(f)L(fe)$ for any two distinct orthonormal $e, f \in \text{Im}\,\mathbb{O}$. These span a quaternionic subalgebra $\mathbb{H}_1$, and N is the involution on $\mathbb{O}$ with eigenvalues 1 on $\mathbb{H}_1$ and -1 on $\mathbb{H}_1^\perp$ since $\mathbb{H}_1^\perp = L(d)\mathbb{H}_1$ for some $d \in \mathbb{O}$ such that $L(d)$ anticommutes with $L(e), L(f), L(ef)$. Given two quaternionic subalgebras $\mathbb{H}_2$ and $\mathbb{H}$ spanned by elements of $B_\mathbb{O}$ (see (13)), we can use such N to map $\mathbb{H}_2$ onto $\mathbb{H}$ as follows. The two subalgebras correspond to two lines in the projective plane over $\mathbb{F}_2$ (*Fano plane*), whose "points" are the seven basis elements of $\text{Im}\,\mathbb{O}$, thus $\mathbb{H}$ and $\mathbb{H}_2$ have precisely one basis element in common, say i. Then $\mathbb{H} = \text{span}\,\{1, i, j, ij\}$ and $\mathbb{H}_2 = \text{span}\,\{1, i, p, ip\}$ for some $p \in B_\mathbb{O} \setminus \{1, i\}$. We claim that the three octonions i, $j + p$, $ij + ip$ span another quaternionic subalgebra $\mathbb{H}_1$. In fact, $(j + p)(ij + ip) = jij + p(ij) + j(ip) + pip = 2i$, note that (p, i, j) is anti-associative and hence $j(ip) = -(ji)p = -p(ij)$. Using the map N for this quaternionic subalgebra $\mathbb{H}_1$ we obtain $N(j + p) = j + p$ while $N(j - p) = -(j - p)$ since $j - p \perp \mathbb{H}_1$. Subtracting the two equations we obtain $N(p) = j$ and similarly $N(ip) = ij$. Moreover, N fixes $i \in \mathbb{H}_1$. Thus $N(\mathbb{H}_2) = \mathbb{H}$.

8. Maximal abelian subspace and roots

The Lie triple of the Grassmannian $G_k(\mathbb{R}^{k+l})$ with $k \le l$ is $V_1 = \mathbb{R}^{k\times l} = \mathbb{R}^k \otimes \mathbb{R}^l$, and its maximal abelian subspace is the set $\Sigma \subset V_1$ of matrices $(D, 0)$ where $D = \text{diag}(t_1, \dots, t_k)$ is any real diagonal matrix. When we consider $\mathbb{R}^k \otimes \mathbb{R}^l = \mathbb{K}\otimes\mathbb{L} = V_1$ as the Lie triple of the Rosenfeld line (see section 5), a basis of Σ is given by

$$B_\Sigma = \Sigma \cap B = \{e \otimes e = ee' : e \in B_\mathbb{K} = B_\mathbb{O} \cap \mathbb{K}\} \tag{15}$$

where $B = B_\mathbb{K} \otimes B_\mathbb{L}$ is the standard basis of $\mathbb{K} \otimes \mathbb{L}$. In the algebra $\mathbb{K} \otimes \mathbb{L}$, the subset Σ is a commutative and associative subalgebra.

Proposition 8.1. *$\Sigma \subset V_1$ is maximal abelian also for V (where $V_1 = \mathbb{K} \otimes \mathbb{L}$ is viewed as a subspace of $V = (\mathbb{K} \otimes \mathbb{L})^2$ in the natural way).*

This is a consequence of the following

Lemma 8.2. *Let B be the canonical orthonormal basis of $V_1 = \mathbb{K} \otimes \mathbb{L} =: \mathbb{K}\mathbb{L}$. Then for any $\alpha, \beta \in B$ we have*

$$[\alpha\mathbf{e}_1, \beta\mathbf{e}_2] = \begin{pmatrix} & -\alpha\bar\beta \\ \beta\bar\alpha & \end{pmatrix} =: A_{\beta\bar\alpha} \neq 0. \tag{16}$$

Proof. We have $\langle A, [\alpha\mathbf{e}_1, \beta\mathbf{e}_2]\rangle = \langle A(\alpha\mathbf{e}_1), \beta\mathbf{e}_2\rangle$ where A runs through the basis (6) of $\mathfrak{k}$. Since A preserves the orthonormal basis B up to sign, this inner product is

[6] In fact we use $G_2 \times G_2 \subset Spin_7 \times Spin_7 \subset Spin_{16}$ where $G_2 = \text{Aut}(\mathbb{O})$.

1 if and only if A is anti-diagonal with the correct entry: $A = \left(\begin{smallmatrix} & -\bar\gamma \\ \gamma & \end{smallmatrix}\right)$ with $\gamma\alpha = \beta$, hence $\gamma = \beta\bar\alpha$. ■

Proof of Proposition 8.1: For any $\delta \in \mathbb{K} \otimes \mathbb{L}$ we have $\delta = \sum t_i\beta_i$ with $\beta_i \in B$ (defined in Lemma 8.2) and $t_i \in \mathbb{R}$, hence

$$[\mathbf{e}_1, \delta\mathbf{e}_2] = \sum_i t_i\,[\mathbf{e}_1, \beta_i\mathbf{e}_2] \overset{(16)}{=} \sum_i t_i \begin{pmatrix} & -\bar\beta_i \\ \beta_i & \end{pmatrix} = \begin{pmatrix} & -\bar\delta \\ \delta & \end{pmatrix}.$$

Thus $\mathbf{e}_1 \in \Sigma$ does not commute with $\delta\mathbf{e}_2$ which shows that the abelian subspace Σ is maximal abelian in $V = (\mathbb{K} \otimes \mathbb{L})^2$. ■

Now we want to compute the common eigenspace decomposition on V for $-\operatorname{ad}(\gamma)^2$, $\gamma \in \Sigma$ (not only $\gamma \in B_\Sigma$); the square roots of their eigenvalues are the *roots of the corresponding symmetric space*, see [7]. Thus we have to compute $-\operatorname{ad}(\alpha)\operatorname{ad}(\beta)$ for any $\alpha, \beta \in B_\Sigma$. The eigenspaces in $V_1 = \mathbb{KL}\mathbf{e}_1$ are already known from the Rosenfeld lines. It remains to compute $-\operatorname{ad}(\alpha)\operatorname{ad}(\beta)$ on $V_2 = \mathbb{KL}\mathbf{e}_2$.

Lemma 8.3. *For all $\gamma \in \Sigma$ and $\omega \in \mathbb{KL}$ we have*

$$-\operatorname{ad}(\gamma\mathbf{e}_1)^2\omega\mathbf{e}_2 = (R(\gamma)^2\omega)\mathbf{e}_2. \tag{17}$$

Hence the roots on V_2 are the common eigenvalues of $R(\gamma)$, $\gamma \in \Sigma$.

Proof. For $\alpha, \beta \in B_\Sigma$ we have

$$[\alpha\mathbf{e}_1, [\beta\mathbf{e}_1, \omega\mathbf{e}_2]] \overset{(16)}{=} [\alpha\mathbf{e}_1, A_{\omega\beta}] = -(\omega\beta)\alpha\mathbf{e}_2$$

(Note that $\bar\beta = \beta$ when $\beta \in B_\Sigma$), thus we have on $V_2 \cong \mathbb{KL}$:

$$\operatorname{ad}(\alpha\mathbf{e}_1)\operatorname{ad}(\beta\mathbf{e}_1) = -R(\alpha)R(\beta). \tag{18}$$

Now let $\gamma = \sum_{e \in B_\mathbb{K}} t_e ee' \in \Sigma$. Applying (18) for $\alpha = ee'$ and $\beta = ff'$ we obtain on V_1:

$$\operatorname{ad}(\gamma)^2 = \sum_{e,f \in B_\mathbb{K}} t_e t_f \operatorname{ad}(ee'\mathbf{e}_1)\operatorname{ad}(ff'\mathbf{e}_1) = -\sum t_e t_f R(ee')R(ff') = -R(\gamma)^2. \quad ■$$

Thus we just need to find the common eigenspaces of $R(\gamma)$, $\gamma \in \Sigma$ on $\mathbb{KL}$. First we look for the eigenspaces inside Σ. We are using the standard bases $B_\mathbb{O} = \{1, i, j, ij, l, il, jl, (ij)l\}$ for $\mathbb{O}$ and $B_\mathbb{K} = B_\mathbb{O} \cap \mathbb{K}$ for $\mathbb{K} \subset \mathbb{O}$. In the case $\mathbb{K} = \mathbb{O}$ we choose $s_i, s_j, s_l \in \{\pm 1\}$ arbitrary and obtain the following eight elements $\nu \in \Sigma$,

$$\begin{aligned} \nu &= (1 + s_i\hat\imath)(1 + s_j\hat\jmath)(1 + s_l\hat l) \\ &= 1 + s_i + s_j + s_is_j + s_l + s_is_l + s_js_l + s_is_js_l = \sum_{e \in B_\mathbb{O}} s_e\hat e \end{aligned} \tag{19}$$

where $\hat\imath := ii'$ etc. and where the other s_e are multiplicative: $s_1 = 1$, $s_{ij} = s_is_j$, $s_{il} = s_is_l$, $s_{jl} = s_js_l$, $s_{ijl} = s_is_js_l$. From (19) we see: multiplying ν by , , or gives ν back, multiplied by s_i, s_j, s_l, respectively (remind that Σ is commutative and $\hat e^2 = 1$ for any $\hat e \in \Sigma \cap B$). By the multiplicativity of the s_e the same is true for arbitrary basis elements: $\nu\hat e = s_e\nu$.

Thus for any $\tau = \sum t_e \hat{e} \in \Sigma$ we have

$$R(\tau)\nu = \lambda(\tau)\nu, \quad \lambda(\tau) = \sum s_e t_e. \tag{20}$$

In this way we obtain eight perpendicular common eigenvectors $\nu \in \Sigma$ for $R(\tau)$, $\tau \in \Sigma$. The corresponding eigenvalues ("roots") are the linear forms $\hat{e} \mapsto s_e$ on Σ. Inspection shows that the number of negative s_e can be only zero or four.

Similarly, in the case $\mathbb{K} = \mathbb{H}$ we use the four vectors

$$\nu = (1 + s_i \hat{\imath})(1 + s_j \hat{\jmath}) = 1 + s_i \hat{\imath} + s_j \hat{\jmath} + s_k \hat{k} \in \Sigma$$

with arbitrary $s_i, s_j \in \{\pm 1\}$ and $s_k = s_i s_j$. The spaces $\mathbb{R}\nu$ decompose Σ into four common eigenspaces for $R(\hat{e})$, $e \in B_{\mathbb{H}}$, where the number of negative s_e is zero or two. Likewise for $\mathbb{K} = \mathbb{C}$ we choose

$$\nu = 1 + s_i \hat{\imath}$$

for $s_i = \pm 1$. These form a basis of eigenvectors for $\hat{\imath}$ (or better for $R(\hat{\imath})$) on Σ, with eigenvalues $s_i = \pm 1$. Last, when $\mathbb{K} = \mathbb{R}$, then $\Sigma = \mathbb{R} \cdot 1$, and the only eigenvector (with multiplicity l) is $\nu = 1$. We omit this case in the following.

Next we observe that $R(f')\nu$ for any $f' \in B_{\mathbb{L}}$ is still a common eigenvector for $R(\hat{e})$, $e \in B_{\mathbb{K}}$. (Note also that $R(f)\nu = \mp R(f')\nu$ since $R(ff')\nu = \pm\nu$.) In fact, $R(f')$ commutes or anticommutes with $R(\hat{e})$ since

$$R(ee')R(f') = R(e')R(f')R(e) = \begin{cases} +R(f')R(ee') & \text{if } e \in \{1, f\}, \\ -R(f')R(ee') & \text{else.} \end{cases}$$

Thus $R(\hat{e})R(f')\nu = \pm s_e R(f')\nu$ with "+" for $e \in \{1, f\}$ and "−" otherwise. The number of orthonormal eigenvectors obtained this way is $k \cdot l = \dim V_2$. Thus they form a basis of V_2 by common eigenvectors for $R(\hat{e})$, $e \in B_{\mathbb{K}}$. The corresponding roots are of the form

$$\lambda = \sum_{e \in B_{\mathbb{K}}} t_e \phi_e \tag{21}$$

where $(\phi_e)_{e \in B_{\mathbb{O}}}$ is the basis of Σ^* dual to $B_\Sigma = \Sigma \cap B$, and $t_e \in \{\pm 1\}$. The eigenvalue t_e of $R(\hat{e})$ corresponding to the eigenvector $R(f')\nu$ satisfies

$$t_e = \begin{cases} s_e & \text{if} \quad \hat{e} \in \{1, \hat{f}\}, \\ -s_e & \text{if} \quad \hat{e} \in B_\Sigma \setminus \{1, \hat{f}\}. \end{cases} \tag{22}$$

There are 2^k linear forms $\sum_{e \in B_{\mathbb{K}}} t_e \phi_e$ with arbitrary $t_e = \pm 1$. Since only λ^2 matters, we may assume $t_1 = 1$ by otherwise changing all t_e to $-t_e$. Moreover, in the cases $\mathbb{OO}$ and $\mathbb{HH}$ the number of negative t_e is even by (22), that is $\prod_e t_e = 1$, which reduces the number of possible linear forms to 2^{k-2} (64 for $\mathbb{OO}$ and 4 for $\mathbb{HH}$). In the remaining cases, $\mathbb{HO}$ and $\mathbb{CL}$, we obtain 2^{k-1} such forms (8 and 2, respectively). Thus all possible linear forms actually occur as roots, with multiplicities $(\dim \mathbb{KL})/(\text{number of possible linear forms})$. This is $64/64 = 1$ for $\mathbb{OO}$, $32/8 = 4$ for $\mathbb{HO}$, $16/4 = 4$ for $\mathbb{HH}$, $16/2 = 8$ for $\mathbb{CO}$, $8/2 = 4$ for $\mathbb{CH}$ and $4/2 = 2$ for $\mathbb{CC}$.

In Table 1 below, we show the root system R_1 and R on $V_1 = \mathbb{K}\otimes\mathbb{L}$ and $V = (\mathbb{K}\otimes\mathbb{L})^2$ with multiplicities m,

(a) first the roots on V_1 for the Rosenfeld line,

(b) then the remaining roots on V_2 for the Rosenfeld plane.

No	$\mathbb{K}\otimes\mathbb{L}$	k	(a) λ	(a) m	(a) R_1	(b) λ	(b) m	(b) R
1	$\mathbb{R}\otimes\mathbb{R}$	1	none	0	$\emptyset$	ϕ_1	1	A_1
2	$\mathbb{R}\otimes\mathbb{C}$	1	$2\phi_1$	1	A_1	ϕ_1	2	BC_1
3	$\mathbb{R}\otimes\mathbb{H}$	1	$2\phi_1$	3	A_1	ϕ_1	4	BC_1
4	$\mathbb{R}\otimes\mathbb{O}$	1	$2\phi_1$	7	A_1	ϕ_1	8	BC_1
5	$\mathbb{C}\otimes\mathbb{C}$	2	$2(\phi_1 \pm \phi_i)$	1	$(A_1)^2$	$\phi_1 \pm \phi_i$	2	$(BC_1)^2$
6	$\mathbb{C}\otimes\mathbb{H}$	2	$2(\phi_1 \pm \phi_i)$	1				
			$2\phi_1, 2\phi_i$	2	B_2	$\phi_1 \pm \phi_i$	4	BC_2
7	$\mathbb{C}\otimes\mathbb{O}$	2	$2(\phi_1 \pm \phi_i)$	1				
			$2\phi_1, 2\phi_i$	6	B_2	$\phi_1 \pm \phi_i$	8	BC_2
8	$\mathbb{H}\otimes\mathbb{H}$	4	$2(\phi_e \pm \phi_f)$	1	D_4	$\sum s_e\phi_e,\ \prod s_e = 1$	4	B_4
9	$\mathbb{H}\otimes\mathbb{O}$	4	$2(\phi_e \pm \phi_f)$	1				
			$2\phi_e$	4	B_4	$\sum s_e\phi_e$	4	F_4
10	$\mathbb{O}\otimes\mathbb{O}$	8	$2(\phi_e \pm \phi_f)$	1	D_8	$\sum s_e\phi_e,\ \prod s_e = 1$	1	E_8

Table 1: Roots of the Rosenfeld planes

9. Inclusions

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathbb{S}^1 \subset \mathbb{RP}^2$	$\mathbb{S}^2 \subset \mathbb{CP}^2$	$\mathbb{S}^4 \subset \mathbb{HP}^2$	$\mathbb{S}^8 \subset \mathbb{OP}^2$
$\mathbb{C}$		$G_2^+\mathbb{R}^4 \subset (\mathbb{CP}^2)^2$	$G_2^+\mathbb{R}^6 \subset G_2\mathbb{C}^6$	$G_2^+\mathbb{R}^{10} \subset \mathbb{COP}^2$
$\mathbb{H}$			$G_4^+\mathbb{R}^8 \subset G_4^+\mathbb{R}^{12}$	$G_4^+\mathbb{R}^{12} \subset \mathbb{HOP}^2$
$\mathbb{O}$				$G_8^\#\mathbb{R}^{16} \subset \mathbb{OOP}^2$

Table 2: $\mathbb{KLP}^1 \subset \mathbb{KLP}^2$

In Table 2 we display the generalized Rosenfeld lines $\mathbb{KLP}^1$ as subsets of the corresponding Rosenfeld planes $\mathbb{KLP}^2$ with Lie triple $(\mathbb{K}\otimes\mathbb{L})^2$. We denote by $G_k\mathbb{C}^n$ the Grassmannian of all complex k-dimensional linear subspaces in $\mathbb{C}^n$ and $G_k^+\mathbb{R}^n$ the Grassmannian of oriented k-dimensional linear subspaces in $\mathbb{R}^n$. Further, let $G_{n/2}^\#(\mathbb{R}^n)$ be the manifold of oriented *balanced splittings*: a balanced splitting of $\mathbb{R}^n$ is a set $\{W, W^\perp\}$ where $W \subset \mathbb{R}^n$ is an $n/2$-dimensional subspace (n even) and $W^\perp$ its orthogonal complement. An orientation of W induces an orientation of $W^\perp$. Clearly, $\{W, W^\perp\} = \{W^\perp, (W^\perp)^\perp\}$. In order to understand the inclusions recall that the identities $Spin_4 = SU_2 \times SU_2$ and $Spin_6 = SU_4$ imply $G_2^+\mathbb{R}^4 = \mathbb{S}^2 \times \mathbb{S}^2$ and $G_2^+\mathbb{R}^6 = G_2\mathbb{C}^4$. The inclusions in horizontal direction are obvious, for those in vertical direction one has to recall that $\mathbb{S}^n = G_1^+\mathbb{R}^{n+1}$ and $G_k^+\mathbb{R}^n = G_{n-k}^+\mathbb{R}^n$. E.g.

in the last column, the inclusions between the smaller spaces ("Rosenfeld lines") are

$$G_8^+(\mathbb{R}^9) \subset G_8^+(\mathbb{R}^{10}) \subset G_8^+(\mathbb{R}^{12}) \subset G_8^\#(\mathbb{R}^{16}).$$

To understand the last inclusion note that an oriented 8-plane $W \subset \mathbb{R}^{12}$ can be considered as an oriented 8-plane in $\mathbb{R}^{16} \supset \mathbb{R}^{12}$ which determines an oriented balanced splitting $\{W, W^\perp\}$ of $\mathbb{R}^{16}$, and the mapping is injective since $W^\perp \not\subset \mathbb{R}^{12}$ when $W \subset \mathbb{R}^{12}$.

In the present paper we have only determined the infinitesimal structure, the Lie triples $\mathbb{K} \otimes \mathbb{L} \subset (\mathbb{K} \otimes \mathbb{L})^2$. All additional information in our Table 2 is taken from [5, Table 3] and [3, p. 294] where the polar subspaces of symmetric spaces are listed.

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