

On Global Okounkov Bodies of Spherical Varieties*

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Abstract. We define and study the *global Okounkov moment cone* of a projective spherical variety X , generalizing both the global Okounkov body and the *moment body* of X defined by Kaveh and Khovanskii. Under mild assumptions on X we show that the global Okounkov moment cone of X is rational polyhedral. As a consequence, also the global Okounkov body of X , with respect to a particular valuation, is rational polyhedral.

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1. Introduction

Started by the work of A. Okounkov, cf. [17], the theory of Okounkov bodies has been widely developed in recent years, in connection with other research topics such as representation theory, asymptotics of linear series and positivity, and tropical geometry, see e.g. [10], [13], [14], [21], [22], [23].

In this short note, we study a convex cone associated to a projective variety X , spherical under the action of a semisimple and simply connected group G . The convex cone, called *global Okounkov moment cone* and denoted by $\tilde{\Delta}_\nu(X)$, is closely related to the *global Okounkov body*, which we denote here by $\Delta_\nu(X)$. The latter is a “global version” of the Okounkov bodies associated to pseudoeffective divisors of projective varieties.

We recall that the construction of $\Delta_\nu(X)$ depends on the choice of a function

$$\nu : \bigsqcup_{\mathcal{L} \in \text{Pic}(X)} H^0(X, \mathcal{L}) \setminus \{0\} \rightarrow \mathbb{N}_0^n,$$

where $n = \dim X$, called a *valuation-like* function.

In the case of a general smooth projective variety X , such functions are often constructed using a flag of subvarieties $X = Y_0 \supset Y_1 \supset \dots \supset Y_n = \{\text{pt}\}$, where Y_i is an irreducible Cartier divisor of Y_{i-1} , smooth at the point Y_n , for all $i \in \{1, \dots, n\}$. For D a Cartier divisor and $s \in H^0(X, \mathcal{O}_X(D))$, one defines $\nu(s) = (\nu_1(s), \dots, \nu_n(s))$ setting first $\nu_1(s) = \text{ord}_{Y_1}(s)$. Then, by restriction on Y_1 , the section s determines

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naturally a section $s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1(s)Y_1))$, and we set $\nu_2(s) = \text{ord}_{Y_2}(s_1)$. Iterating this construction defines $\nu_i(s)$ for all $i \in \{1, \dots, n\}$. Other definitions of valuation-like functions are possible, and sometimes more convenient. This is the case e.g. for flag varieties, where Schubert varieties are natural candidates for the elements of the flag, but the possible failure of their smoothness leads to alternative constructions. One such, which uses Bott-Samelson varieties, is described in [22].

Once the function ν is given the global Okounkov body is defined as the closed convex cone in $\mathbb{R}^n \times N^1(X)$ generated by all pairs of the form $(\nu(s), [D]) \in \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$ where D is effective and s is as above.

In the setting of spherical varieties it is convenient to use valuation-like functions defined using a mixture of the two constructions mentioned above. We use a flag of G -stable subvarieties that ends with a closed G -orbit Y_r of X , to define ν_i for $i \in \{1, \dots, r\}$. Then we use a valuation-like function on Y_r to obtain the full ν . See Section 3 below for details.

In addition, given the importance of the structure of G -module of the space of sections $H^0(X, \mathcal{L})$, it is natural to add to the points of the global Okounkov body a third component. We assume that s belongs to a simple submodule V of $H^0(X, \mathcal{O}_X(D))$ (we show in Section 3 that this assumption is harmless), and, instead of the pair $(\nu(s), [D])$, we consider the triple $(\nu(s), [D], \lambda)$ where λ is the highest weight of V . The closed convex cone generated by such triples is the global Okounkov moment cone $\tilde{\Delta}_{\nu}(X)$, see Definition 3.8 below.

Our construction is related to objects already defined in the literature specifically for spherical varieties, such as the polytopes considered by Alexeev and Brion in [1], the *moment body* of X defined by Kaveh-Khovanskii in [11], the convex bodies studied by Kaveh in [9]. But, in addition to being of global nature, our approach considers valuation-like functions defined geometrically, as opposed to others that are more representation-theoretical.

The moment polytopes of projective spherical varieties (cf. [2]) are also expected to be in close relationship with our convex cones. This is one of our motivations to add the third component in our definition, even if it ends up having no real influence on the geometry of the cones themselves (see Proposition 3.7 below).

Our main result is Theorem 3.21 below, where we show under mild hypotheses on X that $\tilde{\Delta}_{\nu}(X)$ is rational polyhedral. This also implies that $\Delta_{\nu}(X)$ is rational polyhedral, answering in our setting a question posed in [14].

We end this introduction by remarking that our varieties are Mori dream spaces (see [18, Theorem 4.1.1]), as defined by Hu and Keel ([7]). As a consequence, they also admit valuations defining rational polyhedral global Okounkov bodies by the method of Postingshel and Urbinati ([21]) that uses tropical compactifications. It is not clear to us, however, whether our valuation can be related to theirs.

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2. Setting

Let G be a simply connected semisimple complex algebraic group, let $B \subseteq G$ be a Borel subgroup and $T \subseteq B$ a maximal torus. We denote by W the Weyl group of G . The Lie algebras of G, B , and T will be denoted by the corresponding lower case German letter. Given a dominant weight λ , we denote by $V(\lambda)$ the irreducible G -module of highest weight λ .

We recall that an irreducible normal G -variety X is *spherical* if B has an open orbit. Throughout the paper X will be a projective \mathbb{Q} -factorial spherical G -variety of dimension n . We shall further assume that X admits G -invariant Cartier prime divisors E_1, \dots, E_r with $E_i = Z(s_i)$, for a G -invariant section $s_i \in H^0(X, \mathcal{O}_X(E_i))$, with combinatorial normal crossings such that the intersection $E_1 \cap \dots \cap E_r$ is a closed G -orbit Gx_0 . Then $Gx_0 \cong G/P$ for a parabolic subgroup P containing B . We assume that all intersections of the E_i are smooth along the orbit Gx_0 . We also allow the case $r = 0$, in which we assume $X = Gx_0$.

An example where these assumptions are satisfied is the case where X is a *wonderful variety*, see Remark 4.3 and e.g. [20]. We also consider a B -equivariant birational morphism

$$\tau : Z \longrightarrow G/P, \tag{1}$$

where Z is a Bott-Samelson variety defined by a reduced sequence $w = (s_{i_1}, \dots, s_{i_m})$, where $s_{i_j} \in W$ are simple reflections (cf. e.g. [8, Section 13.10]).

Under our assumptions, any line bundle on X (and on any variety we will consider) has a unique G -linearization, by [12]. Therefore we will assume that all line bundles are implicitly equipped with their G -linearization.

3. The global Okounkov moment cone

We define a partial flag of subvarieties

$$X := Y_0 \supset Y_1 \supset \dots \supset Y_r = Gx_0, \text{ of } X \text{ by } Y_i := \bigcap_{j=1}^i E_j,$$

i.e., $Y_i = Z(s_1, \dots, s_i)$, for $i = 1, \dots, r$.

We now define a valuation-like function with values in \mathbb{N}_0^n on the disjoint union $\bigsqcup_{\mathcal{L} \in \text{Pic}(X)} H^0(X, \mathcal{L}) \setminus \{0\}$ of all section spaces of all line bundles by first using the \mathbb{N}_0^r -valued function defined by the partial flag above, and then continuing with the valuation-like function defined by the ‘‘vertical flag’’ on a Bott-Samelson variety (cf. [22]). We thus first define

$$\nu' : \bigsqcup_{\mathcal{L} \in \text{Pic}(X)} H^0(X, \mathcal{L}) \setminus \{0\} \rightarrow \mathbb{N}_0^r$$

by $\nu'(s) := (a_1, \dots, a_r)$, for a nonzero section $s \in H^0(X, \mathcal{L})$, where

$$a_1 := \text{ord}_{Y_1}(s), \quad a_2 := \text{ord}_{Y_2} \left(\frac{s}{s_1^{a_1}} \Big|_{Y_1} \right), \quad \dots, \quad a_r := \text{ord}_{Y_r} \left(\frac{s}{s_1^{a_1} \dots s_{r-1}^{a_{r-1}}} \Big|_{Y_{r-1}} \right). \tag{2}$$

Recall the valuation-like function

$$\tilde{\nu}'' : \bigsqcup_{\mathcal{L} \in \text{Pic}(Z)} H^0(Z, \mathcal{L}) \setminus \{0\} \rightarrow \mathbb{N}_0^m$$

defined by the “vertical flag” of Bott-Samelson-subvarieties of Z with respect to the reduced expression $w = (s_{i_1}, \dots, s_{i_m})$. This is a flag of Bott-Samelson varieties Z_j , where Z_j is defined by the reduced subsequence $(s_{i_{j+1}}, \dots, s_{i_m})$ of w (cf. [22]). Using the birational map $\tau : Z \rightarrow G/P$ (eq. (1)), we then define

$$\nu'' : \bigsqcup_{\mathcal{L} \in \text{Pic}(G/P)} H^0(G/P, \mathcal{L}) \setminus \{0\} \rightarrow \mathbb{N}_0^m, \quad \nu''(s) := \tilde{\nu}''(\tau^*s), \quad s \in H^0(G/P, \mathcal{L}).$$

Setting $(a_{r+1}, \dots, a_n) := \nu''(\frac{s}{s_1^{a_1} \dots s_r^{a_r}} |_{Y_r})$ (cf. (2)), and defining $\nu(s) := (a_1, \dots, a_n)$, we obtain a valuation-like function

$$\nu : \bigsqcup_{\mathcal{L} \in \text{Pic}(X)} H^0(X, \mathcal{L}) \setminus \{0\} \rightarrow \mathbb{N}_0^n. \tag{3}$$

We write $\nu(s) = (\nu_1(s), \dots, \nu_n(s))$, for a section s of some line bundle.

Remark 3.1. It is clear from its definition that ν' is G -invariant, and we recall that $\tilde{\nu}''$ is B -invariant. As a consequence, the function ν is B -invariant.

One defines similarly a valuation-like function ν_{Y_i} for sections of line bundles on Y_i , for any variety Y_i belonging to the flag Y_\bullet .

Remark 3.2. We recall that, since X is spherical, for any line bundle \mathcal{L} on X the space of sections $H^0(X, \mathcal{L})$ is a multiplicity-free G -module, that is, any simple submodule appears with multiplicity 1. This fact is used in several places in this paper, e.g. in the proof of the following proposition.

Proposition 3.3. *Let \mathcal{L} be line bundle on X , and consider the decomposition*

$$H^0(X, \mathcal{L}) = \bigoplus_{\lambda} V(\lambda)$$

of the section space $H^0(X, \mathcal{L})$ into a direct sum of irreducible G -modules.

- (i) *For any $V(\lambda) \subseteq H^0(X, \mathcal{L})$, and any two nonzero sections $s, t \in V(\lambda)$, we have $\nu'(s) = \nu'(t)$.*
- (ii) *If $s \in V(\lambda), t \in V(\mu)$ are nonzero sections for distinct highest weights λ and μ , then $\nu'(s) \neq \nu'(t)$. In particular, $\nu(s) \neq \nu(t)$.*

Proof. For the first part, note that the G -invariance of the Y_i implies that ν' is G -invariant, i.e., $\nu'(gs) = \nu'(s)$, for any nonzero $s \in H^0(X, \mathcal{L})$, and $g \in G$. Moreover, $\nu'(s + t) \geq \min\{\nu'(s), \nu'(t)\}$, for any nonzero $s, t \in H^0(X, \mathcal{L})$ such that $s + t \neq 0$.

Now, if $s \in V(\lambda) \subseteq H^0(X, \mathcal{L})$ is a nonzero section, it is a cyclic vector for the G -module. Hence, if $t \in V(\lambda) \subseteq H^0(X, \mathcal{L})$, we can write t as

$$t = \sum_{i=1}^m c_i g_i s,$$

for some $g_1, \dots, g_m \in G$, and $c_1, \dots, c_m \in \mathbb{C}$.

The G -invariance and the valuation-like properties of ν' then yield

$$\nu'(t) \geq \min\{\nu'(g_i s) \mid i = 1, \dots, m\} = \nu'(s).$$

Reversing the roles of s and t , that is, using t as a G -cyclic vector, yields $\nu'(s) \geq \nu'(t)$. This proves (i).

For (ii), assume that $\nu'(s) = \nu'(t) = (a_1, \dots, a_r)$, for nonzero sections $s \in V(\lambda), t \in V(\mu)$, of \mathcal{L} . By (i), we may without loss of generality assume that s and t are B -semiinvariants of weight λ and μ , respectively.

The definition of ν' , and the fact that Y_i is the common zero set of the G -invariant sections s_1, \dots, s_i , with $s_i \in H^0(X, \mathcal{O}_X(E_i))$, shows that the restriction $\frac{s}{s_1^{a_1} \dots s_r^{a_r}}|_{G/P}$ defines a regular section of the line bundle $\mathcal{M} = \mathcal{L}_{G/P} \otimes \mathcal{O}_{G/P}(-a_1 E_1 - \dots - a_r E_r)$ over G/P . Moreover, since s is B -semiinvariant of weight λ , and the s_i are G -invariant, the restricted section $\frac{s}{s_1^{a_1} \dots s_r^{a_r}}|_{G/P}$ is also B -semiinvariant of weight λ . The same argument, applied to t , shows that $\frac{t}{s_1^{a_1} \dots s_r^{a_r}}|_{G/P}$ is a B -semiinvariant section of \mathcal{M} of weight μ . Since $H^0(G/P, \mathcal{M})$ is an irreducible G -module, this is a contradiction unless $\lambda = \mu$. ■

Definition 3.4. Let

$$\tilde{S}_\nu(X) := \left\{ (\nu(s), [D], \lambda) \in \mathbb{N}_0^n \times N^1(X) \times \Gamma \mid \begin{array}{l} s \in V(\lambda) \setminus \{0\}, \\ V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D)) \end{array} \right\}.$$

Proposition 3.5. The set $\tilde{S}_\nu(X)$ is an additive subsemigroup of $\mathbb{N}_0^n \times N^1(X) \times \Gamma$.

Proof. Let $s \in V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$ and $t \in V(\mu) \subseteq H^0(X, \mathcal{O}_X(E))$ be nonzero sections. We show that $\nu(s) + \nu(t)$ is the value of a section in the G -submodule $V(\lambda + \mu) \subseteq H^0(X, \mathcal{O}_X(D + E))$.

Let $v_\lambda \in V(\lambda)$ be a B -semiinvariant section. Since ν is B -invariant, and the B -module generated by s contains v_λ , we have $\nu(s) \leq \nu(v_\lambda)$. Therefore, if $\nu(s) \neq \nu(v_\lambda)$, then for all $c \in \mathbb{C}$ such that $s + cv_\lambda \neq 0$ we have $\nu(s + cv_\lambda) = \nu(s)$.

By replacing s with v_λ if $\nu(s) = \nu(v_\lambda)$, or with $s + cv_\lambda$ for a suitable c if $\nu(s) \neq \nu(v_\lambda)$, we may assume that the decomposition of s as a sum of weight vectors has a nonzero component of weight λ . Similarly, we may assume that t has a nonzero weight component of weight μ . The product section $s \cdot t \in H^0(X, \mathcal{O}_X(D + E))$ decomposes as a sum

$$s \cdot t = \sum_{\xi} r_{\xi}, \quad \xi \in V(\xi) \subseteq H^0(X, \mathcal{O}_X(D + E)). \tag{4}$$

Since $\nu(r_{\xi}) \neq \nu(r_{\xi'})$ for $\xi \neq \xi'$, by Proposition 3.3, we have that

$$\nu(s \cdot t) = \min\{\nu(r_{\xi})\},$$

where the minimum is taken over the set of highest weights occurring in the decomposition (4). In fact, this minimum is attained by the highest weight ξ_0 for which $\nu'(s \cdot t) = \nu'(r_{\xi_0})$. Since $\nu'(s \cdot t) = \nu'(s) + \nu'(t) = \nu'(v_\lambda) + \nu'(v_\mu) = \nu'(v_\lambda \cdot v_\mu) = \nu'(v_{\lambda + \mu})$, it follows that $\xi_0 = \lambda + \mu$. Hence,

$$(\nu(s), [D], \lambda) + (\nu(t), [E], \mu) = (\nu(r_{\lambda + \mu}), [D + E], \lambda + \mu) \in \tilde{S}_\nu(X),$$

where $r_{\lambda+\mu}$ is the component in $V(\lambda + \mu)$ of the product

$$s \cdot t \in H^0(X, \mathcal{O}_X(D + E)).$$

This proves that $\tilde{S}_\nu(X)$ is a semigroup. ■

Definition 3.6. The *global Okounkov semigroup* of X (with respect to ν) is

$$S_\nu(X) := \{(\nu(s), [D]) \in \mathbb{N}_0^n \times N^1(X) \mid s \in H^0(X, \mathcal{O}_X(D)) \setminus \{0\}\}.$$

The *global Okounkov body* of X (with respect to ν) is the closed convex cone $\Delta_\nu(X) = \overline{\text{cone}(S_\nu(X))}$ generated by $S_\nu(X)$.

Proposition 3.7. The map $p: \tilde{S}_\nu(X) \rightarrow S_\nu(X)$ given by $(\nu(s), [D], \lambda) \mapsto (\nu(s), [D])$ defines an isomorphism of semigroups.

Proof. Clearly, p is a homomorphism of semigroups. For the surjectivity, let $(\nu(s), [D]) \in S_\nu(X)$, so that $s \in H^0(X, \mathcal{O}_X(D))$. The section s then decomposes as a sum $s = \sum_\xi r_\xi$, where $r_\xi \in V(\xi) \subseteq H^0(X, \mathcal{O}_X(D))$. Since, by Proposition 3.3, the values of the r_ξ are distinct, we have $\nu(s) = \nu(r_\lambda)$, for some $\lambda \in \Gamma$ with $V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$. Then, $(\nu(r_\lambda), [D], \lambda) \in \tilde{S}_\nu(X)$, and

$$p(\nu(r_\lambda), [D], \lambda) = (\nu(r_\lambda), [D]) = (\nu(s), [D]).$$

For the injectivity, let $(\nu(s), [D], \lambda)$ and $(\nu(s'), [D'], \lambda')$ be in $\tilde{S}_\nu(X)$ with $\nu(s) = \nu(s')$ and $[D] = [D']$. Then $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$, and by Proposition 3.3 we have $\lambda = \lambda'$. ■

Definition 3.8. Let $\tilde{\Delta}_\nu(X) := \overline{\text{cone}(\tilde{S}_\nu(X))} \subseteq \mathbb{R}^n \oplus N^1(X)_\mathbb{R} \oplus \mathfrak{t}^*$ be the closed convex cone generated by the semigroup $\tilde{S}_\nu(X)$. We call $\tilde{\Delta}_\nu(X)$ the *global Okounkov moment cone* of X (with respect to ν).

Similarly, for any variety Y_i in the flag Y_\bullet , we define $\tilde{\Delta}_{\nu_{Y_i}}(Y_i) \subseteq \mathbb{R}^{n-i} \oplus N^1(Y_i)_\mathbb{R} \oplus \mathfrak{t}^*$ to be the closed convex cone generated by the corresponding semigroup $\tilde{S}_{\nu_{Y_i}}(Y_i)$ for the spherical variety Y_i .

Remark 3.9. All subvarieties Y_i inherit from X the properties needed for the definition of the semigroup $\tilde{S}_{\nu_{Y_i}}(Y_i)$. In particular, being G -stable, it is well-known that they are spherical varieties (see e.g. [18, Theorem 3.1.19]).

Remark 3.10. Since the semigroup $S_\nu(X)$ generating $\Delta_\nu(X)$ also keeps track of the G -modules from which sections with given values come, the cone $\Delta_\nu(X)$ can also be seen as a global version of the moment polytope of X (cf. [2]), with the additional information of the values of ν ; hence the name.

Let $Y := Y_1$ be the divisor of the flag Y_\bullet . The goal now is to prove that the convex cone $\tilde{\Delta}_\nu(X)$ is rational polyhedral if $\tilde{\Delta}_{\nu_Y}(Y)$ is rational polyhedral.

We first observe that, for a line bundle \mathcal{L} on X and a G -submodule $V(\lambda) \subseteq H^0(X, \mathcal{L})$, because of the G -equivariance, the restriction map

$$V(\lambda) \rightarrow H^0(Y, \mathcal{L}|_Y), \quad s \mapsto s|_Y, \tag{5}$$

is either zero or injective. Also recall that spherical varieties have the property of multiplicity-freeness recalled in Remark 3.2. These facts motivate the following definition.

Definition 3.11. (i) Let $J \subseteq \overline{\text{Eff}}(X) \times \mathfrak{t}^*$ be the subset of all pairs $([D], \lambda)$ with $\mathcal{L} = \mathcal{O}_X(D)$ and λ as above, and such that the restriction map (5) is injective, and define $K \subseteq \overline{\text{Eff}}(X) \times \mathfrak{t}^*$ to be the closed convex cone generated by J .

(ii) Let $J_Y \subseteq \overline{\text{Eff}}(Y) \times \mathfrak{t}^*$ be the subset of all pairs $([D|_Y], \lambda)$ with $([D], \lambda) \in J$, and define K_Y to be the closed convex cone generated by J_Y .

Lemma 3.12. *The convex cone K is rational polyhedral.*

Proof. Since X is a Mori dream space, the Cox ring, $\text{Cox}(X)$, of X is a finitely generated \mathbb{C} -algebra. Hence, the subalgebra $\text{Cox}(X)^U$ of unipotent invariants is also finitely generated over \mathbb{C} (cf. [6, Theorem 16.2]). We can then choose a set of generators ζ_1, \dots, ζ_N for $\text{Cox}(X)^U$ which are homogeneous with respect to the grading by divisors as well as with respect to the grading of the U -invariants by the weights, i.e., we can assume that $\zeta_i \in V(\lambda_i)^U \subseteq H^0(X, \mathcal{O}_X(D_i))^U$, for an effective divisor D_i and a weight λ_i . Moreover, we can assume that the restriction maps (5) are injective for $\lambda_1, \dots, \lambda_\ell$, and zero for $\lambda_{\ell+1}, \dots, \lambda_N$.

Now, let $([D], \lambda)$ be a pair for which the map (5) is injective, and let $s \in V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$ be a highest weight section, so $s \in V(\lambda)^U$. Writing

$$s = \sum_{\underline{m}=(m_1, \dots, m_N) \in \mathbb{N}_0^N} c_{\underline{m}} \zeta_1^{m_1} \cdots \zeta_N^{m_N}, \quad c_{\underline{m}} \in \mathbb{C},$$

we see directly that the sum of the terms for (m_1, \dots, m_N) not satisfying $D \sim m_1 D_1 + \cdots + m_N D_N$ is zero. The same holds for the sum of the terms for (m_1, \dots, m_N) not satisfying $\lambda = m_1 \lambda_1 + \cdots + m_N \lambda_N$. Hence, we can write s as a linear combination of monomials in the s_i for which both conditions are satisfied.

We now claim that the sum $s = \sum_{\underline{m} \in \mathbb{N}_0^N} c_{\underline{m}} \zeta_1^{m_1} \cdots \zeta_N^{m_N}$ is even a linear combination of monomials only in $\zeta_1, \dots, \zeta_\ell$, that is, we have $m_{\ell+1} = \cdots = m_N = 0$ for all \underline{m} occurring in the sum. Indeed, if we had $m_i > 0$ for some $i \in \{\ell + 1, \dots, N\}$, the monomial $\zeta_1^{m_1} \cdots \zeta_N^{m_N}$ would restrict to zero on Y , so that the restriction map (5) would be zero for the highest weight module

$$V(m_1 \lambda_1 + \cdots + m_N \lambda_n) \subseteq H^0(X, \mathcal{O}_X(m_1 D_1 + \cdots + m_N D_N)) = H^0(X, \mathcal{O}_X(D)).$$

In particular, this would contradict the fact that $m_1 \lambda_1 + \cdots + m_N \lambda_N = \lambda$. It follows that K is the closed convex cone generated by the points $([D_1], \lambda_1), \dots, ([D_\ell], \lambda_\ell)$. ■

Definition 3.13. Let $\varphi: N^1(X)_{\mathbb{R}} \oplus \mathfrak{t}^* \rightarrow N^1(Y)_{\mathbb{R}} \oplus \mathfrak{t}^*$ be the unique \mathbb{R} -linear map satisfying $\varphi([D], \lambda) = ([D|_Y], \lambda)$ for all pairs $([D], \lambda)$ where D is a divisor and $\lambda \in \mathfrak{t}^*$.

Lemma 3.12 has the following immediate corollary.

Corollary 3.14. *The convex cone K_Y is given by $K_Y = \varphi(K)$. In particular, K_Y is rational polyhedral.*

Definition 3.15. We define the semigroups

$$\tilde{J} := \{(\nu(s), [D], \lambda) \mid s \in V(\lambda) \setminus \{0\} \subseteq H^0(X, \mathcal{O}_X(D)), ([D], \lambda) \in J\}, \quad (6)$$

$$\tilde{J}_Y := \{(\nu_Y(t), [E], \lambda) \mid t \in V(\lambda) \setminus \{0\} \subseteq H^0(Y, \mathcal{O}_Y(E)), ([E], \lambda) \in J_Y\}, \quad (7)$$

and the homomorphism of semigroups

$$\tilde{\varphi}: \tilde{J} \longrightarrow \tilde{J}_Y, \quad (\nu(s), [D], \lambda) \mapsto (\nu_Y(s|_Y), [D|_Y], \lambda).$$

We also denote by $\tilde{K} = \overline{\text{cone}}(\tilde{J})$ and $\tilde{K}_Y = \overline{\text{cone}}(\tilde{J}_Y)$ the closed convex cones generated by \tilde{J} and \tilde{J}_Y , respectively. The map $\tilde{\varphi}$ extends uniquely to a continuous map $\tilde{K} \longrightarrow \tilde{K}_Y$, also denoted by $\tilde{\varphi}$, for simplicity, between the respective closed convex cones.

Since the convex cone K is generated by pairs $([D], \lambda)$ for which the restriction map (5) is injective, the map $\tilde{\varphi}$ between the convex cones is surjective, i.e.,

$$\tilde{K}_Y = \tilde{\varphi}(\tilde{K}).$$

Lemma 3.16. *For any big divisor D on Y there exist a rational weight λ and $a \in \mathbb{Q}^{n-1}$ such that $(a, [D], \lambda)$ is in the relative interior of $\tilde{\Delta}_{\nu_Y}(Y)$.*

Proof. By Proposition 3.7, the relative interior of $\tilde{\Delta}_{\nu_Y}(Y)$ is mapped by p onto the relative interior of $\Delta_{\nu_Y}(Y)$, hence the lemma follows from the fact that D is in the interior of $\overline{\text{Eff}}(Y)$. ■

Definition 3.17. Let $\pi: \mathbb{R}^{n-1} \oplus N^1(Y)_{\mathbb{R}} \oplus \mathfrak{t}^* \longrightarrow N^1(Y)_{\mathbb{R}} \oplus \mathfrak{t}^*$ denote the projection onto the direct summand $N^1(Y)_{\mathbb{R}} \oplus \mathfrak{t}^*$.

Lemma 3.18. *If the convex cone $\tilde{\Delta}_{\nu_Y}(Y)$ is rational polyhedral, then*

$$\tilde{K}_Y = \tilde{\Delta}_{\nu_Y}(Y) \cap \pi^{-1}(K_Y).$$

In particular, the convex cone \tilde{K}_Y is rational polyhedral, being the intersection of two rational polyhedral convex cones.

Proof. Notice that $\tilde{J}_Y = \tilde{\Delta}_{\nu_Y}(Y) \cap \pi^{-1}(J_Y)$ and that the convex cone $\pi^{-1}(K_Y)$ is rational polyhedral and generated by $\pi^{-1}(J_Y)$.

We claim that $\pi^{-1}(J_Y)$ intersects the relative interior of $\tilde{\Delta}_{\nu_Y}(Y)$. Given this claim, the lemma is a consequence of [14, Prop. A.1.] (and its proof).

In order to prove the claim, let D be an ample divisor on X , and let $[E] := [D|_Y]$. By Lemma 3.16, there is a point of the form $(a, [E], \lambda)$ in the relative interior of $\tilde{\Delta}_{\nu_Y}(Y)$. By replacing this triple by a positive integer multiple, if necessary, we may assume that $(a, [E], \lambda)$ is an integral point in $\tilde{\Delta}_{\nu_Y}(Y)$, and, by Serre’s vanishing theorem, that the restriction map $R_D: H^0(X, \mathcal{O}_X(D)) \longrightarrow H^0(Y, \mathcal{O}_Y(E))$ of sections is surjective.

By the G -equivariance of R_D , the weight λ occurs as the highest weight of an irreducible G -submodule $V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$, and the restriction

$$R_D|_{V(\lambda)}: V(\lambda) \longrightarrow H^0(Y, \mathcal{O}_Y(E))$$

is injective, i.e., $([E], \lambda) \in J_Y$. This finishes the proof. ■

The above lemma enables us to deduce the rational polyhedrality of the convex cone \tilde{K} .

Lemma 3.19. *If the convex cone $\tilde{\Delta}_{\nu_Y}(Y)$ is rational polyhedral, then*

$$\tilde{K} = ((\{0\} \times \mathbb{R}_{\geq 0}^{n-1}) \times K) \cap \tilde{\varphi}^{-1}(\tilde{K}_Y) \tag{8}$$

In particular, the convex cone \tilde{K} is rational polyhedral.

Proof. We use the same strategy as in Lemma 3.18. If a point of the form $(\nu(s), [D], \lambda)$ is in \tilde{J} for some $s \in V(\lambda) \setminus \{0\} \subseteq H^0(X, \mathcal{O}_X(D))$ and $([D], \lambda) \in J$, then $\nu(s) \in \{0\} \times \mathbb{R}^{n-1}$ by the definition of J .

This yields

$$\tilde{J} = ((\{0\} \times \mathbb{R}_{\geq 0}^{n-1}) \times J) \cap \tilde{\varphi}^{-1}(\tilde{J}_Y).$$

We claim that $((\{0\} \times \mathbb{N}_{\geq 0}^{n-1}) \times J)$ intersects the relative interior of $\tilde{\varphi}^{-1}(\tilde{K}_Y)$. Given this claim, the lemma follows from [14, Prop. A.1.].

To prove the claim, let $(a, [E], \lambda) \in \tilde{J}_Y$ be in the relative interior of \tilde{K}_Y , choose D such that $([D], \lambda) \in J$ and $[E] = [D|_Y]$, and choose $t \in V(\lambda) \setminus \{0\} \subset H^0(Y, \mathcal{O}_Y(E))$ such that $\nu_Y(t) = a$. Up to replacing $(a, [E], \lambda) \in \tilde{J}_Y$ by a positive integer multiple, we may assume that the restriction map $R_D: H^0(X, \mathcal{O}_X(D)) \rightarrow H^0(Y, \mathcal{O}_Y(E))$ is surjective. Then we have $V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$, and the map R_D sends $V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$ isomorphically to $V(\lambda) \subseteq H^0(Y, \mathcal{O}_Y(E))$. In other words, the section t extends to a section $s \in V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$.

It follows that $\nu(s) = (0, a)$, so $(\nu(s), [D], \lambda) \in (\{0\} \times \mathbb{N}_{\geq 0}^{n-1}) \times J$ and $\tilde{\varphi}(\nu(s), [D], \lambda) = (t, [E], \lambda)$, which finishes the proof. ■

We can now prove the main result, deducing the rational polyhedrality of $\tilde{\Delta}_\nu(X)$ from that of $\tilde{\Delta}_{\nu_Y}(Y)$.

Lemma 3.20. *If the convex cone $\tilde{\Delta}_{\nu_Y}(Y)$ is rational polyhedral, then so is $\tilde{\Delta}_\nu(X)$.*

Proof. Let

$$((0, \eta_1), [D_1], \lambda_1), \dots, ((0, \eta_m), [D_m], \lambda_m) \in \{0\} \times \mathbb{R}_{\geq 0}^{n-1} \times K \subseteq \mathbb{R}^n \oplus N^1(X)_{\mathbb{R}} \oplus \mathfrak{t}^*$$

be integral generators of the convex cone \tilde{K} .

Now, let $s \in V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D))$ be a nonzero section. If s vanishes to order a along Y , we have that $s' := \frac{s}{s^a} \in V(\lambda) \subseteq H^0(X, \mathcal{O}_X(D - aY))$ is a section with nonzero restriction to Y , so that $(\nu(\xi), [D - aY], \lambda) \in \tilde{J}$. Then

$$\begin{aligned} (\nu(s), [D], \lambda) &= (\nu(\xi), [D - aY], \lambda) + ((a, 0, \dots, 0), [aY], 0) \\ &\in \text{cone}\{((0, \eta_1), [D_1], \lambda_1), \dots, ((0, \eta_m), [D_m], \lambda_m), ((1, 0, \dots, 0), [Y], 0)\}. \end{aligned}$$

It follows that $\tilde{\Delta}_\nu(X)$ is equal to the convex cone generated by the elements

$$((0, \eta_1), [D_1], \lambda_1), \dots, ((0, \eta_m), [D_m], \lambda_m), ((1, 0, \dots, 0), [Y], 0). \tag{■}$$

Theorem 3.21. *The convex cones $\tilde{\Delta}_\nu(X)$ and $\Delta_\nu(X)$ are rational polyhedral.*

Proof. We proceed by induction on r (the number of the G -invariant prime Cartier divisors E_1, \dots, E_r , cf. Section 2). If $r = 0$ then $X = G/P$ is G -homogeneous, and $\Delta_\nu(X)$ is rational polyhedral by [22, Corollary 4.11].

The inductive step follows from the fact that $Y = Y_1$ admits the partial flag of subvarieties $Y = Y_1 \supset Y_2 \supset \dots \supset Y_r$, from Lemma 3.20, and the fact that $\Delta_\nu(X)$ is the image of $\tilde{\Delta}_\nu(X)$ under the projection $\mathbb{R}^n \oplus N^1(X)_\mathbb{R} \oplus \mathfrak{t}^* \longrightarrow \mathbb{R}^n \oplus N^1(X)_\mathbb{R}$. ■

Remark 3.22. The inductive proof for the rational polyhedrality follows the ideas from [22] and [16], where it is vital to study the images under the restriction maps for sections of line bundles from a variety to a prime divisor. Whereas the key point in [16] was the surjectivity of these restriction maps for movable line bundles, and in [22] a volume identity for Newton-Okounkov bodies – a numeric and asymptotic version of surjectivity – played the same role, we here instead introduce the cone \tilde{K}_Y (cf. Definition 3.15) that serves as a convex-geometric object that describes the asymptotics of the restriction maps for sections and allows us to perform the induction step.

4. Special cases and examples

The above proof of Theorem 3.21 suggests a recursive method to compute the cones $\tilde{\Delta}_\nu(X)$ and $\Delta_\nu(X)$. The procedure to obtain $\tilde{\Delta}_\nu(X)$ involves ultimately only $\tilde{\Delta}_{\nu_Y}(Y)$, which we can assume given by the recursive hypothesis, together with the cone K . The latter can be derived from the knowledge of generators of the Cox ring of X , as in the proof of Lemma 3.12. For such generators see [3] and [5]. Then $\Delta_\nu(X)$ is obtained by simply projecting $\tilde{\Delta}_\nu(X)$ on the first two components, as in the proof of Theorem 3.21.

In the special case of flag varieties, one can also easily reconstruct the global Okounkov moment cone from the global Okounkov body. Indeed, let X be equal to its closed G -orbit $Y_r = E_1 \cap \dots \cap E_r \cong G/P$. Then, for all globally generated Cartier divisors D of X , the space of sections $H^0(X, \mathcal{O}_X(D))$ is a simple G -module, and we denote by $\omega(D)$ its highest weight. This induces a linear map $\omega: N^1(X)_\mathbb{R} \longrightarrow \mathfrak{t}^*$.

Lemma 4.1. *Assume $X \cong G/P$ is a single G -orbit, and let $(a_i, D_i) \in \mathbb{R}^n \oplus N^1(X)_\mathbb{R}$ for $i \in \{1, \dots, k\}$ be generators of $\Delta_\nu(X)$ as a convex cone. Then $\tilde{\Delta}_\nu(X)$ is the convex cone generated by the elements of the form $(a_i, [D_i], \omega(D_i))$, for all $i \in \{1, \dots, k\}$.*

Proof. Let us consider a general element $(\nu(s), [D])$ of $S_\nu(X)$, where D is a globally generated Cartier divisor, and $s \in H^0(X, \mathcal{O}_X(D))$. Since $H^0(X, \mathcal{O}_X(D)) = V(\omega(D))$, the point $(\nu(s), [D], \omega(D))$ is the unique point of $\tilde{S}_\nu(X)$ having projection in $S_\nu(X)$ equal to $(\nu(s), [D])$. At this point the lemma follows immediately from the fact that the map $D \mapsto \omega(D)$, for D varying in the set of effective Cartier divisors of X , is additive. ■

We generalize the above special case in the next proposition. Recall from section 2 the valuation-like function ν'' defined for the closed G -orbit Y_r of X .

Proposition 4.2. *Suppose that the restriction of line bundles induces an isomorphism $\psi_j: N^1(X)_{\mathbb{R}} \rightarrow N^1(Y_j)_{\mathbb{R}}$ for all $j \in \{1, \dots, r\}$. Let $(a_i, [D_i]) \in \mathbb{R}^{n-r} \oplus N^1(Y_r)_{\mathbb{R}}$ for $i \in \{1, \dots, k\}$ be generators of $\Delta_{\nu'}(Y_r)$. Denote by $[D'_i] \in N^1_0(X)_{\mathbb{R}}$ the element such that $\psi_r([D'_i]) = [D_i]$.*

Then $\tilde{\Delta}_{\nu}(X)$ is generated by the elements $(0, \dots, 0, a_i, [D'_i], \omega([D_i]))$ for all $i \in \{1, \dots, r\}$, together with the elements $(1, \dots, 0, 0, [E_1], 0), \dots, (0, \dots, 1, 0, [E_r], 0)$.

Proof. We use induction on r . The case $r = 0$ is Lemma 4.1, so we prove the induction step. Consider $Y = Y_1$. Since any color of Y is the intersection of Y with a color of X , also Y has the same number of colors of Y_r .

We apply the induction hypothesis and conclude that $\tilde{\Delta}_{\nu_Y}(Y)$ is generated by the elements $(0, \dots, 0, a_i, [D'_i|_Y], \omega([D_i]))$ and the elements $(1, \dots, 0, 0, [F_2], 0), \dots, (0, \dots, 1, 0, [F_r], 0)$, where $[F_j] = [E_j \cap Y]$. By the proof of Lemma 3.20, it is enough to show that the set Γ of the elements $(0, 0, \dots, 0, a_i, [D'_i], \omega([D_i]))$ and the elements $(0, 1, \dots, 0, 0, [E_2], 0), \dots, (0, 0, \dots, 1, 0, [E_r], 0)$ generates the convex cone \tilde{K} .

By the proof of Lemma 3.12, together with [3, Proposition 4.2.3], the cone K is generated by elements of the form $([Z], 0)$ and $([D], \omega([D|_E]))$, where Z varies in the set of G -stable prime divisors of X , and D in the set of B -stable but not G -stable prime divisors of X .

Then K_Y is generated by elements of the forms $([Z|_Y], 0)$ and $([D|_Y], \omega([D|_E]))$, with D and Z as above. Notice that for all $j \in \{2, \dots, r\}$ the class $[F_j]$ is of the form $[Z|_Y]$ with $Z = E_j$, and a similar statement holds for all $i \in \{1, \dots, k\}$ and the class $[D'_i|_Y]$. It follows that $\pi(\tilde{\Delta}_{\nu_Y}(Y)) \subseteq K_Y$, which yields $\tilde{K}_Y = \tilde{\Delta}_{\nu_Y}(Y)$.

At this point the fact that ψ_1 is an isomorphism, together with the formula (8), ensures that \tilde{K} is generated as a convex cone by Γ . ■

Remark 4.3. A family of spherical varieties satisfying the assumption of Proposition 4.2 is the one of *wonderful completions* of symmetric homogeneous spaces of adjoint simple groups. Recall that X is *wonderful* if E_1, \dots, E_r are all its G -stable prime divisors, they have normal crossings, non-empty intersection, and their partial intersections are precisely the G -orbit closures of X . A symmetric space of the form G/G^θ , where G is an adjoint simple group and $\theta: G \rightarrow G$ is an involution, admits a wonderful completion (see [4]).

As a first example, we consider a full flag variety and the valuation defined by a resolution by a Bott-Samelson variety.

Example 4.4. Let $E = GL(3)/B$ be the full flag variety of $GL(3)$, i.e., the variety of all flags $[v] \subset V$, where $v \in \mathbb{C}^3$ and V is a two-dimensional linear subspace of \mathbb{C}^3 . Fix the flag $V_0 \subset V_1 \subset V_2$ of \mathbb{C}^3 , where V_i is generated by the first i vectors of the standard basis of \mathbb{C}^3 . The global Okounkov body $\Delta_{\nu'}(E)$, with respect to a Bott-Samelson resolution of $GL(3)/B$ (given as an incidence variety), is computed in [16, Section 7.5]. It is the convex cone generated by

$(0, 0, 0, [D_1]), (0, 0, 0, [D_2]), (0, 1, 1, [D_1]), (0, 0, 1, [D_2]), (1, 1, 0, [D_2]), (0, 1, 0, [D_1]),$
 where D_1 is the Schubert divisor given by the condition $v \in V_2$, and D_2 is the one given by the condition $V \supseteq V_1$.

We continue with some examples of wonderful varieties and their global Okounkov moment cones. We recall that, if X is wonderful, a basis of $N^1(X)$ is given by the classes of the prime divisors that are B -stable but not G -stable.

Example 4.5. Let $\text{Gr}(n, V)$ be the Grassmannian of n -dimensional subspaces of the vector space V . Consider $X = \mathbb{P}(\mathbb{C}^3) \times \text{Gr}(2, \mathbb{C}^3)$, under the diagonal action of $G = \text{GL}(3)$. It is a smooth spherical variety, with two G -orbits: one is the unique G -stable prime divisor E , given by the incidence condition $v \in V$, for a point $p = ([v], V)$, where $v \in \mathbb{C}^3$ and $V \in \text{Gr}(2, \mathbb{C}^3)$. The other G -orbit is the complement $X \setminus E$. Notice that E is the flag variety of G . Choose the Borel subgroup $B \subset G$ of upper triangular matrices of G , the maximal torus T of diagonal matrices of G , and denote by ω_1, ω_2 the corresponding standard fundamental dominant weights. Let $V_0 \subset V_1 \subset V_2$ be the standard flag as in the above example.

Denote by D'_1 and D'_2 the prime divisors of X given by the same two conditions as in the above example, namely $v \in V_2$ and $V \supseteq V_1$, respectively, for $p = ([v], V) \in X$. Then $[D'_i|_E] = [D_i]$ for all $i \in \{1, 2\}$. Moreover, E is linearly equivalent to $D'_1 + D'_2$ (see e.g. [24, Table 1, case 1A]).

According to Proposition 4.2, the global Okounkov moment cone $\tilde{\Delta}_\nu(X)$ is the convex cone generated by the points $(0, 0, 0, 0, [D'_1], \omega_1)$, $(0, 0, 0, 0, [D'_2], \omega_2)$, $(0, 0, 1, 1, [D'_1], \omega_1)$, $(0, 0, 0, 1, [D'_2], \omega_2)$, $(0, 1, 1, 0, [D'_2], \omega_2)$, $(0, 0, 1, 0, [D'_1], \omega_1)$, and $(1, 0, 0, 0, [D'_1 + D'_2], 0)$.

Remark 4.6. Note that Example 4.5 differs from the toy example of projective space equipped with a flag of linear subspaces, or a product of such spaces with the product flag. In the latter case, for the projective space, the corresponding global Newton-Okounkov body is simply the cone generated by the coordinate axis; in the product case it is generated by two such cones, one for each factor.

Example 4.7. Let X be the *variety of complete conics* under the action of $G = \text{GL}(3)$. It can be described as the closure in $\mathbb{P}^5 \times (\mathbb{P}^5)^*$ of the set of couples (C, C^*) , where C is a smooth conic and C^* is its dual. The variety X is wonderful and has four G -orbits: one open in X , two 1-codimensional orbits with smooth closures denoted resp. by E_1, E_2 , and a closed G -orbit $Y_2 = E_1 \cap E_2$ isomorphic to the full flag variety of G , whose Schubert divisors are denoted D_1 and D_2 as before.

As in the previous example, the variety X has two B -stable prime divisors D'_1, D'_2 , but here $[D'_i|_{Y_2}] = [2D_i]$ for all $i \in \{1, 2\}$ (see e.g. [19, Proposition 11]). Moreover, one can choose the indices so that E_i is linearly equivalent to $2D'_i - D'_{3-i}$ for all $i \in \{1, 2\}$ (see e.g. [24, Table A, case 2]).

Proposition 4.2 yields the global Okounkov moment cone $\tilde{\Delta}_\nu(X)$, as the convex cone generated by the points

$$\begin{aligned} & (0, 0, 0, 0, 0, \frac{[D'_1]}{2}, \omega_1), \quad (0, 0, 0, 0, 0, \frac{[D'_2]}{2}, \omega_2) \\ & (0, 0, 0, 1, 1, \frac{[D'_1]}{2}, \omega_1), \quad (0, 0, 0, 0, 1, \frac{[D'_2]}{2}, \omega_2) \\ & (0, 0, 1, 1, 0, \frac{[D'_2]}{2}, \omega_2), \quad (0, 0, 0, 1, 0, \frac{[D'_1]}{2}, \omega_1) \end{aligned}$$

together with the points $(1, 0, 0, 0, 0, [2D'_1 - D'_2], 0)$ and $(0, 1, 0, 0, 0, [2D'_2 - D'_1], 0)$.

Example 4.8. Many wonderful varieties do not satisfy the assumptions of Proposition 4.2. Let us see how, for them, the recursive nature of our proof of Theorem 3.21 can be exploited to obtain a description of the global Okounkov moment cone. Define $X = G \times^Q F$, where $F = \mathbb{P}^1 \times \mathbb{P}^1$ under the diagonal action of $\mathrm{PSL}(2)$, and Q is the parabolic subgroup of $G = \mathrm{SL}(3)$ containing B_- (the subgroup of lower triangular matrices) and Levi subgroup containing T (the subgroup of diagonal matrices) with simple root α_1 . Here we let Q act on F via its quotient isomorphic to $\mathrm{PSL}(2)$. Then X is a wonderful G -variety (see [15, Section 3.6]), and the subgroup B of the upper triangular matrices acts leaving stable four prime divisors: D_1^+ , D_1^- , D'_2 (which are not G -stable), and Y (G -stable).

It is elementary to show that the divisor Y is isomorphic to the full flag variety of $\mathrm{SL}(3)$, that it is linearly equivalent to $D_1^+ + D_1^- - D'_2$, and that $[D_1^+|_Y] = [D_1^-|_Y] = [D_1]$, $[D'_2|_Y] = [D_2]$, where D_1 and D_2 are as in Example 4.4.

Let us compute $\tilde{\Delta}_\nu(X)$. We have already given $\tilde{\Delta}_{\nu_Y}(Y)$ in Example 4.5. The cone K is generated by $([D_1^+], \omega_1)$, $([D_1^-], \omega_1)$, $([D'_2], \omega_2)$, so the cone K_Y is generated by $([D_1], \omega_1)$ and $([D_2], \omega_2)$. We deduce that $\tilde{K}_Y = \tilde{\Delta}_{\nu_Y}(Y)$. Then \tilde{K} is generated by the elements

$$\begin{aligned} &(0, 0, 0, 0, [D_1^+], \omega_1), & (0, 0, 0, 0, [D_1^-], \omega_1), \\ &(0, 0, 1, 1, [D_1^+], \omega_1), & (0, 0, 1, 1, [D_1^-], \omega_1), \\ &(0, 0, 1, 0, [D_1^+], \omega_1), & (0, 0, 1, 0, [D_1^-], \omega_1), \\ &(0, 0, 0, 0, [D'_2], \omega_2), & (0, 0, 0, 1, [D'_2], \omega_2), \\ &(0, 1, 1, 0, [D'_2], \omega_2), \end{aligned}$$

Therefore $\tilde{\Delta}_\nu(X)$ is generated by all of the above, together with the element

$$(1, 0, 0, 0, [D_1^+ + D_1^- - D'_2], 0).$$

We underline that in this example the natural map $\psi_1: N^1(X)_{\mathbb{R}} \rightarrow N^1(Y_1)_{\mathbb{R}}$ is not an isomorphism, but it is surjective, and that there are examples of wonderful varieties where the maps ψ_j are neither injective nor surjective.

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