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Lusztig-Vogan Bijection for the Complex Group G_2

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Abstract. Let G be a complex reductive Lie group. There is a bijection η between its dominant weights Λ^+ and the set $\mathcal{N}_{\mathcal{O},\tau}$. This bijection is called the Lusztig-Vogan bijection. In this paper, we present the combinatorial description of the Lusztig-Vogan bijection for the exceptional complex Lie group G_2.

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Key Words: Lusztig-Vogan bijection, dominant weight, nilpotent orbit, induced representation.

1. Introduction

Let G be a complex reductive algebraic group, and let B be a Borel subgroup of G with maximal torus T. We denote by $\Lambda = \Lambda(G)$ the *weight lattice* of G with respect to T, and by $\Lambda^+ = \Lambda^+(G)$ the set of *dominant weights* with respect to the positive roots defined by B.

Let $\mathcal{N}$ be the set of all nilpotent elements in the Lie algebra $\mathfrak{g}$ of G. Suppose $X \in \mathcal{N}$, and let $\mathcal{O}_X = \{\mathrm{Ad}(g)X | g \in G\}$ be the nilpotent orbit passing through X, with centralizer $G^X = \{g \in G | \mathrm{Ad}(g)X = X\}$. Let $\mathcal{N}_{\mathcal{O},\tau}$ denote the set of G-conjugacy classes of pairs

$$\{(\mathcal{O}_X, \tau) | X \in \mathcal{N} \text{ and } (\tau, V_\tau) \text{ an irreducible rational representation of } G^X\}.$$

Lusztig [13] conjectured the existence of a bijection $\eta : \mathcal{N}_{\mathcal{O},\tau} \longleftrightarrow \Lambda^+$ using his work on cells in affine Weyl groups. From the point of view of Harish-Chandra modules, Vogan also conjectured a bijection between these two sets. This bijection is usually called the Lusztig-Vogan bijection, and denoted by η.

We recall a brief account about the bijection η from the point of Vogan. Let K be a maximal compact subgroup of G. Let $\mathcal{C}(\mathfrak{g}, K)$ be the abelian category of finitely generated $S(\mathfrak{g}/\mathfrak{k})$-modules N, which satisfies

$$k \cdot (p \cdot n) = (\mathrm{Ad}(k)p) \cdot (k \cdot n), \quad (k \in K, p \in S(\mathfrak{p}), n \in N); \quad \text{and} \quad \mathcal{V}(N) \subset \mathcal{N}_\theta^*.$$

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Here $\mathcal{N}_\theta^*$ stands for the cone of nilpotent elements in $(\mathfrak{g}_\mathbb{C}/\mathfrak{k}_\mathbb{C})^*$ and $\mathcal{V}(N)$ is the associated variety of N [19]. Then $\{N(\mathcal{O}_X,\tau) \triangleq \mathrm{Ind}_{G^X}^G(V_\tau) | (\mathcal{O}_X,\tau) \in \mathcal{N}_{\mathcal{O},\tau}\}$ constitutes a basis for $\mathcal{C}(\mathfrak{g},K)$ in the meaning of the Grothendieck group.

By [1, Theorem 2], the Grothendieck group of $\mathcal{C}(\mathfrak{g},K)$ has another basis $\{\mathrm{Ind}_T^G(\lambda) | \lambda \in \Lambda^+\}$. Combining the two bases gives us equations of the form

$$N(\mathcal{O}_X,\tau) = \sum_{\lambda\in\Lambda^+(G)} m_\lambda(X,\tau)\mathrm{Ind}_T^G(\lambda). \tag{1}$$

We add finite similar representations $N(\mathcal{O}_{X'},\tau')$ with $\mathcal{O}_{X'} \subset \partial\mathcal{O}_X$ to make the largest dominant weight appearing in the right-hand side of (1) as small as possible. Here $\partial\mathcal{O}_X$ is the boundary of $\mathcal{O}_X$. That is,

$$N(\mathcal{O}_X,\tau) + \sum_{\mathcal{O}_{X'}\subset\partial\mathcal{O}_X;\ \tau'\in\widehat{G^{X'}}} c_{X',\tau'} N(\mathcal{O}_{X'},\tau') = \sum_{\lambda\in\Lambda^+(G)} m_\lambda(X,\tau)\mathrm{Ind}_T^G(\lambda). \tag{2}$$

Then we define the map $\eta : \mathcal{N}_{\mathcal{O},\tau} \longrightarrow \Lambda^+(G)$ by

$$\eta(\mathcal{O}_X,\tau) = \text{the largest } \lambda \text{ in the right-hand side of (2) such that } m_\lambda(X,\tau) \neq 0.$$

Such a bijection has been established by Bezrukavnikov in two papers (the bijections in each paper are conjecturally the same) [4, 5]. Bezrukavnikov's second bijection is closely related to Ostrik's conjectural description of the bijection [17, 6]. In the case of $G = GL(n,\mathbb{C})$, Achar [1] described an explicit combinatorial bijection between $\mathcal{N}_{\mathcal{O},\tau}$ and Λ^+ from the Harish-Chandra module perspective. In [18], Rush presents a combinatorial description of this bijection in type A that subsumes and dramatically simplifies Achar's algorithm. In [21], the Lusztig-Vogan bijection for local systems of some special nilpotent orbits is computed explicitly. In [22], the author gives some calculations of the Lusztig-Vogan bijection for some special local systems of exceptional Richardson orbits. In [11], the induced representations of the minimal orbits for classical groups have been studied, which will help to describe the Lusitig-Vogan bijection.

Let X be a nilpotent element. By Jacobson-Morozov theorem, there is a standard triple $\{H,X,Y\}$ satisfying $\phi \triangleq \mathrm{Span}_\mathbb{C}\{H,X,Y\} \cong \mathfrak{sl}_2(\mathbb{C})$. Let $\mathfrak{u}^X := \mathfrak{g}^X \cap [\mathfrak{g},X]$ and $\mathfrak{g}^\phi = \{Z \in \mathfrak{g} | [Z,V] = 0, \text{ for all } V \in \phi\}$.

Proposition 1.1. [7, Lemma 3.7.3] *There is a direct sum decomposition* $\mathfrak{g}^X = \mathfrak{u}^X \oplus \mathfrak{g}^\phi$ *in which* $\mathfrak{u}^X$ *is a nilpotent ideal of* $\mathfrak{g}^X$ *and* $\mathfrak{g}^\phi$ *is reductive.*

By [20, Lemma 7.5], any irreducible rational representation of G^X will be trivial when restricting to the unipotent part of G^X. Let G^X_{red} represent the reductive part of G^X, and let $\widehat{G^X_{\mathrm{red}}}$ be the set which consists of finite-dimensional irreducible representations of G^X_{red}. We will always assume $\tau \in \widehat{G^X_{\mathrm{red}}}$ to stand for an irreducible rational representation of G^X. Specially, when G^X_{red} is simply connected, we will let τ be an irreducible finite-dimensional representation of $\mathfrak{g}^\phi$ by Proposition 1.1.

The bijection η for G_2 first appears in Achar's thesis [1, Table B.3] with very limited details. In this paper, we describe this combinational bijection explicitly. There are five nilpotent orbits for G_2, which have dimensions $0,6,8,10,12$, respectively. We will denote by $\mathcal{O}_n$ to stand for the n-dimensional orbit for $n=0,6,8,10,12$.

In this paper, we use different method from [1] to deal with the orbits $\mathcal{O}_6$ and $\mathcal{O}_{10}$. For $\mathcal{O}_6$, we use the two-step induction of representations to study the structure of $N(\mathcal{O}_6,\tau)$. Meanwhile, the tool of the induction of orbit covers is used to study the space of algebraic sections of the bundle associated with the orbit $\mathcal{O}_{10}$. Together with McGovern's formulas, this will imply the bijection for $\mathcal{O}_{10}$. The tool of the induction of orbit covers is also useful for studying the induced representations [23]. We now present the main results of this article.

Theorem 1.2. *Let $\{\alpha,\beta\}$ be a simple system of G_2 with α the shorter one. Let $\{\lambda_1,\lambda_2\}$ be the two fundamental weights with respect to $\{\alpha,\beta\}$.*

(a) *For the zero orbit $\mathcal{O}_0$, the isotropy subgroup is G_2 itself. For $m,n\in\mathbb{N}$, let $V_{m\lambda_1+n\lambda_2}$ be the algebraic irreducible representation of G_2 with highest weight $m\lambda_1+n\lambda_2$. Then $\eta(\mathcal{O}_0,V_{m\lambda_1+n\lambda_2})=(m+2)\lambda_1+(n+2)\lambda_2$.*

(b) *For the minimal orbit $\mathcal{O}_6$, the isotropy subgroup $G^X_{\mathrm{red}}\cong SL(2,\mathbb{C})$. Let V_k be the irreducible representation of G^X_{red} with the highest weight $k\in\mathbb{N}$. Then*

$$\eta(\mathcal{O}_6,V_k)=\begin{cases}\frac{k+6}{2}\lambda_1 & \text{if } k \text{ is even;}\\ \frac{k+3}{2}\lambda_1+\lambda_2 & \text{if } k \text{ is odd.}\end{cases}$$

(c) *For the 8-dimensional orbit $\mathcal{O}_8$, the isotropy subgroup $G^Y_{\mathrm{red}}\cong SL(2,\mathbb{C})$. Let V_t be the irreducible representation of G^Y_{red} with the highest weight $t\in\mathbb{N}$. Then*

$$\eta(\mathcal{O}_8,V_t)=\begin{cases}\lambda_1+\frac{t+2}{2}\lambda_2 & \text{if } t \text{ is even;}\\ \frac{t+3}{2}\lambda_2 & \text{if } t \text{ is odd.}\end{cases}$$

(d) *For the subregular orbit $\mathcal{O}_{10}$, the reductive part of its isotropy subgroup is isomorphic to the symmetric group S_3. Let the irreducible representations of S_3 be represented by $\{\mathbb{1},\mathrm{sgn},\chi\}$. Then*

$$\begin{cases}\eta(\mathcal{O}_{10},\mathbb{1})=\lambda_1;\\ \eta(\mathcal{O}_{10},\mathrm{sgn})=\lambda_2;\\ \eta(\mathcal{O}_{10},\chi)=2\lambda_1.\end{cases}$$

(e) *For the principle orbit $\mathcal{O}_{12}$, the only algebraic irreducible representation of its isotropy subgroup is the trivial representation $\mathbb{1}$. Then $\eta(\mathcal{O}_{12},\mathbb{1})=0$.*

As a byproduct, we get the following formulas.

Corollary 1.3. *Let $Z\in\mathcal{O}_{10}$, then we have*

(a) $$\mathrm{Ind}^G_{G^Z}(\mathbb{1})=\mathrm{Ind}^G_T[(0)-(\lambda_1)];$$

(b) $$\mathrm{Ind}^G_{G^Z}(\mathrm{sgn})=\mathrm{Ind}^G_T[(\lambda_1)-(\lambda_2)];$$

(c) $$\mathrm{Ind}^G_{G^Z}(\chi)=\mathrm{Ind}^G_T[(\lambda_1)-(2\lambda_1)].$$

This paper is organized as follows. In Section 2, we recall the basic properties about the nilpotent orbits of G_2 and then prove Theorem 1.2(a) and Theorem 1.2(e). In Section 3, we prove Theorem 1.2(b). In Section 4, we prove Theorem 1.2(c). In Section 5, we prove Theorem 1.2(d) and Corollary 1.3.

We describe this bijection η of Theorem 1.2 explicitly in Figure 1. Figure 1 shows clearly that the dominant weights far from the hyperplanes of the fundamental Weyl chamber are attached to $\mathcal{O}_0$; that most weights near the hyperplane P_β are attached to $\mathcal{O}_6$; that most weights near the hyperplane P_α are attached to $\mathcal{O}_8$; and that a few remaining small weights are attached to the orbits $\mathcal{O}_{10}$ and $\mathcal{O}_{12}$.

2. Zero orbit and principal orbit

In this section, we recall some fundamental properties about the nilpotent orbits of the group G_2. Then we describe the combinational Lusztig-Vogan bijection for the zero orbit $\mathcal{O}_0$ and the principal orbit $\mathcal{O}_{12}$.

Let G be a complex simple group of type G_2 with Lie algebra $\mathfrak{g}$. Let $\Delta = \{\alpha, \beta\}$ be a simple system of $\mathfrak{g}$ with α the shorter one. Then its positive roots are $\{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$. The fundamental weights of $\mathfrak{g}$ are $\lambda_1 = 2\alpha+\beta$ and $\lambda_2 = 3\alpha+2\beta$. Then the set of integral dominant weight $\Lambda^+ = \{m\lambda_1+n\lambda_2 | m, n \in \mathbb{N}\}$. Let ρ be the half sum of all positive roots of $\mathfrak{g}$. The Weyl group W of $\mathfrak{g}$ is isomorphic to the Dihedral group and has order 12. Let S_α (resp. S_β) be the reflection with respect to α (resp. β).

There are 5 nilpotent orbits for G_2 by Bala-Carter Theory. For convenience of reference, we list the properties of these orbits in Table 1 reproduced from [12, Table 22.1.5]).

Label	Weight Dynkin diagram ○⇚○		$\dim \mathcal{O}$	$\dim(\mathfrak{g}^X)$	$\dim(\mathfrak{u}^X)$	$\pi_1(\mathcal{O})$
0	0	0	0	14	0	1
A_1	0	1	6	8	5	1
$\tilde{A}_1$	1	0	8	6	3	1
$G_2(a_1)$	0	2	10	4	4	S_3
G_2	2	2	12	2	2	1

Table 1: Nilpotent orbits of G_2

Let G be a complex simply connected simple Lie group with Lie algebra $\mathfrak{g}$. Then the trivial orbit $\mathcal{O}_0 \triangleq \{0\}$ is the unique nilpotent orbit of minimal dimension. Also there exists a nonzero nilpotent orbit of minimal dimension, called the *minimal orbit*, which is contained in the closure of any nonzero nilpotent orbit. This minimal orbit is generated by the action of G on a nonzero vector in the highest root space (with respect to some root system). On the other hand, there is a largest orbit, which is dense in $\mathcal{N}$ and called the *principal orbit* or *regular orbit*. Moreover, in the boundary of the principal orbit, there exists a unique open and dense orbit, called the *subregular orbit*. The subregular orbit is the second largest orbit and its closure contains any other nilpotent orbit than the principal orbit. Specially for G_2, there are five nilpotent orbits $\{\mathcal{O}_0, \mathcal{O}_6, \mathcal{O}_8, \mathcal{O}_{10}, \mathcal{O}_{12}\}$. The minimal orbit is $\mathcal{O}_6$, the principal orbit is $\mathcal{O}_{12}$ and the subregular orbit is $\mathcal{O}_{10}$.

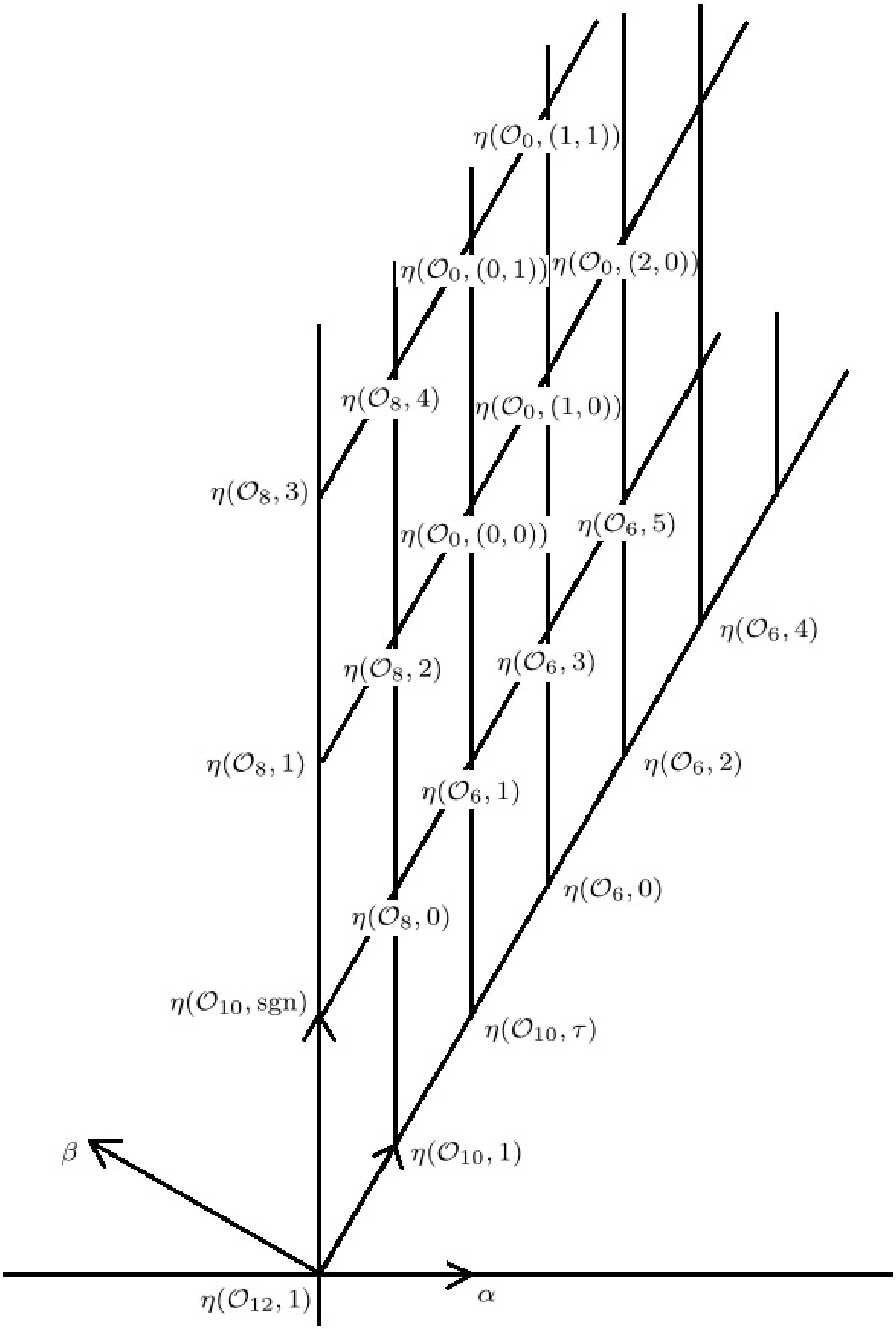

Figure 1: The bijection η for G_2

The bijection η for the zero orbit as well as the principal orbit of any G has been known explicitly. Let Π be a root system of G, and let Δ be a choice of simple system. For $\mu \in \Pi$, let $\check{\mu}$ denote the corresponding coroot. An integral dominant weight λ is called *minuscule* if $|\langle \check{\mu}, \lambda \rangle| \leq 1$ for all $\mu \in \Pi$.

Conversely, λ is said to be *majuscule* if $|\langle \check{\mu}, \lambda \rangle| \geq 2$ for all $\mu \in \Delta$.

Proposition 2.1. [1, Theorem 2.4.2] *Let G be a connected complex reductive group. Let V_λ be the irreducible representation of highest weight λ. When restricted to $\mathcal{O}_0$, the image of η is precisely the set of majuscule weights, and η is given by the formula*

$$\eta(\mathcal{O}_0, V_\lambda) = \lambda + 2\rho.$$

When restricted to the principal orbit, the image of η is precisely the set of minuscule weights. In this case, η gives a bijection between the representations of the center of G and the set of minuscule weights.

Any finite-dimensional irreducible representation of G_2 is attached to an integral dominant weight $m\lambda_1 + n\lambda_2$ with $m, n \in \mathbb{N}$. Then restricting the above proposition to the special case G_2, we have the following theorem.

Theorem 2.2. *For $m, n \in \mathbb{N}$, let $V_{m\lambda_1+n\lambda_2}$ be the algebraic irreducible representation of G_2 with highest weight $m\lambda_1 + n\lambda_2$.*

(a) *For the zero orbit $\mathcal{O}_0$, the isotropy subgroup is G_2 itself. Then*

$$\eta(\mathcal{O}_0, V_{m\lambda_1+n\lambda_2}) = (m+2)\lambda_1 + (n+2)\lambda_2.$$

(b) *For the principle orbit $\mathcal{O}_{12}$, the only algebraic irreducible representation of its isotropy subgroup is the trivial representation $\mathbb{1}$. And the only minuscule weights is zero weight 0. Then $\eta(\mathcal{O}_{12}, \mathbb{1}) = 0$.*

3. Minimal orbit

In this section, we consider the minimal orbit $\mathcal{O}_6$ of G_2. We use two-step induction of representations to compute $N(\mathcal{O}_6, \tau)$ and then get η for $\mathcal{O}_6$.

For any $\gamma \in \Pi$, we write X_γ to stand for a nonzero root vector in the root space of $\mathfrak{g}$ associated with γ. Since $3\alpha + 2\beta$ is the highest root of $\mathfrak{g}$, then $\mathcal{O}_6$ is generated by the action of G on the nonzero root vector $X_{3\alpha+2\beta}$. To be simple, we let $X = X_{3\alpha+2\beta} \in \mathcal{O}_6$. The standard triple ϕ associated to X is

$$\phi = \mathrm{Span}_{\mathbb{C}}\{\mathrm{X}_{3\alpha+2\beta}, \mathrm{X}_{-(3\alpha+2\beta)}, \mathrm{H}_\alpha + 2\mathrm{H}_\beta\}.$$

Then $$\mathfrak{g}^X = \mathfrak{g}^\phi + \mathfrak{u}^X = \mathrm{Span}_{\mathbb{C}}\{\mathrm{X}_\alpha, \mathrm{X}_\beta, \mathrm{X}_{\alpha+\beta}, \mathrm{X}_{2\alpha+\beta}, \mathrm{X}_{3\alpha+\beta}, \mathrm{X}_{3\alpha+2\beta}, \mathrm{X}_{-\alpha}, \mathrm{H}_\alpha\}$$

with $\mathfrak{g}^\phi = \mathrm{Span}_{\mathbb{C}}\{\mathrm{H}_\alpha, \mathrm{X}_\alpha, \mathrm{X}_{-\alpha}\} \cong \mathfrak{sl}_2(\mathbb{C})$. Since the irreducible representations of G^X are parameterized by $\mathbb{N}$, then we let V_k stand for the irreducible representation of G^ϕ with highest weight $k \in \mathbb{N}$. Let P_α be a parabolic subgroup of G with Lie algebra

$$\mathfrak{p}_\alpha = \mathrm{Span}_{\mathbb{C}}\{\mathrm{X}_\alpha, \mathrm{X}_\beta, \mathrm{X}_{\alpha+\beta}, \mathrm{X}_{2\alpha+\beta}, \mathrm{X}_{3\alpha+\beta}, \mathrm{X}_{3\alpha+2\beta}, \mathrm{X}_{-\alpha}, \mathrm{H}_\alpha, \mathrm{H}_\beta\}.$$

Then $\mathfrak{g}^X \subset \mathfrak{p}_\alpha = \mathfrak{l}_\alpha + \mathfrak{u}^X$, and the Levi part of $\mathfrak{p}_\alpha$ is

$$\mathfrak{l}_\alpha = \mathrm{Span}_{\mathbb{C}}\{\mathrm{X}_\alpha, \mathrm{X}_{-\alpha}, \mathrm{H}_\alpha, \mathrm{H}_\beta\} \cong \mathfrak{gl}_2(\mathbb{C}).$$

We have $$\mathrm{Ind}_{G^X}^{G}(V_k) = \mathrm{Ind}_{P_\alpha}^{G}\mathrm{Ind}_{G^X}^{P_\alpha}(V_k).$$

Here $$\mathrm{Ind}_{G^X}^{P_\alpha}(V_k) \cong \mathrm{Ind}_{SL_2(\mathbb{C})}^{GL_2(\mathbb{C})}(V_k)$$

with unipotent part U^X acting trivially.

The following two propositions will be used in our computations.

Proposition 3.1. [3, Theorem 3.10] *Let G be a complex reductive Lie group and let P be a parabolic subgroup of G with Levi factor L. Suppose V_λ to be the irreducible algebraic representation of G of highest weight λ, and E_μ to be representation of L with L-dominant weight μ. Then V_λ^* appears in $(\mathrm{Ind}_P^G)^k(E_\mu^*)$ if and only if $\mu+\rho = w(\lambda+\rho)$, for some $w \in W$ with $l(w) = k$. Here ρ is the half sum of positive roots of G, and $(\mathrm{Ind}_P^G)^k(E_\mu^*)$ is the cohomologically induced representation of G on the sheaf cohomology group $H^k(G/P, E_\mu^*)$ ([3, Definition 3.5]). In this case, V_λ^* appears with multiplicity one. Specially, when $k=0$, $(\mathrm{Ind}_P^G)^0(E_\mu^*) = \mathrm{Ind}_P^G(E_\mu^*)$.*

Proposition 3.2. [2] *Let L be a Levi factor of some parabolic subgroup of G.*

Then
$$\mathrm{Ind}_L^G(\lambda_L) = \sum_{w\in W_L} (-1)^w \mathrm{Ind}_T^G(\lambda_L + \rho_L - w\rho_L).$$
Here W_L is the Weyl group of L and ρ_L is half the sum of positive roots of L.

The structure of the module $N(\mathcal{O}_6, V_k)$ is described as follows.

Theorem 3.3. *Let V_k stand for the irreducible representation of G^X with highest weight $k \in \mathbb{N}$. Write $N(\mathcal{O}_6, V_k) \triangleq \mathrm{Ind}_{G^X}^G(V_k)$. Then we have*

$$\begin{aligned}
&N(\mathcal{O}_6, V_k)\\
&= \mathrm{Ind}_T^G(k\lambda_1) + \mathrm{Ind}_T^G(k\lambda_1+\lambda_2) + \mathrm{Ind}_T^G(k\lambda_1+2\lambda_2) + \mathrm{Ind}_T^G((k-4)\lambda_1+3\lambda_2)\\
&\quad - \mathrm{Ind}_T^G((k-3)\lambda_1+2\lambda_2) - \mathrm{Ind}_T^G((k-3)\lambda_1+3\lambda_2) - \mathrm{Ind}_T^G((k+2)\lambda_1-\lambda_2)\\
&\quad - \mathrm{Ind}_T^G((k+2)\lambda_1) - \mathrm{Ind}_T^G((k+2)\lambda_1+\lambda_2) - \mathrm{Ind}_T^G((k+6)\lambda_1-2\lambda_2)\\
&\quad + \mathrm{Ind}_T^G((k+5)\lambda_1-2\lambda_2) + \mathrm{Ind}_T^G((k+5)\lambda_1-\lambda_2) \qquad (3)
\end{aligned}$$

Proof. First, we have $\mathrm{Ind}_{SL_2(\mathbb{C})}^{GL_2(\mathbb{C})}(V_k) = \sum_{a\in\mathbb{Z}} V_{(a+k,a)}$, as $L \cong GL_2(\mathbb{C})$-representations. By Proposition 3.1, after calculation, we get that

$$\mathrm{Ind}_{G^X}^G(V_k) = \sum_{s\geq 0} V_{(k\lambda_1+s\lambda_2)},$$

as G-representations. By Proposition 3.2, we have

$$\begin{aligned}
N(\mathcal{O}_6, V_k) &= \sum_{s\geq 0} V_{(k\lambda_1+s\lambda_2)} = \sum_{s\geq 0}\sum_{w\in W} (-1)^w \mathrm{Ind}_T^G(k\lambda_1+s\lambda_2+\rho-w\rho)\\
&= \sum_{s\geq 0}[\mathrm{Ind}_T^G(k\lambda_1+s\lambda_2) + \mathrm{Ind}_T^G((k-4)\lambda_1+(s+3)\lambda_2)\\
&\quad + \mathrm{Ind}_T^G((k-3)\lambda_1+(s+4)\lambda_2) + \mathrm{Ind}_T^G((k+2)\lambda_1+(s+2)\lambda_2)\\
&\quad + \mathrm{Ind}_T^G((k+6)\lambda_1+(s-1)\lambda_2) + \mathrm{Ind}_T^G((k+5)\lambda_1+(s-2)\lambda_2)\\
&\quad - \mathrm{Ind}_T^G((k+2)\lambda_1+(s-1)\lambda_2) - \mathrm{Ind}_T^G((k+6)\lambda_1+(s-2)\lambda_2)\\
&\quad - \mathrm{Ind}_T^G((k+5)\lambda_1+s\lambda_2) - \mathrm{Ind}_T^G(k\lambda_1+(s+3)\lambda_2)\\
&\quad - \mathrm{Ind}_T^G((k-4)\lambda_1+(s+4)\lambda_2) - \mathrm{Ind}_T^G((k-3)\lambda_1+(s+2)\lambda_2)]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Ind}_T^G(k\lambda_1) + \mathrm{Ind}_T^G(k\lambda_1+\lambda_2) + \mathrm{Ind}_T^G(k\lambda_1+2\lambda_2) \\
&\quad + \mathrm{Ind}_T^G((k-4)\lambda_1+3\lambda_2) - \mathrm{Ind}_T^G((k-3)\lambda_1+2\lambda_2) \\
&\quad - \mathrm{Ind}_T^G((k-3)\lambda_1+3\lambda_2) - \mathrm{Ind}_T^G((k+2)\lambda_1-\lambda_2) \\
&\quad - \mathrm{Ind}_T^G((k+2)\lambda_1) - \mathrm{Ind}_T^G((k+2)\lambda_1+\lambda_2) - \mathrm{Ind}_T^G((k+6)\lambda_1-2\lambda_2) \\
&\quad + \mathrm{Ind}_T^G((k+5)\lambda_1-2\lambda_2) + \mathrm{Ind}_T^G((k+5)\lambda_1-\lambda_2). \qquad \blacksquare
\end{aligned}
$$

Let $\lfloor k \rfloor$ denote the largest integer not greater than $k \in \mathbb{R}$.

Theorem 3.4. *When $n \geq 1$, we have*

(a)
$$
\begin{aligned}
& N(\mathcal{O}_6, V_{2n+4}) - \sum_{0\le j\le \lfloor (n-1)/3\rfloor} N(\mathcal{O}_0, V_{2(n-3j-1)\lambda_1+(2j+1)\lambda_2}) \\
&\quad + \sum_{0\le j\le \lfloor n/3\rfloor} \Big[N(\mathcal{O}_0, V_{(n-3j)\lambda_1+(2j)\lambda_2}) - N(\mathcal{O}_0, V_{(2(n-3j)+1)\lambda_1+(2j)\lambda_2}) \Big] \\
&= \mathrm{Ind}_T^G(n\lambda_1) - 2\mathrm{Ind}_T^G((n-1)\lambda_1+\lambda_2) - \mathrm{Ind}_T^G((n+5)\lambda_1) \\
&\quad + 2\mathrm{Ind}_T^G((n+3)\lambda_1+\lambda_2) + 2\mathrm{Ind}_T^G((n-1)\lambda_1+2\lambda_2) + \mathrm{Ind}_T^G((n+2)\lambda_1) \\
&\quad - 2\mathrm{Ind}_T^G(n\lambda_1+2\lambda_2) - \mathrm{Ind}_T^G((n+3)\lambda_1)
\end{aligned}
$$

(b)
$$
\begin{aligned}
& N(\mathcal{O}_6, V_{2n+3}) - \sum_{0\le j\le \lfloor n/3\rfloor} N(\mathcal{O}_0, V_{2(n-3j)\lambda_1+(2j)\lambda_2}) \\
&\quad + \sum_{0\le j\le \lfloor (n-2)/3\rfloor} \Big[N(\mathcal{O}_0, V_{(n-3j-2)\lambda_1+(2j+1)\lambda_2}) - N(\mathcal{O}_0, V_{2(n-3j)-3)\lambda_1+(2j+1)\lambda_2}) \Big] \\
&= \mathrm{Ind}_T^G((n-2)\lambda_1+\lambda_2) - \mathrm{Ind}_T^G(n\lambda_1) - \mathrm{Ind}_T^G((n-3)\lambda_1+2\lambda_2) + \mathrm{Ind}_T^G((n+4)\lambda_1) \\
&\quad - 2\mathrm{Ind}_T^G((n+1)\lambda_1+\lambda_2) - \mathrm{Ind}_T^G((n+3)\lambda_1+\lambda_2) - \mathrm{Ind}_T^G((n-2)\lambda_1+3\lambda_2) \\
&\quad + \mathrm{Ind}_T^G((n+1)\lambda_1+2\lambda_2) + 2\mathrm{Ind}_T^G(n\lambda_1+\lambda_2) + \mathrm{Ind}_T^G((n-3)\lambda_1+3\lambda_2).
\end{aligned}
$$

Proof. By Theorem 3.3, we get the formula of $N(\mathcal{O}_6, V_k)$. By Proposition 3.2, we write $N(\mathcal{O}_0, V_{m\lambda_1+n\lambda_2})$ as the sum of the terms as $\mathrm{Ind}_T^G(\text{weight})$. Then we insert them to the left-hand side of the formulas in Theorem 3.4. We use the well-known result $\mathrm{Ind}_T^G(\lambda) = \mathrm{Ind}_T^G(w\lambda)$ for $w \in W_G$. Then after careful and tedious calculation, we prove this theorem. ∎

Theorem 3.5. *We have*

$$
\eta(\mathcal{O}_6, V_k) = \begin{cases} \frac{k+6}{2}\lambda_1 & \text{if } k \text{ is even;} \\ \frac{k+3}{2}\lambda_1 + \lambda_2 & \text{if } k \text{ is odd.} \end{cases}
$$

Proof. When $k = 2n+4$ is even and $n \geq 1$, we see that the largest weight appearing in Theorem 3.4(a) is $(n+5)\lambda_1$. Then in this case, $\eta(\mathcal{O}_6, V_{2n+4}) \leq (n+5)\lambda_1$. Next we will prove that the equality holds. If we want to make $(n+5)\lambda_1$ smaller, then we should add more representations V_λ^G of G. By Proposition 3.2, we have

$$
V_\lambda^G = \sum_{w\in W} (-1)^w \mathrm{Ind}_T^G(\lambda + \rho - w\rho). \tag{4}
$$

If $(n+5)\lambda_1$ appears in the right-hand side of (4), we will get $\lambda+2\rho \geq (n+5)\lambda_1$. Since λ is dominant, then $\lambda+2\rho > (n+5)\lambda_1$. In order to eliminate the term $\lambda+2\rho$, then we should add another $V_{\lambda'}^G$ with $\lambda' > \lambda$. This will give us a larger weight $\lambda'+2\rho$. By induction, we know that $(n+5)\lambda_1$ is the smallest weight. Therefore,

$$\eta(\mathcal{O}_6, V_{2n+4}) = (n+5)\lambda_1.$$

Similarly, when $k = 2n+3$, we have $\eta(\mathcal{O}_6, V_{2n+3}) = (n+3)\lambda_1 + \lambda_2$.
It remains to consider the cases of $k = 0,1,2,3,4$. This will be verified case by case. Here we illustrate the case of $k = 3$.

$$\begin{aligned}
&N(\mathcal{O}_6, V_3) + N(\mathcal{O}_0, V_{0\lambda_1}) \\
&= \mathrm{Ind}_T^G(3\lambda_1) + \mathrm{Ind}_T^G(\lambda_1+2\lambda_2) - \mathrm{Ind}_T^G(2\lambda_1+\lambda_2) - 2\mathrm{Ind}_T^G(\lambda_1+\lambda_2) \\
&- \mathrm{Ind}_T^G(3\lambda_1+\lambda_2) - \mathrm{Ind}_T^G(0) + \mathrm{Ind}_T^G(\lambda_1) + \mathrm{Ind}_T^G(4\lambda_1) + \mathrm{Ind}_T^G(\lambda_2).
\end{aligned}$$

Then the formula in Theorem 3.5 holds for $k = 3$. Similarly, we prove the cases of $k = 0,1,2,4$. Therefore, we prove this theorem. ■

4. Eight-dimensional orbit

In this part, we study the bijection on the orbit $\mathcal{O}_8$. We first define a representation $\Theta(\nu)$ as in (5), which is an approximation to $N(\mathcal{O},\tau)$ when $\nu|_{G^X} = \tau$ with $X \in \mathcal{O}_8$. Then we get η by computing the representation $\Theta(\nu)$.

For any nilpotent orbit $\mathcal{O}$, let $\{X,H,Y\}$ be the Jacobson-Morozov $\mathfrak{sl}_2(\mathbb{C})$-triple with respect to $X \in \mathcal{O}$, and form the H-eigenspace decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i.$$

We take P_X to be the connected subgroup of G with Lie algebra $\oplus_{i\geq 0}\mathfrak{g}_i$ and its Lie algebra

$$\mathfrak{p}_X = \mathfrak{l}_X + \mathfrak{u}_X$$

has Levi factor $\mathfrak{l}_X = \mathfrak{g}_0$ and nilpotent part $\mathfrak{u}_X = \oplus_{i\geq 1}\mathfrak{g}_i$. Let $\mathfrak{d}_X = \oplus_{i\geq 2}\mathfrak{g}_i$.
For any representation (ν, W_ν) of P_X, define a certain representation $\Theta(\nu)$ of G as follows:

$$\Theta(\nu) = \oplus_{q\geq 0;(\sigma,M_\sigma)\in\hat{G}}(-1)^q \mathrm{Hom}_{L_X}(\wedge^q\mathfrak{u}_X, \mathrm{Hom}(M_\sigma, W_\nu\otimes S(\mathfrak{d}_X^*)))\otimes M_\sigma. \quad (5)$$

Proposition 4.1. [1, Proposition 2.1.7] *Given an irreducible representation* $\tau\in\widehat{G^X}$, *if* $\nu|_{G^X}\cong\tau$, *then* $\Theta(\nu)$ *is an approximation to* $N(\mathcal{O}_X,\tau)$, *in the sense that*

$$\Theta(\nu) = N(\mathcal{O}_X,\tau) + \sum_{\mathcal{O}_{X'}\subset\partial\mathcal{O}_X;\tau'\in\widehat{G^{X'}}} c_{(X',\tau')}N(\mathcal{O}_{X'},\tau'),$$

where only finitely many of $c_{(X',\tau')}$ *are nonzero.*

Moreover, about this representation $\Theta(\nu)$, we have the following result.

Proposition 4.2. [1, Proposition 2.2.4] *Let ν be an irreducible representation of L_X with highest weight λ. Write $\sum_r (-1)^r \wedge^r (\mathfrak{u}_X/\mathfrak{d}_X)$ as a sum of irreducible representations; say*

$$\sum_r (-1)^r \wedge^r (\mathfrak{u}_X/\mathfrak{d}_X) \cong \sum_{i=1}^{k} m_i W_{\xi_i}.$$

Then $$\Theta(\nu) \cong \sum_{i=1}^{k} \sum_{w \in W_{L_X}} (-1)^w m_i \mathrm{Ind}_T^G(\lambda + \rho_{L_X} - w(\xi_i + \rho_{L_X})).$$

Now we go back to the case of $\mathcal{O}_8$ of the group G_2. Let $X \triangleq X_{2\alpha+\beta} \in \mathcal{O}_8$. The standard triple ϕ associated to X is

$$\mathrm{Span}_{\mathbb{C}}\{\mathrm{X}_{2\alpha+\beta}, \mathrm{X}_{-(2\alpha+\beta)}, 2\mathrm{H}_\alpha + 3\mathrm{H}_\beta\}.$$

Then $$\mathfrak{g}^X = \mathfrak{g}^\phi + \mathfrak{u}^X = \mathrm{Span}_{\mathbb{C}}\{\mathrm{X}_\beta, \mathrm{X}_{2\alpha+\beta}, \mathrm{X}_{3\alpha+\beta}, \mathrm{X}_{3\alpha+2\beta}, \mathrm{X}_{-\beta}, \mathrm{H}_\beta\}$$

with $\mathfrak{g}^\phi = \mathrm{Span}_{\mathbb{C}}\{\mathrm{X}_\beta, \mathrm{X}_{-\beta}, \mathrm{H}_\beta\} \cong \mathfrak{sl}_2(\mathbb{C})$. So the irreducible representations of G^X are charactered by $t \in \mathbb{N}$. Let P_X be the parabolic subgroup of G with Lie algebra

$$\mathfrak{p}_X = \mathrm{Span}_{\mathbb{C}}\{\mathrm{X}_\alpha, \mathrm{X}_\beta, \mathrm{X}_{\alpha+\beta}, \mathrm{X}_{2\alpha+\beta}, \mathrm{X}_{3\alpha+\beta}, \mathrm{X}_{3\alpha+2\beta}, \mathrm{X}_{-\beta}, \mathrm{H}_\alpha, \mathrm{H}_\beta\}.$$

Then the Levi part of $\mathfrak{p}_X$ is

$$\mathfrak{l}_X = \mathrm{Span}_{\mathbb{C}}\{\mathrm{X}_\beta, \mathrm{X}_{-\beta}, \mathrm{H}_\beta, 2\mathrm{H}_\alpha + 3\mathrm{H}_\beta\} \cong \mathfrak{gl}_2(\mathbb{C}).$$

Also $$\mathfrak{u}_X = \mathrm{Span}_{\mathbb{C}}\{\mathrm{X}_\alpha, \mathrm{X}_{\alpha+\beta}, \mathrm{X}_{2\alpha+\beta}, \mathrm{X}_{3\alpha+\beta}, \mathrm{X}_{3\alpha+2\beta}\}$$

and $$\mathfrak{d}_X = \mathrm{Span}_{\mathbb{C}}\{\mathrm{X}_{2\alpha+\beta}, \mathrm{X}_{3\alpha+\beta}, \mathrm{X}_{3\alpha+2\beta}\}.$$

Then $$\mathfrak{u}_X/\mathfrak{d}_X = \mathbb{C}\{X_\alpha, X_{\alpha+\beta}\}.$$

As $\mathfrak{l}_X \cong \mathfrak{gl}_2(\mathbb{C})$-representations, we have $\mathfrak{u}_X/\mathfrak{d}_X \cong V_{(1,0)}$, where $V_{(1,0)}$ stands for the irreducible $\mathfrak{gl}_2(\mathbb{C})$-representation with highest weight $(1,0)$. Furthermore, we have

$$\sum_i (-1)^i \wedge^i (\mathfrak{u}_X/\mathfrak{d}_X) \cong V_{(0,0)} - V_{(1,0)} + V_{(1,1)},$$

as $\mathfrak{gl}_2(\mathbb{C})$-representations. Let $\nu_a = V_{(a+t,a)}$ with $a \in \mathbb{Z}$ be an irreducible representation of L_X and $\nu_a|_{G^X} = V_t$. Then by Proposition 4.2, we have

$$\begin{aligned}\Theta(\nu_a) = \mathrm{Ind}_T^G[&(a+t,a) - (a+t+1,a-1) + (a+t-1,a-1)\\ &- (a+t,a-2) - (a+t-1,a) + (a+t+1,a-2)].\end{aligned}$$

Write the weights (a,b) of L_X as the form $m\lambda_1 + n\lambda_2$, we have

$$\begin{aligned}\Theta(\nu_a) = \mathrm{Ind}_T^G[&((a-t)\lambda_1 + t\lambda_2) - ((a-t-3)\lambda_1 + (t+2)\lambda_2)\\ &+ ((a-t-1)\lambda_1 + t\lambda_2) - ((a-t-4)\lambda_1 + (t+2)\lambda_2)\\ &- ((a-t+1)\lambda_1 + (t-1)\lambda_2) + ((a-t-5)\lambda_1 + (t+3)\lambda_2)].\end{aligned}$$

Theorem 4.3. *In the $\mathcal{O}_8$ case, we have*

$$\eta(\mathcal{O}_8, V_t) = \begin{cases} \lambda_1 + \frac{t+2}{2}\lambda_2 & \text{if } t \text{ is even;} \\ \frac{t+3}{2}\lambda_2 & \text{if } t \text{ is odd.} \end{cases}$$

Proof. When $t = 2m$ is even, we choose $a = -m + 1$. Then we have

$$\begin{aligned}\Theta(\nu_a) = \mathrm{Ind}_T^G[&((-3m+1)\lambda_1 + (2m)\lambda_2) - ((-3m-2)\lambda_1 + (2m+2)\lambda_2) \\ &+ ((-3m)\lambda_1 + (2m)\lambda_2) - ((-3m-3)\lambda_1 + (2m+2)\lambda_2) \\ &- ((-3m+2)\lambda_1 + (2m-1)\lambda_2) + ((-3m-4)\lambda_1 + (2m+3)\lambda_2)] \\ = \mathrm{Ind}_T^G[&(2\lambda_1 + (m-1)\lambda_2) - (2\lambda_1 + m\lambda_2) + (m\lambda_2) \\ &- ((m+1)\lambda_2) - (\lambda_1 + (m-1)\lambda_2) + (\lambda_1 + (m+1)\lambda_2)]. \end{aligned} \tag{6}$$

Here the action of $S_\beta \cdot S_\alpha \in W$ maps the weight $((-3m+1)\lambda_1 + (2m)\lambda_2)$ (resp. $((-3m-2)\lambda_1 + (2m+2)\lambda_2)$, $((-3m)\lambda_1 + (2m)\lambda_2)$, $((-3m-3)\lambda_1 + (2m+2)\lambda_2)$, $((-3m+2)\lambda_1 + (2m-1)\lambda_2)$, $((-3m-4)\lambda_1 + (2m+3)\lambda_2)$) to $(2\lambda_1 + (m-1)\lambda_2)$ (resp. $(2\lambda_1 + m\lambda_2)$, $(m\lambda_2)$, $((m+1)\lambda_2)$, $(\lambda_1 + (m-1)\lambda_2)$, $(\lambda_1 + (m+1)\lambda_2)$).
So when $m \geq 1$, all the weights appearing in (6) are dominant and the largest one is $(\lambda_1 + (m+1)\lambda_2)$. Also when we choose different $a \in \mathbb{Z}$, we verify that the largest weight appearing in $\Theta(\nu_a)$ is larger than $\lambda_1 + (m+1)\lambda_2$. Then for $m \geq 1$, we have

$$\eta(\mathcal{O}_8, V_{2m}) = \lambda_1 + (m+1)\lambda_2.$$

When $m = 0$, the largest weight in $\Theta(\nu_a)$ is smallest when $a = -1$. That is,

$$\begin{aligned}\Theta(\nu_{-1}) &= \mathrm{Ind}_T^G[(2\lambda_1 - \lambda_2) - (2\lambda_1) + (0\lambda_1) - (\lambda_2) - (\lambda_1 - 1\lambda_2) + (\lambda_1 + \lambda_2)] \\ &= \mathrm{Ind}_T^G[-(2\lambda_1) + (0\lambda_1) - (\lambda_2) + (\lambda_1 + \lambda_2)].\end{aligned}$$

Therefore, we have $$\eta(\mathcal{O}_8, V_0) = \lambda_1 + \lambda_2.$$

When $t = 2m + 1$ is odd, we choose $a = -m$. And then similarly, we will get that

$$\eta(\mathcal{O}_8, V_{2m+1}) = (m+2)\lambda_2.$$ ■

5. Subregular orbit

In this section, we consider $\mathcal{O}_{10}$. We will use a well-known formula of McGovern about the space of regular functions on nilpotent orbit to compute $N(\mathcal{O}_{10}, \mathbb{1})$. And then we use some properties about orbit cover to compute $N(\mathcal{O}_{10}, \mathrm{sgn})$. Then we get η for $\mathcal{O}_{10}$.
Given $X \triangleq X_\beta + X_{2\alpha+\beta} \in \mathcal{O}_{10}$, we have the standard $\mathfrak{sl}_2(\mathbb{C})$-triple $\{X, H, Y\}$ with $H = 2H_\alpha + 4H_\beta$. We define $\mathfrak{p}_X = \mathfrak{l}_X + \mathfrak{u}_X$ and $\mathfrak{d}_X$ attached to $X \in \mathcal{O}_{10}$ as in the beginning of Section 4. Then we get that

$$\mathfrak{l}_X = \mathrm{Span}_{\mathbb{C}}\{\mathrm{H}_\alpha, \mathrm{X}_\alpha, \mathrm{X}_{-\alpha}, \mathrm{H}_\alpha + 2\mathrm{H}_\beta\},$$

and $\mathfrak{u}_X/\mathfrak{d}_X = \{0\}$. In $\mathcal{O}_{10}$ case, we have $\dim \mathfrak{g}^\phi = 0$ and $\pi_1(\mathcal{O}_{10}) = S_3$. Then irreducible rational representations of G^X will be represented by $\{\mathbb{1}, \mathrm{sgn}, \chi\}$. Here $\mathbb{1}$ is the trivial representation of S_3, sgn stands for the sign representation and χ represents the 2-dimensional one.

Proposition 5.1. [15, Theorem 3.1] *As G-representations, we have*

$$\mathrm{Ind}_{G^X}^G(\mathbb{1}) = \mathcal{R}(\mathcal{O}_{10}) \cong \sum_j (-1)^j \mathrm{Ind}_L^G(\wedge^j \mathfrak{g}_1).$$

Here $\mathcal{R}(\mathcal{O}_X)$ stands for the space of regular functions on $\mathcal{O}_X$ and we decree that $\wedge^0 \mathfrak{g}_1 = \mathbb{C}$ even if $\mathfrak{g}_1 = 0$.

Theorem 5.2. *We have $\mathcal{R}(\mathcal{O}_{10}) = \mathrm{Ind}_T^G[(0) - (\lambda_1)]$. Then $\eta(\mathcal{O}_{10}, \mathbb{1}) = \lambda_1$.*

Proof. By Proposition 5.1, we have

$$\mathcal{R}(\mathcal{O}_{10}) = \mathrm{Ind}_{L_X}^G(V_0) = \mathrm{Ind}_{T_X}^G[(0,0) - (1,-1)].$$

Here L_X has Lie algebra $\mathfrak{l}_X \cong \mathfrak{gl}(2,\mathbb{C})$ with Cartan subgroup T_X, and $(0,0)$, $(1,-1)$ are two weights of $\mathfrak{l}_X \cong \mathfrak{gl}(2,\mathbb{C})$. So

$$\mathcal{R}(\mathcal{O}_{10}) = \mathrm{Ind}_T^G[(0) - (2\lambda_1 - \lambda_2)].$$

The element $S_\beta \cdot S_\alpha \in W$ acts on λ_1 by $(S_\beta \cdot S_\alpha)\lambda_1 = (2\lambda_1 - \lambda_2)$. This implies

$$\mathrm{Ind}_{G^X}^G(\mathbb{1}) = \mathcal{R}(\mathcal{O}_{10}) = \mathrm{Ind}_T^G[(0) - (\lambda_1)]. \tag{7}$$

This also implies $\eta(\mathcal{O}_{10}, 1) \leq \lambda_1$. If we add representations $N(\mathcal{O}', \tau')$ with $\mathcal{O}'$ in the boundary of $\mathcal{O}_{10}$, according to Theorem 1.2(a,b,c), the largest weight will be larger than λ_1. This implies $\eta(\mathcal{O}_{10}, \mathbb{1}) = \lambda_1$. ∎

Next we will use properties of induced orbit covers to calculate $\mathrm{Ind}_{G^X}^G(\mathrm{sgn})$ and then get $\eta(\mathcal{O}_{10}, \mathrm{sgn})$. Here we recall some properties about orbit covers and the space of regular functions on them.

An orbit cover of $\mathcal{O}$ for G is a G-space $\widetilde{\mathcal{O}}$ with a G-invariant cover map $\pi : \widetilde{\mathcal{O}} \to \mathcal{O}$. For any $X \in \mathcal{O}$ and a subgroup Π of G^X/G^X_0, we have a natural G-invariant cover map

$$\widetilde{\mathcal{O}} = G/G^X_\Pi \to G/G^X = \mathcal{O},$$

where G^X_Π is the open subgroup of G^X corresponding to Π. Then $\widetilde{\mathcal{O}}$ is an orbit cover, and (X, Π) or $(\mathcal{O}_X, \Pi)$ is called a representation of $\widetilde{\mathcal{O}}$. Obviously, every orbit cover has one and only one such representation up to the conjugation of G.

Now let $P = LU_P$ be a parabolic subgroup, with Levi decomposition $P = LU_P$. Its Lie algebra is $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_P$. Fix an orbit cover $\widetilde{\mathcal{O}}_L$ of L, and let (Y, Π_L) be its representation. By [14, Lusztig-Spaltenstein Theorem], we can choose $X \in \mathcal{O}_G \cap (\mathcal{O}_L + \mathfrak{u}_P)$ such that there is a surjective homomorphism

$$\theta : P^X/P^X_0 \longrightarrow L^Y/L^Y_0.$$

Then $\theta^{-1}(\Pi_L) \subset P^X/P^X_0 \subset G^X/G^X_0$. Hence $(X, \theta^{-1}(\Pi_L))$ is a representation of some orbit cover $\widetilde{\mathcal{O}}_G$ of $\mathcal{O}_G$.

In [10], this orbit cover $(X, \theta^{-1}(\Pi_L))$ of G is called *the parabolic induction* of $\widetilde{\mathcal{O}}_L$, denoted by $\mathrm{Ind}_L^G(\widetilde{\mathcal{O}}_L)$. By [10, Lemma 2.2], this definition is well-defined, and independent of the choice of P with Levi factor L and of $X \in \mathcal{O}_G \cap (\mathcal{O}_L + \mathfrak{u}_P)$.

Theorem 5.3. *Given $X \in \mathcal{O}_{10}$, we have*

$$\mathrm{Ind}_{G^X}^G(\mathrm{sgn}) = \mathrm{Ind}_T^G[(\lambda_1) - (\lambda_2)]. \tag{8}$$

Then $\eta(\mathcal{O}_{10}, \mathrm{sgn}) = (\lambda_2)$.

Proof. Let P be the parabolic subgroup of G with respect to the long root β. Then its Levi part L has Lie algebra

$$\mathrm{Span}_{\mathbb{C}}\{\mathrm{X}_\beta, \mathrm{H}_\beta, \mathrm{X}_{-\beta}, 2\mathrm{H}_\alpha + 3\mathrm{H}_\beta\}.$$

It is well-known that $\mathcal{O}_{10}$ is the induced orbit from the 0-orbit of L. By [9, p. 138], there exist $X \in \mathcal{O}_{10}$ such that $G^X/G_0^X \cong S_3$ and $P^X/P_0^X \cong \mathbb{Z}_3 \subset G^X/G_0^X$. According to the definition of induction of orbit cover, we know that the orbit cover $(\mathcal{O}_{10}, \mathbb{Z}_3)$ is induced from $(\mathcal{O}_0, 1)$ by the parabolic subgroup P. Then by [10, Theorem 2.1], we have that

$$\mathcal{R}((\mathcal{O}_{10}, \mathbb{Z}_3)) = \mathrm{Ind}_L^G(\mathcal{R}((\mathcal{O}_0, 1))).$$

Here $(\mathcal{O}_{10}, \mathbb{Z}_3)$ and $(\mathcal{O}_0, 1)$ stand for the orbit cover in G and L respectively, and $\mathcal{R}(\cdot)$ represents the space of regular functions on. According to the definition of orbit cover, there exist $G_0^X \subset G^* \subset G^X$ such that $G^*/G_0^X \cong \mathbb{Z}_3$ and $(\mathcal{O}_{10}, \mathbb{Z}_3) = G/G^*$.

Then $$\mathcal{R}((\mathcal{O}_{10}, \mathbb{Z}_3)) = \mathrm{Ind}_{G^*}^G(\mathbb{1}) = \mathrm{Ind}_{G^X}^G(\mathbb{1} + \mathrm{sgn}).$$

And $$\mathrm{Ind}_L^G(\mathcal{R}((\mathcal{O}_0, 1))) = \mathrm{Ind}_L^G(\mathbb{1}) = \mathrm{Ind}_T^G[(0) - (-3\lambda_1 + 2\lambda_2)].$$

Also by Theorem 5.2, we have $\mathrm{Ind}_{G^X}^G(\mathbb{1}) = \mathrm{Ind}_T^G[(0) - (\lambda_1)]$. Then we get

$$\begin{aligned}\mathrm{Ind}_{G^X}^G(\mathrm{sgn}) &= \mathrm{Ind}_T^G[(0) - (-3\lambda_1 + 2\lambda_2)] - \mathrm{Ind}_T^G[(0) - (\lambda_1)] \\ &= \mathrm{Ind}_T^G[(\lambda_1) - (-3\lambda_1 + 2\lambda_2)]].\end{aligned}$$

The element $S_\alpha \cdot S_\beta \cdot S_\alpha$ in Weyl group W acts on λ_2 by

$$(S_\alpha \cdot S_\beta \cdot S_\alpha)\lambda_2 = (-3\lambda_1 + 2\lambda_2).$$

So we have $$\mathrm{Ind}_{G^X}^G(\mathrm{sgn}) = \mathrm{Ind}_T^G[(\lambda_1) - (\lambda_2)].$$

Since $\lambda_1 \prec \lambda_2$, it implies $\eta(\mathcal{O}_{10}, \mathrm{sgn}) = (\lambda_2)$. ∎

Let A be the space of regular functions on the universal (sixfold) cover of $\mathcal{O}_{10}$. We will get a formula about the G_2-representation A.

Lemma 5.4. *As G_2-representations, we have*

$$A = \mathrm{Ind}_T^G[(0)] + \mathrm{Ind}_T^G[\lambda_1] - \mathrm{Ind}_T^G[\lambda_2] - \mathrm{Ind}_T^G[(2\lambda_1)]. \tag{9}$$

Proof. By [16, P.174], we know that as G_2-representations,

$$A = \sum_{m,n \geq 0} (m+1)(n+1) V_{(m\lambda_1 + n\lambda_2)}. \tag{10}$$

We use Proposition 3.2 to write $V_{(m\lambda_1+n\lambda_2)}$ as the sum of the terms $\mathrm{Ind}_T^G(\text{weight})$. Then we use the property $\mathrm{Ind}_T^G(\lambda) = \mathrm{Ind}_T^G(w\lambda)$ for $w \in W_G$ to eliminate as many terms as possible. After calculation, we can simplify (10) and get Formula (9). ∎

Theorem 5.5. *Given $X \in \mathcal{O}_{10}$, we have*

$$\mathrm{Ind}_{G^X}^G(\chi) = \mathrm{Ind}_T^G[(\lambda_1) - (2\lambda_1)].$$

Then $\eta(\mathcal{O}_{10}, \chi) = (2\lambda_1)$.

Proof. By Lemma 5.4, we get

$$A = \mathrm{Ind}_T^G[(0)] + \mathrm{Ind}_T^G[\lambda_1] - \mathrm{Ind}_T^G[\lambda_2] - \mathrm{Ind}_T^G[(2\lambda_1)].$$

We know that $A = \mathrm{Ind}_{G^X}^G(\mathbb{1}) + \mathrm{Ind}_{G^X}^G(\mathrm{sgn}) + \mathrm{Ind}_{G^X}^G(\chi)$. By (7) and (8), this implies

$$\mathrm{Ind}_{G^X}^G(\chi) = \mathrm{Ind}_T^G[\lambda_1] - \mathrm{Ind}_T^G[(2\lambda_1)].$$

Therefore $$\eta(\mathcal{O}_{10}, \chi) = (2\lambda_1).$$ ■

Remark 5.6. Theorem 1.2 is proved by combining Theorem 2.2, Theorem 3.5, Theorem 4.3, Theorem 5.2, Theorem 5.3 and Theorem 5.5. Meanwhile, Corollary 1.3 is directly implied by Theorem 5.2, Theorem 5.3 and Theorem 5.5.

Remark 5.7. Our explicit description of the Lusztig-Vogan bijection η for the group G_2 provides a brief glimpse of the combinational study of the Lusztig-Vogan bijection for other groups.

For the group G_2, we observe that the dominant weights far from the hyperplanes of the fundamental Weyl chamber are attached to $\mathcal{O}_0$; that most weights near the hyperplane P_β are attached to $\mathcal{O}_6$; that most weights near the hyperplane P_α are attached to $\mathcal{O}_8$; and that a few remaining small weights are attached to the orbits $\mathcal{O}_{10}$ and $\mathcal{O}_{12}$. Furthermore, we also observe that

$$\begin{cases} \eta(\mathcal{O}_6, V_k) \prec \eta(\mathcal{O}_6, V_{k+1}), & k \geq 0; \\ \eta(\mathcal{O}_8, V_t) \prec \eta(\mathcal{O}_8, V_{t+1}), & t \geq 0. \end{cases}$$

Here $\prec$ stands for the usual partial order of weights and $\mu \prec \nu$ means that $\nu - \mu$ is a sum of positive roots or $\mu = \nu$. We conjecture that this observation should be true for any other groups. That is, for a given nilpotent orbit, there should be some order '$<$' of the irreducible representations of its isotropy subgroup, such that when two representations satisfy $V_\tau < V_{\tau'}$, then $\eta(\mathcal{O}, V_\tau) \prec \eta(\mathcal{O}, V_{\tau'})$. Then we can determine the Lusztig-Vogan bijection through three steps:

(a) By using the method of two-step induction of representations and the tool of the induction of orbit covers, we determine the value of η for some nilpotent orbits and some special representations;

(b) For a fixed orbit $\mathcal{O}$, find the order '$<$' of the irreducible representations of its isotropy subgroup.

(c) By the definition of the Lusztig-Vogan bijection, we determine the value of η for any $(\mathcal{O}, V_\tau)$.

This will give a clue to describe the Lusztig-Vogan bijection explicitly for any other groups.

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