Journal of Lie Theory
Volume **29** (2019) 1045–1070

Holomorphic Functions of Exponential Type on Connected Complex Lie Groups

Oleg Yu. Aristov

Communicated by K.-H. Neeb

Abstract. Holomorphic functions of exponential type on a complex Lie group G (introduced by Akbarov) form a locally convex algebra, which is denoted by $\mathcal{O}_{exp}(G)$. Our aim is to describe the structure of $\mathcal{O}_{exp}(G)$ in the case when G is connected. The following topics are auxiliary for the claimed purpose but of independent interest: (1) a characterization of linear complex Lie group (a result similar to that of Luminet and Valette for real Lie groups); (2) properties of the exponential radical when G is linear; (3) an asymptotic decomposition of a word length function into a sum of three summands (again for linear groups). The main result presents $\mathcal{O}_{exp}(G)$ as a complete projective tensor of three factors, corresponding to the length function decomposition. As an application, it is shown that if G is linear then the Arens-Michael envelope of $\mathcal{O}_{exp}(G)$ is the algebra of all holomorphic functions.

Mathematics Subject Classification: 22E10, 22E30, 32A38, 46F05.

Key Words: Complex Lie group, linear group, holomorphic function of exponential type, Arens-Michael envelope, submultiplicative weight, length function, exponential radical.

1. Introduction

A holomorphic function on a complex Lie group G is said to be of exponential type if it is majorized by a submultiplicative weight (a non-negative locally bounded function ω satisfying $\omega(gh) \le \omega(g)\omega(h)$ for all $g, h \in G$). Akbarov in [1] introduced this notion for a compactly generated Stein group. In fact, the definition can be used as well for a general complex Lie group G. The set $\mathcal{O}_{exp}(G)$ of all holomorphic functions of exponential type is a $\widehat{\otimes}$-algebra (i.e., a complete Hausdorff locally convex topological algebra with jointly continuous multiplication) with respect to the pointwise multiplication (see Lemma 5.2). Considering basic examples, Akbarov showed that $\mathcal{O}_{exp}(\mathbb{C}^m)$ coincides with the classical space of entire functions of exponential type, i.e., having at most order 1 and finite type, on $\mathbb{C}^m$. (The terminology is taken from this example.) On the other hand, he proved that $\mathcal{O}_{exp}(\mathrm{GL}_m(\mathbb{C}))$ is $\mathcal{R}(\mathrm{GL}_m(\mathbb{C}))$, the algebra of regular functions in the sense of algebraic geometry. The main objective of this text is to give an explicit description of $\mathcal{O}_{exp}(G)$ for an arbitrary connected complex Lie group G.

Our interest is motivated by investigation of holomorphic reflexivity for some topological Hopf algebras initiated in [ibid.]. The essential question in this direction

ISSN 0949–5932 / \$2.50

whether or not the natural map from $\mathcal{O}_{exp}(G)$ to $\mathcal{O}(G)$, the algebra of all holomorphic functions on G, is an Arens-Michael envelope, in other words, whether or not $\mathcal{O}(G)$ is topologically isomorphic to the completion of $\mathcal{O}_{exp}(G)$ with respect to the topology determined by all possible continuous submultiplicative prenorms. It is claimed in [ibid., Lem. 6.6] that this is true if G is affine algebraic and connected. Unfortunately, the argument contains a gap. (In the proof, two maps, ρ and $\widetilde{\rho}$, are considered but it is not clear why $\widetilde{\rho}$ extends ρ.) This defect can be fixed if one shows that $\mathcal{R}(G) \to \mathcal{O}_{exp}(G)$ has dense range. This is possible but the only way to prove density that I know is by rude force, i.e., applying the main structural result of this article (Theorem 5.10); cf. also Theorem 5.12.

To distinguish functions of exponential type inside $\mathcal{O}(G)$ we need to understand asymptotic behavior of a word length function. The reason is that any submultiplicative weight has the form $g \mapsto e^{\ell(g)}$, where ℓ is a length function, and any length function is dominated at infinity by a word length function. We will be concerned with growth rate of any (eq., every) word length function in Section 4.

Two normal closed subgroups of G, the linearizer and the exponential radical, appear naturally in the exposition. To appreciate why they are important consider the following examples.

If H is the 3-dimensional complex Heisenberg group, which can be presented as

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad (a, b, c \in \mathbb{C}), \tag{1}$$

then, as not hard to see, a word length function is equivalent to $|a| + |c| + |b|^{1/2}$; in particular, all polynomials in a, b, and c are of exponential type. Similar asymptotic behavior for an arbitrary simply connected nilpotent group G is found in [34] and [18]. In [2], which can be considered as the first part of this text, these results are used to determine the structure of $\mathcal{O}_{exp}(G)$ for a general simply connected nilpotent complex Lie group G.

On the other hand, consider the quotient H/N of the Heisenberg group over the discrete central subgroup given by

$$N := \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (n \in \mathbb{Z}).$$

It is identified with $\mathbb{C}^\times \times \mathbb{C}^2$ endowing with the group law

$$(z, a, c) \cdot (z', a', c') := (zz'e^{ca'}, a + a', c + c').$$

It is easy to see that the coordinate functions a and c are of exponential type. Nevertheless z is not of exponential type. Indeed, take the word length function ℓ associated with the symmetric neighbourhood of the identity

$$U := \{1/2 \le |z| \le 2,\ |a|,\ |c| \le 1\}.$$

If z is dominated by e^ℓ, then there are constants $C, D \ge 0$ with $|z(g)| \le e^{C\ell(g)+D}$ for all $g \in H/N$. Set $h = (1, 0, 1)$ and $g = (1, 1, 0)$. Then $h^n g^n = (e^{n^2}, n, n)$; hence $z(h^n g^n) = e^{n^2}$. Since h and g are in U, we get $\ell(h^n g^n) \le 2n$. Hence $e^{n^2} \le e^{2Cn+D}$ for all $n \in \mathbb{N}$, a contradiction.

Moreover, it can be shown that $\mathcal{O}_{\exp}(H/N)$ is isomorphic to $\mathcal{O}_{\exp}(\mathbb{C}^2)$, so the dimension degenerates. A crucial observation to generalize this argument is that H/N is not linear as a complex Lie group. (Actually H/N is a standard example of a non-linear complex Lie group.) In fact, if G is connected, then we have an isomorphism $\mathcal{O}_{exp}(G/\operatorname{Lin}_{\mathbb{C}}(G)) \cong \mathcal{O}_{exp}(G)$, where $\operatorname{Lin}_{\mathbb{C}}(G)$ is the linearizer of G (the intersection of kernels of all finite-dimensional holomorphic representations); see Theorem 5.3. For the proof, we need an auxiliary result: *If holomorphic homomorphisms of a connected complex Lie group G to invertibles of Banach algebras separate points, then G is linear* (see Theorem 2.2).

Further, consider another example: the simply connected 2-dimensional non-abelian Lie group S, i.e., $\mathbb{C}^2$ with the multiplication

$$(s,t)\cdot(s',t') := (s+s', te^{s'}+t') \tag{2}$$

(see Example 5.13). It is not hard to see that any word length function on S is equivalent to $|s| + \log|t|$, where $(s,t)\in S$. This decomposition corresponds to the presentation

$$\mathcal{O}_{exp}(S) \cong \mathcal{O}_{exp}(\mathbb{C}) \otimes \mathcal{R}(\mathbb{C})\,.$$

To transfer this observation to the general case we need the notion of exponential radical. The idea of exponential radical dates back to Guivarc'h [12]. It was rediscovered and named by Osin in [28], where the simply connected solvable case is considered. In [8], Cornulier modified Osin's definition in a way more convenient to general connected Lie groups. The main property of the exponential radical is that it is a strictly exponentially distorted subgroup, which means logarithmic growth for the restriction of a word length function (e.g., the exponential radical of the group S defined in (2) is $\{(0,t)\colon\, t\in\mathbb{C}\}$). For a complex Lie group, the exponential radical is easier to describe than in the real case and we consider it carefully in Section 3.

This paper is organized as follows. In Section 2, we discuss the linearizer of a connected complex Lie group. Section 3 contains properties of exponential radical. Section 4 is devoted to an asymptotic decomposition of a word length function into a sum of three summands. Section 5 contains main results on holomorphic functions of exponential type and Arens-Michael envelopes, as well as some examples.

The author is grateful to K.-H. Neeb for valuable comments and the anonymous referee for useful suggestions.

2. Characterization of the linearizer

Our main references to the structure theory of Lie groups are [17, 20, 16]. Our terminology and notation in this area are principally from [16]. For a complex Lie group G, the intersection of kernels of all finite-dimensional holomorphic representations is called the *linearizer* and is denoted by $\operatorname{Lin}_{\mathbb{C}}(G)$. Also, G is *linear* if it admits a faithful finite-dimensional holomorphic representation (eq., $\operatorname{Lin}_{\mathbb{C}}(G)=\{1\}$). Further, G is *linearly complex reductive* if there exists a compact real Lie group K such that G is the universal complexification of K [16, Def. 15.2.7] (it is called 'reductive' in [20]; cf. [ibid. Th. 4.31]). An *integral subgroup* H of G is a subgroup that is generated by $\exp\mathfrak{h}$ for a subalgebra $\mathfrak{h}$ of the Lie algebra of G; in this case we write $H=\langle\exp\mathfrak{h}\rangle$. If H is a closed subgroup and G is connected, then H^* denotes the smallest complex integral subgroup containing H [16, Defs. 9.4.10, 15.2.11]. We write (G,G) for the subgroup generated by the commutators $ghg^{-1}h^{-1}$ for $g,h\in G$. The connected component of 1 and the center of G are denoted by G_0 and $Z(G)$, resp.

It is known that a connected real Lie group with Levi-Malcev decomposition $G = RS$, where R is the solvable radical and S is a semisimple Levi factor, is linear iff R and S are both linear (as real Lie groups) [17, Th. XVIII.4.2]. Since a connected semisimple complex Lie group is always linear [17, Th. XVII.3.2], the following proposition is an analogue of this theorem in the complex case. The author was unable to find a proof of this result in the literature.

Theorem 2.1. *A connected complex Lie group is linear iff its radical is linear.*

Proof. The necessity is evident. To prove the sufficiency take a connected complex Lie group G and consider a Levi-Malcev decomposition $G = RS$, where R is the solvable radical and S is a semisimple Levi subgroup. Suppose that R is linear. Then $R = B \rtimes L$, where B is simply connected solvable and L is linearly complex reductive [16, Th. 16.3.7].

Our first goal is to replace B by a normal subgroup of G with the same properties. It is not hard to show that we can assume that $(S,L)=\{1\}$ and there is a normal integral subgroup B_1 in G such that $R = B_1L$, $(R,R)\subset B_1$, and $B_1\cap L$ is discrete. (The argument is almost the same as for the real case in [17, Th. XVIII.4.2, paragraphs 1–3 of the proof]. The only step which is different is that we have to apply the fact that a holomorphic finite-dimensional representation of a complexified torus is completely reducible [16, Th. 15.2.10] instead of that a finite-dimensional representation of a real torus is completely reducible.)

We claim that $R = B_1 \rtimes L$ and B_1 is simply connected closed (cf. the second part of the proof for [17, Th. XVIII.4.2]). Indeed, R is a linear complex Lie group and $L = K^*$ for some maximal compact subgroup K in R. Hence $L\cap(R,R)=\{1\}$ [16, Th. 16.3.7] and (R,R) is closed in R [20, Pr. 4.37]. (Remark that the latter is true for all linear real Lie groups [16, Cor. 16.2.8].) Also, (R,R) is normal in R, so is $L(R,R)$. Since R is solvable, L is abelian and contains a maximal torus of R. Further, $L(R,R)$ is integral and contains a maximal torus, so, by [16, Cor. 14.5.6], we get that $L(R,R)$ is closed in R. Remind that $(R,R)\subset B_1$, so we can consider a homomorphism of Lie groups $\sigma\colon B_1/(R,R)\to R/(L(R,R))$. Since $R = B_1L$, we obtain that σ is surjective.

On the other hand, $R = B\rtimes L$ and L is abelian; hence(R,R) is contained in B and $R/(L(R,R))$ is isomorphic to $B/(R,R)$. Moreover, (R,R) is a connected subgroup in the simply connected group B; therefore (R,R) and $R/(L(R,R))$ are simply connected. Further,

$$\operatorname{Ker}\sigma = B_1/(R,R)\cap (L(R,R))/(R,R)$$

is discrete because $B_1\cap L$ is discrete. Whence $\operatorname{Ker}\sigma$ is trivial since $R/(L(R,R))$ is simply connected. Thus σ is an isomorphism. Hence $B_1/(R,R)$ is simply connected; so is B_1. Besides, it follows from

$$B_1/(R,R)\cap (L(R,R))/(R,R)=\{1\}$$

that $B_1\cap L(R,R)\subset (R,R)$. Consequently $B_1\cap L\subset L\cap(R,R)=\{1\}$; in particular, B_1 is closed. The claim is proved.

Since $(S,L)=\{1\}$, the set LS is an integral subgroup in G. By [16, Th. 16.3.7], to complete the proof we need to show that LS is linearly complex reductive and $G=B_1\rtimes LS$. First, we claim that G is a Stein group. Indeed, since R is linear, it is a Stein group. On the other hand, $Z(G)_0$ is a closed subgroup of R, so $Z(G)_0$ is a Stein group. It follows from the Matsushima-Morimoto theorem [24, Th. XIII.5.9] that a connected complex Lie group is a Stein group iff the connected component of 1 in the center is a Stein group. The claim is proved.

Further, LS is an integral subgroup in a Stein group, therefore it is holomorphically separable, hence, by the Matsushima-Morimoto theorem, it is a Stein group. Since L is abelian, we get $Z(LS)=LZ(S)$. The center of a connected semisimple complex Lie group is finite [20, Cor. 4.17], so $Z(LS)_0/L$ is finite. Then $Z(LS)_0$ is toroidal, i.e., $Z(LS)_0=T^*$ for some maximal torus T [16, Pr. 15.3.9]. Besides, $Z(LS)_0$ is a Stein group because it is a closed subgroup of a Stein group. It follows from [16, Pr. 15.3.4] that a toroidal Stein group is a complexified torus. Since LS is connected, application of [16, Th. 15.2.9] yields that LS is linearly complex reductive.

By dimension argument, $B_1\cap LS$ is discrete; hence it is central. Let $g\in B_1\cap LS$. Then $g\in LZ(S)$ and $Z(S)$ is finite, whence there is $k\in\mathbb{N}$ such that $g^k\in L$. Therefore $g^k\in B_1\cap L=\{1\}$. The center of B_1 is simply connected, so the only element of finite order is 1. Thus $B_1\cap LS=\{1\}$ and we have finally that $G=B_1\rtimes LS$. ■

It is a standard fact that a Banach space valued function that is weakly holomorphic is also holomorphic with respect to the norm; see, e.g. [14, Th. 2.1.3]. So we can say freely that a homomorphism $\pi: G\to \mathrm{GL}(A)$, where G is a complex Lie group and $\mathrm{GL}(A)$ is the group of invertible elements of a unital Banach algebra A, is *holomorphic* if for any continuous linear functional x on A the function $G\to\mathbb{C}\colon g\mapsto\langle x,\pi(g)\rangle$ is holomorphic.

Theorem 2.2. *The linearizer* $\mathrm{Lin}_{\mathbb{C}}(G)$ *of a connected complex Lie group* G *coincides with*

$$\bigcap_{\pi}\{\mathrm{Ker}\,\pi\colon\ \pi\colon G\to \mathrm{GL}(A)\}\,,$$

where A *runs all unital Banach algebras and* π *runs all possible holomorphic homomorphisms.*

Remark 2.3. Luminet and Vallete proved a result that gives a characterization of the real linearizer for a real Lie group as the intersection of the kernels of all norm continuous homomorphisms to the invertible groups of unital Banach algebras [23, Th. A, (i)⇔(v)]. (Also, this is true for more general class of continuous inverse algebras [3].) Since any holomorphic homomorphism from a complex Lie group is holomorphic with respect to the norm, it is norm continuous. Thus we can consider Theorem 2.2, in which the norm continuity assumption is redundant, as an analogue of the result of Luminet and Vallete.

For the proof, we need two lemmas and the notation $\mathrm{Rad}\,A$ for the Jacobson radical of a Banach algebra A. Recall that $a\in A$ is called *topologically nilpotent* if $\|a^n\|^{1/n}\to 0$.

Lemma 2.4. *Let G be a connected solvable complex Lie group, A a unital Banach algebra, and $\pi : G \to \mathrm{GL}(A)$ a holomorphic homomorphism. Then for each $g \in (G,G)$ there is a topologically nilpotent $r \in A$ such that $\pi(g) = 1 + r$.*

Proof. Consider $\mathrm{GL}(A)$ as a (complex) Banach Lie group [25, Exm. III.1.11(b)] and π as a Banach Lie group homomorphism. Identifying the Lie algebra of $\mathrm{GL}(A)$ with A we can write the exponential map as

$$\exp\colon A \to \mathrm{GL}(A)\colon a \mapsto \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

[ibid., Rem. IV.2.2]. Denote by $\mathfrak{g}$ the Lie algebra of G. Then applying the Lie functor to π we get a Banach Lie algebra homomorphism $\mathrm{L}_\pi : \mathfrak{g} \to A$ such that $\pi \exp = \exp \mathrm{L}_\pi$. It follows from results of [33] (see also [4, Th. 24.1]) that $\mathrm{L}_\pi[\mathfrak{g},\mathfrak{g}] \subset \mathrm{Rad}\, A_0$, where A_0 is the closed associative unital subalgebra of A generated by the solvable Lie subalgebra $\mathrm{L}_\pi\mathfrak{g}$ of A. For any ξ in $[\mathfrak{g},\mathfrak{g}]$ we have

$$\exp \mathrm{L}_\pi(\xi) = \sum_{n=0}^{\infty} \frac{\mathrm{L}_\pi(\xi)^n}{n!}\,.$$

So $\exp \mathrm{L}_\pi(\xi) = 1 + r$ for some $r \in \mathrm{Rad}\, A_0$ (because $\mathrm{Rad}\, A_0$ is closed). Since G is connected, the subgroup (G,G) is generated by $\exp[\mathfrak{g},\mathfrak{g}]$ [16, Pr. 11.2.3]. Therefore $\pi(g)$ has the same form for any $g \in (G,G)$. Finally, note that each element of the Jacobson radical of a Banach algebra is topologically nilpotent [13, Th. 2.1.33]. ∎

Recall that a Banach algebra is said to be *classically semisimple* if it isomorphic to a finite product of full matrix algebras over $\mathbb{C}$. We denote by $\mathscr{E}(K)'$ the algebra of distribution on a compact Lie group K.

Lemma 2.5. *Let K be a compact Lie group and let A be a unital Banach algebra. If $\phi : \mathscr{E}(K)' \to A$ is a continuous homomorphism with dense range, then A is classically semisimple.*

Proof. The linear space T of matrix coefficients of finite-dimensional representations of K is a (non-unital) subalgebra of $\mathscr{E}(K)'$. It follows from the Peter-Weyl Theorem that T is dense in $L^2(K)$. But $L^2(K)$ is dense in $\mathscr{E}(K)'$; therefore, $\phi(T)$ is dense in A. So we can take $t \in T$ such that $\phi(t)$ is sufficiently close to 1 to be invertible.
It is well known (e.g. [15, Th. 27.21]) that each element of T is contained in a finite sum of complemented minimal two-sided ideals and each such ideal is a full matrix algebra. Therefore there exists a central idempotent $p \in T$ such that $tp = t$. Then $1 = \phi(t)^{-1}\phi(t) = \phi(t)^{-1}\phi(t)\phi(p) = \phi(p)$. So $\phi(Tp)$ is dense in A. Since Tp is finite-dimensional, this image is closed; hence $\phi(Tp) = A$. Thus A is a quotient of the classically semisimple algebra Tp, so A is classically semisimple itself. ∎

Proof of Theorem 2.2. Since $\bigcap_\pi \mathrm{Ker}\,\pi \subset \mathrm{Lin}_{\mathbb{C}}(G)$, it suffices to consider the case where $\bigcap_\pi \mathrm{Ker}\,\pi = \{1\}$ and prove that G is linear. By Theorem 2.1, we can assume also that G is solvable. If follows from [16, Th. 16.3.7(v)⇒(iii)] that we need to check that K^* is linear and $K^* \cap (G,G) = \{1\}$ for any maximal compact subgroup K of G (where K^* is the smallest complex integral subgroup containing K [16, Def. 15.2.11]).

First, we claim that G is a Stein group. Indeed, $\bigcap_\pi \operatorname{Ker}\pi = \{1\}$ implies that coefficients of holomorphic homomorphisms to Banach algebras, which are holomorphic functions, separate points of G. Thus G is holomorphically separable; hence it is a Stein group [24, Th. XIII.5.9].

From [16, Lem. 14.3.3] it follows that a maximal compact subgroup K of a solvable Lie group is abelian. Then K^* is an abelian integral subgroup. Whence K^* contains a maximal torus; so it is closed [16, Cor. 4.5.6]. Being a closed submanifold of a Stein manifold, K^* is a Stein manifold [24, Th. XIII.5.2]. Therefore, K^* has no compact factors. Hence, by [16, Pr. 15.3.4(i)], K^* is a universal complexification of K.

Now, let A be a unital Banach algebra and $\pi\colon G \to \mathrm{GL}(A)$ a holomorphic homomorphisms. The restriction of π to K is infinitely differentiable, hence it can be extended to a continuous homomorphism $\phi\colon \mathscr{E}(K)' \to A$. Denote by C the closure of $\phi(\mathscr{E}(K)')$ in A; then $\pi(K) \subset \mathrm{GL}(C)$. Lemma 2.5 implies that C is classically semisimple Banach algebra. It is evident that C is commutative; therefore it is a finite sum of copies of $\mathbb{C}$. So $\mathrm{GL}(C)$ is a finite-dimensional Lie group. By the definition of universal complexification, the homomorphism $K \to \mathrm{GL}(C)$ is extended to a holomorphic homomorphism $K^* \to \mathrm{GL}(C)$, which coincides with the restriction of π by the uniqueness property.

Let $g \in K^* \cap (G,G)$. By Lemma 2.4, there exists a topologically nilpotent $r \in A$ such that $\pi(g) = 1 + r$. Since $\pi(g) \in C$, we have $r \in C$. But C is semisimple and commutative; so the only topologically nilpotent element in C is 0 [13, Pr. 2.1.34]. Therefore $\pi(g) = 1$. Since π is arbitrary, it follows from the assumption that $g = 1$. ■

3. Exponential radical of a linear complex group

Recall that a (real) Lie group G is of polynomial growth iff its Lie group $\mathfrak{g}$ is of *Type R* [29, 6.25, 6.39], i.e., the eigenvalues of $\operatorname{ad}\xi$ are contained in $i\mathbb{R}$ for all $\xi \in \mathfrak{g}$. The following theorem describes the exponential radical of simply connected solvable Lie groups; it is a combination of results from [12] and [28]. (See the definition of a strictly exponentially distorted subgroup in (7) below.)

Theorem 3.1. *Let G be a simply connected solvable Lie group and let $\mathfrak{g}$ be the Lie algebra of G.*

(A) *Then there exist a closed normal subgroup E such that G/E is the largest polynomial growth quotient of G and a Lie ideal $\mathfrak{e}$ in $\mathfrak{g}$ of G such that $\mathfrak{g}/\mathfrak{e}$ is the largest Type R quotient of $\mathfrak{g}$. Moreover, $E = \langle \exp \mathfrak{e} \rangle$.*

(B) *The subgroup E is nilpotent and stable under automorphisms of G; the ideal $\mathfrak{e}$ is nilpotent and stable under automorphisms of $\mathfrak{g}$.*

(C) *The subgroup E is strictly exponentially distorted in G.*

Following [8, Def. 6.2], we say that the *exponential radical* of a connected (real or complex) Lie group G is the closed normal subgroup E such that G/E is the largest quotient of G that is a *P-decomposed* Lie group, i.e., locally isomorphic to a direct product of a group of polynomial growth and a semisimple group.

The following several paragraphs contain details omitted in [ibid.] concerning existence of the exponential radical and decomposition its Lie algebra in the general connected case. First, we give a definition on the Lie algebra level.

Definition 3.2. We say that a (real or complex) Lie algebra is *R-decomposed* if it is a direct sum of a semisimple algebra and an algebra of Type R.

Obviously, G is P-decomposed iff its Lie algebra $\mathfrak{g}$ is R-decomposed. Note that both properties have alternative descriptions: a connected Lie group is P-decomposed iff it has the Rapid Decay Property [6] and a unimodular Lie algebra is R-decomposed iff it is a B-algebra in the sense of Varopoulos [35, Sect. 1.8].
The proof of the following lemma is straightforward.

Lemma 3.3. *A Lie algebra is R-decomposed iff any maximal semisimple subalgebra without compact summands is complemented and each complement is of Type R.*

Suppose temporarily that $\mathfrak{g}$ is a real Lie algebra. Let $\mathfrak{r}$ be the solvable radical of $\mathfrak{g}$ and $\mathfrak{s}$ a Levi complement. Write

$$\mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc},$$

where $\mathfrak{s}_c$ is compact and $\mathfrak{s}_{nc}$ is maximal semisimple without compact summands. Denote by $\mathfrak{e}_r$ the ideal in $\mathfrak{r}$ such that $\mathfrak{r}/\mathfrak{e}_r$ is the largest quotient of $\mathfrak{r}$ that is R-decomposed and set

$$\mathfrak{e} := \mathfrak{e}_r + [\mathfrak{s}_{nc}, \mathfrak{r}] \,. \tag{3}$$

Lemma 3.4. *The subspace $\mathfrak{e}$ is a nilpotent ideal in $\mathfrak{g}$.*

Proof. First, note that $[\mathfrak{s}_{nc}, \mathfrak{r}]$ is an ideal in $\mathfrak{g}$ (see the Lie group form in [8, Lem. 6.8]). Since $\mathfrak{e}_r$ is an ideal in $\mathfrak{r}$ that is stable under automorphisms of $\mathfrak{r}$ (Theorem 3.1(B)), it is stable under derivations, so $\mathfrak{e}_r$ is an ideal in $\mathfrak{g}$. Therefore $\mathfrak{e}$ is an ideal in $\mathfrak{g}$.
Let $\mathfrak{r}_\infty$ be the intersection of the lower central series of $\mathfrak{r}$. Since $\mathfrak{r}/\mathfrak{r}_\infty$ is nilpotent, it is of Type R; so $\mathfrak{e}_r \subset \mathfrak{r}_\infty$. Note that $\mathfrak{r}_\infty \subset [\mathfrak{r}, \mathfrak{r}]$, which is nilpotent. On the other hand, by [16, Cor. 5.4.15], we have that $[\mathfrak{g}, \mathfrak{r}]$ is a nilpotent ideal of $\mathfrak{g}$. So $\mathfrak{e}_r$ and $[\mathfrak{s}_{nc}, \mathfrak{r}]$ are both contained in nilpotent ideals; hence they are nilpotent themselves. Therefore $\mathfrak{e}$ is nilpotent. ∎

Lemma 3.5. *The Lie algebra $\mathfrak{g}/\mathfrak{e}$ is the largest quotient of $\mathfrak{g}$ that is R-decomposed.*

Proof. First, we claim that for a finite family of ideals $\mathfrak{j}_1, \ldots, \mathfrak{j}_n$ such that each $\mathfrak{g}/\mathfrak{j}_k$ is R-decomposed, all algebras $\mathfrak{g}/(\cap \mathfrak{j}_k)$ is also R-decomposed (cf. [12] for Type R algebras). Indeed, note that the property to be R-decomposed is inherited by finite sums and subalgebras but $\mathfrak{g}/(\cap \mathfrak{j}_k)$ is isomorphic to the range of $\mathfrak{g} \to \bigoplus_k \mathfrak{g}/\mathfrak{j}_k$.
Suppose that $\mathfrak{j}$ is an ideal in $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{j}$ is R-decomposed. To show that $\mathfrak{j} \subset \mathfrak{e}$ note that $\mathfrak{g}/\mathfrak{r}$ is R-decomposed, so $\mathfrak{g}/(\mathfrak{r} \cap \mathfrak{j})$ is R-decomposed. So we can assume that $\mathfrak{j} \subset \mathfrak{r}$. Further, $\mathfrak{r} + \mathfrak{s}_c$ is a semidirect sum and an ideal in $\mathfrak{g}$. Moreover, $\mathfrak{g} = (\mathfrak{r} \rtimes \mathfrak{s}_c) \rtimes \mathfrak{s}_{nc}$. Then $\mathfrak{g}/\mathfrak{j}$ is isomorphic to $(\mathfrak{r}/\mathfrak{j} \rtimes \mathfrak{s}_c) \rtimes \mathfrak{s}_{nc}$. Therefore $\mathfrak{s}_{nc}$ is maximal in $\mathfrak{g}/\mathfrak{j}$ as a semisimple subalgebra without compact summands. Since $\mathfrak{g}/\mathfrak{j}$ is R-decomposed, Lemma 3.3 implies that $\mathfrak{s}_{nc}$ is complemented and $\mathfrak{r}/\mathfrak{j} \rtimes \mathfrak{s}_c$ is of Type R. In particular, $[\mathfrak{s}_{nc}, \mathfrak{r}] \subset \mathfrak{j}$. Note that $\mathfrak{r}/\mathfrak{j}$ is the radical of $\mathfrak{r}/\mathfrak{j} \rtimes \mathfrak{s}_c$; so $\mathfrak{r}/\mathfrak{j}$ is of Type R. By the definition of $\mathfrak{e}_\mathfrak{r}$, we have $\mathfrak{e}_\mathfrak{r} \subset \mathfrak{j}$. It follows from (3) that $\mathfrak{e} \subset \mathfrak{j}$.
On the other hand, since $\mathfrak{e} \subset \mathfrak{r}$, we obtain $\mathfrak{g}/\mathfrak{e} \cong (\mathfrak{r}/\mathfrak{e} \rtimes \mathfrak{s}_c) \rtimes \mathfrak{s}_{nc}$. At the same time, $[\mathfrak{s}_{nc}, \mathfrak{r}] \subset \mathfrak{e}$; therefore the action of $\mathfrak{s}_{nc}$ on $\mathfrak{r}/\mathfrak{e} \rtimes \mathfrak{s}_c$ is trivial. Thus $\mathfrak{g}/\mathfrak{e} \cong (\mathfrak{r}/\mathfrak{e} \rtimes \mathfrak{s}_c) \oplus \mathfrak{s}_{nc}$.

Since $\mathfrak{r}/\mathfrak{e}$ and $\mathfrak{s}_c$ are of Type R, it follows from [29, Pr. 6.28] that $\mathfrak{r}/\mathfrak{e} \rtimes \mathfrak{s}_c$ is of Type R. ■

The following corollary follows easily from Lemma 3.5.

Corollary 3.6. *For any connected Lie group G, the exponential radical exists and coincides with the closure of the integral subgroup $\langle \exp \mathfrak{e} \rangle$, where $\mathfrak{e}$ is defined by* (3) *for the Lie algebra $\mathfrak{g}$ of G.*

Our next goal is to show that, for a complex Lie algebra $\mathfrak{g}$, the decomposition (3) has the simplified form

$$\mathfrak{e} = \mathfrak{r}_\infty + [\mathfrak{s}, \mathfrak{r}]\,, \tag{4}$$

where $\mathfrak{r}_\infty$ denotes the intersection of the lower central series of $\mathfrak{r}$. It is evident that in this case $\mathfrak{s}_{nc} = \mathfrak{s}$ but the equality $\mathfrak{e}_\mathfrak{r} = \mathfrak{r}_\infty$ needs a little more work.

Lemma 3.7. *A complex Lie algebra is of Type R iff it is nilpotent.*

Proof. Suppose that $\mathfrak{g}$ is a complex Lie algebra is of Type R, i.e., the eigenvalues of $\operatorname{ad} \xi$ are contained in $i\mathbb{R}$ for all $\xi \in \mathfrak{g}$. In particular, the eigenvalues of $\operatorname{ad}(i\xi)$ are also contained in $i\mathbb{R}$. Therefore, each eigenvalue is 0; so, by Engel's Theorem, $\mathfrak{g}$ is nilpotent. On the other hand, each nilpotent Lie algebra is of Type R. ■

Corollary 3.8. *A connected complex Lie group is of polynomial growth iff it is nilpotent.*

Lemma 3.9. *Let $\mathfrak{g}$ be a solvable complex Lie algebra and $\mathfrak{e}$ a real ideal such that $\mathfrak{g}/\mathfrak{e}$ is the largest Type R quotient of $\mathfrak{g}$. Then $\mathfrak{e} = \mathfrak{g}_\infty$.*

Proof. First, we show that $\mathfrak{e}$ is a complex ideal. Note that $i\mathfrak{e}$ is a real ideal. We claim that $\mathfrak{g}/i\mathfrak{e}$ is of Type R. Indeed, we have to show that $[\xi, \eta] - \lambda\eta \in i\mathfrak{e}$ for some $\xi, \eta \in \mathfrak{g}$ and $\lambda \in \mathbb{C}$ implies $\lambda \in i\mathbb{R}$. Multiplying by i, we obtain $[\xi, i\eta] - \lambda i\eta \in \mathfrak{e}$. Since $\mathfrak{g}/\mathfrak{e}$ is of Type R, we have that $\lambda \in i\mathbb{R}$.
Further, $\mathfrak{g}/\mathfrak{e}$ is the largest Type R quotient of $\mathfrak{g}$; so $\mathfrak{e} \subset i\mathfrak{e}$. Therefore $\mathfrak{e} = i\mathfrak{e}$ and $\mathfrak{e}$ is a complex ideal.
Since $\mathfrak{g}/\mathfrak{e}$ is a complex Lie algebra of Type R, Lemma 3.7 implies that $\mathfrak{g}/\mathfrak{e}$ is nilpotent. On the other hand, $\mathfrak{g}/\mathfrak{g}_\infty$ is the largest nilpotent quotient of $\mathfrak{g}$. Therefore, $\mathfrak{g}_\infty \subset \mathfrak{e}$. Finally, since $\mathfrak{g}/\mathfrak{g}_\infty$ is of Type R, we have $\mathfrak{e} \subset \mathfrak{g}_\infty$. ■

Corollary 3.10. *If $\mathfrak{g}$ is a complex Lie algebra, then* (4) *is satisfied.*

Now we return to the Lie group level.

Proposition 3.11. *The exponential radical of a connected complex Lie group G coincides with the normal complex Lie subgroup E such that G/E is the largest quotient of G that is locally isomorphic to a direct product of a nilpotent and semisimple complex Lie group.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{e}$ defined by (3). Corollary 3.10 implies that $\mathfrak{e} = \mathfrak{r}_\infty + [\mathfrak{s}, \mathfrak{r}]$; thus $\mathfrak{e}$ is a complex ideal in $\mathfrak{g}$. Since $E = \overline{\langle \exp \mathfrak{e} \rangle}$, it is a complex Lie subgroup of G. Therefore G/E is a complex Lie group. It follows from Corollary 3.8 that any P-decomposed complex Lie group is locally isomorphic to a direct product of a nilpotent and a semisimple complex group. It is not hard to see that G/E is the largest quotient with this property. ■

Now we consider the exponential radical of a connected linear complex Lie group. Recall that every such group has the form $B \rtimes L$, where B is complex simply connected solvable and L is linearly complex reductive [16, Th. 16.3.7].

Theorem 3.12. *Let G be a connected complex Lie group of the form $B \rtimes L$, where B is complex simply connected solvable and L is linearly complex reductive. Suppose that the exponential radical of G is $\{1\}$. Then the action of L on B is trivial, so $G = B \times L$.*

Proof. Consider a Levi-Malcev decomposition $L = TS$, where T is the solvable radical of L and S is a semisimple Levi subgroup. Since both T and S are integral subgroups, it suffice to prove that the actions of the corresponding Lie subalgebras $\mathfrak{t}$ and $\mathfrak{s}$ on the Lie algebra $\mathfrak{b}$ of B are trivial.

Obviously, $\mathfrak{b} + \mathfrak{t}$ is a semidirect sum. Since the Lie algebra $\mathfrak{l}$ of L is reductive, $\mathfrak{t}$ is central in $\mathfrak{l}$ [16, Pr. 5.7.3]. Then $\mathfrak{b} \rtimes \mathfrak{t}$ is a solvable ideal; hence, $B \rtimes T$ is a closed normal solvable subgroup. Moreover, it follows from Proposition 3.11 that the solvable radical R of G is nilpotent. Therefore $B \rtimes T$ is nilpotent (in fact, $B \rtimes T = R$).

Since T is linearly complex reductive, $\mathfrak{b}$ is a completely reducible module with respect to the adjoint action of T [ibid., 15.2.10]. On the other hand, since R connected and nilpotent, the adjoint action of R on its Lie algebra $\mathfrak{r}$ is unipotent [27, Ch. 2, Th. 1.6]. In particular, the action of T on $\mathfrak{b}$ is unipotent. Thus the action of T on $\mathfrak{b}$ is trivial, so is the action of $\mathfrak{t}$ on $\mathfrak{b}$.

On the other hand, Proposition 3.11 implies that the action of $\mathfrak{s}$ on $\mathfrak{r}$ (in particular, on $\mathfrak{b}$) is trivial. ■

Proposition 3.13. *Let G be a connected linear complex Lie group. Then the exponential radical of G is simply connected nilpotent and coincides with* $\exp \mathfrak{e}$.

Proof. Since $\mathfrak{e} \subset [\mathfrak{g}, \mathfrak{r}]$ and (G, R) is an integral subgroup, the normal integral subgroup $\langle \exp \mathfrak{e} \rangle$ is contained in (G, R) [16, Lem. 11.2.2].

Let K be a maximal compact subgroup of G. Since G is a linear group, we have $K^* \cap (G, R) = \{1\}$ [16, Th. 16.3.7]. Then the normal subgroup $\langle \exp \mathfrak{e} \rangle$ intersects K trivially; hence, it intersects all maximal compact subgroups of G trivially. Whence it is closed and simply connected [16, Cor. 14.5.6, Th. 14.3.11]. Thus $E = \langle \exp \mathfrak{e} \rangle$. Finally, by Lemma 3.4, $\mathfrak{e}$ is nilpotent; therefore $\langle \exp \mathfrak{e} \rangle = \exp \mathfrak{e}$. ■

Finally, we can prove the main result of the section.

Theorem 3.14. *Let $G = B \rtimes L$, where B is simply connected solvable and L is linearly complex reductive, and let E be the exponential radical of G. Then $E \subset B$ and $G/E \cong B/E \times L$.*

Proof. Note that $\mathfrak{r}_\infty \subset [\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]$ and $[\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]$. Since, by Corollary 3.10, $\mathfrak{e} = \mathfrak{r}_\infty + [\mathfrak{s}, \mathfrak{r}]$ and, by [16, Lem. 5.6.4(ii)], $\operatorname{rad}[\mathfrak{g}, \mathfrak{g}] = \mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]$, we have $\mathfrak{e} \subset \operatorname{rad}[\mathfrak{g}, \mathfrak{g}]$. Proposition 3.13 implies that $E \subset \operatorname{Rad}(G, G)$. It follows from [20, Pr. 4.44] that the subgroup B contains the representation radical, which is, by definition, is the intersection of all kernels of semisimple holomorphic representations of G. The assumption of the theorem yields that that G is linear [16, Th. 16.3.7], so the representation radical of G coincides with $\operatorname{Rad}(G, G)$ [20, Cor. 4.39]. Hence, $E \subset B$.

Consider the action of L on B/E such that $G/E \cong B/E \rtimes L$. Since G is linear, it follows from Proposition 3.13 that E is simply connected. Therefore, B/E is simply connected and, evidently, solvable. On the other hand, by Proposition 3.11, the exponential radical of G/E is trivial. Thus Theorem 3.12 implies that $G/E \cong B/E \times L$. ■

4. Asymptotic behavior of a word length function

Recall that a *length function* on a locally compact group G is a locally bounded function $\ell \colon G \to \mathbb{R}$ such that

$$\ell(gh) \le \ell(g) + \ell(h) \qquad (g, h \in G).$$

Note that we does not assume in general that ℓ is *symmetric*, i.e., $\ell(e) = 0$ and $\ell(g^{-1}) = \ell(g)$ for all $g \in G$.

If G is compactly generated, i.e., there is a relatively compact generating set U ($\bigcup_{n=0}^{\infty} U^n = G$, where $e \in U$ and $U^0 := \{e\}$), then the function defined by

$$\ell_U(g) := \min\{n \colon\ g \in U^n\}. \tag{5}$$

is a length function. Any such length function is called a *word length function.*

For positive functions τ_1 and τ_2 on a set X, we say that τ_1 *dominated by* τ_2 (at infinity) and write $\tau_1 \lesssim \tau_2$ if there are $C, D > 0$ such that

$$\tau_1(x) \le C\tau_2(x) + D \qquad (x \in X).$$

Moreover, if $\tau_1 \lesssim \tau_2$ and $\tau_2 \lesssim \tau_1$, then we say that τ_1 and τ_2 are *equivalent* (at infinity) and write $\tau_1 \simeq \tau_2$. Note that the length functions equivalence means that we have a bijective quasi-isometry between the corresponding metric spaces.

This section is devoted to the proof of the following result.

Theorem 4.1. *Suppose that G is a connected linear complex Lie group. Fix a decomposition $G = B \rtimes L$, where B is complex simply connected solvable and L is linearly complex reductive. Let E be the exponential radical of G, $\pi : B \to B/E$ the quotient homomorphism, $\mathfrak{b}$ and $\mathfrak{e}$ the Lie algebras of B and E, resp., and $\mathfrak{v}$ a complementary subspace to $\mathfrak{h} \cap \mathfrak{e}$ in $\mathfrak{e}$, where $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{b}$. Then*

$$\tau \colon \mathfrak{e} \times \mathfrak{v} \times L \to G \colon (\eta, \xi, l) \mapsto \exp(\eta)\exp(\xi)\, l$$

is a biholomorphic equivalence of complex manifolds such that

$$\ell(\exp(\eta)\exp(\xi)l) \simeq \log(1 + \ell_0(\exp\eta)) + \ell_1(\pi(\exp(\xi))) + \ell_2(l) \tag{6}$$

on $\mathfrak{e} \times \mathfrak{v} \times L$, where $\eta \in \mathfrak{e}$, $\xi \in \mathfrak{v}$, and $l \in L$, and

$$\log(1 + \ell_0(\exp\eta)) \simeq \log(1 + \|\eta\|) \quad \text{on } \mathfrak{e},$$

where $\|\cdot\|$ is a norm on $\mathfrak{e}$ and ℓ, ℓ_0, ℓ_1, and ℓ_2 are word length functions on G, E, B/E, and L, resp.

We begin with a decomposition of a word length function on a semidirect product.

Proposition 4.2. *Let $G = N \rtimes H$ be a semidirect product of compactly generated locally compact groups. Suppose that ℓ and ℓ_1 are word length functions on G and H, resp. Then*

$$\ell(nh) \simeq \ell(n) + \ell_1(h) \quad \text{on } N \times H \quad (n \in N,\ h \in H)\,.$$

For the proof, we need three lemmas.

Lemma 4.3. ([32, Th. 1.1.21] or [1, Th. 4.3(a)]) *Each (in particular, each symmetric) word length function dominates all length functions. As a corollary, all word length functions are equivalent.*

Lemma 4.4. *Let $\pi : G \to G_1$ be a continuous homomorphism of compactly generated locally compact groups and let ℓ and ℓ_1 be word length functions on G and G_1, resp.*

(A) *Then $\ell_1(\pi(g)) \lesssim \ell(g)$ on G.*

(B) *If, in addition, $\sigma\colon G_1 \to G$ is a continuous homomorphism such that $\pi\sigma = 1$, then $\ell_1(g_1) \simeq \ell(\sigma(g_1))$ on G_1.*

Proof. By Lemma 4.3, we can choose relatively compact generating sets U and U_1 determining ℓ and ℓ_1, resp., at our request. If we put $\pi(U) \subset U_1$, then $\ell_1(\pi(g)) \le \ell(g)$ for each $g \in G$, which proves part (A). If we put $\sigma(U_1) \subset U$, then $\ell(\sigma(g_1)) \le \ell_1(g_1)$ for each $g_1 \in G_1$. Thus part (B) follows part (A). ■

Lemma 4.5. (cf. [9, Lem. 4.3]) *Let G be a group and N a normal subgroup. Suppose that there are symmetric length functions ℓ and ℓ_1 on G and G/N, resp., and a subset X in G such that*

$$\ell_1(\pi(x)) \simeq \ell(x) \quad \text{on } X \quad \text{and} \qquad \ell_1(\pi(g)) \lesssim \ell(g) \quad \text{on } G\,,$$

where $\pi\colon G \to G/N$ is the quotient homomorphism. Then

$$\ell(nx) \simeq \ell(n) + \ell_1(\pi(x)) \quad \text{on } N \times X \quad (n \in N,\ x \in X)\,.$$

Proof. Since $\ell(x) \lesssim \ell_1(\pi(x))$ on X, we have

$$\ell(nx) \le \ell(n) + \ell(x) \lesssim \ell(n) + \ell_1(\pi(x))\,.$$

On the other hand, $\ell(n) \le \ell(nx) + \ell(x^{-1}) = \ell(nx) + \ell(x)$ and $\ell_1(\pi(x)) = \ell_1(\pi(nx)) \lesssim \ell(nx)$. Therefore,

$$\begin{aligned}\ell(n) + \ell_1(\pi(x)) &\le \ell(nx) + \ell(x) + \ell_1(\pi(x))\\ &\lesssim \ell(nx) + 2\ell_1(\pi(x)) \lesssim 3\ell(nx) \simeq \ell(nx)\,.\end{aligned}$$ ■

Proof of Proposition 4.2. Consider a homomorphism $\sigma : H \to G$ splitting the quotient homomorphism $\pi\colon G \to H$. Lemma 4.4 implies that $\ell_1(h) \simeq \ell(\sigma(h))$ on H. Since $\ell_1(\pi(g))$ is a length function on G, by virtue of Lemma 4.3, we have $\ell_1(\pi(g)) \lesssim \ell(g)$ on G. Evidently, we can assume that ℓ and ℓ_1 are symmetric and, finally, apply Lemma 4.5 with $X = \sigma(H)$. ■

Further, we need asymptotic behavior of the restriction of a word length function on the exponential radical.

Proposition 4.6. *Let G be a connected linear complex Lie group, E the exponential radical of G, and $\mathfrak{e}$ the Lie algebra of E. If $\|\cdot\|$ is a norm on $\mathfrak{e}$ and ℓ is a word length function on G, then*

$$\ell(\exp(\eta)) \simeq \log(1+\|\eta\|) \quad \text{on } \mathfrak{e}\,.$$

For the proof, we need the following lemma.

Lemma 4.7. *Let G be a simply connected nilpotent (real or complex) Lie group with the Lie algebra $\mathfrak{g}$. If $\|\cdot\|$ is a norm on $\mathfrak{g}$ and ℓ is a word length function on G then*

$$\log(1+\|\xi\|) \simeq \log(1+\ell(\exp\xi)) \qquad (\xi \in \mathfrak{g})\,.$$

Proof. It is sufficient to show that $\ell(\exp\xi) \lesssim \|\xi\| \lesssim \ell(\exp\xi)^k$ for some constant $k \ge 1$. But this estimate is well known and follows, e.g., from [11, Lem. II.1] or [18, (4.2)]. ∎

Recall that a closed subgroup H with word length function ℓ_0 is said to be *strictly exponentially distorted* in a locally compact group G with word length function ℓ if

$$\log(1+\ell_0(h)) \simeq \ell(h) \quad \text{on } H\,. \tag{7}$$

Proof of Proposition 4.6. Let ℓ_0 be a word length function on E. Since the exponential radical is strictly exponentially distorted [8, Th. 6.5], we obtain

$$\log(1+\ell_0(g)) \simeq \ell(g) \quad \text{on } E \quad (g \in E)\,.$$

At the same time, by Lemma 4.7,

$$\log(1+\ell_0(\exp\eta)) \simeq \log(1+\|\eta\|) \qquad (\eta \in \mathfrak{e})\,.$$ ∎

Let E, B and L be as in Theorem 3.14. Note that $B \to B/E$ (and $G \to G/E$) can be non-split; so Proposition 4.2 is not applicable in the direct way. To find a decomposition of a word length function on G in the general case we use the following trick from [8]. Consider a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{b}$; then, in particular, $\mathfrak{h}$ is nilpotent and $\mathfrak{h}+\mathfrak{r}_\infty = \mathfrak{b}$ [5, Ch. 7, § 2, N. 1, Cor. 3]. Note that $\mathfrak{h}+\mathfrak{e}=\mathfrak{b}$ and choose a complementary subspace $\mathfrak{v}$ of $\mathfrak{h}\cap\mathfrak{e}$ in $\mathfrak{e}$.

The following proposition is a variant of manifold splitting. It can be proved as in [16, Lem. 14.3.6] but the succeeding proof uses the Product Formula.

Proposition 4.8. *The map $\tau : \mathfrak{e}\times\mathfrak{v} \to B : (\eta,\xi) \mapsto \exp(\eta)\exp(\xi)$ is a biholomorphic equivalence of complex manifolds.*

Proof. First, let us prove that τ is surjective. Since B is simply connected solvable, the subgroup $H := \langle \exp\mathfrak{h}\rangle$ is simply connected and closed [16, Prop. 11.2.15]. Further, E is normal in B; so the subgroup EH is a subgroup in B. The equality $\mathfrak{h}+\mathfrak{e}=\mathfrak{b}$ implies that EH is dense in B. Applying [16, Prop. 11.2.15] again, we get that EH is closed. Therefore $B = EH$.

It follows from Proposition 3.13 that E is simply connected nilpotent. The subgroup H is also simply connected nilpotent. Therefore, for any element of B, there is a decomposition $\exp(\eta)\exp(\zeta)$, where $\eta \in \mathfrak{e}$ and $\zeta \in \mathfrak{h}$. Write $\zeta = \nu + \xi$, where $\nu \in \mathfrak{h} \cap \mathfrak{e}$ and $\xi \in \mathfrak{v}$. Then, by the Product Formula [16, Pr. 9.2.14(1)],

$$\exp(\zeta) = \lim_{n\to\infty} (\exp(\nu/n)\exp(\xi/n))^n .$$

The claim is that for each $n \in \mathbb{N}$ there is $g \in E$ such that

$$(\exp(\nu/n)\exp(\xi/n))^n = g\exp(\xi) .$$

We proceed by induction. If $n = 1$, then the claim is obvious. Suppose that it is true for $n-1$. Set $\nu' := (n-1)\nu/n$ and $\xi' := (n-1)\xi/n$ and write

$$\left(\exp\frac{\nu'}{n-1}\exp\frac{\xi'}{n-1}\right)^{n-1} = g'\exp(\xi')$$

for some $g' \in E$. Then

$$\begin{aligned}(\exp(\nu/n)\exp(\xi/n))^n &= (\exp(\nu/n)\exp(\xi/n))^{n-1}\exp(\nu/n)\exp(\xi/n)\\ &= (\exp(\nu'/(n-1))\exp(\xi'/(n-1)))^{n-1}\exp(\nu/n)\exp(\xi/n)\\ &= g'\exp(\xi')\exp(\nu/n)\exp(\xi/n) = g\exp(\xi) ,\end{aligned}$$

where $g := g'\exp(\xi')\exp(\nu/n)\exp(-\xi')$ is in E. The claim is proved.

Thus $\exp(\zeta) = \lim_{n\to\infty} g_n \exp(\xi)$, where $g_n \in E$. Since E is closed, $\exp(\zeta)$ is in the range of τ, so is $\exp(\eta)\exp(\zeta)$. Hence τ is surjective.

Secondly, prove that τ is injective. Suppose that $\tau(\eta,\xi) = \tau(\eta',\xi')$ for some $\eta,\eta' \in \mathfrak{e}$ and $\xi,\xi' \in \mathfrak{v}$. Let $p : \mathfrak{b} \to \mathfrak{b}/\mathfrak{e}$ and $\pi : B \to B/E$ be the quotient maps. The exponential map is natural; so

$$\exp_{B/E}(p(\xi)) = \pi\tau(\eta,\xi) = \pi\tau(\eta',\xi') = \exp_{B/E}(p(\xi')) .$$

Since $\mathfrak{b}/\mathfrak{e}$ is nilpotent, $\exp_{B/E}$ is bijective. Therefore $p(\xi) = p(\xi')$. Finally, note that p is injective on $\mathfrak{v}$, hence $\xi = \xi'$. Thus $\exp(\eta) = \exp(\eta')$. Since the exponential map is natural and $E \to B$ is injective, $\exp_E(\eta) = \exp_E(\eta')$. But $\mathfrak{e}$ is nilpotent, whence $\exp_E$ is bijective; so $\eta = \eta'$.

Finally, $\mathfrak{e} \times \mathfrak{v}$ and B are both $\mathbb{C}^m$ for some m. So we can treat τ as a holomorphic injection $\mathbb{C}^m \to \mathbb{C}^m$. Then, by [19, Th. 8.5], τ is a biholomorphic equivalence onto the range of τ, which coincides with $\mathbb{C}^m$. This concludes the proof. ∎

Lemma 4.9. *Let ℓ and ℓ_1 be word length functions on G and G/E, resp. Then $\ell(g) \simeq \ell_1(\pi(g))$ on $\exp\mathfrak{v}$.*

Proof. First, note that $\ell_1(\pi(g)) \lesssim \ell(g)$ on G by Lemma 4.4(A).

To prove $\ell(g) \lesssim \ell_1(\pi(g))$ on $\exp\mathfrak{v}$ we follow [8, Lem. 5.2] with small modifications. Let π denote the projection $G \to G/E$. Fix a compact symmetric generating subset S of $H := \exp\mathfrak{h}$ and denote by ℓ_2 the corresponding word length function on H. We can assume that ℓ_1 is the word length function on G/E corresponding to $\pi(S)$. If $g \in H$, then write $\pi(g)$ as an element of minimal length with respect to $\pi(S)$, i.e., $\pi(g) = \pi(s_1)\cdots\pi(s_m)$, where $s_1,\ldots,s_m \in S$. Set $h = s_1\cdots s_m$. We can assume that ℓ is a word length function corresponding to a compact generating set containing S. Then $\ell(h) \le m = \ell_1(\pi(g))$. As $h^{-1}g$ belongs to the exponential radical E, which is strictly exponentially distorted by [28, Th. 1.1(3)], we get that $\ell(h^{-1}g) \lesssim \log(1+\ell_0(h^{-1}g))$ for $g \in H$, where ℓ_0 is a word length function on E. (Here we consider h as a function of g) Therefore,

$$\ell(g) \le \ell(h) + \ell(h^{-1}g) \lesssim \ell_1(\pi(g)) + \log(1+\ell_0(h^{-1}g)) \quad (g \in H)\,. \tag{8}$$

Write $g = \exp\xi$ and $h = \exp\eta$ for $\xi, \eta \in \mathfrak{h}$. Fix a norm $\|\cdot\|$ on $\mathfrak{b}$, the Lie algebra of B. Since E is nilpotent, we can use Lemma 4.7 for the restriction $\|\cdot\|$ on $\mathfrak{e}$ and ℓ_0. Hence,

$$\log(1+\ell_0(\exp(-\eta)\exp\xi)) \simeq \log(1+\|\exp^{-1}(\exp(-\eta)\exp\xi)\|)\,.$$

(Here we consider η as a function of ξ.) Since H is nilpotent, its group law is given by a polynomial and we have an upper bound

$$\|\exp^{-1}(\exp(-\eta)\exp\xi)\| \le A(1+\|\xi\|)^k(1+\|\eta\|)^k$$

for some constants $A, k \ge 1$. This implies

$$\log(1+\ell_0(\exp(-\eta)\exp\xi))) \lesssim \log(1+\|\eta\|) + \log(1+\|\xi\|)\,.$$

Further,

$$\ell_2(h) \le m = \ell_1(\pi(g)) \le \ell(g) \le \ell_2(g)$$

(since ℓ_2 is a word length function on H, it dominates each length function). As H is nilpotent, we can apply Lemma 4.7 for the restriction $\|\cdot\|$ on $\mathfrak{h}$ and ℓ_2 and get

$$\log(1+\|\eta\|) \lesssim \log(1+\ell_2(h)) \lesssim \log(1+\ell_2(g)) \lesssim \log(1+\|\xi\|)\,.$$

Therefore we have from (8) that on H

$$\ell(g) \lesssim \ell_1(\pi(g)) + \log(1+\|\xi\|)\,.$$

Now suppose that $g = \exp\xi$ for some $\xi \in \mathfrak{v}$. If $\|\cdot\|'$ is a norm on $\mathfrak{b}/\mathfrak{e}$ and $p\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{e}$ is the quotient map, then $\|p(\xi)\|' \simeq \|\xi\|$ on $\mathfrak{v}$. Since $\exp(p(\xi)) = \pi(\exp\xi)$, we have from the application of Lemma 4.7 for $\|\cdot\|'$ and ℓ_1 that

$$\log(1+\|\xi\|) \simeq \log(1+\|p(\xi)\|') \lesssim \log(1+\ell_1(\pi(g))) \lesssim \ell_1(\pi(g))\,.$$

Thus $\ell(g) \lesssim \ell_1(\pi(g))$ on $\exp\mathfrak{v}$. ∎

We are now in a position to prove the decomposition result.

Proof of Theorem 4.1. It follows from Proposition 4.8 that $\mathfrak{e} \times \mathfrak{v} \to B$ and $\mathfrak{e} \times \mathfrak{v} \times L \to G$ are biholomorphic equivalences.

We consider now the quotient map $\sigma\colon G \to G/E$ and the length function $\widetilde{\ell}(h,l) := \ell_1(h) + \ell_2(l)$ on $B/E \times L$. According to Theorem 3.14, we can identify G/E with $B/E \times L$ and π with $\sigma \times 1\colon B \rtimes L \to B/E \times L$. Proposition 4.6 implies that it sufficient to show that

$$\ell(\exp(\eta)\exp(\xi)l) \simeq \ell(\exp(\eta)) + \widetilde{\ell}(\sigma(\exp(\xi)l)) \quad \text{on } \mathfrak{e} \times \mathfrak{v} \times L\,. \tag{9}$$

By Lemma 4.9, we have $\ell(\exp(\xi)) \simeq \ell_1(\pi(\exp(\xi)))$ on $\mathfrak{v}$. Besides, Proposition 4.2 yields that $\ell(bl) \simeq \ell(b) + \ell_2(l)$ on $B \times L$. Combining these relations, we get $\ell(\exp(\xi)l) \simeq \widetilde{\ell}(\sigma(\exp(\xi)l))$ on $\mathfrak{v} \times L$. On the other hand, each length function is dominated by a word length function, so $\widetilde{\ell}(\sigma(g)) \lesssim \ell(g)$ on G. Thus both conditions of Lemma 4.5 are satisfied for ℓ and $\widetilde{\ell}$ with $X = (\exp\mathfrak{v})L$; so application of this lemma completes the proof of (9). ∎

5. Holomorphic functions of exponential type

Recall that a *submultiplicative weight* on a locally compact group G is a non-negative locally bounded function $\omega\colon G\to\mathbb{R}$ such that

$$\omega(gh)\le\omega(g)\omega(h)\qquad(g,h\in G)\,.$$

Akbarov proposed the term 'semicharacter' in [1] but we follow [2].

A holomorphic function f on a complex Lie group G is said to be of *exponential type*, if there is a submultiplicative weight ω satisfying $|f(g)|\le\omega(g)$ for all $g\in G$. The linear space (in fact, a locally convex algebra and even a topological Hopf algebra) of all holomorphic function f of exponential type on G is denoted by $\mathcal{O}_{exp}(G)$ [1, Sect. 5.3.1].

Consider the Fréchet space $\mathcal{O}(G)$ of holomorphic functions on a complex Lie group G and its strong dual space $\mathscr{A}(G):=\mathcal{O}(G)'$ endowed with the convolution multiplication. In fact, $\mathscr{A}(G)$ is a $\widehat{\otimes}$-algebra with respect to the convolution, i.e., it is a complete Hausdorff locally convex topological algebra with jointly continuous multiplication; it is called the *algebra of analytic functionals* on G [22].

If A is a unital Banach algebra and $\pi\colon G\to\mathrm{GL}(A)$ is a holomorphic homomorphism, then π uniquely extends to a unital continuous homomorphism $\bar\pi\colon\mathscr{A}(G)\to A$ with

$$\langle x,\bar\pi(a')\rangle=\langle a',\pi_x\rangle\qquad(a'\in\mathscr{A}(G),\,x\in A'),\tag{10}$$

where $\pi_x\in\mathcal{O}(G)$ is defined by

$$\pi_x(g):=\langle x,\pi(g)\rangle\qquad(g\in G).\tag{11}$$

On the other hand, for a unital continuous homomorphism $\bar\pi\colon\mathscr{A}(G)\to A$,

$$\pi(g):=\bar\pi(\delta_g)\qquad(g\in G)\tag{12}$$

defines a holomorphic homomorphism $\pi\colon G\to\mathrm{GL}(A)$, satisfying (10) [22].

Proposition 5.1. *Let f be a function on a complex Lie group G. The following conditions are equivalent:*

(1) $f\in\mathcal{O}_{exp}(G)$.

(2) *There exist a unital Banach algebra A, a holomorphic homomorphism $\pi\colon G\to\mathrm{GL}(A)$, and $x\in A'$ such that*

$$f(g):=\langle x,\pi(g)\rangle\qquad(g\in G).\tag{13}$$

(3) *f is a coefficient of a holomorphic representation in some Banach space.*

Recall that an *Arens-Michael envelope* of a $\widehat{\otimes}$-algebra A is a pair $(\widehat{A},\iota_A)$, where $\widehat{A}$ is an Arens-Michael algebra and ι_A is a continuous homomorphism $A\to\widehat{A}$ such that for any Arens-Michael algebra B and for each continuous homomorphism $\phi\colon A\to B$ there exists a unique continuous homomorphism $\widehat{\phi}:\widehat{A}\to B$ with $\phi=\widehat{\phi}\iota_A$ [13, Chap. 5].

Proof. (1)⇒(2). Suppose that $f \in \mathcal{O}_{exp}(G)$. Put $\mathscr{A}_{exp}(G) := \mathcal{O}_{exp}(G)'$ and consider f as a functional on $\mathscr{A}_{exp}(G)$. The natural map $\nu : \mathscr{A}(G) \to \mathscr{A}_{exp}(G)$ is an Arens-Michael envelope. (This result is proved in [1, Th. 5.2] for Stein groups but the argument in the general case is similar.) Therefore there exists a continuous submultiplicative prenorm $\|\cdot\|$ on $\mathscr{A}(G)$ and $C > 0$ such that $|\langle f, \nu(a')\rangle| \le C\,\|a'\|$ for all $a' \in \mathscr{A}(G)$. Denote by A the Banach algebra that is the completion of $\mathscr{A}(G)$ with respect to $\|\cdot\|$ and by $\bar\pi$ the corresponding continuous homomorphism $\mathscr{A}(G) \to A$. Then the functional f factors on some $x \in A'$ such that $\langle f, \nu(a')\rangle = \langle x, \bar\pi(a')\rangle$ for all a'. Consider a holomorphic homomorphism $\pi\colon G \to \mathrm{GL}(A)$ defined by (12). Since $\langle f, \delta_g\rangle = f(g)$, we get $f(g) = \langle x, \bar\pi(\delta_g)\rangle = \langle x, \pi(g)\rangle$.

(2)⇒(1). Suppose that $\pi\colon G \to \mathrm{GL}(A)$ is a holomorphic homomorphism and $x \in A'$. Any function f of the form (13), being a composition of a holomorphic and linear map, is holomorphic. Obviously, the function $g \mapsto \|\pi(g)\|$, where $\|\cdot\|$ is the norm on A, is submultiplicative and continuous; therefore it is a submultiplicative weight; moreover,

$$g \mapsto \max\{\|x\|, 1\}\,\|\pi(g)\|$$

is a submultiplicative weight. Since $|f(g)| \le \|x\|\,\|\pi(g)\|$ for all g, we have that f is of exponential type.

(2)⇔(3). It is sufficient to note that a function of the form (13) is a coefficient of the representation that is a composition of π and the regular representation A on itself. ∎

Recall that $\mathcal{O}_{exp}(G)$ is endowed with an inductive locally convex topology via identifying

$$\mathcal{O}_{exp}(G) = \varinjlim_{\omega} \mathcal{O}_\omega(G)\,,$$

where ω runs all submultiplicative weights on G and $\mathcal{O}_\omega(G)$ is a Banach space defined by

$$\mathcal{O}_\omega(G) := \Big\{ f \in \mathcal{O}(G) \colon |f|_\omega := \sup_{g\in G} \omega(g)^{-1}|f(g)| < \infty \Big\}. \tag{14}$$

Note that this definition can be applied also for any complex manifold and any locally bounded function with values in $[1, +\infty)$.

It is proved in [1, Th. 4.5] that $\mathcal{O}_{exp}(G)$ is a projective stereotype algebra (at least, for a compactly generated Stein group G). But $\mathcal{O}_{exp}(G)$ is also an algebra in more traditional category of functional analysis as it is seen from the following lemma.

Lemma 5.2. *If G is a complex Lie group, then $\mathcal{O}_{exp}(G)$ is a $\widehat{\otimes}$-algebra with respect to the point-wise multiplication.*

Proof. Since a product of two submultiplicative weights is a submultiplicative weight, $\mathcal{O}_{exp}(G)$ is an algebra. Since the strong dual of Fréchet space is complete and $\mathcal{O}_{exp}(G)$ is the strong dual of the Fréchet space $\mathscr{A}_{exp}(G)$ [2, Pr. 2.12], we get that $\mathcal{O}_{exp}(G)$ is complete.

It remains to show that the multiplication is jointly continuous. Note that $\mathcal{O}_{exp}(G)$ is endowed with the inductive topology, i.e., the family of all absolutely convex subsets U in $\mathcal{O}_{exp}(G)$ such that $U \cap \mathcal{O}_\omega(G)$ is open for each submultiplicative weight ω on G is a base of neighbourhoods of 0.

Let U be an open subset in $\mathcal{O}_{exp}(G)$. Then for each submultiplicative weight ω there is $C_\omega \ge 0$ such that $U_\omega := \{|f(g)| < C_\omega \omega(g)\,\forall\, g \in G\}$ is contained in $U \cap \mathcal{O}_\omega(G)$. It is easy to see that $\bigcup_\omega U_\omega$ is an open subset in $\mathcal{O}_{exp}(G)$ and contained in U. For each submultiplicative weight ω, the function $g \mapsto \omega(g)^{1/2}$ is also a submultiplicative weight. Set $V_\omega := \{|f(g)| < C_\omega^{1/2} \omega(g)^{1/2}\,\forall\, g \in G\}$; then $\bigcup_\omega V_\omega$ is open in $\mathcal{O}_{exp}(G)$. It is obvious that $f_1, f_2 \in V_\omega$ implies $f_1 f_2 \in U_\omega$. Thus the multiplication is jointly continuous. ∎

It is not hard to see that any holomorphic homomorphism $\phi\colon G \to H$ of complex Lie group induces a $\widehat{\otimes}$-algebra homomorphism $\widetilde{\phi}:\mathcal{O}_{exp}(H) \to \mathcal{O}_{exp}(G)$ given by $[\widetilde{\phi}(f)](g) := f(\phi(g))$. Since the dual map $\widetilde{\phi}' : \mathscr{A}_{exp}(G) \to \mathscr{A}_{exp}(H)$ coincides with the image of the homomorphism $\mathscr{A}(G) \to \mathscr{A}(H)$ under the Arens-Michael envelope functor, $\widetilde{\phi}'$ is also a $\widehat{\otimes}$-algebra homomorphism.

Now we can prove that it suffices to study holomorphic functions of exponential type only on linear groups.

Theorem 5.3. *Let G be a connected complex Lie group and let $\sigma : G \to G/\operatorname{Lin}_{\mathbb{C}}(G)$ be the quotient homomorphism. Then*

(A) $\widetilde{\sigma}'\colon \mathscr{A}_{exp}(G) \to \mathscr{A}_{exp}(G/\operatorname{Lin}_{\mathbb{C}}(G))$ *is a $\widehat{\otimes}$-algebra isomorphism.*

(B) $\widetilde{\sigma}\colon \mathcal{O}_{exp}(G/\operatorname{Lin}_{\mathbb{C}}(G)) \to \mathcal{O}_{exp}(G)$ *is a $\widehat{\otimes}$-algebra isomorphism.*

(C) *Each functions in $\mathcal{O}_{exp}(G)$ is constant on cosets of* $\operatorname{Lin}_{\mathbb{C}}(G)$.

Proof. (A) It follows from Theorem 2.2 that each holomorphic homomorphism $G \to \mathrm{GL}(A)$ factors on σ. On the other hand, (10) and (12) give a bijection between the set of holomorphic homomorphisms from G to $\mathrm{GL}(A)$ and the set of unital continuous homomorphisms from $\mathscr{A}(G)$ to A. The same, of course, is true for $G/\operatorname{Lin}_{\mathbb{C}}(G)$. Therefore each unital continuous homomorphism $\mathscr{A}(G) \to A$, where A is a unital Banach algebra, factors on $\mathscr{A}(G) \to \mathscr{A}(G/\operatorname{Lin}_{\mathbb{C}}(G))$. Since $\mathscr{A}(G) \to \mathscr{A}_{exp}(G)$ and $\mathscr{A}(G/\operatorname{Lin}_{\mathbb{C}}(G)) \to \mathscr{A}_{exp}(G/\operatorname{Lin}_{\mathbb{C}}(G))$ are Arens-Michael envelopes [1, Th. 6.2], we see $\widetilde{\sigma}'$ is a topological isomorphism.

(B) It is sufficient to note that the strong dual of $\widetilde{\sigma}'$ coincides with σ.
Part (C) follows (B) immediately. ∎

Remark 5.4. Nevertheless, even for simplest examples there are functions of exponential type that are not coefficients of finite-dimensional holomorphic representations. For example, if $G = \mathbb{C}$, then any finite-dimensional holomorphic representation has the form $z \mapsto \exp(zT)$ for some complex matrix T. It is nor hard to see from the Jordan decomposition of T that all matrix coefficients belongs to the algebra generated by z and $\{e^{\lambda z} : \lambda \in \mathbb{C}\}$. This algebra is smaller than $\mathcal{O}_{exp}(\mathbb{C})$.

For a complex manifold M and a locally bounded function $\upsilon\colon M \to [1, +\infty)$, denote by V_υ the closure of

$$\text{a.c.}\{\upsilon(x)^{-1}\delta_x : x \in M\}$$

in $\mathscr{A}(M) := \mathcal{O}(M)'$, where a.c. denotes the absolutely convex hull. Let $\|\cdot\|_\upsilon$ be the Minkowski functional of V_υ and $\mathscr{A}_\upsilon(M)$ the completion of $\mathscr{A}(M)$ with respect to $\|\cdot\|_\upsilon$. Also, denote by $\mathscr{A}_{\upsilon^\infty}(M)$ the completion of $\mathscr{A}(M)$ with respect to the sequence of prenorms $(\|\cdot\|_{\upsilon^n};\, n \in \mathbb{N})$, where $\upsilon^n(x) := \upsilon(x)^n$.

Also, we put $\mathcal{O}_{\upsilon^\infty}(M) := \bigcup_{n\in\mathbb{N}} \mathcal{O}_{\upsilon^n}(M)$ and consider $\mathcal{O}_{\upsilon^\infty}(M)$ with the inductive limit topology. We denote by $E \mathbin{\widehat{\otimes}} F$ the complete projective tensor product of locally convex spaces E and F.

Proposition 5.5. *Let M_1 and M_2 be complex manifolds and let $\upsilon_1\colon M_1 \to [1,+\infty)$ and $\upsilon_2\colon M_2 \to [1,+\infty)$ be locally bounded functions. Set $\upsilon(x_1,x_2) := \upsilon_1(x_1)\upsilon_2(x_2)$.*

(A) *Then the natural map $\rho : \mathscr{A}(M_1) \otimes \mathscr{A}(M_2) \to \mathscr{A}(M_1 \times M_2)$ induces the topological isomorphism of Banach spaces*

$$\mathscr{A}_{\upsilon_1}(M_1) \mathbin{\widehat{\otimes}} \mathscr{A}_{\upsilon_2}(M_2) \cong \mathscr{A}_{\upsilon}(M_1 \times M_2).$$

and the topological isomorphism of Fréchet spaces

$$\mathscr{A}_{\upsilon_1^\infty}(M_1) \mathbin{\widehat{\otimes}} \mathscr{A}_{\upsilon_2^\infty}(M_2) \cong \mathscr{A}_{\upsilon^\infty}(M_1 \times M_2).$$

(B) *If, in addition, each $\mathscr{A}_{\upsilon_i^\infty}(M_i)$ is nuclear ($i = 1,2$), then the natural map $\mathcal{O}(M_1) \otimes \mathcal{O}(M_2) \to \mathcal{O}(M_1 \times M_2)$ induces the topological isomorphism of locally convex spaces*

$$\mathcal{O}_{\upsilon_1^\infty}(M_1) \mathbin{\widehat{\otimes}} \mathcal{O}_{\upsilon_2^\infty}(M_2) \cong \mathcal{O}_{\upsilon^\infty}(M_1 \times M_2).$$

Proof. (A) Since $\|\cdot\|_{\upsilon_i}$ is the Minkowski functional of V_{υ_i} ($i = 1,2$), it follows from [31, III.6.3] that the projective tensor prenorm $\|\cdot\|_{\upsilon_1} \mathbin{\widehat{\otimes}} \|\cdot\|_{\upsilon_2}$ on $\mathscr{A}(M_1)\otimes\mathscr{A}(M_2)$ is the Minkowski functional of

$$S := \text{a.c.}\{\mu_1 \otimes \mu_2 \in \mathscr{A}(M_1) \otimes \mathscr{A}(M_2) : \mu_1 \in V_{\upsilon_1},\ \mu_2 \in V_{\upsilon_2})\}.$$

Since $\|\cdot\|_{\upsilon_1}$ and $\|\cdot\|_{\upsilon_2}$ are continuous on $\mathscr{A}(M_1)$ and $\mathscr{A}(M_2)$, resp., $\|\cdot\|_{\upsilon_1} \mathbin{\widehat{\otimes}} \|\cdot\|_{\upsilon_2}$ is extended to a continuous prenorm on $\mathscr{A}(M_1) \mathbin{\widehat{\otimes}} \mathscr{A}(M_2)$, which coincides with the Minkowski functional of $\overline{\rho(S)}$ via the topological isomorphism $\mathscr{A}(M_1) \mathbin{\widehat{\otimes}} \mathscr{A}(M_2) \to \mathscr{A}(M_1 \times M_2)$. It is not hard to see that $\overline{\rho(S)}$ equals to V_υ, the closure of

$$\text{a.c.}\{\upsilon(x_1,x_2)^{-1}\delta_{(x_1,x_2)} : (x_1,x_2) \in M_1 \times M_2\}$$

in $\mathscr{A}(M_1 \times M_2)$. Thus $\|\cdot\|_{\upsilon_1} \mathbin{\widehat{\otimes}} \|\cdot\|_{\upsilon_2}$ and $\|\cdot\|_{\upsilon}$ are identical.
Further, consider the projective system

$$(\mathscr{A}_{\upsilon_1^n}(M_1) \mathbin{\widehat{\otimes}} \mathscr{A}_{\upsilon_2^m}(M_2) \colon (n,m) \in \mathbb{N}^2)$$

(with naturally defined connecting maps) in the category of Fréchet spaces. Since the diagonal is cofinal in $\mathbb{N}^2$, we have

$$\varprojlim_{(n,m)\in\mathbb{N}^2} (\mathscr{A}_{\upsilon_1^n}(M_1) \mathbin{\widehat{\otimes}} \mathscr{A}_{\upsilon_1^m}(M_2)) \cong \varprojlim_{n\in\mathbb{N}} (\mathscr{A}_{\upsilon_1^n}(M_1) \mathbin{\widehat{\otimes}} \mathscr{A}_{\upsilon_2^n}(M_2)).$$

Writing $\varprojlim_{(n,m)}$ as an iterated projective limit and using the fact that projective tensor products of Fréchet spaces commute with projective limits, we get

$$\Big(\varprojlim_{n\in\mathbb{N}} \mathscr{A}_{\upsilon_1^n}(M_1)\Big) \mathbin{\widehat{\otimes}} \Big(\varprojlim_{m\in\mathbb{N}} \mathscr{A}_{\upsilon_1^m}(M_2)\Big) \cong \varprojlim_{(n,m)\in\mathbb{N}^2} (\mathscr{A}_{\upsilon_1^n}(M_1) \mathbin{\widehat{\otimes}} \mathscr{A}_{\upsilon_1^m}(M_2)).$$

Thus
$$\mathscr{A}_{v_1^\infty}(M_1)\mathbin{\widehat{\otimes}}\mathscr{A}_{v_2^\infty}(M_2)\cong(\varprojlim_{n\in\mathbb{N}}\mathscr{A}_{v_1^n}(M_1))\mathbin{\widehat{\otimes}}(\varprojlim_{m\in\mathbb{N}}\mathscr{A}_{v_1^m}(M_2))$$
$$\cong\varprojlim_{n\in\mathbb{N}}(\mathscr{A}_{v_1^n}(M_1)\mathbin{\widehat{\otimes}}\mathscr{A}_{v_2^n}(M_2))\cong\varprojlim_{n\in\mathbb{N}}\mathscr{A}_{v^n}(M_1\times M_2)\cong\mathscr{A}_{v^\infty}(M_1\times M_2)\,.$$

(B) Now suppose that each $\mathscr{A}_{v_i^\infty}(M_i)$ is nuclear for $i=1$ and 2. Then the space $\mathscr{A}_{v_1^\infty}(M_1)\mathbin{\widehat{\otimes}}\mathscr{A}_{v_2^\infty}(M_2)$ is also nuclear. Consequently these spaces are reflexive and we have from [2, Lem. 2.11] that
$$\mathscr{A}_{v_i^\infty}(M_i)'\cong\mathcal{O}_{v_i^\infty}(M_i)\quad\text{and}\quad\mathscr{A}_{v^\infty}(M_1\times M_2)'\cong\mathcal{O}_{v^\infty}(M_1\times M_2).$$
Recall that for any nuclear Fréchet spaces E and F, the natural linear map $E'\otimes F'\to(E\mathbin{\widehat{\otimes}}F)'$ induces the topological isomorphism $E'\mathbin{\widehat{\otimes}}F'\cong(E\mathbin{\widehat{\otimes}}F)'$. Thus
$$\mathcal{O}_{v^\infty}(M_1\times M_2)\cong\mathscr{A}_{v^\infty}(M_1\times M_2)'\cong(\mathscr{A}_{v_1^\infty}(M_1)\mathbin{\widehat{\otimes}}\mathscr{A}_{v_2^\infty}(M_2))'$$
$$\cong\mathscr{A}_{v_1^\infty}(M_1)'\mathbin{\widehat{\otimes}}\mathscr{A}_{v_2^\infty}(M_2)'\cong\mathcal{O}_{v_1^\infty}(M_1)\mathbin{\widehat{\otimes}}\mathcal{O}_{v_2^\infty}(M_2)\,.$$
■

Recall that each submultiplicative weight has the form $\omega(g)=e^{\ell(g)}$, where ℓ is a length function. This correspondence allows to apply results of Section 4. In decomposition (6), three types of length function appear:

- $\log(1+\ell)$, where ℓ is a word length function on a simply connected nilpotent Lie group;
- a word length function on a simply connected nilpotent Lie group itself;
- a word length function on a connected linearly complex reductive Lie group.

We look at these cases separately.

If G is an affine algebraic complex group, then we consider the algebra $\mathcal{R}(G)$ of regular (in the sense of algebraic geometry) functions as a $\widehat{\otimes}$-algebra with respect to the strongest locally convex topology. Note that a simply connected nilpotent complex Lie group G is algebraic and $\mathcal{R}(G)$ is just the algebra of polynomials.

Lemma 5.6. *Let G be an affine simply connected nilpotent complex Lie group and $\omega(g):=1+\ell(g)$, where ℓ is a word length function on G. Then $\mathcal{O}_{\omega^\infty}(G)=\mathcal{R}(G)$ as locally convex algebras.*

Proof. If $\|\cdot\|$ is a norm on the Lie algebra $\mathfrak{g}$ of G, then, by Lemma 4.7, we have $\log(1+\ell(\exp\eta))\simeq\log(1+\|\eta\|)$ on $\mathfrak{g}$. So $f\in\mathcal{O}_{\omega^\infty}(G)$ iff it is bounded by a polynomial in norm. Hence $\mathcal{O}_{\omega^\infty}(G)=\mathcal{R}(G)$. Furthermore, the topology on $\mathcal{R}(G)$ coincides with the inductive topology of $\mathcal{O}_{\omega^\infty}(G)$. ■

The case of a word length function on a simply connected nilpotent complex Lie group is considered in [2]. The following result is [ibid. Th. 3.2].

Theorem 5.7. *Let G be a simply connected nilpotent complex Lie group with Lie algebra $\mathfrak{g}$, and let $(t_1,\dots,t_m)$ be the canonical coordinates of the first kind associated with an $\mathscr{F}$-basis in $\mathfrak{g}$, where $\mathscr{F}$ is the lower central series. Then*
$$\mathcal{O}_{exp}(G)=\left\{f\in\mathcal{O}(G):\begin{array}{l}\exists C>0,\ \exists r\in\mathbb{R}_+\ \text{such that}\\ |f(t_1,\dots,t_m)|\le Ce^{r\max_i|t_i|^{1/w_i}}\ \forall t_1,\dots,t_m\end{array}\right\}$$
and we have $\mathcal{O}_{exp}(G)\cong\varinjlim_{r\in\mathbb{R}_+}\mathcal{O}_{\eta^r}(G)$ as locally convex spaces, where $\eta(t_1,...,t_m):=e^{\max_i|t_i|^{1/w_i}}$, and the Banach space $\mathcal{O}_{\eta^r}(G)$ is defined as in (14).

To consider the linearly complex reductive case we need the following result, which is well known.

Theorem 5.8. *Let L be connected linearly complex reductive. Then any holomorphic homomorphism of L into a complex algebraic group H is polynomial.*

The proof is similar to [26, Th. 3.3.4]. The only step which is different is that although we cannot claim that $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ but nevertheless any reductive subalgebra in a complex Lie algebra is algebraically closed.

Note that any connected linearly complex reductive group L is an affine algebraic group. This can be obtain, e.g., by application of [20, Ths. 2.23, 5.10] from the fact that the algebra of real analytic representative functions on K, where L is a universal complexification of K, is finitely generated [7, Ch. VI, § VII].

Theorem 5.9. *Suppose that L is connected linearly complex reductive. Then $\mathcal{O}_{exp}(L)=\mathcal{R}(L)$ as a locally convex algebra.*

Proof. Since L is an affine algebraic, we have $\mathcal{R}(G)\subset\mathcal{O}_{exp}(G)$ [1, (5.31)].

Consider a compact subgroup K such that L is the universal complexification of K. Note that the map $\mathscr{E}(K)'\to\mathscr{A}(L)$, which is dual to $\mathcal{O}(L)\to\mathscr{E}(K)$, has dense range [21, Pr. 4]. Then it follows from Lemma 2.5 that the closure of the range of any homomorphism of $\mathscr{A}(L)$ to a Banach algebra is classically semisimple. In particular, if ω is a submultiplicative weight on L, then $\mathscr{A}_\omega(L)$ is finite-dimensional. By [2, Lem. 2.10], we get $\mathcal{O}_\omega(L)\cong\mathscr{A}_\omega(L)'$; hence $\mathcal{O}_\omega(L)$ is finite-dimensional. Thus $\mathcal{O}_{exp}(L)$ is an inductive limit of finite-dimensional spaces, hence the topology on $\mathcal{O}_{exp}(L)$ is the strongest locally convex topology.

By Proposition 5.1, any $f\in\mathcal{O}_{exp}(L)$ is a coefficient of a holomorphic homomorphism to the invertibles of a Banach algebra. As pointed out above, we can assume that this Banach algebra is classically semisimple; so f is a coefficient of some holomorphic finite-dimensional representation. Moreover, Theorem 5.8 implies that this representation is polynomial; therefore $f\in\mathcal{R}(L)$. So $\mathcal{O}_{exp}(L)\subset\mathcal{R}(L)$; thus $\mathcal{O}_{exp}(L)=\mathcal{R}(L)$ and the topologies coincide. ∎

Now we prove our main result.

Theorem 5.10. *Let G be a connected linear complex Lie group and E is the exponential radical of G. Fix a decomposition $G\cong B\rtimes L$, where B is simply connected nilpotent and L is linearly complex reductive. Then the map τ defined in Theorem 4.1 induces an isomorphism of $\widehat{\otimes}$-algebras*

$$\mathcal{R}(E)\mathbin{\widehat{\otimes}}\mathcal{O}_{exp}(B/E)\mathbin{\widehat{\otimes}}\mathcal{R}(L)\to\mathcal{O}_{exp}(G)\,,$$

where $\mathcal{O}_{exp}(B/E)$ is described in Theorem 5.7.

Proof. Since B/E and E are nilpotent and simply connected, the exponential maps on B/E and E are biholomorphic equivalences. So we can consider τ as a map from $E\times B/E\times L$ to G (up to the identification of $\mathfrak{v}$ with $\mathfrak{b}/\mathfrak{e}$).

Let ℓ, ℓ_0, ℓ_1, and ℓ_2 denote word length functions on G, E, B/E, and L, resp.

Define submultiplicative weights

$$\omega(g) := e^{\ell(g)}, \quad \omega_0(e) := 1 + \ell_0(e), \quad \omega_1(h) := e^{\ell_1(h)}, \quad \omega_2(l) := e^{\ell_2(l)}$$

on G, E, B/E, and L, resp. Note that $\mathcal{O}_{exp}(G)$ is topologically isomorphic to $\mathcal{O}_{\omega^\infty}(G)$ (see [1, Th. 4.3] or [2, Pr. 2.8]) . From the length function equivalence given in Theorem 4.1, we have that $\mathcal{O}_{\omega^\infty}(G)$ is topologically isomorphic to the space $\mathcal{O}_{\upsilon^\infty}(E \times B/E \times L)$, where $\upsilon(e,h,l) := \omega_0(e)\omega_1(h)\omega_2(l)$.

Lemma 5.6 implies that $\mathcal{O}_{\omega_0^\infty}(E) = \mathcal{R}(E)$ as locally convex spaces. Then $\mathscr{A}_{\omega_0^\infty}(E)$ is a space of formal power series, which is nuclear. Furthermore, being the Arens-Michael envelopes of $\mathscr{A}(B/E)$ and $\mathscr{A}(L)$, the Fréchet algebras $\mathscr{A}_{\omega_1^\infty}(B/E)$ and $\mathscr{A}_{\omega_2^\infty}(L)$ are also nuclear [1, Th. 5.10]. Applying Proposition 5.5, we have

$$\mathcal{O}_{\upsilon^\infty}(E \times B/E \times L) \cong \mathcal{O}_{\omega_0^\infty}(E) \mathbin{\widehat{\otimes}} \mathcal{O}_{\omega_1^\infty}(B/E) \mathbin{\widehat{\otimes}} \mathcal{O}_{\omega_2^\infty}(L).$$

Finally, Theorem 5.9 implies that $\mathcal{O}_{\omega_2^\infty}(L) = \mathcal{O}_{exp}(L) = \mathcal{R}(L)$. ■

Thus the combination of Theorems 5.3, 5.7, and 5.10 gives a complete description of the algebra $\mathcal{O}_{exp}(G)$ for an arbitrary connected complex Lie group G.

Now we turn to an application of these results to a question on the Arens-Michael envelope of $\mathcal{O}_{exp}(G)$, which initially motivated this research.

Corollary 5.11. *The map τ defined in Theorem 4.1 induces an homomorphism*

$$\theta \colon \mathcal{R}(E \times B/E \times L) \to \mathcal{O}_{exp}(G)$$

that has dense range.

Proof. Only density is left to prove. Theorem 5.7 implies that $\mathcal{R}(B/E)$ (the polynomials) is dense in $\mathcal{O}_{exp}(B/E)$. Therefore the image of $\mathcal{R}(E)\otimes\mathcal{R}(B/E)\otimes\mathcal{R}(L)$ under our homomorphism is dense in $\mathcal{O}_{exp}(G)$. ■

For a connected complex Lie group G, we consider the natural embeddings $j : \mathcal{O}_{exp}(G) \to \mathcal{O}(G)$ and $j_0 : \mathcal{O}_{exp}(G/\operatorname{Lin}_{\mathbb{C}}(G)) \to \mathcal{O}(G/\operatorname{Lin}_{\mathbb{C}}(G))$. Also, we remind the reader that $\widetilde{\sigma} \colon \mathcal{O}_{exp}(G/\operatorname{Lin}_{\mathbb{C}}(G)) \to \mathcal{O}_{exp}(G)$ from Theorem 5.3 is a topological isomorphism.

Theorem 5.12. *Let G be a connected complex Lie group. Then*
(A) *$j_0\widetilde{\sigma}^{-1} \colon \mathcal{O}_{exp}(G) \to \mathcal{O}(G/\operatorname{Lin}_{\mathbb{C}}(G))$ is an Arens-Michael envelope.*
(B) *G is linear iff it is a Stein group and $j \colon \mathcal{O}_{exp}(G) \to \mathcal{O}(G)$ is an Arens-Michael envelope.*

Proof. First, we show the condition from part (B) is necessary. Suppose that G is linear. Then it is clearly a Stein group. Let B, L, and E be as above and note that $E \times B/E \times L$ is an affine algebraic variety; so the obvious embedding

$$\iota \colon \mathcal{R}(E \times B/E \times L) \to \mathcal{O}(E \times B/E \times L)$$

is an Arens-Michael envelope [30, Ex. 3.6]. Identifying $\mathcal{O}(E \times B/E \times L)$ with $\mathcal{O}(G)$, we obtain that $\iota = j\theta$, where θ is defined in Corollary 5.11. The homomorphism θ, having dense range, is an epimorphism. It follows from [2, Lem. 2.3] that the factorization $\iota = j\theta$ of the Arens-Michael envelope homomorphism ι on the epimorphism θ implies that j is also an Arens-Michael envelope homomorphism.

Next, to prove part (A) note that $G/\operatorname{Lin}_{\mathbb{C}}(G)$ is linear. The above argument shows that j_0 is an Arens-Michael envelope, so is $j_0\widetilde{\sigma}^{-1}$.

Finally, we demonstrate the sufficiency from Part (B). Suppose that G is a Stein group and j is an Arens-Michael envelope. Since so is $j_0\widetilde{\sigma}^{-1}$, the universal property of the Arens-Michael enveloping functor implies that $\mathcal{O}(G/\operatorname{Lin}_{\mathbb{C}}(G)) \to \mathcal{O}(G)$ (induced by the quotient map $G \to G/\operatorname{Lin}_{\mathbb{C}}(G)$) is a topological isomorphism of Stein algebras. By Forster's Duality Theorem [10], the quotient map $G \to G/\operatorname{Lin}_{\mathbb{C}}(G)$ is a biholomorphic equivalence, therefore $\operatorname{Lin}_{\mathbb{C}}(G)$ is trivial. ■

We finish with examples.

Example 5.13. Let $\mathfrak{g}$ be the 2-dimensional solvable complex Lie algebra with basis $\{e_1, e_2\}$ and commutation relation $[e_1, e_2] = e_2$. Then $\mathfrak{e} = [\mathfrak{g}, \mathfrak{g}] = \mathbb{C}e_2$. Consider the simply connected complex Lie group G with $\mathfrak{g}$ as the Lie algebra (cf. (2)). Let (s, t) be the canonical coordinates of second type on G, i.e.,

$$g = \exp(se_1)\exp(te_2) \qquad (g \in G)\,.$$

In these coordinates, any $f \in \mathcal{O}_{exp}(G)$ has the form

$$f(s,t) = \sum_n f_n(s) t^n\,,$$

where each f_n is an entire function of exponential type on $\mathbb{C}$, i.e.,

$$|f_n(s)| \le C e^{r|s|} \qquad (s \in \mathbb{C})$$

for some $C > 0$ and $r \in \mathbb{R}_+$. Note that this decomposition can also be obtained from [30, Prop. 5.2].

Another group with the same Lie algebra is the '$az + b$'-group G_1, consisting of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \qquad (a \in \mathbb{C}^\times,\, b \in \mathbb{C})\,.$$

Unlike G, the group G_1 is algebraic and $\mathcal{O}_{\exp}(G_1) = \mathcal{R}(G_1)$, i.e., a function of exponential type has a decomposition $f(a,b) = \sum_n f_n(a) b^n$, where all f_n are Laurent polynomials in a.

Example 5.14. Let $\mathfrak{g}$ be the 6-dimensional complex Lie algebra with basis $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ and commutation relations

$$\begin{gathered} [e_1, e_2] = e_3, \\ [e_2, f_1] = f_1,\ [e_2, f_2] = f_2,\ [e_2, f_3] = 2f_3, \\ [e_3, f_1] = f_1,\ [e_3, f_2] = -f_2,\ [e_3, f_3] = 0, \\ [f_1, f_2] = f_3 \end{gathered}$$

the undefined brackets being zero. To see that $\mathfrak{g}$ is a Lie algebra it is sufficient to note that $\mathfrak{g}$ is an iterated semidirect sum $\mathfrak{h}_1 \ltimes (\mathfrak{h}_2 \ltimes \mathfrak{h}_3)$, where $\mathfrak{h}_1 = \operatorname{span}\{e_1\}$, $\mathfrak{h}_2 = \operatorname{span}\{e_2, e_3\}$, and $\mathfrak{h}_3 = \operatorname{span}\{f_1, f_2, f_3\}$.

Then e_3, f_1, f_2, f_3 is a basis for $\mathfrak{g}_2 := [\mathfrak{g}, \mathfrak{g}]$ but f_1, f_2, f_3 is a basis for $\mathfrak{g}_\infty = \mathfrak{g}_3 := [\mathfrak{g}, \mathfrak{g}_2]$. Since $\mathfrak{g}$ is solvable, we have $\mathfrak{e} = \mathfrak{g}_\infty$. So $\mathfrak{e}$ and $\mathfrak{g}/\mathfrak{e}$ are both isomorphic to the 3-dimensional complex Heisenberg algebra.
Let G be the simply connected complex Lie group having $\mathfrak{g}$ as the Lie algebra. Consider the coordinates $(s_1, s_2, s_3, t_1, t_2, t_3)$ defined by

$$g = \exp(s_1 e_1 + s_2 e_2 + s_3 e_3) \exp(t_1 f_1 + t_2 f_2 + t_3 f_3) \qquad (g \in G)$$

and identify G with $\mathbb{C}^6$. Thus any $f \in \mathcal{O}_{exp}(G)$ has the form

$$f(s_1, s_2, s_3, t_1, t_2, t_3) = \sum_{n_1, n_2, n_3} f_{n_1, n_2, n_3}(s_1, s_2, s_3)\, t_1^{n_1} t_2^{n_2} t_3^{n_3},$$

where each f_{n_1, n_2, n_3} is an entire function such that

$$|f_{n_1, n_2, n_3}(s_1, s_2, s_3)| \le C e^{r \max\{|s_1|, |s_2|, |s_3|^{1/2}\}} \qquad (s_1, s_2, s_3 \in \mathbb{C})$$

for some $C > 0$ and $r \in \mathbb{R}_+$ (cf. [2, Exm. 3.4]).

Example 5.15. Fix $n \in \mathbb{N}$ and consider the standard action of $\mathrm{SL}_n(\mathbb{C})$ on $\mathbb{C}^n$. Set $G := \mathbb{C}^n \rtimes \mathrm{SL}_n(\mathbb{C})$. The Lie algebra of G is the semidirect sum $\mathfrak{g} = \mathbb{C}^n \rtimes \mathfrak{sl}_n(\mathbb{C})$. Then the radical $\mathfrak{r} \cong \mathbb{C}^n$ and $\mathfrak{sl}_n(\mathbb{C})$ is a Levi complement. It is easy to see that $\mathfrak{r}_\infty = 0$ but $\mathfrak{e} \cong \mathbb{C}^n$; so $\mathfrak{r}_\infty \ne \mathfrak{e}$. Thus $\mathcal{O}_{exp}(G) = \mathcal{R}(\mathbb{C}^n \times \mathrm{SL}_n(\mathbb{C}))$, i.e., every holomorphic function of exponential type is a polynomial in coordinates on $\mathbb{C}^n$ and matrix elements of $\mathrm{SL}_n(\mathbb{C})$.

The reader can also find another example in [2, Example 3.5].

References

[1] S. S. Akbarov: *Holomorphic functions of exponential type and duality for Stein groups with algebraic connected component of identity*, Fundam. Prikl. Mat. 14(1) (2008) 3–178; English translation: J. Math. Sci. 162(4) (2009) 459–586.

[2] O. Yu. Aristov: *Arens-Michael envelopes of nilpotent Lie algebras, functions of exponential type, and homological epimorphisms*, submitted, arXiv:1810.13213 (2018).

[3] D. Beltiţă, K.-H. Neeb: *Finite-dimensional Lie subalgebras of algebras with continuous inversion*, Studia Math. 185(3) (2008) 249–262.

[4] D. Beltiţă, M. Şabac: *Lie Algebras of Bounded Operators*, Operator Theory: Advances and Applications 120, Birkäuser, Basel (2001).

[5] N. Bourbaki: *Lie Groups and Lie Algebras, Chapters 7–9*, Springer, Berlin (2005).

[6] I. Chatterji, C. Pittet, L. Saloff-Coste: *Connected Lie groups and property RD*, Duke Math. J. 137(3) (2007) 511–536.

[7] C. Chevalley: *Theory of Lie Groups. I*, Princeton Mathematical Series 8, Princeton University Press, Princeton (1946).

[8] Y. de Cornulier: *Dimension of asymptotic cones of Lie groups*, J. Topology 1 (2008) 342–361.

[9] Y. de Cornulier: *Asymptotic cones of Lie groups and cone equivalences*, Illinois J. Math. 55(1) (2011) 237–259.

[10] O. Forster: *Zur Theorie der Steinschen Algebren und Moduln*, Math. Zeitschrift 97 (1967) 376–405.

[11] Y. Guivarc'h: *Croissance polynomiale et périodes des fonctions harmoniques*, Bull. Soc. Math. France 101 (1973) 333–379.

[12] Y. Guivarc'h: *Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire*, Astérisque 74 (1980) 47–98.

[13] A. Ya. Helemskii: *Banach and Polynormed Algebras: General Theory, Representations, Homology*, Oxford University Press, Oxford (1993); original Russian edition: Nauka, Moscow (1989).

[14] M. Hervé: *Analyticity in Infinite-dimensional Spaces*, de Gruyter, Berlin (1989).

[15] E. Hewitt, K. A. Ross: *Abstract Harmonic Analysis I*, Springer, Berlin (1979).

[16] J. Hilgert, K.-H. Neeb: *Structure and Geometry of Lie Groups*, Springer, Berlin (2011).

[17] G. Hochschild: *The structure of Lie groups*, Holden Day, San Francisco (1965).

[18] R. Karidi: *Geometry of balls in nilpotent Lie groups*, Duke Math. J. 74(2) (1994) 301–317.

[19] L. Kaup, B. Kaup: *Holomorphic Functions of Several Variables. An Introduction to the Fundamental Theory*, de Gruyter Studies in Mathematics 3, de Gruyter, Berlin (2011).

[20] D. H. Lee: *The Structure of Complex Lie Groups*, Chapman Hall/CRC, Boca Raton (2002).

[21] G. V. Litvinov: *Group algebras of analytic functionals, and their representations*, Dokl. Akad. Nauk SSSR 190(4) (1970) 769–771.

[22] G. V. Litvinov: *Group representations in locally convex spaces, and topological group algebras* (Russian), Trudy Sem. Vektor. Tenzor. Anal. 16 (1972) 267–349. English translation: Selecta Math. Soviet. 7(2) (1988) 101–182.

[23] D. Luminet, A. Valette: *Faithful uniformly continuous representations of Lie groups*, J. London Math. Soc. 49 (1994) 100–108.

[24] K.-H. Neeb: *Holomorphy and Convexity in Lie Theory*, de Gruyter Expositions in Mathematics 28, de Gruyter, Berlin (2000).

[25] K.-H. Neeb: *Infinite-Dimensional Lie Groups*, 3rd cycle, Monastir (Tunisie) (2005) pp. 76.

[26] A. L. Onishchik, E. B. Vinberg: *Lie Groups and Algebraic Groups*, Springer Series in Soviet Mathematics, Springer (1990). Original Russian edition: Nauka, Moscow (1988).

[27] A. L. Onishchik, E. B. Vinberg (eds.): *Lie Groups and Lie Algebras III: Structure of Lie Groups and Lie Algebras*, Encyclopaedia of Mathematical Sciences 41, Springer, Berlin (1994). Original Russian edition: VINITI, Moscow (1990).

[28] D. V. Osin: *Exponential radicals of solvable Lie groups*, J. Algebra 248 (2002) 790–805.

[29] A. L. T. Paterson: *Amenability*, Mathematical Surveys and Monographs 29, American Mathematical Society, Providence (1988).

[30] A. Yu. Pirkovskii: *Arens-Michael envelopes, homological epimorphisms, and relatively quasi-free algebras* (Russian), Tr. Mosk. Mat. Obs. 69 (2008) 34–125; English translation: Trans. Moscow Math. Soc. (2008) 27–104.

[31] H. H. Schaefer, M. P. H. Wolff: *Topological Vector Spaces*, Springer, Berlin (1999).

[32] L. B. Schweitzer: *Dense m-convex Fréchet subalgebras of operator algebra crossed products by Lie groups*, Internat. J. Math. 4(4) (1993) 601–673.

[33] Yu. V. Turovskii: *On spectral properties of elements of normed algebras and invariant subspaces*, Funktsional. Anal. i Prilozhen 18(2) (1984) 77–78; English translation: Funct. Anal. Appl. 18(2) (1984) 152–154.

[34] N. T. Varopoulos, L. Saloff-Coste, T. Coulhon: *Analysis and Geometry on Groups*, Cambridge Tracts in Mathematics 100, Cambridge University Press, Cambridge(1992).

[35] N. T. Varopoulos: *Analysis on Lie groups*, Rev. Mat. Iberoamericana 12 (1996) 791–917.

Oleg Yu. Aristov
Obninsk, 249038
Russian Federation
aristovoyu@inbox.ru

Received April 16, 2019
and in final form July 18, 2019