

## Representations of Simple Hom-Lie Algebras

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**Abstract.** The purpose of this paper is to study representations of simple multiplicative Hom-Lie algebras. First, we provide a new proof using Killing form for the characterization theorem of simple Hom-Lie algebras given by Chen and Han, then discuss the representations structure of simple multiplicative Hom-Lie algebras. Moreover, we study weight modules and root space decompositions of simple multiplicative Hom-Lie algebras, characterize weight modules and provide examples of representations of  $\mathfrak{sl}_2$ -type Hom-Lie algebras.

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*Key Words:* Hom-Lie algebra, simple Hom-Lie algebra, representation, weight module.

### 1. Introduction

Nowadays, one of the most modern trends in mathematics has to do with representations and deformations. These two topics are important tools in most parts of Mathematics and Physics. Hom-type algebras arised first in examples of  $q$ -deformations of algebras of vector fields, like Witt and Virasoro algebras, where the usual derivation is replaced by a  $\sigma$ -derivation. Motivated by these examples, Hartwig, Larsson and Silvestrov developed from the algebraic point of view in [7] the deformation theory using  $\sigma$ -derivations and introduced a new category of algebras called Hom-Lie algebras. A Hom-Lie algebra is a triple  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  in which the bracket satisfies a twisted Jacobi identity along the linear map  $\alpha$ . It should be pointed out that Lie algebras form a subclass of Hom-Lie algebras, i.e. when  $\alpha$  equal to the identity map. Various classical algebraic structures were considered and generalized within a similar framework like Hom-Lie superalgebras in [14].

Representations of Hom-Lie algebras were introduced and studied in [18], see also [15]. Based on this, we aim in this paper to discuss simple Hom-Lie algebra representations. Simple Hom-Lie algebras were characterized in [4], where the authors showed that they are obtained by a Yau twist of semisimple Lie algebras. This key observation is used here to built a representation theory of simple Hom-Lie algebras. Moreover examples are provided by a study of Hom-type  $\mathfrak{sl}(2)$ -modules.

The paper is organized as follows. In Section 2, we review basic definitions and relevant constructions about Hom-Lie algebras. Then in Section 3 we recall some fundamental results about the structure of simple multiplicative Hom-Lie algebras

and provide a new proof of the main theorem using Killing form. Section 4 is dedicated to the construction of multiplicative simple Hom-Lie algebras representations, we show that there is a correspondence between representations of multiplicative simple Hom-Lie algebras and representations of the induced semisimple Lie algebras using invertible twisting maps. In Section 5, we introduce and discuss the notion of simple multiplicative Hom-Lie algebra weight-modules. Finally in Section 6 we study and classify Hom- $\mathfrak{sl}(2)$ -modules.

## 2. Basics

In this section, we provide some preliminaries, basic definitions and relevant constructions about Hom-Lie algebras and related structures. Throughout this paper all algebras and vector spaces are considered over  $\mathbb{K}$ , an algebraically closed field of characteristic 0.

**Definition 2.1.** A *Hom-Lie algebra* is a triple  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  consisting of a vector space  $\mathfrak{g}$ , a bilinear map  $[\cdot, \cdot]_\alpha : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$  and a linear map  $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$  that satisfies:

$$[x, y]_\alpha = -[y, x]_\alpha, \quad \forall x, y \in \mathfrak{g} \quad (\text{skewsymmetry})$$

$$\bigcirc_{x,y,z} [\alpha(x), [y, z]_\alpha]_\alpha = 0, \quad \forall x, y, z \in \mathfrak{g} \quad (\text{Hom-Jacobi identity}).$$

A Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  is said to be *multiplicative* if  $\alpha$  is an algebra morphism, i.e.

$$\alpha([x, y]_\alpha) = [\alpha(x), \alpha(y)]_\alpha, \quad \forall x, y \in \mathfrak{g}.$$

It is said *regular* if  $\alpha$  is an algebra automorphism.

**Definition 2.2.** Let  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  be a Hom-Lie algebra. A subspace  $\mathfrak{h}$  of  $\mathfrak{g}$  is called Hom-Lie *subalgebra* if  $[\mathfrak{h}, \mathfrak{h}]_\alpha \subseteq \mathfrak{h}$  and  $\alpha(\mathfrak{h}) \subseteq \mathfrak{h}$ . In particular, a Hom-Lie subalgebra  $\mathfrak{h}$  is said to be an *ideal* of  $\mathfrak{g}$  if  $[\mathfrak{h}, \mathfrak{g}]_\alpha \subseteq \mathfrak{h}$ . The Hom-Lie algebra  $\mathfrak{g}$  is called *abelian* if  $[x, y] = 0, \forall x, y \in \mathfrak{g}$ .

**Definition 2.3.** Let  $(\mathfrak{g}_1, [\cdot, \cdot]_{\alpha_1}, \alpha_1)$  and  $(\mathfrak{g}_2, [\cdot, \cdot]_{\alpha_2}, \alpha_2)$  be two Hom-Lie algebras. A linear map  $\varphi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$  is a *Hom-Lie algebra morphism* if for all  $x, y \in \mathfrak{g}_1$

$$\varphi([x, y]_{\alpha_1}) = [\varphi(x), \varphi(y)]_{\alpha_2} \quad \text{and} \quad \varphi \circ \alpha_1 = \alpha_2 \circ \varphi.$$

In particular, they are isomorphic if  $\varphi$  is a bijective linear map.

A linear map  $\varphi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$  is said to be a *weak Hom-Lie algebra morphism* if for all  $x, y \in \mathfrak{g}_1$ , we have only  $\varphi([x, y]_{\alpha_1}) = [\varphi(x), \varphi(y)]_{\alpha_2}$ .

There is a key construction introduced by D. Yau that gives rise to a Hom-Lie algebra starting from a Lie algebra and a Lie algebra homomorphism [19].

**Proposition 2.4** (Yau Twist). *Let  $\mathfrak{g} = (\mathfrak{g}, [\cdot, \cdot])$  be a Lie algebra and  $\alpha : \mathfrak{g} \longrightarrow \mathfrak{g}$  be a Lie algebra homomorphism. Then,  $\mathfrak{g}_\alpha = (\mathfrak{g}, [\cdot, \cdot]_\alpha := \alpha([\cdot, \cdot]), \alpha)$  is a Hom-Lie algebra.*

**Proof.** It is straightforward to prove that the new bracket  $[\cdot, \cdot]_\alpha$  satisfies the Hom-Jacobi identity. ■

**Remark 2.5.** More generally, let  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  be a Hom-Lie algebra and  $\gamma : \mathfrak{g} \rightarrow \mathfrak{g}$  be a weak Hom-Lie algebra morphism. Then  $\mathfrak{g}_\gamma = (\mathfrak{g}, [\cdot, \cdot]_\gamma := \gamma([\cdot, \cdot]), \gamma \circ \alpha)$  is a new Hom-Lie algebra.

**Definition 2.6.** Let  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  be a Hom-Lie algebra. If there exists a Lie algebra  $(\mathfrak{g}, [\cdot, \cdot])$  such that  $[x, y]_\alpha = \alpha([x, y]) = [\alpha(x), \alpha(y)]$ , for all  $x, y \in \mathfrak{g}$ , then  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  is said to be of *Lie-type* and  $(\mathfrak{g}, [\cdot, \cdot])$  is called the induced Lie algebra of  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ .

**Lemma 2.7.** Let  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  be a regular Hom-Lie algebra. Then  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  is of *Lie-type* with the induced Lie algebra  $(\mathfrak{g}, [\cdot, \cdot])$ , where  $[x, y] = \alpha^{-1}([x, y]_\alpha), \forall x, y \in \mathfrak{g}$ .

**Proof.** Let  $[x, y] = \alpha^{-1}([x, y]_\alpha)$  for any  $x, y \in \mathfrak{g}$ . Since  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  is multiplicative then,  $\alpha([x, y]_\alpha) = [\alpha(x), \alpha(y)]_\alpha$ . It implies  $\alpha^2([x, y]) = \alpha([\alpha(x), \alpha(y)])$  and thus  $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ . In the following we shall prove that  $(\mathfrak{g}, [\cdot, \cdot])$  is a Lie algebra. The skewsymmetry of  $[\cdot, \cdot]$  is obvious. Now, we prove that it satisfies the Jacobi identity. Indeed, let  $x, y, z \in \mathfrak{g}$ , we have

$$\bigcirc_{x,y,z} [x, [y, z]] = \bigcirc_{x,y,z} \alpha^{-1}[x, \alpha^{-1}[y, z]_\alpha]_\alpha = \bigcirc_{x,y,z} \alpha^{-2}[\alpha(x), [y, z]_\alpha]_\alpha = 0.$$

It follows that  $(\mathfrak{g}, [\cdot, \cdot])$  is a Lie algebra. ■

The concept of representation of a Hom-Lie algebra was introduced in [18], see also [15].

**Definition 2.8.** A *representation* of a multiplicative Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  on a vector space  $V$  with respect to  $\beta \in \text{End}(V)$  is a linear map  $\rho_\beta : \mathfrak{g} \rightarrow \text{End}(V)$ , such that for any  $x, y \in \mathfrak{g}$ , the following conditions are satisfied:

$$\rho_\beta(\alpha(x)) \circ \beta = \beta \circ \rho_\beta(x), \tag{1}$$

$$\rho_\beta([x, y]_\alpha) \circ \beta = \rho_\beta(\alpha(x))\rho_\beta(y) - \rho_\beta(\alpha(y))\rho_\beta(x). \tag{2}$$

Hence  $(V, \rho_\beta, \beta)$  is called a  $\mathfrak{g}$ -*module* via the action  $x.v = \rho_\beta(x)v, \forall x \in \mathfrak{g}, v \in V$ .

We have the following property.

**Proposition 2.9.** Let  $(V, \rho_\beta, \beta)$  be a representation of a simple multiplicative Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  with  $\beta$  invertible. Then, for all  $n \in \mathbb{N}$  we have,

$$(1) \quad \rho_\beta(\alpha^n(x)) = \beta^n \rho_\beta(x) \beta^{-n}.$$

$$(2) \quad \rho_\beta(\alpha^n[x, y])\beta = \rho_\beta(\alpha^{n+1}(x))\rho_\beta(\alpha^n(y)) - \rho_\beta(\alpha^{n+1}(y))\rho_\beta(\alpha^n(x)).$$

**Proof.** (1) is straightforward by induction using identity (1), and (2) is proved by induction using identity (2) of Definition 2.8. ■

**Definition 2.10.** (1) For a  $\mathfrak{g}$ -module  $(V, \rho_\beta, \beta)$ , if a subspace  $V_1 \subseteq V$  is invariant under  $\rho_\beta(x) \forall x \in \mathfrak{g}$  and under  $\beta$  then  $(V_1, \rho_\beta, \beta|_{V_1})$  is called a  $\mathfrak{g}$ -*submodule* of  $(V, \rho_\beta, \beta)$ .

(2) A  $\mathfrak{g}$ -module  $(V, \rho_\beta, \beta)$  is called *irreducible* if it has precisely two  $\mathfrak{g}$ -submodules (itself and 0) and it is called *completely reducible* if  $V = V_1 \oplus \dots \oplus V_s$ , where  $(V_i, \rho_\beta, \beta|_{V_i})$  are irreducible  $\mathfrak{g}$ -submodules.

**Theorem 2.11** ([13]). *Let  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  be a Lie-type Hom-Lie algebra with  $(\mathfrak{g}, [\cdot, \cdot])$  the induced Lie algebra.*

- (1) *Let  $(V, \rho_\beta, \beta)$  be a representation of the Hom-Lie algebra, where  $\beta$  is invertible. Then  $(V, \rho) = (V, \beta^{-1} \circ \rho_\beta)$  is a representation of the Lie algebra  $(\mathfrak{g}, [\cdot, \cdot])$ .*
- (2) *Suppose that  $(V, \rho)$  is a representation of the Lie algebra  $(\mathfrak{g}, [\cdot, \cdot])$ . If there exists  $\beta \in \text{End}(V)$  such that*

$$\beta \circ \rho(x) = \rho(\alpha(x)) \circ \beta, \forall x \in \mathfrak{g}, \forall v \in V, \quad (3)$$

*then  $(V, \rho_\beta = \beta \circ \rho, \beta)$  is a representation of the Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ .*

**Definition 2.12.** Let  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  be a Lie-type Hom-Lie algebra with  $(\mathfrak{g}, [\cdot, \cdot])$  the induced Lie algebra. A representation  $(V, \rho_\beta, \beta)$  of  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  is called of Lie-type if  $\rho_\beta = \beta \circ \rho$ , where  $\rho$  is the representation of the induced Lie algebra  $(\mathfrak{g}, [\cdot, \cdot])$ . It is called regular if the representation  $(V, \rho := \beta^{-1} \circ \rho_\beta)$  of the induced Lie algebra  $(\mathfrak{g}, [\cdot, \cdot])$  is irreducible.

The previous theorem provides a relationship between representations of Lie-type Hom-Lie algebras and those of their induced Lie algebras.

### 3. Structure of simple multiplicative Hom-Lie algebras

In [4], the authors have proved that multiplicative simple Hom-Lie algebras are of Lie-type and their induced Lie algebras are semisimple. Moreover they discussed the dimension problem and showed that there is an  $n$ -dimensional simple Hom-Lie algebra for any integer  $n$  larger than 2. We should mention also the following relevant references [15],[13] and [20].

**Definition 3.1.** A Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  is called *simple* if it has no proper ideals and  $[\mathfrak{g}, \mathfrak{g}]_\alpha = \mathfrak{g}$ . It is called *semisimple* Hom-Lie algebra if  $\mathfrak{g}$  is a direct sum of simple ideals.

We have the following two propositions.

**Proposition 3.2** ([15]). *Let  $(\mathfrak{g}, [\cdot, \cdot])$  be a simple Lie algebra and  $\alpha \in \text{Aut}(\mathfrak{g})$ . Then,  $\mathfrak{g}_\alpha = (\mathfrak{g}, \alpha([\cdot, \cdot]), \alpha)$  is a simple Hom-Lie algebra.*

**Proposition 3.3.** *Simple multiplicative Hom-Lie algebras are regular Hom-Lie algebras.*

**Proof.** Let  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  be a simple Hom-Lie algebra, then  $[\mathfrak{g}, \mathfrak{g}]_\alpha = \mathfrak{g}$ . Suppose that  $\ker(\alpha) \neq 0$ . Then,  $\alpha(\ker(\alpha)) = 0$  and  $\alpha([\ker(\alpha), \mathfrak{g}]_\alpha) = 0$ . So  $\ker(\alpha)$  is a non trivial ideal of  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ . This contradicts the simplicity of  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ , except when  $\ker(\alpha) = 0$ . Hence,  $\alpha$  is an automorphism. ■

The following theorem summarizes results given in [4] about a characterization of simple multiplicative Hom-Lie algebras. We provide a new and different proof based on Killing form.

**Theorem 3.4.** ([4]) *Let  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  be a simple multiplicative Hom-Lie algebra. Then the induced Lie algebra  $(\mathfrak{g}, [\cdot, \cdot] = \alpha^{-1}([\cdot, \cdot]_\alpha))$  is semisimple and its  $n$  simple ideals are isomorphic mutually besides  $\alpha$  acts simply transitively on simple ideals of  $\mathfrak{g}$ . Furthermore  $\mathfrak{g}$  can be generated by a simple ideal  $\mathfrak{g}_1$  of the Lie algebra  $(\mathfrak{g}, [\cdot, \cdot])$  and  $\alpha \in \text{Aut}(\mathfrak{g})$ . Taking  $\alpha, \gamma \in \text{Aut}(\mathfrak{g})$  such that  $\alpha^n$  and  $\gamma^n$  leaves each simple ideal invariant and  $\alpha^n(\mathfrak{g}_1) = \mathfrak{g}_1$  (or  $\gamma^n(\mathfrak{g}_1) = \mathfrak{g}_1$ ). Then we have*

- (1)  $\mathfrak{g} = \mathfrak{g}_1 \oplus \alpha(\mathfrak{g}_1) \oplus \alpha^2(\mathfrak{g}_1) \oplus \dots \oplus \alpha^{n-1}(\mathfrak{g}_1)$   
 (or  $\mathfrak{g} = \mathfrak{g}_1 \oplus \gamma(\mathfrak{g}_1) \oplus \gamma^2(\mathfrak{g}_1) \oplus \dots \oplus \gamma^{n-1}(\mathfrak{g}_1)$ ).
- (2)  $\alpha$  and  $\gamma$  are conjugate on  $\mathfrak{g} \Leftrightarrow \alpha^n$  and  $\gamma^n$  are conjugate on  $\mathfrak{g}_1$ .

**Proof.** The Killing form  $K : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$  of  $(\mathfrak{g}, [\cdot, \cdot])$  is non-degenerate. In fact, let  $\mathfrak{k} = \{x \in \mathfrak{g} / K(x, y) = 0, \forall y \in \mathfrak{g}\}$  its kernel. It is clear that  $\mathfrak{k}$  is an ideal of  $\mathfrak{g}$ , since  $K([x, y], z) = K(x, [y, z]) = 0, \forall x \in \mathfrak{k}, y, z \in \mathfrak{g}$ . Since  $\alpha$  is an automorphism and  $K(\alpha(x), \alpha(y)) = K(x, y)$ , then  $K(\alpha(x), y) = K(x, \alpha^{-1}(y)) = 0, \forall x \in \mathfrak{k}, y \in \mathfrak{g}$  and  $\alpha(\mathfrak{k}) \subset \mathfrak{k}$ . Then  $\mathfrak{k}$  is an ideal of the multiplicative simple Hom-Lie algebra  $\mathfrak{g}$  and then  $\mathfrak{k} = 0$  and  $K$  is non-degenerate. We deduce that  $(\mathfrak{g}, [\cdot, \cdot])$  is a semisimple Lie algebra and then a direct sum of its simple ideals. Let  $\mathfrak{g}_1$  be a minimal proper ideal of the induced Lie algebra  $(\mathfrak{g}, [\cdot, \cdot])$ . In particular  $\mathfrak{g}_1$  is a simple ideal. Let  $n$  be the minimal integer such that  $\alpha^{n-1}(\mathfrak{g}_1) \neq \mathfrak{g}_1$  and  $\alpha^n(\mathfrak{g}_1) = \mathfrak{g}_1$ . The algebra  $\mathfrak{b} = \mathfrak{g}_1 \oplus \alpha(\mathfrak{g}_1) \oplus \dots \oplus \alpha^{n-1}(\mathfrak{g}_1)$  is an ideal of the simple multiplicative Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ . Then  $\mathfrak{g} = \mathfrak{b} = \mathfrak{g}_1 \oplus \alpha(\mathfrak{g}_1) \oplus \dots \oplus \alpha^{n-1}(\mathfrak{g}_1)$ . ■

**Definition 3.5.** A multiplicative Hom-Lie algebra is called *semisimple* if its induced Lie algebra is semisimple.

**Proposition 3.6.** *A multiplicative semisimple Hom-Lie algebra is a direct sum of multiplicative simple Hom-Lie algebras.*

**Proof.** For  $\mathfrak{g}$  semisimple Hom-Lie algebra, the Killing form of the induced Lie algebra will be non-degenerate and from the previous definition the induced Lie algebra is semisimple. We take a minimal ideal  $\mathfrak{g}_1$  which will be a simple ideal. Let  $n$  be the minimal integer such that  $\alpha^{n-1}(\mathfrak{g}_1) \neq \mathfrak{g}_1$  and  $\alpha^n(\mathfrak{g}_1) = \mathfrak{g}_1$ . The algebra  $\mathfrak{b} = \mathfrak{g}_1 \oplus \alpha(\mathfrak{g}_1) \oplus \dots \oplus \alpha^{n-1}(\mathfrak{g}_1)$  is a simple ideal of the semisimple multiplicative Lie algebra. Let  $\mathfrak{b}'$  be the subspace of  $\mathfrak{g}$  orthogonal (with respect to  $K$ ) to  $\mathfrak{b}$ . As  $K$  is invariant,  $\mathfrak{b}'$  is an ideal of  $\mathfrak{g}$ . In fact, let  $x \in \mathfrak{b}, y \in \mathfrak{b}', z \in \mathfrak{g}$ , by invariance of  $K$ , we have  $K(x, [z, y]) = K([x, z], y) = 0$  since  $[x, z] \in \mathfrak{b}$ . Moreover, since  $\alpha(\mathfrak{b}) = \mathfrak{b}, K(x, \alpha(y)) = K(\alpha^{-1}(x), y) = 0$ , then  $\mathfrak{b}'$  is an ideal of the Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ .

By minimality of  $\mathfrak{b}$ , the intersection  $\mathfrak{b} \cap \mathfrak{b}'$  can only be  $(0)$  or  $\mathfrak{b}$ . We can prove that the second case cannot occur. If not  $K(x, y) = 0, \forall x, y \in \mathfrak{b}$  and  $x = \sum_{i=1}^k [x_i, y_i]$  since  $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{b}$ . Then for all  $z \in \mathfrak{g}$ , using the invariance of  $K$  and  $\mathfrak{b}$  an ideal, we have

$$K(x, z) = K\left(\sum_{i=1}^k [x_i, y_i], z\right) = \sum_{i=1}^k K([x_i, y_i], z) = \sum_{i=1}^k K(x_i, [y_i, z]) = 0,$$

which contradicts the fact that  $K$  is non-degenerate. Hence  $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{b}'$ . The restriction of  $K$  to  $\mathfrak{b}' \times \mathfrak{b}'$  is an invariant non-degenerate bilinear form.

The proof is completed by induction on the dimension of  $\mathfrak{g}$ . ■

4. Representations of simple multiplicative Hom-Lie algebras

We aim in this section to characterize representations of simple multiplicative Hom-Lie algebras and provide the relationship with those of their induced semisimple Lie algebras.

**Proposition 4.1.** *Let  $(V, \rho_\beta, \beta)$  be a representation of a simple multiplicative Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ . Then,  $Im(\beta)$  and  $Ker(\beta)$  are submodules of  $V$  for the algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ . Moreover, we have an isomorphism of  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ -modules  $\bar{\beta}: V/Ker(\beta) \rightarrow Im(\beta)$ .*

**Proof.** Let  $v \in Ker(\beta)$ ,  $\rho_\beta(\alpha(x)) \circ \beta(v) = 0 = \beta(\rho_\beta(x)(v)) \forall x \in \mathfrak{g}$ . Then,  $\rho_\beta(\alpha(x))(v) \in Ker(\beta)$ . So  $Ker(\beta)$  is a submodule of  $V$ . Now let  $v \in Im(\beta)$ . There exists  $w \in V$  such that  $v = \beta(w)$ . Since  $\alpha$  is an automorphism  $\forall x \in \mathfrak{g}$ ,  $\rho_\beta(x)(v) = \rho_\beta(\alpha(\alpha^{-1}(x)))\beta(w) = \beta(\rho_\beta(\alpha^{-1}(x)w))$ . So  $\rho_\beta(x)(v) \in Im(\beta) \forall x \in \mathfrak{g}$  and therefore  $Im(\beta)$  is a submodule of  $V$ . ■

**Corollary 4.2.** *If  $(V, \rho_\beta, \beta)$  is an irreducible representation of a simple multiplicative Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ . Then  $\beta$  is invertible.*

**Proposition 4.3.** *Let  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  be a simple multiplicative Hom-Lie algebra and  $(V, \rho_\beta, \beta)$  a representation with  $\beta$  invertible. If  $(V, \rho := \beta^{-1} \circ \rho_\beta)$  is irreducible representation of the induced Lie algebra. Then  $(V, \rho_\beta, \beta)$  is irreducible representation of the multiplicative simple Hom-Lie algebra.*

**Proof.** Assume that  $(V, \rho_\beta, \beta)$  is reducible. Then, there exists  $W \neq \{0_V\}$  a subspace of  $V$  such that  $(V, \rho_\beta, \beta |_W)$  is a submodule of  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ . That is  $\beta(W) \subset W$  and  $\rho_\beta(x)W \subset W, \forall x \in \mathfrak{g}$ . Hence,  $\beta \circ \rho(x)W \subset W, \forall x \in \mathfrak{g}$  and then  $\rho(x)W \subset \beta^{-1}(W) \subset W, \forall x \in \mathfrak{g}$  and so  $W$  is a submodule for  $(V, \rho)$  which is a contradiction. ■

**Proposition 4.4.** *Let  $\mathfrak{g}$  be a simple Lie algebra and  $(W, \rho)$  be a representation of  $\mathfrak{g}$ . Then, for  $\alpha \in Aut(\mathfrak{g})$ ,  $(W, \tilde{\rho} := \rho \circ \alpha^{-k})$  is a representation of  $\alpha^k(\mathfrak{g})$ . Moreover, there exists  $\beta \in GL(W)$  such that  $\tilde{\rho}(\alpha^k(x)) = \beta^k \circ \rho_k(x) \circ \beta^{-k}, \forall x \in \mathfrak{g}$ .*

**Proof.** Let  $(W, \rho)$  be a representation of  $\mathfrak{g}$  on  $W$  such that the following diagram is commutative

$$\begin{array}{ccc}
 \mathfrak{g} & \xrightarrow{\rho} & End(W) \\
 \alpha^k \downarrow & & \downarrow T \\
 \alpha^k(\mathfrak{g}) & \xrightarrow{\tilde{\rho}} & End(W)
 \end{array} \tag{4}$$

This means that there exists  $T$  such that  $\tilde{\rho} \circ \alpha^k = T \circ \rho$ . Using Skolem-Noether Theorem [2], there exists  $S \in GL(W)$  such that  $\forall x \in \mathfrak{g}$ ,  $T \circ \rho(x) = S \circ \rho(x) \circ S^{-1}$ . Then, we have  $\tilde{\rho} \circ \alpha^k(x) = S \circ \rho(x) \circ S^{-1}$ .

By basic linear algebra theory, there exists  $\beta \in GL(W)$  such that  $S = \beta^k$ . Then the commutativity becomes  $\tilde{\rho} \circ \alpha^k(x) = \beta^k \circ \rho(x) \circ \beta^{-k}, \forall x \in \mathfrak{g}$ . ■

**Proposition 4.5.** ([2]) *Let  $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_n$  be a semisimple Lie algebra and let  $(V_i, \rho_i)$  be a  $\mathfrak{g}_i$ -module,  $\forall 1 \leq i \leq n$ . Then, an irreducible representation of  $\mathfrak{g}$  is given by  $(V, \rho)$  where  $V = V_1 \otimes \dots \otimes V_n$  and  $\rho$  is given for  $x = (x_1, \dots, x_n)$  by*

$$\begin{aligned} \rho : \mathfrak{g} &\longrightarrow \text{End}(V_1 \otimes \dots \otimes V_n) \\ x &\longmapsto \rho(x) = \sum_{i=1}^n \text{Id}_{V_1} \otimes \dots \otimes \rho_i(x_i) \otimes \dots \otimes \text{Id}_{V_n}. \end{aligned}$$

Using the classification of simple multiplicative Hom-Lie algebras by Chen and Han in Theorem 3.4, Theorem 2.11 of Li [13], and Proposition 4.4, we provide a construction of representations of a simple multiplicative Hom-Lie algebra.

**Theorem 4.6.** *Let  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  be a simple multiplicative Hom-Lie algebra and  $\mathfrak{g}_1$  be a simple ideal generating the induced semisimple Lie algebra such that we have  $\mathfrak{g} = \mathfrak{g}_1 \oplus \alpha(\mathfrak{g}_1) \oplus \dots \oplus \alpha^{n-1}(\mathfrak{g}_1)$ . Let  $(V_0, \rho_0), (V_1, \rho_1), \dots, (V_{n-1}, \rho_{n-1})$  be  $n$  representations of  $\mathfrak{g}_1$ .*

- (1) *There exist  $\beta_0 \in GL(V_0), \beta_1 \in GL(V_1), \dots, \beta_{n-1} \in GL(V_{n-1})$  depending on  $\alpha$  such that  $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ , where  $V = V_0 \otimes \dots \otimes V_{n-1}$ , defined for all  $X = (x_0, \alpha(x_1), \dots, \alpha^{n-1}(x_{n-1})) \in \mathfrak{g}$  by*

$$\rho(X) = \sum_{k=0}^{n-1} \text{Id}_{V_0} \otimes \dots \otimes \left( \beta_k^k \circ \rho_k(x_k) \circ \beta_k^{-k} \right) \otimes \dots \otimes \text{Id}_{V_{n-1}}, \tag{5}$$

*is an irreducible representation of the induced Lie algebra  $(\mathfrak{g}, [\cdot, \cdot])$  on  $V$ .*

- (2) *Let  $\beta := \beta_0 \otimes \dots \otimes \beta_{n-1} \in GL(V)$  and let  $\rho_\beta : \mathfrak{g} \rightarrow \text{End}(V)$  defined by*

$$\rho_\beta(X) = \beta \circ \rho(X) = \sum_{k=0}^{n-1} \beta_0 \otimes \dots \otimes \beta_k^{k+1} \circ \rho_k(x_k) \circ \beta_k^{-k} \otimes \dots \otimes \beta_{n-1}. \tag{6}$$

*Then the triple  $(V, \rho_\beta, \beta)$  is a regular representation of the simple multiplicative Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ .*

**Proof.** Let  $\rho : \mathfrak{g}_1 \rightarrow \text{End}(W)$  be a representation of  $\mathfrak{g}_1$ . By Proposition 4.4, for all  $k \in \mathbb{N}$ , there exists  $\beta_k \in GL(W)$  and a representation  $\tilde{\rho}_k$  of  $\alpha^k(\mathfrak{g}_1)$  on  $W$  given by  $\tilde{\rho}_k(\alpha^k(x)) = \beta_k^k \circ \rho_k(x) \circ \beta_k^{-k}$ .

Now consider  $n$  representations  $(V_0, \rho_0), (V_1, \rho_1), \dots, (V_{n-1}, \rho_{n-1})$  of  $\mathfrak{g}_1$ , where  $\mathfrak{g} = \mathfrak{g}_1 \oplus \alpha(\mathfrak{g}_1) \oplus \dots \oplus \alpha^{n-1}(\mathfrak{g}_1)$  and  $V = V_0 \otimes \dots \otimes V_{n-1}$ . Then, according to Proposition 4.4 and Proposition 4.5, there exists  $\beta_0 \in GL(V_0), \dots, \beta_{n-1} \in GL(V_{n-1})$  such that  $\rho : \mathfrak{g} \rightarrow \text{End}(V)$  defined by

$$\begin{aligned} &\rho(x_0, \alpha(x_1), \dots, \alpha^{n-1}(x_{n-1})) \\ &= \rho_0(x_0) \otimes \text{Id}_{V_1} \otimes \dots \otimes \text{Id}_{V_{n-1}} + \text{Id}_{V_0} \otimes \left( \beta_1 \circ \rho_1(x_1) \circ \beta_1^{-1} \right) \otimes \dots \otimes \text{Id}_{V_{n-1}} \\ &\quad + \dots + \text{Id}_{V_0} \otimes \text{Id}_{V_1} \otimes \dots \otimes \left( \beta_{n-1}^{n-1} \circ \rho_{n-1}(x_{n-1}) \circ \beta_{n-1}^{-n+1} \right) \\ &= \sum_{k=0}^{n-1} \text{Id}_{V_0} \otimes \dots \otimes \left( \beta_k^k \circ \rho_k(x_k) \circ \beta_k^{-k} \right) \otimes \dots \otimes \text{Id}_{V_{n-1}}, \end{aligned}$$

is an irreducible representation of the induced Lie algebra  $(\mathfrak{g}, [\cdot, \cdot])$  on  $V$ , which completes the proof of (1).

For (2), set  $\beta := \beta_0 \otimes \dots \otimes \beta_{n-1}$ . Condition (3) is satisfied, i.e. for  $(v_0 \otimes \dots \otimes v_{n-1}) \in V$ , we have

$$\begin{aligned} & \beta \circ \rho(X)(v_0 \otimes \dots \otimes v_{n-1}) \\ &= \beta \circ \sum_{k=0}^{n-1} Id(v_0) \otimes \dots \otimes \beta_k^k \circ \rho_k(x_k) \circ \beta_k^{-k}(v_k) \otimes \dots \otimes Id(v_{n-1}) \\ &= \sum_{k=0}^{n-1} \beta_0(v_0) \otimes \dots \otimes \beta_k^{k+1} \circ \rho_k(x_k) \circ \beta_k^{-k}(v_k) \otimes \dots \otimes \beta_{n-1}(v_{n-1}) \\ &= \sum_{k=0}^{n-1} \beta_0(v_0) \otimes \dots \otimes \beta_k^{k+1} \circ \rho_k(x_k) \circ \beta_k^{-k} \circ \beta_k \circ \beta_k^{-1}(v_k) \otimes \dots \otimes \beta_{n-1}(v_{n-1}) \\ &= \sum_{k=0}^{n-1} \beta_0(v_0) \otimes \dots \otimes \beta_k^{k+1} \circ \rho_k(x_k) \circ \beta_k^{-(k+1)} \circ \beta_k(v_k) \otimes \dots \otimes \beta_{n-1}(v_{n-1}) \\ &= \sum_{k=0}^{n-1} \beta_0(v_0) \otimes \dots \otimes \rho_k(\alpha^{k+1}(x_k)) \circ \beta_k \otimes \dots \otimes \beta_{n-1}(v_{n-1}) \\ &= \rho(\alpha(X)) \circ \beta(v_0 \otimes \dots \otimes v_{n-1}). \end{aligned}$$

So  $(V, \rho_\beta = \beta \circ \rho, \beta)$  is a representation of the simple multiplicative Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ , which is regular if the representation of the induced Lie algebra is irreducible. ■

**Proposition 4.7.** *Let  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  and  $(\mathfrak{g}, [\cdot, \cdot]_\gamma, \gamma)$  be two isomorphic simple multiplicative Hom-Lie algebras generated by a simple ideal  $\mathfrak{g}_1$ . Let  $(\mathfrak{g}, [\cdot, \cdot]'_\alpha)$  and  $(\mathfrak{g}, [\cdot, \cdot]'_\gamma)$  their induced Lie algebras. If  $(V, \rho_\beta, \beta)$  (resp.  $(V, \rho_\delta, \delta)$ ) is an irreducible representation of  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  (resp.  $(\mathfrak{g}, [\cdot, \cdot]_\gamma, \gamma)$ ) on the same vector space  $V$ . Then  $\delta$  and  $\beta$  are conjugated by  $S \in GL(V)$ .*

**Proof.** Since  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  and  $(\mathfrak{g}, [\cdot, \cdot]_\gamma, \gamma)$  are isomorphic, then there exists a linear automorphism  $\varphi$  of  $\mathfrak{g}$  such that  $\varphi \circ \alpha = \gamma \circ \varphi$  and  $\varphi([x, y]_\alpha) = [\varphi(x), \varphi(y)]_\gamma$ . On the induced Lie algebras  $(\mathfrak{g}, [\cdot, \cdot]'_\alpha)$  and  $(\mathfrak{g}, [\cdot, \cdot]'_\gamma)$ ,  $\varphi$  is still an isomorphism of semisimple Lie algebras, that is, it satisfies  $\varphi([x, y]'_\alpha) = [\varphi(x), \varphi(y)]'_\gamma$ .

Let  $(V, \rho_\beta, \beta)$  and  $(V, \rho_\delta, \delta)$  be two representations on the vector space  $V$  of respectively  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  and  $(\mathfrak{g}, [\cdot, \cdot]_\gamma, \gamma)$ . The corresponding induced representations  $(V, \rho'_\alpha)$  and  $(V, \rho'_\gamma)$  respectively of  $(\mathfrak{g}, [\cdot, \cdot]'_\alpha)$  and  $(\mathfrak{g}, [\cdot, \cdot]'_\gamma)$  are equivalent. So there exists a linear automorphism  $T$  of  $End(V)$  such that  $\rho'_\gamma \circ \varphi = T \circ \rho'_\alpha$ . By Skolem-Noether Theorem [3], there exists an  $S \in GL(V)$  such that  $T = Ad_S$ . Therefore,

$$\forall x \in \mathfrak{g}, \rho'_\gamma(\varphi(x)) = S \circ \rho'_\alpha(x) \circ S^{-1}. \tag{7}$$

Using Theorem 2.11, Theorem 3.4 and Proposition 2.9,

$$\begin{aligned} \delta^2 \circ \rho'_\gamma(\varphi(x)) \circ \delta^{-1} &= \rho_\delta(\gamma(\varphi(x))) = \rho_\delta(\varphi(\alpha(x))) = \delta \circ \rho'_\gamma(\varphi(\alpha(x))) \\ &= \delta \circ S \circ \rho'_\alpha(\alpha(x)) \circ S^{-1} = \delta \circ S \circ \beta \circ \rho'_\alpha(x) \circ \beta^{-1} \circ S^{-1}. \end{aligned}$$

So we get 
$$\delta^2 \circ \rho'_\gamma(\varphi(x)) = \delta \circ S \circ \beta \circ \rho'_\alpha(x) \circ \beta^{-1} \circ S^{-1}. \tag{8}$$

Moreover, using (7) we get

$$\delta^2 \circ \rho'_\gamma(\varphi(x)) = \delta^2 \circ S \circ \rho'_\alpha(x) \circ S^{-1} \circ \delta^{-1}. \tag{9}$$

Comparing (8) and (9), we will obtain  $S \circ \beta = \delta \circ S$ . Therefore  $\delta = S \circ \beta \circ S^{-1}$ . ■

### 5. Weight modules of simple multiplicative Hom-Lie algebras

We introduce and discuss in the following the root space decomposition of simple multiplicative Hom-Lie algebras, weight modules and Verma modules.

#### 5.1. Root space decomposition of simple multiplicative Hom-Lie algebras

Let  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  be a simple multiplicative Hom-Lie algebra and let  $\mathfrak{g}_1$  be a simple ideal generating the induced Lie algebra  $(\mathfrak{g}, [\cdot, \cdot])$  such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \alpha(\mathfrak{g}_1) \dots \oplus \alpha^{n-1}(\mathfrak{g}_1).$$

Let  $\mathfrak{h}_1$  be a Cartan subalgebra of  $\mathfrak{g}_1$  (we write CSA for abbreviation). The decomposition of  $\mathfrak{g}_1$  into root spaces relatively to  $\mathfrak{h}_1$  is given by  $\mathfrak{g}_1 = \mathfrak{h}_1 \oplus \bigoplus_{\eta \in \Delta_1} (\mathfrak{g}_1)_\eta$ , where  $\Delta_1$  is the set of roots. Since  $\alpha$  is an automorphism of  $\mathfrak{g}$ . For all  $\eta \in (\mathfrak{g}_1)_\eta, x \in \mathfrak{g}_1$  and  $h \in \mathfrak{h}_1$ , we have  $\alpha^k([h, x]) = \eta(h)\alpha^k(x), \forall 0 \leq k \leq n - 1$ . Namely,  $[\alpha^k(h), \alpha^k(x)] = \eta(h)\alpha^k(x)$  and so  $[\alpha^k(h), \alpha^k(x)] = \eta \circ \alpha^{-k}(\alpha^k(h))\alpha^k(x)$ . Then,  $\alpha^k((\mathfrak{g}_1)_\eta)$  is a root space of  $\alpha^k(\mathfrak{g}_1)$  and we have  $\alpha^k((\mathfrak{g}_1)_\eta) = (\alpha^k(\mathfrak{g}_1))_{\eta \circ \alpha^{-k}}$ . The set of roots of  $\alpha^k(\mathfrak{g}_1)$  is given by  $\Delta_{k+1} = \Delta_1 \circ \alpha^{-k} = \{\eta \circ \alpha^{-k}, \eta \in \Delta_1\}$ .

**Remark 5.1.** If  $\alpha^n$  is an outer automorphism, then  $\alpha^n(\mathfrak{h}_1) = \mathfrak{h}_1$ , otherwise,  $\alpha^n(\mathfrak{h}_1) \neq \mathfrak{h}_1$ .

A consequence of the discussion above is the following proposition.

**Proposition 5.2.** Let  $\mathfrak{g} = \mathfrak{g}_1 \oplus \alpha(\mathfrak{g}_1) \dots \oplus \alpha^{n-1}(\mathfrak{g}_1)$  be the induced Lie algebra of a simple multiplicative Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ . Then

(1)  $\mathfrak{h} = \mathfrak{h}_1 \oplus \alpha(\mathfrak{h}_1) \oplus \dots \oplus \alpha^{n-1}(\mathfrak{h}_1)$  is a CSA of  $\mathfrak{g}$  and  $\Delta = \bigsqcup_{k=0}^{n-1} \Delta_{k+1}$  is the set of roots with respect to  $\mathfrak{h}$  and the root space decomposition is given by

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{k=0}^{n-1} \bigoplus_{\eta \in \Delta_{k+1}} \alpha^k(\mathfrak{g}_1)_\eta.$$

(2) Let  $\gamma$  be another automorphism which satisfies  $\mathfrak{g} = \mathfrak{g}_1 \oplus \gamma(\mathfrak{g}_1) \dots \oplus \gamma^{n-1}(\mathfrak{g}_1)$  and  $\gamma^n(\mathfrak{g}_1) = \mathfrak{g}_1$ . Then, there exists  $\varphi \in \text{Aut}(\mathfrak{g})$  such that  $\varphi \circ \alpha = \gamma \circ \varphi$ , ( $\gamma = \varphi \circ \alpha \circ \varphi^{-1}$ ) and  $\gamma^k(\mathfrak{g}_1) = \gamma^k(\mathfrak{h}_1) \oplus \bigoplus_{\eta \in \Delta_1 \circ \gamma^{-k}} (\gamma^k(\mathfrak{g}_1))_\eta$ .

Set  $\Delta'_{k+1} = \Delta_1 \circ \gamma^{-k} = \Delta_1 \circ \varphi \circ \alpha^{-k} \circ \varphi^{-1}$  and  $\mathfrak{h}' = \mathfrak{h}_1 \oplus \gamma(\mathfrak{h}_1) \oplus \dots \oplus \gamma^{n-1}(\mathfrak{h}_1)$ .

Hence  $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{k=0}^{n-1} \bigoplus_{\eta \in \Delta'_{k+1}} (\gamma^k(\mathfrak{g}_1))_\eta)$  is the root space decomposition of  $\mathfrak{g}$  with respect to  $\mathfrak{h}'$ .

#### 5.2. Weight modules of simple multiplicative Hom-Lie algebras

Let  $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_n$  be a semisimple Lie algebra. Let  $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \dots \oplus \mathfrak{h}_n$  be a CSA of  $\mathfrak{g}$  and  $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$  its triangular decomposition. Let  $\Delta$  be the set of

roots with respect to  $\mathfrak{h}$ . Let  $\rho : \mathfrak{g} \rightarrow \text{End}(V)$  be a representation of  $\mathfrak{g}$ . Let  $\lambda \in \mathfrak{h}^*$  and  $V_\lambda = \{v \in V : \rho(h)v = \lambda(h)v, \forall h \in \mathfrak{h}\}$ . If  $V_\lambda \neq \{0\}$ ,  $\lambda$  is called a weight of  $V$  and  $V_\lambda$  is called a weight subspace of  $V$  of weight  $\lambda$ . Denote the set of all weights by  $P(V)$ . If  $V = \bigoplus_{\lambda \in P(V)} V_\lambda$  is a direct sum of its weight subspaces then, we say that  $V$  is a weight module. Every weight module of  $\mathfrak{g}$  is a sum of modules of the form  $V_1 \otimes \dots \otimes V_n$  where  $V_i$  is a module of the simple factor  $\mathfrak{g}_i$  of  $\mathfrak{g}$ . Let  $V = V_1 \otimes \dots \otimes V_n$ , then the set of weights of  $V$  is given by:

$$P(V) = P(V_1) \oplus \dots \oplus P(V_n) = \left\{ \sum_{k=1}^n \lambda_i / \lambda_i \in P(V_i) \right\}$$

and we have 
$$V = \bigoplus_{(\lambda_1, \dots, \lambda_n) \in P(V)} (V_1)_{\lambda_1} \otimes \dots \otimes (V_n)_{\lambda_n}.$$

Let  $(W, \rho)$  be a representation of  $\mathfrak{g}_1$ . By Theorem 4.6 and Proposition 4.4, we associate a representation  $(W, \tilde{\rho})$  of  $\alpha^k(\mathfrak{g}_1)$  such that there exists  $\beta \in GL(W)$  satisfying  $\tilde{\rho}(\alpha^k(x)) = \beta^k \circ \rho_k \circ \beta^{-k}$ .

We give a weight module version of Theorem 4.6.

**Proposition 5.3.** *Let  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  be a simple multiplicative Hom-Lie algebra and  $\mathfrak{g} = \mathfrak{g}_1 \oplus \alpha(\mathfrak{g}_1) \oplus \alpha^2(\mathfrak{g}_1) \oplus \dots \oplus \alpha^{n-1}(\mathfrak{g}_1)$  be its induced Lie algebra,  $(\alpha^n(\mathfrak{g}_1) = \mathfrak{g}_1)$ .*

- (1) *Consider  $n$  weight modules  $(V_0, \rho_0), (V_1, \rho_1), \dots, (V_{n-1}, \rho_{n-1})$  of  $\mathfrak{g}_1$ . Then, for all integer  $k = 0, \dots, n-1$ , there exists a representation  $(V_k, \tilde{\rho}_k)$  of  $\alpha^k(\mathfrak{g}_1)$  on  $V_k$  and  $\beta_k \in GL(V_k)$  satisfying  $\tilde{\rho}_k(\alpha^k(x)) = \beta_k^k \circ \rho_k(x) \circ \beta_k^{-k}$  for all  $x \in \mathfrak{g}_1$ . Then  $V_k$  is a weight module of  $\alpha^k(\mathfrak{g}_1)$  and  $\beta_k^k((V_k)_\lambda)$  is a weight subspace of  $V_k$  of weight  $\lambda \circ \alpha^{-k}$ .*
- (2) *Let  $P(V_i)$  be the set of weights of  $V_i$  with respect to  $\mathfrak{h}_1$ . The set of weights of  $V = \bigotimes_{i=0}^{n-1} V_i$  with respect to  $\mathfrak{h} = \mathfrak{h}_1 \oplus \alpha(\mathfrak{h}_1) \oplus \dots \oplus \alpha^{n-1}(\mathfrak{h}_1)$  is given by*

$$P_{\mathfrak{h}}(V) = \left\{ \sum_{k=0}^{n-1} \lambda_i \circ \alpha^{-k} / \lambda_i \in P(V_i) \right\}.$$

The weight decomposition of  $V$  is given by

$$V = \bigoplus_{\lambda \in P(V)} \left( (V_0)_{\lambda_0} \otimes (V_1)_{\lambda_1 \circ \alpha^{-1}} \otimes \dots \otimes (V_{n-1})_{\lambda_{n-1} \circ \alpha^{1-n}} \right).$$

**Proof.** Let  $(V_k, \rho_k)$  be a weight module of  $\mathfrak{g}_1$ . Then  $V_k = \bigoplus_{\lambda \in P(V_k)} (V_k)_\lambda$ , where  $(V_k)_\lambda = \{v \in V_k / \rho_k(h)v = \lambda(h)v, h \in \mathfrak{h}_1\}$ . By Proposition 4.4, there exists  $\tilde{\rho}_k$  and  $\beta_k \in GL(V_k)$  such that for  $h \in \mathfrak{h}_1, v \in (V_k)_\lambda$  we have

$$\tilde{\rho}_k(\alpha^k(h))\beta_k^k(v) = \beta_k^k \rho_k(h)v = \beta_k^k(\lambda(h)v) = \lambda(h)\beta_k^k(v) = \lambda \circ \alpha^{-k}(\alpha^k(h))(\beta_k^k(v)).$$

So  $\beta_k^k((V_k)_\lambda) = (V_k)_\lambda$  is a weight subspace of  $\beta_k^k(V_k) = V_k$  with respect to the CSA  $\alpha^k(\mathfrak{h}_1)$  of  $\alpha^k(\mathfrak{g}_1)$  of weight  $\lambda \circ \alpha^{-k}$ . Hence we have  $\beta_k^k((V_k)_\lambda) = (\beta_k^k(V_k))_{\lambda \circ \alpha^{-k}}$  and

$$V_k = \beta_k^k(V_k) = \bigoplus_{\lambda \in P(V_k)} \beta_k^k((V_k)_\lambda) = \bigoplus_{\lambda \in P(V_k)} (\beta_k^k(V_k))_{\lambda \circ \alpha^{-k}} = \bigoplus_{\lambda \in P(V_k)} (V_k)_{\lambda \circ \alpha^{-k}}.$$

Thus, a weight decomposition of  $V$  is given by

$$V = \bigoplus_{\lambda \in P(V)} \left( (V_0)_{\lambda_0} \otimes (V_1)_{\lambda_1 \circ \alpha^{-1}} \otimes \dots \otimes (V_{n-1})_{\lambda_{n-1} \circ \alpha^{1-n}} \right). \quad \blacksquare$$

**Remark 5.4.** Since  $\alpha^n$  is an automorphism of  $\mathfrak{g}_1$ , if  $\alpha^n$  is an outer automorphism then  $\alpha^n(\mathfrak{h}_1) = \mathfrak{h}_1$  and  $\beta^n(V_{\lambda_k}) = V_{\lambda_k}, 0 \leq k \leq n - 1$ .

If  $\alpha^n$  is an inner automorphism then  $\alpha^n(\mathfrak{h}_1) \neq \mathfrak{h}_1$  and  $\beta^n(V_{\lambda_k}) = V_{\lambda_k \circ \alpha^{-n}}$ .

**Definition 5.5.** (1) A module of a simple multiplicative Hom-Lie algebra is called a *weak weight module* if it is a weight module of the induced semisimple Lie algebra. By a *weak weight subspace* of a module of a simple multiplicative Hom-Lie algebra we understand a weight subspace of the weight module of the induced Lie algebra.

(2) A strong weight module  $(V, \rho_\beta, \beta)$  of  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  is a weight module of  $(\mathfrak{g}, [\cdot, \cdot])$  such that  $\beta$  transforms weight subspaces to weight subspaces.

Let  $\mathfrak{g}$  be a simple Lie algebra. Then, every irreducible finite dimensional module of  $\mathfrak{g}$  is a highest weight module and its highest weight is a dominant weight. Let  $P^+$  be the set of dominant weights of  $\mathfrak{g}$  and  $\lambda \in P^+$ . Denote such irreducible module by  $V(\lambda)$  and  $v_\lambda$  its highest weight vector.

**Proposition 5.6.** *Keeping the same hypothesis as in Proposition 5.3 and let  $(V, \rho_\beta, \beta)$  be a finite dimensional irreducible module of  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ . Then  $(V, \rho_\beta, \beta)$  is a weak weight module.*

*Moreover, there exist  $(\lambda_0, \lambda_1, \dots, \lambda_{n-1})$  dominant weights for  $\mathfrak{g}_1$  such that*

$$V = V(\lambda_0) \otimes V(\lambda_1 \circ \alpha^{-1}) \otimes \dots \otimes V(\lambda_{n-1} \circ \alpha^{-(n-1)})$$

*is an irreducible  $\mathfrak{g}$ -weight module of highest weight  $(\lambda_0, \lambda_1 \circ \alpha^{-1}, \dots, \lambda_{n-1} \circ \alpha^{1-n})$ .*

*Furthermore, using Theorem 4.6, there exist  $\beta_0 \in GL(V(\lambda_0)), \beta_1 \in GL(V(\lambda_1)), \dots, \beta_{n-1} \in GL(V(\lambda_{n-1}))$  such that  $\beta_k^k(V(\lambda_k)) = V(\lambda_k \circ \alpha^{-k})$ .*

**Proof.** Let us assume that  $V(\lambda_k)$  is a highest weight module of highest weight  $\lambda_k \in \mathfrak{h}_1^*$  of  $\mathfrak{g}_1$  and highest weight vector  $v_{\lambda_k}, (v_{\lambda_k} \neq 0)$ . Since

$$\alpha^k(\mathfrak{g}_1) = \alpha^k(\mathfrak{n}_1^+) \oplus \alpha^k(\mathfrak{h}_1) \oplus \alpha^k(\mathfrak{n}_1^-),$$

by Proposition 4.4 we have for all  $x \in \mathfrak{n}^+$ ,

$$\tilde{\rho}_k(\alpha^k(x)) \circ \beta_k^k(v_{\lambda_k}) = \beta_k^k \circ \rho_k(x)(v_{\lambda_k}) = \beta_k^k(\vec{0}) = \vec{0},$$

So  $\beta_k^k(v_{\lambda_k})$  is a highest weight vector of highest weight  $\lambda_k \circ \alpha^{-k}$  of  $\beta_k^k(V(\lambda_k)) = V(\lambda_k \circ \alpha^{-k})$  as  $\alpha^k(\mathfrak{g}_1)$ -module and we have  $(v_{\lambda_0} \otimes \beta_1(v_{\lambda_1}) \otimes \dots \otimes \beta_{n-1}^{n-1}(v_{\lambda_{n-1}}))$  is a highest weight vector of  $V$  considered as  $\mathfrak{g}$ -module.

Let  $\beta = \beta_0 \otimes \dots \otimes \beta_{n-1}$  and set  $\rho_\beta := \beta \circ \rho$ , where  $\rho$  is the representation of the induced semisimple Lie algebra. Then,  $(V, \rho_\beta, \beta)$  is a representation of the simple multiplicative Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ . It turns out that  $\beta_k$  do not necessary transforms weight subspaces to weight subspaces. If it is the case then the weight module of the induced Lie algebra becomes weight module for the multiplicative Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$ . The same holds for highest weight modules. ■

Let  $M(\lambda)$  be a Verma module of  $\mathfrak{g}$  of highest weight  $\lambda \in P$ . If there exists  $\alpha \in \Delta_+$  (the set of positive roots) such that  $\lambda(\alpha) \in \mathbb{N}$ , then  $M(\lambda)$  is reducible and there exists a maximal submodule  $\overline{M(\lambda)}$  of  $M(\lambda)$  such that  $V(\lambda) := M(\lambda)/\overline{M(\lambda)}$  is irreducible. Otherwise,  $\lambda(\alpha) \in \mathbb{R} \setminus \mathbb{N}$  and  $M(\lambda)$  is irreducible.

**Remark 5.7.** Let  $M(\lambda_0), \dots, M(\lambda_{n-1})$  be  $n$  Verma modules of  $\mathfrak{g}_1$ . Then  $M(\lambda) := M(\lambda_0) \otimes M(\lambda_1 \circ \alpha^{-1}) \otimes \dots \otimes M(\lambda_{n-1} \circ \alpha^{-(n-1)})$  is a Verma module of  $\mathfrak{g} = \mathfrak{g}_1 \oplus \alpha(\mathfrak{g}_1) \oplus \dots \oplus \alpha^{n-1}(\mathfrak{g}_1)$  of highest weight  $\lambda = (\lambda_0, \lambda_1 \circ \alpha^{-1}, \dots, \lambda_{n-1} \circ \alpha^{-(n-1)})$ .

Let  $(M(\lambda), \rho)$  the corresponding representation of  $\mathfrak{g}$ . Using Theorem 4.6, there exists  $\beta \in GL(M(\lambda))$  such that,  $(M(\lambda), \rho_\beta, \beta)$  is a weak weight module of  $(\mathfrak{g}, [\cdot, \cdot]_\alpha, \alpha)$  where  $\rho_\beta = \beta \circ \rho$ .

### 6. Applications

The simple Lie algebra  $\mathfrak{sl}(2)$  is the smallest simple Lie algebra which plays a distinguish role in Lie theory. In this section, we provide examples and study representations of simple Hom-Lie algebras of  $\mathfrak{sl}(2)$ -type.

#### 6.1. Representations of the $\mathfrak{sl}(2)$ -type Hom-Lie algebras

We consider the usual Lie algebra  $\mathfrak{sl}(2)$  generated by  $\{e, f, h\}$  and defined by the brackets  $[h, e] = 2e$ ,  $[h, f] = -2f$ ,  $[e, f] = h$ . We call  $\mathfrak{sl}(2)$ -type Hom-Lie algebras or Hom- $\mathfrak{sl}(2)$ , the Hom-Lie algebras obtained by applying the Yau twist to  $\mathfrak{sl}(2)$ . The twisted algebras are obtained along algebra morphisms which are automorphisms in the case of  $\mathfrak{sl}(2)$ . They are determined in [10], see also [5]. In this section we will study in details representations of  $\mathfrak{sl}(2)$ -type Hom-Lie algebras. The Hom- $\mathfrak{sl}(2)$ -modules will be constructed using Theorem 2.11. One needs first to consider the set of all morphisms on  $\mathfrak{sl}(2)$  and then seek for twistings of  $\mathfrak{sl}(2)$ -modules.

**Lemma 6.1.** *Every diagonal Twist of  $\mathfrak{sl}(2)$  is given by a morphism  $\alpha$  defined with respect to the basis  $\{e, f, h\}$  by  $\alpha(e) = \lambda e$ ,  $\alpha(f) = \lambda^{-1}f$ ,  $\alpha(h) = h$ , where  $\lambda$  is a nonzero parameter in  $\mathbb{K}$ . Let  $(V, \rho_\beta, \beta)$  be a representation of  $\text{Hom-}\mathfrak{sl}(2) = (\mathfrak{sl}(2), [\cdot, \cdot]_\alpha = \alpha([\cdot, \cdot]), \alpha)$  where  $V$  is an  $(n + 1)$ -dimensional vector space with basis  $\{v_0, \dots, v_n\}$ . Then  $\beta(v_i) = \lambda^{-i}b_0v_i$ ,  $0 \leq i \leq n$ , and  $b_0 \in \mathbb{K}$ .*

**Proof.** Straightforward calculations show that the  $\mathfrak{sl}(2)$ -type Hom-Lie algebras with diagonal twist are given by morphisms  $\alpha$  of the form  $\alpha(e) = \lambda e$ ,  $\alpha(f) = \lambda^{-1}f$ ,  $\alpha(h) = h$ , where  $\lambda$  is a parameter different from 0 in  $\mathbb{K}$ . Therefore Hom- $\mathfrak{sl}(2)$  is equipped with the following bracket,

$$[h, e]_\alpha = 2\lambda e, \quad [h, f]_\alpha = -2\lambda^{-1}f, \quad [e, f]_\alpha = h, \quad \text{where } \lambda \neq 0, 1.$$

Let  $\rho$  be a representation of  $\mathfrak{sl}(2, \mathbb{C})$  on an  $(n + 1)$ -dimensional vector space  $V$  generated by  $\{v_0, \dots, v_n\}$ . It is defined as follows

$$\begin{cases} \rho(e)v_i = (n - i + 1)v_{i-1}, \quad \forall i = 1, \dots, n, \quad \rho(e)v_0 = 0, \\ \rho(f)v_i = (i + 1)v_{i+1}, \quad \forall i = 0, \dots, n - 1, \quad \rho(f)v_n = 0, \\ \rho(h)v_i = (n - 2i)v_i, \quad \forall i = 0, \dots, n. \end{cases}$$

In the following, we twist the previous representation with respect to Hom- $\mathfrak{sl}(2)$ . Let  $\beta \in \text{End}(V)$  and set  $\beta(v_i) = \sum_{j=0}^n a_{ij}v_j$ . We construct maps  $\beta$  that satisfy Condition (3): First, we apply  $\rho$  to  $e$ , then in the LHS we get

$$\rho(\alpha(e))\beta(v_i) = \lambda \sum_{j=1}^n (n - j + 1)a_{ij}v_{j-1} \quad \text{and} \quad \beta(\rho(e)v_i) = (n - i + 1) \sum_{j=0}^n a_{i-1,j}v_j$$

on the RHS. For  $j = n$ , the LHS vanishes;  $(n - i + 1)a_{i-1,n} = 0$  and so we get

$$a_{0,n} = \dots = a_{n-1,n} = 0. \tag{10}$$

Considering the equality, one has  $(n - i + 1) \sum_{j=0}^{n-1} a_{i-1,j}v_j = \lambda \sum_{j=1}^n (n - j + 1)a_{ij}v_{j-1}$ .  
 Rewriting the equality we get

$$\begin{aligned} & (n - i + 1)(a_{i-1,0}v_0 + a_{i-1,1}v_1 + \dots + a_{i-1,i-1}v_{i-1} + \dots + a_{i-1,n-1}v_{n-1}) \\ & = \lambda(na_{i1}v_0 + (n - 1)a_{i1}v_1 + \dots + (n - (i - 1))a_{ii}v_{i-1} + \dots + a_{in}v_{n-1}). \end{aligned}$$

More precisely, we have

$$\begin{aligned} & (n - i + 1)a_{i-1,0}v_0 + (n - i + 1)a_{i-1,1}v_1 + \dots + (n - i + 1)a_{i-1,i-1}v_{i-1} + \\ & \quad + \dots + (n - i + 1)a_{i-1,n-1}v_{n-1} \\ & = \lambda na_{i1}v_0 + \lambda(n - 1)a_{i1}v_1 + \dots + \lambda(n - (i - 1))a_{ii}v_{i-1} + \dots + \lambda a_{in}v_{n-1}. \end{aligned}$$

By identification we get the following system of  $n$  equations

$$\left\{ \begin{array}{l} (n - i + 1)a_{i-1,0} = \lambda na_{i1} \\ (n - i + 1)a_{i-1,1} = \lambda(n - 1)a_{i2} \\ (n - i + 1)a_{i-1,2} = \lambda(n - 2)a_{i3} \\ \vdots \\ (n - i + 1)a_{i-1,i-1} = \lambda(n - i + 1)a_{ii} \\ (n - i + 1)a_{ii} = \lambda(n - i)a_{i+1,i+1} \\ \vdots \\ (n - i + 1)a_{i-1,n-1} = \lambda a_{in}. \end{array} \right.$$

Setting  $i = 1$ , we get  $a_{00} = \lambda a_{11}$  and  $a_{12} = \dots = a_{1n}$  and from (10) we get

$$a_{12} = \dots = a_{1n} = 0. \tag{11}$$

For  $i = 2$  we have  $\left\{ \begin{array}{l} (n - 1)a_{10} = \lambda na_{21} \\ (n - 1)a_{11} = \lambda(n - 1)a_{22} \\ (n - 1)a_{12} = \lambda(n - 2)a_{23} \\ \vdots \\ (n - 1)a_{1,n-1} = \lambda a_{2n}. \end{array} \right.$

Using (10) and (11) we obtain

$$a_{21} = a_{23} = \dots = a_{2,n-1} = 0. \tag{12}$$

It remains to verify that  $a_{20} = 0$ .

For  $i = 3$  we have  $\left\{ \begin{array}{l} (n - 2)a_{20} = \lambda na_{31} \\ (n - 2)a_{21} = \lambda(n - 1)a_{32} \\ a_{22} = \lambda a_{33} \\ \vdots \\ (n - 2)a_{2,n-1} = \lambda a_{3n} \end{array} \right.$

Similarly, (12) leads to  $a_{34} = \dots = a_{3n} = 0$ . One may show that  $a_{30}$  and  $a_{31}$  also vanish, and so on.

The case  $i = n$  leads to  $a_{n-1,0} = \lambda n a_{n1}$ ,  $a_{n-1,1} = \lambda(n-1)a_{n2}, \dots, a_{n-1,n-1} = \lambda a_{nn}$ . To check that the coefficients satisfy  $a_{n1} = a_{n2} = \dots = a_{n,n-1} = 0$  we use the identity (3) with generator  $f$ . We have  $\rho(\alpha(f))\beta(v_i) = \lambda^{-1} \sum_{j=0}^{n-1} a_{ij}(j+1)v_{j+1}$  and  $\beta(\rho(f)v_i) = (i+1) \sum_{j=0}^n a_{i+1,j}v_j$ . Writing the equality, we obtain

$$\begin{aligned} \lambda^{-1}a_{i0}v_1 + 2\lambda^{-1}a_{i1}v_2 + 3\lambda^{-1}a_{i2}v_3 + \dots + n\lambda^{-1}a_{i,n-1}v_n \\ = (i+1)a_{i+1,0}v_0 + \dots + (i+1)a_{i+1,n}v_n. \end{aligned}$$

Then we get the following system of  $n$  equations

$$\begin{cases} a_{i+1,0} = 0, a_{i+1,1} = 0 \\ 2\lambda^{-1}a_{i1}v_2 = (i+1)a_{i+1,2}v_2 \\ \vdots \\ n\lambda^{-1}a_{i,n-1}v_n = (i+1)a_{i+1,n-1}v_n \end{cases}$$

Solving the system, we obtain a diagonal matrix, where

$$\lambda^{-1}a_{ii} = a_{i+1,i+1}, \forall 1 \leq i \leq n.$$

Setting  $a_{00} = b_0$ , we have  $a_{ii} = \lambda^{-i}b_0$  for  $i = 1, \dots, n$ . ■

We proved that  $\beta$  is given by a diagonal matrix, that is for all  $v_i \in V$ ,  $\beta(v_i) = \lambda^{-i}b_0v_i$ , where  $b_0$  is a nonzero scalar. In the following, we characterize the action with respect to  $\beta$ , see also [13].

**Theorem 6.2.** *The representations  $(V, \rho_\beta, \beta)$  of  $\mathfrak{sl}(2)$ -type Hom-Lie algebras on a vector space  $V$  equipped with a basis  $\{v_0, \dots, v_n\}$  are given by  $\beta$  such that  $\beta(v_i) = \lambda^{-i}b_0v_i, b_0 \neq 0$  and the actions defined by*

$$\begin{aligned} \rho_\beta(e)v_i &= (n-i+1)\lambda^{-i+1}b_0v_{i-1}, \quad i = 1, \dots, n. \\ \rho_\beta(f)v_i &= (i+1)\lambda^{-i-1}b_0v_{i+1}, \quad i = 0, \dots, n-1. \\ \rho_\beta(h)v_i &= (2i-n)\lambda^{-i}b_0v_i, \quad i = 0, \dots, n. \end{aligned}$$

**Proof.** This is straightforward using Theorem 2.11. ■

From the foregoing theorem we can classify the irreducible  $\mathfrak{sl}(2)$ -weight modules as follows.

**Theorem 6.3.** *There are precisely four types of irreducible Hom- $\mathfrak{sl}(2)$ -modules. The actions are described in the following.*

(1) *The finite-dimensional irreducible modules with basis  $\{v_0, v_1, \dots, v_n\}$  and where*

$$\begin{aligned} h.v_i &= (2i-n)\lambda^{-i}b_0v_i, \quad 0 \leq i \leq n, \quad e.v_i = \lambda^{-i-1}b_0v_{i+1}, \quad 0 \leq i \leq n, \quad e.v_n = 0, \\ f.v_i &= i\lambda^{-i+1}b_0(n+1-i)v_{i-1}, \quad 0 < i \leq n, \quad f.v_0 = 0, \quad \text{with } \beta(v_i) = \lambda^{-i}b_0v_i. \end{aligned}$$

- (2) *The irreducible infinite-dimensional lowest weight Hom- $\mathfrak{sl}(2)$ -modules, with a basis of  $h$ -eigenvectors  $\{v_0, v_1, \dots\}$  and nonnegative integer  $\tau$ , such that*

$$h.v_i = \lambda^{-i}b_0(\tau + 2i)v_i, \quad i \geq 0, \quad e.v_i = \lambda^{-i-1}b_0v_{i+1}, \quad i \geq 0,$$

$$f.v_i = -i\lambda^{-i+1}b_0(\tau + i - 1)v_{i-1}, \quad i > 0, \quad f.v_0 = 0, \quad \text{with } \beta(v_i) = \lambda^{-i}b_0v_i.$$

- (3) *The irreducible infinite-dimensional highest weight Hom- $\mathfrak{sl}(2)$ -modules, with a basis of  $h$ -eigenvectors  $\{v_0, v_1, \dots\}$  and  $\tau \in \mathbb{Z} \cap ]-\infty, 0[$ , such that*

$$h.v_i = \lambda^{-i}b_0(\tau - 2i)v_i, \quad i \geq 0, \quad f.v_i = \lambda^{-i-1}b_0v_{i+1}, \quad i \geq 0,$$

$$e.v_i = i\lambda^{-i+1}b_0(\tau - i + 1)v_{i-1}, \quad i > 0, \quad e.v_0 = 0, \quad \text{with } \beta(v_i) = \lambda^{-i}b_0v_i.$$

- (4) *The irreducible infinite-dimensional Hom- $\mathfrak{sl}(2)$ -modules (the Hom-module of intermediate series) with a basis  $\{\dots, v_{-2}, v_{-1}, v_0, v_1, v_2, \dots\}$  such that*

$$h.v_i = \lambda^{-i}b_0(\tau + 2i)v_i, \quad i \in \mathbb{Z},$$

$$e.v_i = \lambda^{-i-1}b_0v_{i+1}, \quad \text{if } i \geq 0, \quad f.v_i = \lambda^{-i+1}b_0v_{i-1}, \quad \text{if } i \leq 0,$$

$$e.v_i = \frac{1}{4}(\mu - (\tau + 2i + 1)^2 + 1)\lambda^{-i-1}b_0v_{i+1}, \quad \text{if } i < 0,$$

$$f.v_i = \frac{1}{4}(\mu - (\tau + 2i - 1)^2 + 1)\lambda^{-i+1}b_0v_{i-1}, \quad \text{if } i > 0,$$

with  $\beta(v_i) = \lambda^{-i}b_0v_i$  and  $\tau \neq \sqrt{\mu + 1}$ ,  $\mu, \tau \in \mathbb{Z}$ .

**6.2. General method to compute Hom- $\mathfrak{sl}(2)$ -modules**

Let  $\rho_\beta : \mathfrak{sl}(2) \rightarrow \text{End}(V)$  be the representation of the Hom- $\mathfrak{sl}(2)$  and consider the action of the generators  $\{e, f, h\}$  as follows:

$$\rho_\beta(e)v_i = \mu_i v_{i-1}, \quad \rho_\beta(f)v_i = \gamma_i v_{i+1}, \quad \rho_\beta(h)v_i = \nu_i v_i, \quad \text{with } \beta(v_i) = \eta_i v_i.$$

The next step is to calculate the parameters  $\nu_i, \gamma_i, \mu_i$ , and  $\eta_i$  so that  $\rho_\beta$  be a Hom- $\mathfrak{sl}(2)$ -module on a  $(n + 1)$ -dimensional vector space  $V$ . Straightforward computation using Definition 2.8 gives for each  $i = 0, \dots, n$  the following 5 equations

$$\left\{ \begin{array}{l} (1) \quad \nu_i \eta_i = \lambda \gamma_i \mu_{i+1} - \lambda^{-1} \mu_i \gamma_{i-1} \\ (2) \quad \mu_i \eta_i = \frac{1}{2\lambda} (\mu_i \nu_{i-1} - \lambda \nu_i \mu_i) \\ (3) \quad \gamma_i \eta_i = -\frac{\lambda}{2} (\gamma_i \nu_{i+1} - \lambda^{-1} \nu_i \gamma_i) \\ (4) \quad \eta_{i-1} \mu_i = \lambda \eta_i \mu_i \\ (5) \quad \eta_{i+1} \gamma_i = \lambda^{-1} \eta_i \gamma_i \end{array} \right.$$

Conditions (4) and (5) lead to  $\eta_i = (\lambda^{-1})^i \eta_0, i \neq 0$ . Then

$$(1) \Rightarrow a\gamma_i \mu_{i+1} - \lambda^{-1} \mu_i \gamma_{i-1} = \nu_i (\lambda^{-1})^i \eta_0.$$

$$(2) \Rightarrow \frac{1}{2\lambda} \nu_{i-1} - \frac{1}{2} \nu_i = (\lambda^{-1})^i \eta_0.$$

$$(3) \Rightarrow -\frac{\lambda}{2} \nu_{i+1} + \frac{1}{2} \nu_i = (\lambda^{-1})^i \eta_0.$$

It follows that  $-\frac{1}{2} \nu_{i+1} + \frac{1}{2\lambda} \nu_i = (\lambda^{-1})^{i+1} \eta_0$ . Then,  $\frac{1}{\lambda} \nu_{i-1} - \nu_i = 2(\lambda^{-1})^i \eta_0$ . Thus,  $\nu_i = \frac{1}{\lambda^i} (\nu_0 - 2i\eta_0)$ . Now for  $\mu_i$  and  $\gamma_i$  we get

$$\begin{aligned}
\gamma_i \mu_{i+1} &= \lambda^{-2} \mu_i \gamma_{i-1} + \nu_i (\lambda^{-1})^i \eta_0 \\
&= \lambda^{-2} \left( a^{-2} \mu_{i-1} \gamma_{i-2} + \nu_{i-1} (a^{-1})^{i-1} \eta_0 \right) + \nu_i (a^{-1})^i \eta_0 \\
&= \lambda^{-2 \times 2} \left( \lambda^{-2} \mu_{i-2} \gamma_{i-3} + \nu_{i-2} (\lambda^{-1})^{i-2} \eta_0 \right) + (\lambda^{-1})^2 \nu_{i-1} (\lambda^{-1})^{i-1} \eta_0 + \nu_i (\lambda^{-1})^i \eta_0 \\
&= \lambda^{-2 \times 3} \mu_{i-2} \gamma_{i-3} + \lambda^{-2 \times 2} \nu_{i-2} (\lambda^{-1})^{i-2} \eta_0 + (\lambda^{-2}) \nu_{i-1} (\lambda^{-1})^{i-1} \eta_0 + \nu_i (\lambda^{-1})^i \eta_0 \\
&= \lambda^{-2 \times 3} \mu_{i-2} \gamma_{i-3} + \nu_{i-2} (\lambda^{-1})^{i+2} \eta_0 + \nu_{i-1} (\lambda^{-1})^{i+1} \eta_0 + \nu_i (\lambda^{-1})^i \eta_0 \\
&\dots \\
&= \lambda^{-2i} \mu_1 \gamma_0 + \eta_0 \sum_{k=0}^{i-1} \nu_{i-k} (\lambda^{-1})^{i+k} \left( \nu_0 - 2(i-k) \eta_0 \right) \\
&= \lambda^{-2i} \mu_1 \gamma_0 + \eta_0 \sum_{k=0}^{i-1} \lambda^{-2i} \left( \nu_0 - 2i \eta_0 + 2k \eta_0 \right) \\
&= \lambda^{-2i} \mu_1 \gamma_0 + \lambda^{-2i} \eta_0 \left( \sum_{k=0}^{i-1} \nu_0 - \sum_{k=0}^{i-1} 2i \eta_0 + \sum_{k=0}^{i-1} 2k \eta_0 \right) \\
&= \lambda^{-2i} \mu_1 \gamma_0 + \lambda^{-2i} \eta_0 \left( i \nu_0 - 2i^2 \eta_0 + 2i \frac{i-1}{2} \eta_0 \right) \\
&= \lambda^{-2i} \left( \mu_1 \gamma_0 + i \eta_0 \left( \nu_0 - (i+1) \eta_0 \right) \right).
\end{aligned}$$

Therefore, we get the following connections characterizing the parameters such that  $\rho_\beta$  is a Hom- $\mathfrak{sl}(2)$ -module on a  $(n+1)$ -dimensional vector space  $V$ :

$$\eta_i = (\lambda^{-1})^i \eta_0, i \neq 0 \quad \text{and} \quad \gamma_i \mu_{i+1} = \lambda^{-2i} \left( \mu_1 \gamma_0 + i \eta_0 (\nu_0 - (i+1) \eta_0) \right).$$

Setting  $\eta_0 = b_0, \mu_1 = 1, \gamma_0 = n$  leads to the same result obtained in Theorem 6.2.

## References

- [1] I. N. Bernstein, I. M. Gelfand, S. I. Gelfand: *Structure of representations generated by vectors of weight vectors*, Functional Analysis Appl. 5 (1971) 1–8.
- [2] N. Bourbaki: *Éléments de Mathématiques, Livre II: Algèbre, Chapitre 8: Modules et anneaux semi-simples*, Hermann (1958).
- [3] N. Bourbaki: *Éléments de Mathématiques, Chapitre 7 et 8: Groupes et algèbres de Lie*, Springer, Berlin (2007).
- [4] X. Chen, W. Han: *Classification of multiplicative simple Hom-Lie algebras*, J. Lie Theory 26 (2016) 767–775.
- [5] O. Elchinger: *Formalité Liées aux Algèbres Enveloppantes et Etudes des Algèbres Hom-(co)Poisson*, Thèse de Doctorat, Université de Haute Alsace, Mulhouse (2012).
- [6] W. Fulton, J. Harris: *Representation Theory. A First Course*, Springer, Berlin (1991).
- [7] J. Hartwig, D. Larsson, S. Silvestrov: *Deformations of Lie algebras using  $\sigma$ -derivations*, J. Algebra 295 (2006) 314–361.
- [8] N. Hu:  *$q$ -Witt algebras,  $q$ -Lie algebras,  $q$ -holomorph structure and representations*, Algebra Colloq. 6(1) (1999) 51–70.

- [9] J. E. Humphreys: *Introduction to Lie Algebras and Representation Theory*, Springer, New York (1972).
- [10] N. Jacobson: *Lie algebras*, Interscience Publishers, New York (1962).
- [11] V. G. Kac: *Infinite dimensional Lie algebras*, 3rd edition, Cambridge University Press, Cambridge (1990).
- [12] A. W. Knap: *Representation Theory of Semisimple Groups*, Princeton University Press, Princeton (1986).
- [13] X. Li: *Representations of 3-dimensional simple multiplicative Hom-Lie Algebras*, Advances Math. Physics (2013), ID 938901, 7 pp.
- [14] A. Makhlouf, F. Ammar: *Hom-Lie superalgebras and Hom-Lie admissible superalgebras*, J. Algebra 324 (2010) 1513–1528.
- [15] A. Makhlouf, S. Benayadi: *Hom-Lie algebras with symmetric invariant non-degenerate bilinear form*, J. Geometry Physics 76 (2014) 38-60.
- [16] A. Makhlouf, S. Silvestrov: *Hom-algebras structures*, J. Gen. Lie Theory Appl. 2 (2008) 51–64.
- [17] A. Makhlouf, S. Silvestrov: *Hom-algebras and hom-coalgebras*, J. Algebra Appl. 9(4) (2010) 553–589.
- [18] Y. Sheng: *Representation of Hom-Lie algebras*, J. Algebras Representation Theory 15(6) (2012) 1081–1098.
- [19] D. Yau: *Enveloping algebras of Hom-Lie algebras*, J. Generalized Lie Theory Appl. 2 (2008) 95–108.
- [20] D. Yau: *Module Hom-algebras*, arXiv:0812.4695v1 (2008).
- [21] D. Yau: *Hom-Algebras and Homology*, J. Lie Theory 19 (2009) 409–421.

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