Journal of Lie Theory
Volume **29** (2019) 1167–1192

A Differentiable Monoid of Smooth Maps on Lie Groupoids

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Communicated by K.-H. Neeb

Abstract. We investigate a monoid of smooth mappings on the space of arrows of a Lie groupoid and its group of units. The group of units turns out to be an infinite-dimensional Lie group which is regular in the sense of Milnor. Furthermore, this group is closely connected to the group of bisections and the geometry of the Lie groupoid. Under suitable conditions, i.e. if the source map of the Lie groupoid is proper, one also obtains a differentiable structure on the monoid and can identify the bisection group as a Lie subgroup of its group of units. Finally, relations between the (sub-)groupoids associated to the underlying Lie groupoid and subgroups of the monoid are obtained.
The key tool driving the investigation is a generalisation of a result by A. Stacey. In the present article, we establish this so-called Stacey-Roberts Lemma. It asserts that pushforwards of submersions are submersions between the infinite-dimensional manifolds of mappings. The Stacey-Roberts Lemma is of independent interest as it provides tools to study submanifolds of and geometry on manifolds of mappings.

Mathematics Subject Classification: Primary: 58B25; secondary: 22E65, 22A22, 22A15, 58D05, 58D15, 58H05.

Key Words: Lie groupoid, topological semigroup, Stacey–Roberts Lemma, submersion, group of bisections, infinite-dimensional Lie group, regular Lie group, manifold of mappings, topological groupoid, fine very strong topology.

Introduction and statement of results

In the present article we investigate a monoid $S_\mathcal{G}$ of smooth mappings on the arrow space of a Lie groupoid $\mathcal{G}$. Namely, for a Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ with source map α and target map β we define the monoid

$$S_\mathcal{G} := \{f \in C^\infty(G,G) \mid \beta \circ f = \alpha\} \quad \text{with product } (f \star g)(x) := f(x)g\big(xf(x)\big)$$

Our motivation to investigate this monoid is twofold. On one hand, $S_\mathcal{G}$ carries over the investigation of a similar monoid for continuous maps of a topological monoid in [1] to the smooth category. While in the continuous setting one always obtains a topological monoid, the situation is more complicated in the smooth category. This is due to the fact that on non-compact spaces the C^∞-compact open topology has to be replaced by a Whitney type topology (cf. [16, 20] for surveys). On the other hand, it has been noted already in [1] that $S_\mathcal{G}$ is closely connected to the group of bisections $\operatorname{Bis}(\mathcal{G})$ of the underlying Lie groupoid $\mathcal{G}$. Roughly speaking a bisection gives rise to

ISSN 0949–5932 / \$2.50

an element in the unit group of $S_{\mathcal{G}}$ by associating to it its right-translation (we recall these constructions in Subsection 4.2). This points towards the geometric significance of the monoid of $S_{\mathcal{G}}$ and its group of units as bisections are closely connected to the geometry of the underlying Lie groupoid (cf. [30, 32, 31], see also [1]). Viewing the monoid $S_{\mathcal{G}}$ and its group of units as an extension of the bisections, one is prompted to the question, how much of the structure and geometry of the Lie groupoid $\mathcal{G}$ can be recovered from them.

Throughout the article it will be necessary to work with smooth maps on infinite-dimensional spaces and manifolds beyond the Banach setting. For this we work with the so-called Bastiani's calculus [4].[1] We refer to [24, 5, 11, 26] for streamlined expositions. The Bastiani setting has the advantage that it is compatible with the underlying topological framework. For the case at hand, the topological framework is provided by endowing $S_{\mathcal{G}}$ with a canonical function space topology inherited from the space of smooth functions. We then prove that $S_{\mathcal{G}}$ becomes a submanifold of the infinite-dimensional manifold of smooth mappings which turns $S_{\mathcal{G}}$ into a differentiable (and thus also topological) monoid if the source projection α of $\mathcal{G}$ is a proper map. An indispensable ingredient of the construction is a generalisation of a result by A. Stacey, the so-called Stacey–Roberts Lemma which is of independent interest.

Building on our investigation of the monoid $S_{\mathcal{G}}$ we turn to its group of units $S_{\mathcal{G}}(\alpha)$. It turns out that $S_{\mathcal{G}}(\alpha)$ is an infinite-dimensional Lie group (even if $\mathcal{G}$ is not α-proper!) and we investigate the Lie theoretic properties of $S_{\mathcal{G}}(\alpha)$ in Section 3. Section 4 investigates certain (topological) subgroups of $S_{\mathcal{G}}(\alpha)$, e.g. $\mathrm{Bis}(\mathcal{G})$, which are motivated by the geometric interplay of $\mathcal{G}$ and $S_{\mathcal{G}}(\alpha)$. Finally, we investigate the connection of these subgroups and certain subgroupoids of $\mathcal{G}$ in Section 5.

We will now explain the results in this paper in greater detail. First recall that the space of smooth mappings $C^\infty(G, G)$ for a finite-dimensional manifold can be made an infinite-dimensional manifold with respect to the fine very strong topology (see Appendix A for more details). To turn $S_{\mathcal{G}}$ and $S_{\mathcal{G}}(\alpha)$ into submanifolds of this infinite-dimensional manifold, we need an auxiliary result:

Stacey-Roberts Lemma. *Let X, Y be paracompact finite-dimensional manifolds and S a finite-dimensional manifold, possibly with boundary or corners. Consider $\theta\colon X \to Y$ a smooth submersion. Then the pushforward*

$$\theta_*\colon C^\infty(S, X) \to C^\infty(S, Y), \quad f \mapsto \theta \circ f,$$

is a smooth submersion.[2]

This result is of independent interest and has been used in recent investigations of Lie groupoids of mappings [28] and shape analysis on homogeneous spaces [7]. Note that the Stacey-Roberts Lemma generalises an earlier result by Stacey (see [33, Corollary 5.2]). In ibid. Stacey deals with a setting of generalised infinite-dimensional calculus and states that the pushforward of a submersion is a “smooth regular map” if the

[1]In Bastiani calculus a map f is smooth if at every point of it domain f admits directional derivatives which are again continuous. As the chain rule holds, manifolds etc. can be defined as usual.

[2]One has to be careful with the notion of submersion between infinite-dimensional manifolds, as there is no inverse function theorem at our disposal. For our purposes a submersion is a map which admits submersion charts (this is made precise in Definition 2.3 below).

source manifold (i.e. S for the space $C^\infty(S,X)$) is compact. From this it is hard to discern the Stacey-Roberts Lemma in the Bastiani setting even if S is compact. It was first pointed out to the authors by Roberts that Stacey's result yields a submersion in the sense of Definition 2.3 (see [28, Theorem 5.1]). The statement of the Stacey-Roberts Lemma is deceptively simple, as the proof is quite involved (cf. Appendix B). To our knowledge a full proof (building on Stacey's ideas) is up to this point missing in the literature (this is true even for the compact case). Using the Stacey-Roberts Lemma, one obtains a submanifold structure on $S_\mathcal{G}$ and moreover:

Theorem A. *If the Lie groupoid $\mathcal{G}$ is α-proper, i.e. the source map is a proper map, then $S_\mathcal{G}$ consists only of proper mappings. Further, the manifold structure induced by $C^\infty(G,G)$ turns $(S_\mathcal{G},\star)$ into a differentiable monoid.*

Note that this in particular implies that $(S_\mathcal{G},\star)$ is a topological monoid with respect to the topology induced by the fine very strong topology. Due to the first part of Theorem A, $S_\mathcal{G}$ contains only proper mappings if $\mathcal{G}$ is α-proper and our proof uses that the joint composition map $\mathrm{Comp}(f,g) := f\circ g$ is continuous (or differentiable) only if we restrict g to proper maps.
One can prove (see [1]) that the monoid $(S_\mathcal{G},\star)$ is isomorphic to a submonoid of $\big(C^\infty(G,G),\circ_{\mathrm{op}}\big)$ (where "$\circ_{\mathrm{op}}$" is composition in opposite order) by means of

$$R\colon S_\mathcal{G}\to C^\infty(G,G),\quad f\mapsto \big(x\mapsto xf(x)\big).$$

This monoid monomorphism enables the treatment of the group of units of $S_\mathcal{G}$ as an infinite-dimensional Lie group. It identifies the units of $S_\mathcal{G}$ with the group

$$\big(S_\mathcal{G}(\alpha) := \{f\in S_\mathcal{G}\mid R(f)\in \mathrm{Diff}(G)\},\star\big).$$

Thus $S_\mathcal{G}(\alpha)$ can be identified with the subgroup $R_{S_\mathcal{G}(\alpha)} := \{\varphi\in\mathrm{Diff}(G)\mid \beta\circ\varphi=\beta\}$ of $\mathrm{Diff}(G)$. Another application of the Stacey-Roberts Lemma shows that $R_{S_\mathcal{G}(\alpha)}$ is a closed Lie subgroup of $\mathrm{Diff}(G)$, whence:

Theorem B. *Let $\mathcal{G}$ be a Lie groupoid, then the group of units $S_\mathcal{G}(\alpha)$ of the monoid $(S_\mathcal{G},\star)$ is a Lie group with respect to the submanifold structure induced by $C^\infty(G,G)$. Its Lie algebra is isomorphic to the Lie subalgebra*

$$\mathfrak{g}_\beta := \{X\in\mathfrak{X}_c(G)\mid T\beta\circ X(g) = 0_{\beta(g)}\forall g\in G\}$$

of the Lie algebra $\big(\mathfrak{X}_c(G),[\cdot,\cdot]\big)$ of compactly supported vector fields with the usual bracket of vector fields.

To understand the Lie theoretic properties of the Lie group $\big(S_\mathcal{G}(\alpha),\star\big)$ recall the notion of regularity for Lie groups: Let H be a Lie group with identity element $\mathbf{1}$, and $r\in\mathbb{N}_0\cup\{\infty\}$. Denote the tangent map of the right translation $\rho_h\colon H\to H$, $x\mapsto xh$ by $v.h := T_\mathbf{1}\rho_h(v)\in T_hH$ for $v\in T_\mathbf{1}(H) =: \mathbf{L}(H)$. Following [13], H is C^r*-semiregular* if for each C^r-curve $\gamma\colon[0,1]\to\mathbf{L}(H)$ the initial value problem

$$\begin{cases}\eta'(t) &= \gamma(t).\eta(t)\\ \eta(0) &= \mathbf{1}\end{cases}\tag{1}$$

has a (necessarily unique) C^{r+1}-solution $\mathrm{Evol}(\gamma) := \eta\colon[0,1]\to H$. If in addition

$$\mathrm{evol}\colon C^r\big([0,1],\mathbf{L}(H)\big)\to H,\quad \gamma\mapsto\mathrm{Evol}(\gamma)(1)$$

is smooth, we call H a C^r-regular Lie group. If H is C^r-regular and $r \leq s$, then H is also C^s-regular. A C^∞-regular Lie group H is called *regular (in the sense of Milnor)* – a property first defined in [24]. Every finite dimensional Lie group is C^0-regular (cf. [26]). Several important results in infinite-dimensional Lie theory are only available for regular Lie groups (see [24, 26, 13], cf. also [17] and the references therein). Our results subsume the following (cf. Corollary 3.10):

Theorem C. *The Lie group $\big(S_{\mathcal{G}}(\alpha), \star\big)$ is C^1-regular.*

Under mild topological assumptions on the manifold of arrows, one can even sharpen Theorem C to obtain C^0-regularity of the Lie group $\big(S_{\mathcal{G}}(\alpha), \star\big)$. This seemingly minor improvement is quite important as it establishes Lie theoretic and geometric tools such as the strong Trotter formula for the group, cf. [15]. It is currently unknown whether similar results hold for C^1-regular Lie groups.

We then study (topological) subgroups of $S_{\mathcal{G}}(\alpha)$ which arise from the action of $S_{\mathcal{G}}$

$$x.f = R(f)(x) = xf(x), \quad x \in G, f \in S_{\mathcal{G}}.$$

(see e.g. [1, Proposition 6]) and the interaction of the geometry of $\mathcal{G}$ with $S_{\mathcal{G}}$. For example we investigate subgroups of $S_{\mathcal{G}}(\alpha)$ which arise as setwise stabilizer $F_H\big(S_{\mathcal{G}}(\alpha)\big)$ of a subgroupoid $H \subseteq G$ of $\mathcal{G}$

Furthermore, we identify the group of bisections $\mathrm{Bis}(\mathcal{G})$ of a Lie groupoid $\mathcal{G}$ as a subgroup of $S_{\mathcal{G}}(\alpha)$. The group $\mathrm{Bis}(\mathcal{G})$ plays an important role in the study of Lie groupoids (cf. [32, 35]). It is well known (see e.g. [19]) that bisections give rise to right-translations of the groupoid $\mathcal{G}$ which identifies them with elements in $S_{\mathcal{G}}(\alpha)$. Moreover, if the source projection α is proper, the identification yields a Lie group morphism $\Psi\colon \mathrm{Bis}(\mathcal{G}) \to S_{\mathcal{G}}(\alpha)$ (cf. [3, 30] for the Lie group structure on the bisections)

Finally, we exploit the similarity of $\mathrm{Bis}(\mathcal{G})$ and $S_{\mathcal{G}}(\alpha)$ to investigate connections between subgroups of $S_{\mathcal{G}}(\alpha)$ and Lie subgroupoids of $\mathcal{G}$. The underlying Lie groupoid $\mathcal{G}$ is determined (as a set or a manifold) is under certain assumptions determined its group of bisections, see e.g. [32] (see also [35]). We consider related question for $S_{\mathcal{G}}(\alpha)$ and characterise elements in $S_{\mathcal{G}}(\alpha)$ by studying their graphs.

1. Notation and preliminaries

Before we begin, let us briefly recall some conventions that we are using in this paper.

Definition 1.1. (1) We write $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \{0, 1, \dots\}$. The symbols G, M, X, Y will denote paracompact finite-dimensional manifolds over the real numbers $\mathbb{R}$. We assume that all topological spaces in this article are Hausdorff.

(2) (Spaces of smooth mappings) We write $C^\infty_{fS}(X, Y)$ for the space of smooth mappings endowed with the *fine very strong topology.* Details on this fine very strong topology and the coarser very strong topology are recalled in Appendix A. Subsets of $C^\infty_{fS}(X, Y)$ will always be endowed with the induced subspace topology. Note that the set of all smooth diffeomorphisms $\mathrm{Diff}(X)$ is an open subset of $C^\infty_{fS}(X, X)$.

As all spaces of smooth mappings in the present paper will be endowed with this topology, whence we usually suppress the subscript. We recall in Appendix A that this turns $C^\infty(X, Y)$ into an infinite-dimensional manifold and $\mathrm{Diff}(X)$ into an infinite-dimensional Lie group with respect to the composition of mappings.

We refer to [19] for an introduction to (finite-dimensional) Lie groupoids. The notation for Lie groupoids and their structural maps also follows [19].

(3) (Lie groupoids) Let $\mathcal{G} = (G \rightrightarrows M)$ be a finite-dimensional Lie groupoid over M with source projection $\alpha\colon G \to M$ and target projection $\beta\colon G \to M$. Note that α and β are submersions, whence the set of composable arrows

$$G^{(2)} = \big\{(x,y) \in G \times G \mid \alpha(x) = \beta(y)\big\} \subseteq G \times G$$

is a submanifold. We denote by $m\colon G^{(2)} \to G$ the partial multiplication and let $\iota\colon G \to G$ be the inversion. In the following we will always identify M with an embedded submanifold of G via the unit map $1\colon M \to G$ (in the following we suppress the unit map in the notation without further notice).

Remark 1.2. One obtains a Lie groupoid $G^{(2)}$ by defining:

$$\beta^2(x,y) = (x,y)(xy,y^{-1}) = (x,\beta(y)) = \big(x,\alpha(x)\big), \text{ target map}$$
$$\alpha^2(x,y) = (xy,y^{-1})(x,y) = (xy,y^{-1}y) = \big(xy,\alpha(xy)\big) \text{ source map.}$$

The object space is then $(G^{(2)})^0 = \big\{\big(x,\alpha(x)\big) \mid x \in G\big\}$ and the set of composable pairs, $(G^{(2)})^{(2)}$, is given by $\big\{\big((x,y),(z,w)\big) :\ z = xy\big\}$ with partial multiplication $(x,y)(xy,w) = (x,yw)$ and inverse $(x,y)^{-1} = (xy,y^{-1})$. The groupoid $G^{(2)}$ is isomorphic to the subgroupoid of the pair groupoid $G \times G \rightrightarrows G$ consisting of pairs of arrows with the same target, whence it is a Lie groupoid.

2. A monoid of smooth maps of a Lie groupoid

We are interested in a monoid which was first studied in [1] in the context of topological groupoids. Here we study a related monoid, replacing continuous mappings with smooth ones and topological groupoids by Lie groupoids.

Definition 2.1. Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid, then we define the set

$$S_{\mathcal{G}} := \big\{f \in C^\infty(G,G) \mid \beta \circ f = \alpha\big\},$$

which becomes a monoid with respect to the binary operation

$$(f \star g)(x) = f(x)g\big(xf(x)\big), \quad f,g \in S_{\mathcal{G}},\ x \in G.$$

Identifying $M \subseteq G$ via the unit map, α becomes the unit element of $S_{\mathcal{G}}$.

Remark 2.2. Obviously, one can define another monoid $S'_{\mathcal{G}}$ of smooth self maps by switching the roles of α and β in the definition. However, as shown in [1, Proposition 1] this leads to an isomorphic monoid, whence it suffices to consider only $S_{\mathcal{G}}$.

Instead of directly investigating the $S_{\mathcal{G}}$ as a topological monoid, we establish a differentiable structure which, under suitable conditions, turns $(S_{\mathcal{G}},\star)$ into a differentiable monoid. As smooth maps in our setting are continuous[3], this will turn $(S_{\mathcal{G}},\star)$ into a topological monoid.

[3]Since we are working beyond Banach manifolds (even beyond the Fréchet setting) this property is not automatic and depends on the generalisation of calculus used. See e.g. [17] for a calculus with smooth discontinuous maps.

2.1. Interlude: The Stacey-Roberts Lemma

To obtain a manifold structure, observe that $S_{\mathcal{G}}$ is the preimage of $\{\alpha\} \subseteq C^\infty(G,M)$ under the pushforward $\beta_*\colon C^\infty(G,G) \to C^\infty(G,M), f \mapsto \beta \circ f$. A theorem of Stacey [33, Corollary 5.2] asserts that q_* is a regular map if q is a regular map between finite dimensional manifolds and G is compact. In our language that should yield the submersion property of the pushforward. In fact, the main theorem [33, Theorem 5.1] establishes a more general result in a very general setting. Unfortunately, the authors were neither able to completely follow its proof, nor to directly translate it to the concept of submersion adequate to our setting (see below). Therefore we follow the proof strategy of Stacey to a certain point, but we adapted it to the Bastiani setting and generalise it for non-compact source manifolds.

Definition 2.3 (Submersions in infinite dimensions, [14])**.** Let X, Y be (infinite-dimensional) manifolds and $q\colon X \to Y$ be smooth. Thenl q is a *submersion*, if for each $x \in X$, there exists a chart $\Psi\colon U_\Psi \to V_\Psi \subseteq E$ of X with $x \in U_\Psi$ and a chart $\psi\colon U_\psi \to V_\psi \subseteq F$ of Y with $q(U_\Psi) \subseteq U_\psi$ and $\psi \circ q \circ \Psi^{-1} = \pi|_{V_\Psi}$ for $\pi\colon E \to F$ continuous linear with a continuous linear right inverse $\sigma\colon F \to E$, (i.e. $\pi \circ \sigma = \mathrm{id}_F$).

Due to the construction of a manifold of mappings (Appendix A), the proof of the following lemma is highly non trivial and we need several auxiliary results from Appendix B.

Lemma 2.4 (Stacey-Roberts)**.** *Let M, X, Y be finite-dimensional, paracompact manifolds and $\theta\colon X \to Y$ be a smooth submersion. Then the pushforward $\theta_*\colon C^\infty(M,X) \to C^\infty(M,Y)$ is a smooth submersion in the sense of Definition* 2.3*.*

Proof. We already know by A.12 that θ_* is smooth, whence we only have to construct submersion charts for θ_*. Let now $\mathcal{V} \subseteq TX$ be the vertical subbundle given fibre-wise by $\operatorname{Ker} T_p\theta$. Lemma B.3 allows us to choose a smooth horizontal distribution $\mathcal{H} \subseteq TX$ (i.e. a smooth subbundle such that $TX = \mathcal{V} \oplus \mathcal{H}$) and local additions η_X on X and η_Y on Y such that the following diagram commutes:

$$\begin{array}{ccccc} TX = \mathcal{V} \oplus \mathcal{H} & \xleftarrow{\supseteq} & \Omega_X & \xrightarrow{\eta_X} & X \\ \downarrow{\scriptstyle 0 \oplus T\theta|_{\mathcal{H}}} & & & & \downarrow{\scriptstyle \theta} \\ TY & \xleftarrow{\supseteq} & \Omega_Y & \xrightarrow{\eta_Y} & Y \end{array} \tag{2}$$

Now fix $f \in C^\infty(M,X)$ and construct submersion charts for f as follows. By Remark A.11 the manifold structures of $C^\infty(M,X)$ and $C^\infty(M,Y)$ do not depend on the choice of local addition, whence we can use η_X and η_Y without loss of generality. We use the two canonical charts $\varphi_f\colon U_f \to \mathcal{D}_f(M,TX) \cong \Gamma_c(f^*TX)$ and $\varphi_{\theta_*(f)}\colon U_{\theta_*(f)} \to \mathcal{D}_{\theta_*(f)}(M,TY) \cong \Gamma_c(\theta_*(f)^*TY)$ and identify (cf. [20, Proof of Proposition 4.8 and Remark 4.11 2.])

$$\Gamma_c(f^*TX) = \Gamma_c(f^*(\mathcal{V} \oplus \mathcal{H})) \cong \Gamma_c(f^*\mathcal{V}) \oplus \Gamma_c(f^*\mathcal{H}). \tag{3}$$

Further, by the universal property of the pullback bundle $T\theta$ induces a bundle morphism $\Theta\colon f^*TX \to \theta_*(f)^*TY$ over the identity of M. Note that on the subbundle $f^*\mathcal{H}$ the morphism Θ restricts to a bundle isomorphism $B\colon f^*\mathcal{H} \to \theta_*(f)^*TY$ (as B is a bundle morphism over the identity of M which is fibre-wise an isomorphism).

We denote by $I_f\colon \theta_*(f)^*TY \to f^*TX$ the bundle map obtained from composing the inverse B^{-1} with the canonical inclusion $i^f_{\mathcal{H}}\colon f^*\mathcal{H} \to f^*TX$ (of smooth bundles). Further, by a slight abuse of notation we will also denote by $(I_f)_*$ and $(T\theta)_*$ pushforwards of the bundle mappings on the spaces of sections $\Gamma_c(f^*TX)$ and $\Gamma_c(f^*TY)$. Note that these pushforwards are smooth and linear maps (as the vector space structure is given by the fibre-wise operations).

Recall from Remark A.11 that the the inverses φ_f^{-1} and $\varphi_{\theta_*(f)}^{-1}$ are given by postcomposition with the local addition. Hence, we can combine (2), (3) and the definition of I_f to obtain a commutative diagram (where restrictions to open sets are suppressed):

$$\begin{array}{ccccc} \Gamma_c\big(f^*(\mathcal{H}\cap\Omega_X)\big) & \xrightarrow{(i^f_{\mathcal{H}})_*} & \Gamma_c\big(f^*(\Omega_X)\big) & \xrightarrow{\varphi_f^{-1}} & C^\infty(M,X) \\ \uparrow{\scriptstyle (B_*^{-1})} & & {\scriptstyle (I_f)_*}\uparrow\downarrow{\scriptstyle (T\theta)_*} & & \downarrow{\scriptstyle \theta_*} \\ \Gamma_c\big(\theta_*(f)^*\Omega_Y\big) & = & \Gamma_c\big(\theta_*(f)^*\Omega_Y\big) & \xrightarrow{\varphi_{\theta_*(f)}^{-1}} & C^\infty(M,Y) \end{array} \tag{4}$$

We conclude that on the canonical charts for f, the map θ satisfies

$$\varphi_{\theta_*(f)} \circ \theta_* \circ \varphi_f^{-1} = (T\theta)_*|_{\Omega_X}$$

for the continuous linear map $(T\theta)_*$. Vice versa, we deduce from (4) that $(I_f)_*$ is the continuous linear right inverse of $(T\theta)_*$ turning $(\varphi_f, \varphi_{\theta_*(f)})$ into submersion charts. ■

Remark 2.5. (1) Note that the source manifold M in the statement of Proposition 2.4 played no direct rôle in the proof. In particular, Lemma 2.4 stays valid if we replace M by a manifold with boundary or with corners (e.g. a compact interval). We refer to [20, Chapter 10] for the description of the manifold structure on $C^\infty(M,N)$ if M is a manifold with corners.
(2) Even if θ is assumed to be a surjective submersion, the pushforward θ_* will in general not be surjective. For example, consider a surjective submersion $\theta\colon X \to Y$ which does not define a fibration and let Y be connected. Then curves to Y will in general not lift globally (since θ does not admit a complete Ehresmann connection, see [10]).

The Stacey-Roberts Lemma now turns $S_{\mathcal{G}}$ into a submanifold of $C^\infty_{fS}(G,G)$.

Lemma 2.6. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. Then $S_{\mathcal{G}}$ is a closed and split submanifold of $C^\infty_{fS}(G,G)$.*

Proof. Note that $S_{\mathcal{G}}$ is the preimage of the singleton $\{\alpha\} \subseteq C^\infty(G,M)$ under the submersion $\beta_*\colon C^\infty(G,G) \to C^\infty(G,M)$. Thus the statement follows from the usual results on preimages of submanifolds under submersions, see [14, Theorem C]. ■

The differentiability of the monoid operations needs some preparations.

Lemma 2.7. *Let $\mathcal{G}$ be a Lie groupoid then the following mappings are smooth:*

(1) $\Gamma\colon S_{\mathcal{G}} \to C^\infty(G, G^{(2)})$ *with* $\Gamma(f)(x) := \big(x, f(x)\big)$, *for* $x \in G$.

(2) $R\colon S_{\mathcal{G}} \to C^\infty(G,G)$ *by* $R(f)(x) := R_f(x) := xf(x), x \in G$.

Proof. (1) By definition of $S_{\mathcal{G}}$ the map Γ makes sense. Let $\Delta\colon G \to G \times G$ be the diagonal map. By [20, Proposition 10.5], this map induces a diffeomorphism

$$C^\infty(G,G) \times C^\infty(G,G) \to C^\infty(G, G \times G), \quad (f,g) \mapsto (f,g) \circ \Delta.$$

Restricting the diffeomorphism to the closed submanifold

$$C^\infty(G,G) \times S_{\mathcal{G}} \subseteq C^\infty(G,G) \times C^\infty(G,G),$$

it factors to a map $\delta\colon C^\infty(G,G) \times S_{\mathcal{G}} \to C^\infty(G, G^{(2)})$. Since $G^{(2)} \subseteq G \times G$ is an embedded submanifold, [20, Proposition 10.8] implies that $C^\infty(G, G^{(2)})$ is a closed and split submanifold of $C^\infty(G, G \times G)$, whence δ is smooth. Then $\Gamma(f) = \delta(\mathrm{id}_G, f)$ is smooth as $S_{\mathcal{G}} \to C^\infty(G,G) \times S_{\mathcal{G}}, f \mapsto (\mathrm{id}_G, f)$ is smooth.

(2) Since $R = m_* \circ \Gamma$ where m_* is the pushforward of the (smooth) multiplication $m\colon G^{(2)} \to G$ of $\mathcal{G}$. Thus A.12 shows that R is smooth. ∎

Definition 2.8. Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid then we define

$$R_{S_{\mathcal{G}}} := R(S_{\mathcal{G}}) = \{R_f \mid f \in S_{\mathcal{G}}\} \subseteq C^\infty(G,G).$$

As [1, Lemma 1] shows, R satisfies $R(f \star g) = R(g) \circ R(f)$ for all $f, g \in S_{\mathcal{G}}$ and is injective. In the following, we denote by "$\circ_{\mathrm{op}}$" composition in opposite order, i.e. $f \circ g = g \circ_{\mathrm{op}} f$. Then $(R_{S_{\mathcal{G}}}, \circ_{\mathrm{op}})$ is a submonoid of $(C^\infty(G,G), \circ_{\mathrm{op}})$ and with respect to this structure R induces a monoid monomorphism.

Remark 2.9. Note that if $\mathcal{G}$ is a group bundle then $S_{\mathcal{G}} = R_{S_{\mathcal{G}}}$ and if $S_{\mathcal{G}} \cap R_{S_{\mathcal{G}}} \neq \emptyset$ then G is a group bundle. Indeed $f \in S_{\mathcal{G}}$ implies that $\alpha(x) = \beta(f(x))$ for every $x \in G$. On the other hand if $f \in R_{S_{\mathcal{G}}}$ then $f = R_g$ for some $g \in S_{\mathcal{G}}$, so $\beta(x) = \beta(xg(x)) = \beta(f(x))$, hence $S_{\mathcal{G}} \cap R_{S_{\mathcal{G}}} \neq \emptyset$ implies that $\alpha(x) = \beta(x)$ for every $x \in G$ which means that G is a group bundle. Therefore if $\mathcal{G}$ is not a group bundle, then $S_{\mathcal{G}}$ and $R_{S_{\mathcal{G}}}$ are disjoint subsets of $C^\infty(G,G)$.

Remark 2.10 (The isomorphism $R\colon S_{\mathcal{G}} \to R_{S_{\mathcal{G}}}$). By Definition 2.8 the monoids $(R_{S_{\mathcal{G}}}, \circ_{\mathrm{op}})$ and $(S_{\mathcal{G}}, \star)$ are isomorphic as (set) monoids. The inverse of the isomorphism R is given by $R^{-1}(\psi)(x) := x^{-1}\psi(x)$ for $\psi \in R_{S_{\mathcal{G}}}$ and $x \in G$. We further note that $R(\alpha) = \mathrm{id}_G$ and we have the identity

$$R_{S_{\mathcal{G}}} = \{f \in C^\infty(G,G) \mid \beta \circ f = \beta\} = \beta_*^{-1}(\{\beta\}).$$

As a consequence of the Stacey-Roberts Lemma 2.4 also $R_{S_{\mathcal{G}}}$ is a closed and split submanifold of $C^\infty(G,G)$. Now Lemma 2.7 implies that R induces a diffeomorphism between the submanifolds $S_{\mathcal{G}}$ and $R_{S_{\mathcal{G}}}$. To see this note that $R^{-1} = m_* \circ (\iota, \mathrm{id}_{S_{\mathcal{G}}}) \circ \Delta$ (where the notation is as in Lemma 2.7), whence also R^{-1} is smooth.

Thus the monoids $(S_{\mathcal{G}}, \star)$ and $(R_{S_{\mathcal{G}}}, \circ_{\mathrm{op}})$ are isomorphic via the diffeomorphism $R|^{R_{S_{\mathcal{G}}}}$. In particular, $(R_{S_{\mathcal{G}}}, \circ_{\mathrm{op}})$ is a differentiable (topological) monoid if and only if $(S_{\mathcal{G}}, \star)$ is a differentiable (topological) monoid. This will enable us to investigate the group of invertible elements of $S_{\mathcal{G}}$ in Section 3.

2.2. Differentiable monoids for source-proper groupoids

To establish differentiability properties of the monoid operation "$\star$", we would like to use smoothness of the composition

$$\circ\colon C^\infty(Y,Z) \times C^\infty(X,Y) \to C^\infty(X,Z), (f,g) \mapsto f \circ g.$$

However, this mapping will in general not be smooth (not even continuous!) if X is non-compact. Following A.12, we can obtain a smooth map if we restrict to the subset of smooth proper maps Proposition(X, Y). It turns out that the $S_{\mathcal{G}}$ will be contained in this set if the Lie groupoid satisfies the following condition.

Remark 2.11. The Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ is *α-proper* (or *source-proper*) if the source map is a proper map, i.e. $\alpha^{-1}(K)$ is compact for each $K \subseteq M$ compact (cf. [6, §10.3 Proposition 7]). Note that this entails that β is a proper map, since $\beta = \alpha \circ \iota$ and ι is a diffeomorphism.

The concept of α-proper groupoids appears for example in the integration of Poisson manifolds of compact type, see [8].

Example 2.12. Let G be a Lie group acting smoothly on M. We denote by $G \ltimes M$ the corresponding action Lie groupoid. Then $G \ltimes M$ is α-proper if and only if G is compact. Note that α-properness is a stronger condition than being a proper Lie groupoid (which in the $G \ltimes M$ example would only force the group action to be proper).

Example 2.13. Recall from [27, 25] that every paracompact, smooth and effective orbifold can be represented by a so-called atlas groupoid. To this end, one constructs from an atlas a proper étale Lie groupoid (see [25, Proposition 5.29]). Following this procedure for a locally finite orbifold atlas, one sees that the atlas groupoid will even be source proper. Note that an atlas which is not locally finite yields a non-source proper Lie groupoid. Since all atlas groupoids of a fixed orbifold are Morita equivalent, source properness is not stable under Morita equivalence.

Remark 2.14. In general, it is not enough to require that the α-fibres, $\alpha^{-1}(x)$ for $x \in M$ are compact to obtain an α-proper groupoid. However, it is sufficient to require that the α-fibres of $\mathcal{G}$ are compact and connected, to obtain an α-proper groupoid. This can be seen as follows: Consider a quotient map $f\colon X \to Y$ between Hausdorff locally compact spaces such that every component of $f^{-1}(x)$ is compact. By [34, p. 103] Then $f = l \circ \varphi$ uniquely factors into a proper map $\varphi\colon X \to M$ (using that closed maps with $f^{-1}(x)$ compact for all x are proper by [6, §10.2 Theorem 1]) onto a quotient space M and a map $l\colon M \to Y$. If we assume that the fibres $f^{-1}(x)$ are connected then the quotient space M coincides with Y and thus φ coincides with f.

Lemma 2.15. *For an α-proper Lie groupoid $\mathcal{G}$ one has $S_{\mathcal{G}} \subseteq$ Proposition(G, G) and $R(S_{\mathcal{G}}) \subseteq$ Proposition(G, G).*

Proof. Let us first fix $f \in S_{\mathcal{G}}$ and prove that f is a proper map. It suffices to prove that $f^{-1}(K)$ is compact for each compact $K \subseteq G$. Now by definition of $S_{\mathcal{G}}$ we have $f^{-1}(K) \subseteq \alpha^{-1}\big(\beta(K)\big)$. Hence by α-properness, $f^{-1}(K)$ is compact as a closed subset of the compact set $\alpha^{-1}\big(\beta(K)\big)$.
Similarly, proceeding for $R(f)$ as in Remark 2.10 implies that $R_f^{-1}(K) \subseteq \beta^{-1}\big(\beta(K)\big)$. Now $\mathcal{G}$ is α-proper, whence β is proper (as $\beta = \alpha \circ \iota$) and therefore $R_f^{-1}(K)$ as a closed subset of a compact set $\beta^{-1}\big(\beta(K)\big)$ which again is compact. ■

We can now prove the main theorem of this section which establishes continuity and differentiability of the monoid $S_{\mathcal{G}}$ for α-proper Lie groupoids.

Theorem 2.16. *Let $\mathcal{G} = (G \rightrightarrows M)$ be an α-proper Lie groupoid. Then $(S_\mathcal{G}, \star)$ with the manifold structure from Lemma 2.6 is a differentiable monoid.*

Proof. Recall from Definition 2.1 that the monoid multiplication can be written as

$$f \star g = m_* \circ \Big(f, \mathrm{Comp}\big(g, R(f)\big)\Big) \quad f, g \in S_\mathcal{G},$$

where $\mathrm{Comp}(f, g) = f \circ g$ is the joint composition map. By Lemma 2.7 R is a smooth map which maps $S_\mathcal{G}$ into the open subset $\mathrm{Proposition}(G, G)$ by Lemma 2.15 and α-properness of $\mathcal{G}$. Since $\mathrm{Comp}\colon C^\infty(G, G) \times \mathrm{Proposition}(G, G) \to C^\infty(G, G)$ is smooth by A.12 the monoid operation $\star$ is smooth as a map into the (closed) submanifold $S_\mathcal{G}$ (cf. [14, p. 10]). ■

Remark 2.17. We established continuity and differentiability of the multiplication in $S_\mathcal{G}$ by exploiting the continuity of the joint composition map. If the Lie groupoid $\mathcal{G}$ is not α-proper, $S_\mathcal{G}$ contains mappings which are not proper, e.g. α, and the joint composition is in general discontinuous outside of $\mathrm{Proposition}(G, G)$ (see e.g. [16, Example 2.1]. Thus the proof of Theorem 2.16 breaks down and we do not expect that $S_\mathcal{G}$ will in general be a topological monoid. Note however, that the subgroup of units in $S_\mathcal{G}$ (studied in Section 3 below) will always become a Lie group even if $\mathcal{G}$ is not α-proper.

Remark 2.18. Analogous results to the ones established in this section can be established for the coarser very strong topology. One can prove that $S_\mathcal{G}$ is a topological monoid in the very strong topology if $\mathcal{G}$ is α-proper. As the very strong topology does not turn $C^\infty(G, G)$ into a manifold, there is no differentiable structure on $S_\mathcal{G}$.

3. The Lie group of invertible elements

In this section we consider the group of invertible elements of the monoid $(S_\mathcal{G}, \star)$. This group turns out to be a Lie group (with the subspace topology of the (fine) very strong topology). Let us first derive an alternative description of the invertible elements.

Definition 3.1. For a Lie groupoid $\mathcal{G}$ define the group

$$\big(S_\mathcal{G}(\alpha) := \big\{f \in S_\mathcal{G} \mid R(f) \in \mathrm{Diff}(G)\big\}, \star\big).$$

Further, we set $R_{S_\mathcal{G}(\alpha)} := R\big(S_\mathcal{G}(\alpha)\big)$.

Remark 3.2. In Remark 2.10 we have seen that R is a monoid isomorphism between $(S_\mathcal{G}, \star)$ and

$$\big(R_{S_\mathcal{G}} = \big\{f \in C^\infty(G, G) \mid \beta \circ f = \beta\big\}, \circ\big).$$

As the units in $R_{S_\mathcal{G}}$ are clearly given by $R_{S_\mathcal{G}(\alpha)} = R_{S_\mathcal{G}} \cap \mathrm{Diff}(G)$, we see that $\big(S_\mathcal{G}(\alpha), \star\big)$ indeed is the group of units of the monoid $S_\mathcal{G}$.
Now the group $\mathrm{Diff}(G)$ is open in $C^\infty_{fS}(G, G)$ and Lemma 2.7 2.7 implies that R is smooth, $S_\mathcal{G}(\alpha) = R^{-1}\big(\mathrm{Diff}(G)\big)$ is an open submanifold of $S_\mathcal{G}$.

The fact that $R_{S_{\mathcal{G}}(\alpha)} \subseteq \mathrm{Diff}(G) \subseteq \mathrm{Proposition}(G,G)$ will enable us to show that for an arbitrary Lie groupoid $\mathcal{G}$ the group of units in $S_{\mathcal{G}}$ is an infinite-dimensional Lie group. As $S_{\mathcal{G}}(\alpha)$ is isomorphic as a group and a submanifold to $R_{S_{\mathcal{G}}(\alpha)}$ we will first establish a Lie group structure on $R_{S_{\mathcal{G}}(\alpha)}$. To this end recall that for a finite-dimensional manifold G, $\mathrm{Diff}(G)$ is an open subset of $C^\infty_{fS}(G,G)$. By [20, Theorem 11.11] this structure turns $(\mathrm{Diff}(G),\circ)$ (or equivalently $(\mathrm{Diff}(G),\circ_{\mathrm{op}})$) into an infinite-dimensional Lie group whose Lie algebra is given by the compactly supported vector fields $\mathfrak{X}_c(G)$ whose Lie bracket is the negative of the bracket of vector fields (cf. [29]).

Proposition 3.3. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. Then $R_{S_{\mathcal{G}}(\alpha)}$ is a closed Lie subgroup of* $(\mathrm{Diff}(G),\circ)$.

Proof. Being a submersion is a local condition, whence $\beta_*|_{\mathrm{Diff}(G)}$ is a submersion as a consequence of the Stacey-Roberts Lemma 2.4. Clearly $R_{S_{\mathcal{G}}(\alpha)} = R_{S_{\mathcal{G}}} \cap \mathrm{Diff}(G) = (\beta_*|_{\mathrm{Diff}(G)})^{-1}(\{\beta\})$ is a closed submanifold of $\mathrm{Diff}(G)$ and this structure coincides with the manifold structure obtained from $\mathrm{Diff}(G) \cap R_{S_{\mathcal{G}}}$. Thus $R_{S_{\mathcal{G}}(\alpha)}$ is a closed submanifold of $\mathrm{Diff}(G)$, whence it suffices to check that it is a subgroup of $(\mathrm{Diff}(G),\circ)$: Let $f,g \in R_{S_{\mathcal{G}}(\alpha)}$, then clearly $\beta \circ (f \circ g) = \beta$. Moreover, for the inverse f^{-1} of f in $\mathrm{Diff}(G)$ we have $\beta \circ f^{-1} = (\beta \circ f) \circ f^{-1} = \beta$ and thus $f^{-1} \in R_{S_{\mathcal{G}}(\alpha)}$ if and only if $f \in R_{S_{\mathcal{G}}(\alpha)}$. Summing up $R_{S_{\mathcal{G}}(\alpha)}$ is a closed Lie subgroup of $(\mathrm{Diff}(G),\circ)$.[4] ∎

To identify the Lie algebra of $R_{S_{\mathcal{G}}(\alpha)}$ (as a subalgebra of $\mathbf{L}((\mathrm{Diff}(G),\circ)) = \mathfrak{X}_c(G)$) we need some preparations.

Remark 3.4. Consider the subset $T^\beta G = \bigcup_{g\in G} T_g(\beta^{-1}(\beta(g))$ of TG. Note that for all $v \in T^\beta_g G$ the definition implies $T\beta(v) = 0_{\beta(g)} \in T_\beta(g)M$, i.e. fibre-wise we have $T^\beta_g G = \ker T_g\beta$. Since β is a submersion, the same is true for $T\beta$. Computing in submersion charts, the kernel of $T_g\beta$ is a direct summand of the model space of TG. Furthermore, the submersion charts of $T\beta$ yield submanifold charts for $T^\beta G$ whence $T^\beta G$ becomes a split submanifold of TG. Restricting the projection of TG, we thus obtain a subbundle $\pi_\beta \colon T^\beta G \to G$ of the tangent bundle TG.

Proposition 3.5. *The Lie algebra of $(R_{S_{\mathcal{G}}(\alpha)},\circ)$ is given by the Lie subalgebra*

$$\mathfrak{g}_\beta := \left\{X \in \mathfrak{X}_c(G) \mid X(G) \subseteq T^\beta G\right\}$$

of $\big(\mathfrak{X}_c(G),[\cdot,\cdot]\big)$, where $[\cdot,\cdot]$ is the negative of the usual bracket of vector fields

Remark 3.6. It is a priori clear that $\mathfrak{g}_\beta$ is a Lie subalgebra of $\big(\mathfrak{X}_c(G),[\cdot,\cdot]\big)$ since $X \in \mathfrak{g}_\beta$ if and only if X is β-related to the zero-vector field on M. (cf. [19, Lemma 3.5.5] for a similar construction in the context of the Lie algebroid of $\mathcal{G}$). However, this will also follow from our proof below.

Proof of Proposition 3.5. Denote by $C^1_*\big([0,1],\mathrm{Diff}(G)\big)$ the set of all continuously differentiable curves $c\colon [0,1] \to \mathrm{Diff}(G)$ with $c(0) = \mathrm{id}_G$. Since $R_{S_{\mathcal{G}}(\alpha)}$ is a closed Lie subgroup of $\mathrm{Diff}(G)$ by Proposition 3.3, its Lie algebra $\mathbf{L}(R_{S_{\mathcal{G}}(\alpha)})$ can be

[4]Note that in contrast to the finite dimensional case, closed subgroups of infinite-dimensional Lie groups need not be Lie groups (cf. [26, Remark IV.3.17] for an example), whence it was essential to prove that $R_{S_{\mathcal{G}}(\alpha)}$ is a closed submanifold of $\mathrm{Diff}(G)$.

computed by [26, Proposition II.6.3] as the differential tangent set

$$\mathbf{L}^d(R_{S_{\mathcal{G}}(\alpha)}) := \big\{\dot{c}(0) \in \mathfrak{X}_c(G) \mid c \in C^1_*\big([0,1], \mathrm{Diff}(G)\big), c\big([0,1]\big) \subseteq R_{S_{\mathcal{G}}(\alpha)}\big\}.$$

Let us consider a C^1-curve $c\colon [0,1] \to R_{S_{\mathcal{G}}(\alpha)}$ with $c(0) = \mathrm{id}_G$. By defintion of $R_{S_{\mathcal{G}}(\alpha)}$ we have $\beta_* \circ c(t) = \beta$ for all $t \in [0,1]$ and the right hand side does not depend on t. Hence, with the notation of Remark A.11, we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \beta_* \circ c(t) = (T\beta)_*(\dot{c}(0)) = \mathbf{0} \in \mathcal{D}_\beta(G, TM).$$

Here we used the formula $T(\beta_*) = (T\beta)_*$ from [20, Corollary 10.14] and conclude that the compactly supported vector field $\dot{c}(0)$ takes its values in $T^\beta G$, whence $\mathbf{L}^d(R_{S_{\mathcal{G}}(\alpha)}) \subseteq \mathfrak{g}_\beta$.
For the converse let $X \in \mathfrak{g}_\beta$. Recall that the flow map $\mathrm{Fl}_1\colon \mathfrak{X}_c(G) \to \mathrm{Diff}_c(G)$ to time 1 is the Lie group exponential of $\mathrm{Diff}(G)$ (see [21, 4.6]). Thus we can exponentiate X to the smooth one-parameter curve $c_X\colon [0,1] \to \mathrm{Diff}_c(G), t \mapsto \mathrm{Fl}_1(tX)$ with $c_X(0) = \mathrm{id}_G$ and $\dot{c}_X(0) = X$. To see that c_X takes its values in $R_{S_{\mathcal{G}}(\alpha)}$, note that $c_X(0) \in R_{S_{\mathcal{G}}(\alpha)}$ and $R_{S_{\mathcal{G}}(\alpha)}$ is a closed Lie subgroup. Thus it suffices to prove that the derivative of $\beta_* \circ c_X$ vanishes. However, arguing again as above we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \beta_* \circ c_X(t) = (T\beta)_* \dot{c}_X(0) = (T\beta)_*(X) = \mathbf{0} \in \mathcal{D}_\beta(G, TM)$$

since X takes its values in $T^\beta G$ as $X \in \mathfrak{g}_\beta$. Summing up $\mathbf{L}^d(R_{S_{\mathcal{G}}(\alpha)}) = \mathfrak{g}_\beta$ and the Lie bracket coincides with the restriction of the bracket on $\mathfrak{X}_c(G)$ to the subspace. ■

Our next aim is to establish regularity in the sense of Milnor for the subgroup $R_{S_{\mathcal{G}}(\alpha)}$. The idea here is again to leverage that $R_{S_{\mathcal{G}}(\alpha)}$ is a closed Lie subgroup of $\mathrm{Diff}(G)$.

Remark 3.7 (Regularity of $\mathrm{Diff}(G)$)**.** For a paracompact manifold G it is known (see [13, Corollary 13.7]) that $\mathrm{Diff}(G)$ is C^1-regular (even C^0-regular if G is σ-compact), i.e. for all C^1-curves $\eta\colon [0,1] \to \mathfrak{X}_c(G)$ the equation

$$\begin{cases} \gamma'(t) &= \eta(t).\gamma(t) := T\rho_{\gamma(t)}(\eta(t)), \\ \gamma(0) &= \mathrm{id}_G \end{cases} \tag{5}$$

has a unique solution γ_η which depends smoothly on η. In the case of $\mathrm{Diff}(G)$, the solution of (5) is the flow

$$\mathrm{Fl}^\eta\colon [0,1] \to \mathrm{Diff}(G), \quad t \mapsto \mathrm{Fl}^\eta(t, \cdot)$$

of the time dependent vector field η, i.e. for every $x \in G$, $t \mapsto \mathrm{Fl}^\eta(t)(x)$ is the integral curve of $\eta^\wedge\colon [0,1] \times G \to TG, (t,y) \mapsto \eta(t)(y)$ starting at x. Thus for $x \in G$ and every $t_0 \in [0,1]$ we have

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0} \mathrm{Fl}^\eta(t,x) = \eta\big(t_0, \mathrm{Fl}^\eta(t_0,x)\big) \tag{6}$$

For G compact, a proof of this can be found in [24, p. 1046]. It is also true in the non-compact case as a special version of [29, Section 5.4] (in particular [29, Proof of Theorem 5.4.11]) where the case of non-compact orbifolds is treated.[5]

Lemma 3.8. *Let $\eta\colon [0,1] \to \mathfrak{g}_\beta \subseteq \mathfrak{X}_c(G)$ be a C^k-curve, where $k=0$ if G is σ-compact and $k=1$ if G is only paracompact. Then the solution γ_η of (5) factors to a map $\gamma_\eta\colon [0,1] \to R_{S_\mathcal{G}(\alpha)}$.*

Proof. We set $k=0$ if G is σ-compact and $k=1$ if G is only paracompact. Since $\mathrm{Diff}(G)$ is C^k-regular the solution $\mathrm{Evol}(\eta) \in C^{k+1}\big([0,1], \mathrm{Diff}(G)\big)$ of (5) exists in $\mathrm{Diff}(G)$. Now $\mathrm{Evol}(\eta)(0) = \mathrm{id}_G$ and $R_{S_\mathcal{G}(\alpha)}$ is the closed subgroup of all elements g in $\mathrm{Diff}(G)$ which satisfy $\beta \circ g = \beta$. Thus it suffices to prove that $\beta_* \circ \mathrm{Evol}(\eta)$ is constant in t, i.e. we will show that its derivative vanishes in $TC^\infty(G,M)$.

Due to [20, Lemma 10.15] (the proof in loc.cit. for C^∞-curves carries over verbatim to C^1-curves) the derivative of $c(t) := \beta_* \circ \mathrm{Evol}(\eta)$ at t_0 vanishes in $T_{c(t_0)}C^\infty(G,M) \cong \mathcal{D}_{c(t_0)}(G,TM)$ if and only if we have $T_{t_0}\big(\mathrm{ev}_x \circ c(t)\big) = 0$ for each $x \in G$, where $\mathrm{ev}_x\colon C^\infty(G,M) \to M$, $f \mapsto f(x)$. Using now that $\mathrm{ev}_x \circ \beta_* = \beta \circ \mathrm{ev}_x$, we compute

$$T_{t_0}\,\mathrm{ev}_x \circ c(t) = T_{t_0}\beta \circ \mathrm{ev}_x\left(\mathrm{Fl}^\eta(t,\cdot)\right) = T\beta\left(\left.\frac{\partial}{\partial t}\right|_{t=t_0} \mathrm{Fl}^\eta(t,x)\right)$$
$$\overset{(6)}{=} T\beta\Big(\eta\big(t, \mathrm{Fl}^\eta(t,x)\big)\Big) = 0$$

since η takes its image in $T^\beta G$. ∎

Proposition 3.9. *$(R_{S_\mathcal{G}(\alpha)}, \circ)$ is C^1-regular (C^0-regular for G is σ-compact).*

Proof. Due to Lemma 3.8 we know that $R_{S_\mathcal{G}(\alpha)}$ is C^1-semiregular (C^0-semiregular if G is σ-compact). Since $R_{S_\mathcal{G}(\alpha)}$ is a closed Lie subgroup of $\mathrm{Diff}(G)$, the assertion then follows from [32, Lemma B.5]. ∎

By Definition 2.8 the groups $(R_{S_\mathcal{G}(\alpha)}, \circ_{\mathrm{op}})$ and $(S_\mathcal{G}(\alpha), \star)$ are isomorphic via R and this isomorphism is a diffeomorphism with respect to the natural manifold structures. As $(R_{S_\mathcal{G}(\alpha)}, \circ_{\mathrm{op}})$ and $(R_{S_\mathcal{G}(\alpha)}, \circ)$ are anti-isomorphic as Lie groups (e.g. by composing R with the group inversion), we conclude from Propositions 3.3, 3.5 and 3.9 the following.

Corollary 3.10. *Let $\mathcal{G}$ be a Lie groupoid, then the group $(S_\mathcal{G}(\alpha), \star)$ is a C^1-regular Lie group whose Lie algebra is anti-isomorphic to the algebra $\mathfrak{g}_\beta$. If the manifold G is σ-compact, then $(S_\mathcal{G}(\alpha), \star)$ is even C^0-regular.*

In the C^0-regular case the validity of the strong Trotter formula follows from [15]. It is currently unknown whether similar results hold for C^1-regular Lie groups.

4. Topological subgroups of the unit group

In this section we investigate certain subgroups of the Lie group $S_\mathcal{G}(\alpha)$. Albeit these groups will often be closed topological groups, we remark that not every closed

[5] To our knowledge [29] is the only source currently in print where a full proof of the regularity of $\mathrm{Diff}(G)$ for the non-compact G in our setting can be found. See however [26, Theorem III.4.1.] and [13, Corollary 13.7] which together with an unpublished preprint by Glöckner paved the way for the treatment in [29]. Cf. also [21, 4.6] for related considerations.

subgroup of an infinite-dimensional Lie group is a Lie subgroup (cf. [26, Remark IV.3.17]). Hence we do not know whether these groups are Lie subgroups.

4.1. The action of the unit group and its stabilizers

We first consider the group action of $S_{\mathcal{G}}(\alpha)$ on $\mathcal{G}$ and subgroups related to the stabilization of Lie subgroupoids under this action.

Remark 4.1. Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. By [1, Proposition 6] the monoid $S_{\mathcal{G}}$ acts on G from the right by

$$x.f = R_f(x) = xf(x), \quad x \in G, f \in S_{\mathcal{G}}.$$

Indeed $x.\alpha = x$ and $x.(f \star g) = (x.f).g$ for $x \in G$ and $f, g \in S_{\mathcal{G}}$. Note that this action is faithful, i.e. only the identity element acts trivially. It is easy to check that the right action of $S_{\mathcal{G}}$ is transitive if and only if G is a group, i.e. $\mathcal{G} = (G \rightrightarrows *)$.

Definition 4.2. Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and H be an arbitrary subset of G. Define

$$I_H(S_{\mathcal{G}}) := \{f \in S_{\mathcal{G}} : f(H) \subseteq H\}, \quad I_H\big(S_{\mathcal{G}}(\alpha)\big) := \{f \in S_{\mathcal{G}}(\alpha) : f(H) \subseteq H\}$$

and the *setwise stabilizer* $F_H\big(S_{\mathcal{G}}(\alpha)\big) := \{f \in S_{\mathcal{G}}(\alpha) : R_f(H) = H\}$ *of* H.

Remark 4.3. (1) Using the group homomorphism $R \colon S_{\mathcal{G}}(\alpha) \to \mathrm{Diff}(G)$, it is easy to check that for a subgroupoid H of G, $F_H\big(S_{\mathcal{G}}(\alpha)\big)$ is a subgroup of $S_{\mathcal{G}}(\alpha)$. In addition, $I_H(S_{\mathcal{G}})$ and $I_H\big(S_{\mathcal{G}}(\alpha)\big)$ are then two submonoids of $S_{\mathcal{G}}$.

(2) Since point evaluations are continuous with respect to the (fine) very strong topology (see Corollary A.9), if H is closed, then the sets $I_H\big(S_{\mathcal{G}}(\alpha)\big), I_H(S_{\mathcal{G}})$ and $F_H\big(S_{\mathcal{G}}(\alpha)\big)$ are closed in $S_{\mathcal{G}}$.

(3) For $f \in I_H\big(S_{\mathcal{G}}(\alpha)\big)$ and $y \in H$ there exist $x \in G$ with $R_f(x) = xf(x) = y$. However, since it is not clear that $f(x) \in H$, the inverse g of f in $S_{\mathcal{G}}(\alpha)$ does not necessarily satisfy $g(H) \subseteq H$. As H may not be invariant under composition with g, the monoid $I_H\big(S_{\mathcal{G}}(\alpha)\big)$ may not be closed under the inversion map.

We hope to investigate the geometry of this action in future work.

4.2. The subgroup of bisections

The Lie group $S_{\mathcal{G}}(\alpha)$ is naturally related to the group $\mathrm{Bis}(\mathcal{G})$ of bisections of the Lie groupoid $\mathcal{G}$. Let us first recall the definition of this group.

Definition 4.4 (Bisections of a Lie groupoid [19, Definition 1.4.1])**.** A *bisection* of a Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ is a smooth map $\sigma \colon M \to G$ which satisfies

$$\beta \circ \sigma = \mathrm{id}_M \text{ and } \alpha \circ \sigma \in \mathrm{Diff}(M).$$

The set of all bisections $\mathrm{Bis}(\mathcal{G})$ is a Lie group [3, Proposition A] with respect to

$$(\sigma_1 \star \sigma_2)(x) = \sigma_1(x)\sigma_2\big((\alpha \circ \sigma_1)(x)\big) \text{ and } \sigma^{-1} = \Big(\sigma \circ (\alpha \circ \sigma)^{-1}(x)\Big)^{-1} \quad \forall x \in M. \tag{7}$$

Remark 4.5. (1) Note that we use the symbol "$\star$" both for the multiplication in $\mathrm{Bis}(\mathcal{G})$ and in $S_{\mathcal{G}}$. However, there should be no room for confusion as it will always be clear from the setup which product is meant.

(2) Switching the rôles of α and β in Definition 4.4 we obtain another group of bisections which is isomorphic to $\operatorname{Bis}(\mathcal{G})$. It is thus just a matter of convention which group is studied and we choose to define $\operatorname{Bis}(\mathcal{G})$ here to fit to the definition of $S_{\mathcal{G}}$ in Definition 2.1 (cf. [19, 30] where the opposite convention is adopted).

To see the relation between $\operatorname{Bis}(\mathcal{G})$ and $S_{\mathcal{G}}(\alpha)$, we have to adapt the definition of the left-translations used in [19, Definition 1.4.1] to right-translations.

Definition 4.6. A *right Translation* of the Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ is a pair $(\varphi, \varphi_0) \in \operatorname{Diff}(G) \times \operatorname{Diff}(M)$, such that $\alpha \circ \varphi = \varphi_0 \circ \alpha$, $\beta \circ \varphi = \beta$, and each $\varphi_x \colon G_x \to G_{\varphi_0(x)}$ is R_g for some $g \in G^x_{\varphi_0(x)}$, where $R_g(y) = yg$.

Example 4.7 ([19, p.22])**.** For $\sigma \in \operatorname{Bis}(\mathcal{G})$, we obtain a right-translation $(R_\sigma, \alpha\circ\sigma)$, where

$$R_\sigma \colon G \to G, \quad R_\sigma(g) := g\sigma(\alpha(g)).$$

Conversely if (φ, φ_0) is a right-translation on $\mathcal{G}$, then $\sigma \colon M \to G$ with $\sigma(x) = x^{-1}\varphi(x)$ defines a bisection of $\mathcal{G}$.

Lemma 4.8. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. Every right-translation (φ, φ_0) on $\mathcal{G}$ defines an element $\psi(x) := x^{-1}\varphi(x), x \in G$ of $S_{\mathcal{G}}(\alpha)$ which is constant on G_u for every $u \in M$. Finally, the restriction of ψ to M is a bisection.*

Proof. Consider a right-translation (φ, φ_0) and define $\psi \colon G \to G$ with $\psi(x) = x^{-1}\varphi(x)$. Then $\beta\big(\varphi(x)\big) = \beta(x) = \alpha(x^{-1})$ implies that ψ makes sense. Since the maps φ, inversion and multiplication of $\mathcal{G}$ are smooth, ψ is smooth. Also $\beta\circ\psi = \alpha$, that is $\psi \in S_{\mathcal{G}}$, in fact $\psi \in S_{\mathcal{G}}(\alpha)$, since $R_\psi = \varphi$ is a diffeomorphism.
For the second part of the assertion, let $u \in M \subseteq G$ and $g \in G_u$. We observe that the restriction of φ on G_u is of the form R_θ for some $\theta \in G$. Hence $\psi(g) = g^{-1}\varphi(g) = g^{-1}g\theta = \theta$ and $\psi|_{G_u}$ is constant.
Finally, $(\beta\circ\psi)(u) = \beta\big(u^{-1}\varphi(u)\big) = \beta(u) = u$ for every $u \in M$, that is $\psi|_M$ is a right-inverse for β. Also $(\alpha \circ \psi)(u) = \alpha\big(\varphi(u)\big) = \varphi_0\big(\alpha(u)\big) = \varphi_0(u)$, hence $\alpha \circ \psi|_M = \varphi_0$ is a diffeomorphism. Therefore $\psi|_M$ is a bisection. ■

Remark 4.9. Note that every diffeomorphism $\varphi \colon G \to G$ with $\beta\circ\varphi = \beta$, induces an element $R^{-1}(\varphi) \colon G \to G, \psi(x) = x^{-1}\varphi(x)$ of $S_{\mathcal{G}}(\alpha)$. However, the restriction of $R(\varphi)$ on M may not be a bisection, see Example 4.12 below.

In the following we show that $\operatorname{Bis}(\mathcal{G})$ is isomorphic to a subgroup of $S_{\mathcal{G}}(\alpha)$.

Theorem 4.10. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid, then the map*

$$\Psi \colon \big(\operatorname{Bis}(\mathcal{G}), \star\big) \to \big(S_{\mathcal{G}}(\alpha), \star\big), \quad \sigma \mapsto \sigma \circ \alpha \tag{8}$$

is a group monomorphism. If in addition $\mathcal{G}$ is α-proper, Ψ is smooth.

Proof. Let $\sigma \in \operatorname{Bis}(\mathcal{G})$, then we consider the induced right-translation $(R_\sigma, \alpha\circ\sigma)$ (cf. Example 4.7). By Lemma 4.8 the right-translation $(R_\sigma, \alpha \circ \sigma)$ produces an element $\psi_\sigma \in S_{\mathcal{G}}(\alpha)$ by

$$\psi_\sigma(x) = x^{-1}R_\sigma(x) = x^{-1}x\sigma\big(\alpha(x)\big) = \sigma\big(\alpha(x)\big). \tag{9}$$

Lemma 4.8 states that $\psi_\sigma|_M$ is a bisection, and indeed $\psi_\sigma|_M = \sigma$. Therefore every bisection of $\mathcal{G}$ can be extended smoothly to yield an element of $C^\infty(G,G)$ which belongs to $S_\mathcal{G}(\alpha)$. Obviously the map $\Psi\colon \operatorname{Bis}(\mathcal{G}) \to S_\mathcal{G}(\alpha)$ with $\Psi(\sigma) = \psi_\sigma$ is injective. It is also a group homomorphism. To check this, let $g \in G$, $\sigma,\tau \in \operatorname{Bis}(\mathcal{G})$ and compute

$$\begin{aligned}\big(\Psi(\sigma\star\tau)\big)(g) &= \psi_{\sigma\star\tau}(g) \overset{(9)}{=} (\sigma\star\tau)\big(\alpha(g)\big) \overset{(7)}{=} \sigma\big(\alpha(g)\big)\tau\Big(\big(\alpha\circ\sigma\big)\big(\alpha(g)\big)\Big)\\ &= \psi_\sigma(g)\tau\Big(\alpha\big(\sigma(\alpha(g))\big)\Big) = \psi_\sigma(g)\tau\Big(\alpha\big(g\sigma\big(\alpha(g)\big)\big)\Big) = \psi_\sigma(g)\psi_\tau\big(g\psi_\sigma(g)\big)\\ &= (\psi_\sigma\star\psi_\tau)(g) = \big(\Psi(\sigma)\star\Psi(\tau)\big)(g).\end{aligned}$$

To prove the last assertion, assume now that $\mathcal{G}$ is an α-proper Lie groupoid, then α is proper, whence A.12 shows that Ψ is smooth as a mapping to $C^\infty_{fS}(M,G)$. Following [3, Proposition A] (cf. [30, 16]), $\operatorname{Bis}(\mathcal{G})$ is a submanifold of $C^\infty_{fS}(M,G)$ (and with this structure a Lie group), whence Ψ is smooth as a mapping to $\operatorname{Bis}(\mathcal{G})$. ■

Remark 4.11. If α is non-proper, the morphism Ψ will in general fail to be continuous, cf. [16, Example 2.1].

Example 4.12. (Pair groupoid) Let M be a manifold and $G = M\times M$, then the Pair groupoid $P(M)$ is defined as follows: $\big((x,y),(z,w)\big) \in P(M)^{(2)}$ if and only if $z = y$ and $(x,y)(y,w) = (x,w)$, $(x,y)^{-1} = (y,x)$. We have $\alpha(x,y) = y$ and $\beta(x,y) = x$ and units $(x,x), x \in M$. For this Lie groupoid, the monoid $S_{P(M)}$ consists of the maps

$$H\colon M\times M \to M\times M,\quad H(x,y) = \big(y,h(x,y)\big),\quad h\in C^\infty(M\times M,M).$$

The group $S_{P(M)}(\alpha)$ consists of all $H \in S_{P(M)}$ where for every $x\in M$ the map $h(x,-)\colon M \to M, y\mapsto h(x,y)$ is a diffeomorphism. It is easy to check that for $H\in S_{P(M)}(\alpha)$ the restriction of H on M is a bisection if and only if the restriction of h on the diagonal of $M\times M$ induces a diffeomorphism between M and the diagonal.

If we take the submanifold $M = \mathbb{R}\setminus\{0\}$ of $\mathbb{R}$ and consider the pair groupoid $P(M) = (G \rightrightarrows M)$, then $H\colon G\times G \to G, (x,y)\mapsto (y,xy)$ is an element of $S_\mathcal{G}(\alpha)$ whose its restriction $(x,x)\mapsto (x,x^2)$ on the base of $P(M)$ does not induce a diffeomorphism of the diagonal onto M. Thus H does not restrict to a bisection.

Though the bisection group can be realised as a subgroup of $S_\mathcal{G}(\alpha)$, we do not know whether this group embeds into $S_\mathcal{G}(\alpha)$ in a nice way. In the next section we investigate the subset of elements which are contained in the graph of elements in $S_\mathcal{G}$.

5. Connecting the unit group with Lie groupoids

We let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and investigate subsets of $G\times G$ which arise as the union of graphs of elements in $S_\mathcal{G}$. Our interest is twofold, as we want to

(1) connect the monoid $S_\mathcal{G}$ to the power set monoid of G,

(2) investigate the reconstruction of $G^{(2)}$ (as a set) from graphs of $S_\mathcal{G}$. These questions are related to similar questions for (sub-)groups of bisections of the Lie groupoid $\mathcal{G}$ (see e.g. [35, 32] and Remark 5.6 below).

However, as a first step let us establish a characterisation of elements in $S_{\mathcal{G}}(\alpha)$ via their graphs. Note that if $f \in C^\infty(G,G)$, then $f \in S_{\mathcal{G}}$ if and only if $\Gamma_f \subseteq G^{(2)}$, where $\Gamma_f := \{(x, f(x)) \in G \times G \mid x \in G\}$ is the graph of f. Remark 1.2 shows that $(G^{(2)})^0 = \Gamma_\alpha$. It is well known, cf. [14, p. 10] that the graphs of smooth functions are submanifolds, whence Γ_f and $(G^{(2)})^0$ are submanifolds of $G \times G$ for every $f \in S_{\mathcal{G}}$.

Remark 5.1 (Relating $S_{\mathcal{G}}$ and the power set monoid)**.** For a groupoid $\mathcal{G} = (G \rightrightarrows M)$ we denote by $P(G)$ the powerset of G. Together with the multiplication of sets (for two subset U, V of G, $U.V := \{xy \mid (x,y) \in (U \times V) \cap G^{(2)}\}$) is a monoid. The base space M is the unit of $P(G)$. Since $G^{(2)}$ is again a groupoid, $P(G^{(2)})$ with the multiplication of sets is a monoid. If $f, g \in S_{\mathcal{G}}$, then $\Gamma_f, \Gamma_g \subseteq G^{(2)}$. Therefore by the groupoid structure of $G^{(2)}$ and by [1, Proposition 8] $\Gamma_f.\Gamma_g = \Gamma_{f \star g}$.

Definition 5.2. Let $\mathcal{G}$ be a Lie groupoid. For every $A \subseteq S_{\mathcal{G}}$, set $\Gamma(A) = \bigcup_{f \in A} \Gamma_f$. By the definition of $S_{\mathcal{G}}$, $\Gamma(A) \subseteq G^{(2)}$ for every $A \subseteq S_{\mathcal{G}}$.

Lemma 5.3. *If A is a subgroup of $S_{\mathcal{G}}(\alpha)$, then $\Gamma(A)$ forms a wide subgroupoid of $G^{(2)}$.*

Proof. Note that $(G^2)^0 = \{(x, \alpha(x)) : x \in G\} = \Gamma_\alpha \subset \Gamma(A)$ whence the subset $\Gamma(A) \subseteq G^{(2)}$ can only be made into a wide subgroupoid. To prove that $\Gamma(A)$ is a subgroupoid of $G^{(2)}$, it is enough to show that it is closed under multiplication and inverse.
Let $\gamma_1, \gamma_2 \in \Gamma(A)$ and $(\gamma_1, \gamma_2) \in (G^{(2)})^{(2)}$. Then there exists $f_1, f_2 \in A$ with $\gamma_1 \in \Gamma_{f_1}, \gamma_2 \in \Gamma_{f_2}$, hence $\gamma_1\gamma_2 \in \Gamma_{f_1}.\Gamma_{f_2} = \Gamma_{f_1 \star f_2} \subset \Gamma(A)$. Also if $\gamma = (x, f(x)) \in \Gamma(A)$ where $f \in A$, then $\gamma^{-1} = (xf(x), f(x)^{-1})$ by the definition of inversion map on $G^{(2)}$. Now if g be the inverse of f in A and $y = xf(x)$, then by [1, Proposition 3] $g(y) = (f(x))^{-1}$. Hence $\gamma^{-1} \in \Gamma_g \subset \Gamma(A)$. ∎

For the setwise stabilizers of subgroupoids, Definition 4.2, the association in the Lemma 5.3 is invariant under isomorphism.

Proposition 5.4. *Let $\mathcal{G} = (G \rightrightarrows M)$ and $\mathcal{G}' = (G' \rightrightarrows M)$ be isomorphic via the map $\psi \in \mathrm{Diff}(G, G')$ inducing to the identity on the base. Let $H \rightrightarrows M$ be a Lie subgroupoid of $\mathcal{G}$ and $K := \psi(H)$, then $\psi \times \psi \colon G \times G \to G \times G$ induces an isomorphism of topological groupoids*

$$\Phi \colon \Gamma\Big(F_H\big(S_{\mathcal{G}}(\alpha)\big)\Big) \to \Gamma\Big(F_K\big(S_{\mathcal{G}'}(\alpha')\big)\Big), \quad \big(x, f(x)\big) \mapsto \big(\psi(x), \psi \circ f(x)\big).$$

Proof. Direct computation shows that the map $\Psi \colon F_H\big(S_{\mathcal{G}}(\alpha)\big) \to F_K\big(S_{G'}(\alpha')\big)$, $f \mapsto \psi \circ f \circ \psi^{-1}$ is a group isomorphism. Since ψ, ψ^{-1} are proper maps, Proposition A.8 implies that Ψ is an isomorphism of topological groups. Rewriting

$$\Phi\big(x, f(x)\big) = \big(\psi(x), \Psi(f)(\psi(x))\big),$$

one easily checks that Φ is an isomorphism of topological groupoids. ∎

Example 5.5. Let G be a Lie group and $\mathcal{G} = (G \rightrightarrows *)$, then as a set $S_{\mathcal{G}}$ is equal to $C^\infty(G,G)$ and also $(S_{\mathcal{G}}, \star)$ and $\big(C^\infty(G,G), \circ_{\mathrm{op}}\big)$ are two isomorphic monoids (cf. also Remark 2.9). Further, $S_{\mathcal{G}}(\alpha)$ is isomorphic to $\mathrm{Diff}(G)$. Therefore in this case $\Gamma(S_{\mathcal{G}}) = G \times G = G^{(2)}$ and if G is connected then the Homogeneity Lemma [23,

p. 22] implies that $\Gamma(S_{\mathcal{G}}(\alpha)) = G \times G = G^{(2)}$. We note that $S_{\mathcal{G}}(\alpha) \cap \mathrm{Diff}(G)$ consists of all smooth map $f \in C^\infty(G,G)$ for which f and R_f are diffeomorphisms. The authors do not know whether in general $S_{\mathcal{G}}(\alpha) \cap \mathrm{Diff}(G) = \emptyset$ or not. However, this set does not inherit a useful structure from either group (as e.g. the unit elements of both groups are not contained in the intersection).

Remark 5.6. If $\sigma \in \mathrm{Bis}(\mathcal{G})$, then $\Gamma_\sigma \subseteq (M \times G) \cap G^{(2)}$. In [35] (cf. also the later [32, Proposition 2.7]) the authors proved that if G is a $\beta-$connected Lie groupoid, then for every $g \in G$ there exists a bisection $\sigma \in \mathrm{Bis}(\mathcal{G})$ with $\sigma\big(\beta(g)\big) = g$, hence $\bigcup_{\sigma\in\mathrm{Bis}(\mathcal{G})} \Gamma_\sigma = (M \times G) \cap G^{(2)}$. This leads to the following question:

Question 5.7. For which Lie groupoids does one have $\Gamma(S_{\mathcal{G}}) = G^{(2)}$? More general, is it possible to characterize the minimal subset A of $S_{\mathcal{G}}$ which satisfies in $\Gamma(A) = G^{(2)}$? If $S_{\mathcal{G}}$ satisfies $\Gamma(S_{\mathcal{G}}) = G^{(2)}$, does this imply $\Gamma\big(S_{\mathcal{G}}(\alpha)\big) = G^{(2)}$?

Example 5.8. (1) It is well known (see e.g. [23, Homogeneity Lemma, p.22]) that if M is a connected smooth manifold then for any two points $x, y \in M$ there exists a diffeomorphism $\phi\colon M \to M$ such that $\phi(x) = y$. Therefore for the pair groupoid $P(M)$ of a connected smooth manifold M for $\big((x_0,y_0)(y_0,z_0)\big) \in P(M)^{(2)}$ there exists a diffeomorphism $\phi_0\colon M \to M$ such that $\phi_0(y_0) = z_0$. Then Example 4.12 shows that $F\colon M\times M \to M\times M$ with $F(x,y) = \big(y, \phi_0(y)\big)$ belongs to $S_{P(M)}(\alpha)$ and $\big((x_0,y_0)(y_0,z_0)\big) \in \Gamma_F$. Therefore $\Gamma\big(S_{P(M)}(\alpha)\big) = P(M)^{(2)}$.
Conversely one can argue as in [32, Remark 2.18 b)] to obtain: For a disconnected manifold M where the connected componennts are not diffeomorphic, one has $\Gamma\big(S_{P(M)}(\alpha)\big) \neq P(M)^{(2)}$.
(2) Let $\mathcal{G}_{H\ltimes M} = (G := H \times M \to M$ be the action groupoid associated to a Lie group action of H on M. Then for every $h_0, k_0 \in H$ there exists a diffeomorphism $\phi_0 \in \mathrm{Diff}(H)$ with $\phi_0(h_0) = k_0$. Therefore let $\big((h_0,x_0)(k_0,y_0)\big) \in \mathcal{G}^{(2)}_{H\ltimes M}$, if we define $F\colon G \to G$ with $F(h,x) = \Big(\phi_0(h), \big(\phi_0(h)\big)^{-1}.x\Big)$ then it is easy to check that $F \in S_{\mathcal{G}_{H\ltimes M}}$ and $\big((h_0,x_0)(k_0,y_0)\big) \in \Gamma_F$, hence $\Gamma(S_{\mathcal{G}_{H\ltimes M}}) = \mathcal{G}^{(2)}_{H\ltimes M}$. The authors do not know whether $\Gamma\big(S_{\mathcal{G}_{H\ltimes M}}(\alpha)\big) = \mathcal{G}^{(2)}_{H\ltimes M}$.

So far, the results on $\Gamma(A)$ have been mostly algebraic. In general, it is unclear whether $\Gamma(A)$ carries a submanifold structure. However, one can prove the following.

Lemma 5.9. *If $A \subseteq S_{\mathcal{G}}$ is a compact set, then $\Gamma(A)$ is a closed set in $G^{(2)}$.*

Proof. Let $(x_\gamma, y_\gamma)_{\gamma\in I}$ be a net in $\Gamma(A)$ which convergence to $(x,y) \in G^{(2)}$. Then for every $\gamma \in I$ there exists $f_\gamma \in A$ with $y_\gamma = f_\gamma(x_\gamma)$. Bz compactness of A, replace the net with a converging subnet. Thus the net (f_γ) converges in the fine very strong topology to a function $f \in A$. As this topology in finer that the compact-open topology, (f_γ) converges uniformly to f on compact subsets of G. Since $\{x_\gamma\}_{\gamma\in I} \cup \{x\}$ is compact, $f_\gamma(x_\gamma)$ converges to $f(x) = z$ and consequently $(x,y) \in \Gamma_f$. ■

Question 5.10. Is it possible to characterize all subsets A of $S_{\mathcal{G}}$ which $\Gamma(A)$ is a submanifold of the manifold $G \times G$?

Acknowledgements. This research was partially supported by the European Unions Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 691070. We thank D. M. Roberts (Adelaide) for interesting discussions leading to Lemma 2.4. Finally, we wish to thank the anonymous referee for insightful comments which helped improve the paper.

A. Topology and manifold structures on spaces of smooth functions

In this section we recall some essentials concerning the space of smooth mappings $C^\infty(X,Y)$, where X,Y are finite-dimensional manifolds. On non-compact manifolds the compact-open C^∞-topology is not strong enough to control the behaviour of smooth functions. Instead one has to use a finer topology which allows one to control the behaviour of functions on locally finite covers of the manifold by compact sets. To this end let us briefly recall the construction of this topology (cf. e.g. [20, 16]).

Remark A.1 (Conventions)**.** In the following X, Y and Z denote smooth paracompact and finite-dimensional manifolds. For a multiindex $\{i_1,\dots i_n\}$ and $f\colon \mathbb{R}^n \supseteq U \to \mathbb{R}^m$ (with $U \subseteq \mathbb{R}^n$ open) sufficiently differentiable, we denote by $\frac{\partial}{\partial x_{i_1}\dots\partial x_{i_n}} f$ the iterated partial derivative.

Definition A.2 (Elementary neighborhood)**.** Let $r \in \mathbb{N}_0$ and $f\colon X \to Y$ smooth, (U,ϕ) a chart on X, (V,ψ) a chart on Y, $A \subseteq U$ compact such that $f(A) \subseteq V$, and $\epsilon > 0$. Define the set $\mathcal{N}^r\big(f; A, (U,\phi), (V,\psi), \epsilon\big)$ as

$$\left\{ h \in C^\infty(M,E) \;\middle|\; \begin{array}{l} h(A) \subseteq V, \\ \displaystyle\sup_{a\in A}\ \sup_{0\le j\le r} \left\| \frac{\partial^j}{\partial x_{s_1}\dots\partial x_{s_j}} \left(\psi\circ h\circ\phi^{-1} - \psi\circ f\circ\phi^{-1}\right)(a) \right\| < \epsilon \end{array} \right\}.$$

We call this set an *elementary C^r-neighborhood of f in $C^\infty(X,Y)$*.

Definition A.3 (Basic neighborhood)**.** Let $f\colon X \to Y$ be a smooth map. A *basic neighborhood of f in $C^\infty(X,Y)$* is a set of the form

$$\bigcap_{i\in\Lambda} \mathcal{N}^{r_i}\big(f; A_i, (U_i,\phi_i), (V_i,\psi_i), \epsilon_i\big),$$

where Λ is a possibly infinite indexing set, for all i the other parameters are as in Definition A.2, and $\{A_i\}_{i\in\Lambda}$ is locally finite.

Definition A.4 (Very strong topology)**.** The *very strong topology on* $C^\infty(X,Y)$ is the topology with basis the basic neighborhoods in $C^\infty(X,Y)$ and we denote the space $C^\infty(X,Y)$ with the very strong topology will by $C^\infty_{vS}(X,Y)$.

To consider $C^\infty(X,Y)$ as an infinite-dimensional manifold a stronger topology is needed. This is achieved by considering the following refinement.

Definition A.5 (The fine very strong topology)**.** Define an equivalence relation $\sim$ on $C^\infty(X,Y)$ by declaring that

$$f \sim g \quad :\Leftrightarrow \quad \overline{\big\{y \in M \mid f(y) \neq g(y)\big\}} \text{ is compact}.$$

Now refine the very strong topology on $C^\infty(X,Y)$ by demanding that the equivalence classes are open in $C^\infty(X,Y)$. In other words, equip $C^\infty(X,Y)$ with the topology generated by the very strong topology and the equivalence classes. This is the *fine very strong topology* on $C^\infty(X,Y)$. We write $C^\infty_{fS}(X,Y)$ for $C^\infty(X,Y)$ equipped with the fine very strong topology.

Remark A.6. In [20] the very strong topology is called $\mathcal{D}$-topology and the fine very strong topology is called $\mathcal{FD}$-topology.

Definition A.7. Recall that a $f \in C^\infty(X,Y)$ is *proper* if and only if the preimage of a compact set K in Y is compact in X. Further the set $\mathrm{Proposition}(X,Y) \subseteq C^\infty(X,Y)$ is open in the (fine) very strong topology.

Proposition A.8 ([16, Theorem 2.5 and Proposition 2.7]). *Endow $C^\infty(X,Y)$ and $\mathrm{Proposition}(X,Y)$ either with the very strong topology or the fine very strong topology. Let $f\colon Y \to Z$ be a smooth map, then the following mappings are continuous*

$$f_*\colon C^\infty(X,Y) \to C^\infty(X,Z), \quad h \mapsto f\circ h \quad \text{for } f \in C^\infty(Y,Z),$$
$$\mathrm{Comp}\colon C^\infty(Y,Z) \times \mathrm{Proposition}(X,Y) \to C^\infty(X,Z), \quad (f,g) \mapsto f\circ g.$$

Corollary A.9. *Endow $C^\infty(X,Y)$ either with the very strong or fine very strong topology. Then the evaluation $\mathrm{ev}\colon C^\infty(X,Y)\times X \to Y, (f,x)\mapsto f(x)$ is continuous.*

Proof. The one-point manifold $*$ is compact, and $\mathrm{ev}(f,x) = \mathrm{Comp}(f, * \mapsto x)$. ■

A.1. The manifold structure of the space of smooth mappings

Let us briefly sketch the well known construction of the (infinite-dimensional) manifold of mappings $C^\infty(X,Y)$. Denote for a manifold N by $\mathbf{0}\colon N \to TN$ the zero-section.

Definition A.10. A *local addition* on a manifold N is a smooth map $\Sigma\colon TN \supseteq \Omega \to N$ on an open neighborhood of the zero-section, such that

(1) $(\pi_{TN}, \Sigma)\colon \Omega \to U \subseteq N\times N$ induces a diffeomorphism onto an open neighborhood U of the diagonal in $N\times N$,
(2) $\Sigma \circ \mathbf{0} = \mathrm{id}_N$.

If N is finite dimensional, a local addition can always be constructed as the exponential map associated to a Riemannian metric.

Remark A.11 (Smooth structure of $C^\infty(X,Y)$). Consider $f\colon X\to Y$ and define

$$U_f := \left\{h \in C^\infty(X,Y) \mid h\sim f, (f(x),h(x)) \in U,\ \forall x\in X\right\}$$
$$\mathcal{D}_f(X,TY) := \left\{h\in C^\infty(X,TY) \mid \pi_Y\circ h = f, \text{ and } h \sim \mathbf{0}\circ f\right\}$$

where $\pi_Y\colon TY \to Y$ is the bundle projection. This yields a chart for $C^\infty(X,Y)$ around f by $\varphi_f\colon U_f \to \mathcal{D}_f(X,TY)$, $h\mapsto (\pi_Y,\Sigma)^{-1}\circ(f,h)$. Now $\mathcal{D}_f(X,TY)$ is isomorphic to the locally convex space $\Gamma_c(f^*TY)$ of compactly supported sections with values in the pullback bundle f^*TY (cf. [20, 1.17]). The inverse of the chart is given by the formula $\varphi_f^{-1}(h) = \Sigma \circ h$. These charts turn $C^\infty_{fS}(X,Y)$ into an infinite-dimensional manifold, [20, Theorem 10.4]. We will always endow the space $C^\infty_{fS}(X,Y)$ with this manifold structure and note that it does not depend on the choice of local addition, [20, Proof of Theorem 10.4].

Remark A.12. By [20, Corollary 10.14] the pushforward

$$\theta_*\colon C^\infty_{fS}(X,Y) \to C^\infty(X,Z), \quad f \mapsto \theta \circ f.$$

of smooth $\theta\colon Y \to Z$ is smooth. Further, [20, Theorem 11.4] asserts that

$$\mathrm{Comp}\colon C^\infty(Y,Z) \times \mathrm{Proposition}(X,Y) \to C^\infty(X,Z), \quad (f,g) \mapsto f \circ g$$

is smooth. In particular, $\mathrm{ev}\colon C^\infty_{fS}(X,Y) \times X \to Y$, $(f,x) \mapsto f(x)$ is smooth.

B. Details for the proof of the Stacey-Roberts Lemma

For the rest of this section, we let X, Y be finite-dimensional, paracompact manifolds and $\theta\colon X \to Y$ be a smooth submersion.

Lemma B.1 (Extracted from [33, Proof of Theorem 5.1])**.** *There is a smooth map $X_\theta\colon TX \to \mathfrak{X}_c(X)$ into the compactly supported vector fields on X such that*

$$X_\theta(v)\big(\pi(v)\big) = v \quad \forall v \in TX,\ \pi\colon TX \to X \textit{ bundle projection} \tag{10}$$

$$T\theta\big(X_\theta(v)(x)\big) = 0 \quad \textit{if } v \in \operatorname{Ker} T\theta \tag{11}$$

Proof. For brevity we set $d := \dim X$ and $k := \dim Y$.

Step 1: *Preparing a variant partition of unity.* Choose submersion charts for each $p \in X$, i.e. charts $\varphi_p\colon X \supseteq U_p \to \mathbb{R}^d = \mathbb{R}^{d-k} \times \mathbb{R}^k$ and $\psi_{\theta(p)}\colon Y \supseteq U_{\theta(p)} \to \mathbb{R}^k$ with

(a) $\theta(U_p) \subseteq U_{\theta(p)}$, $\varphi_p(p) = 0$ and $\psi_{\theta(p)}\big(\theta(p)\big) = 0$,

(b) $\psi^{-1}_{\theta(p)} \circ \theta \circ \varphi_p^{-1} = \mathrm{pr}_2$, where $\mathrm{pr}_2\colon \mathbb{R}^d = \mathbb{R}^{d-k} \times \mathbb{R}^k \to \mathbb{R}^k, (x,y) \mapsto y$.

By construction the chart domains $\{U_p\}$ cover X, whence we can construct a variant of a partition of unity subordinate to the covering $\{U_p\}$. Since X is paracompact and locally compact, we can choose (see [18, II, §3, Proposition 3.2]) a locally finite subcover $\{W_i\}_{i\in I}$ of relatively compact open sets[6] subordinate to the cover $\{U_p\}$. We choose and fix for each $i \in I$ a $p_i \in X$ with $\overline{W}_i \subseteq U_{p_i}$. A trivial variation of the usual construction of a partition of unity (cf. e.g. [18, II, §3, Theorem 3.3]) we obtain a family of smooth functions $\{\rho_i\colon X \to \mathbb{R}, i \in I\}$ such that

(1) The supports of the ρ_i are compact and contained in $W_i \subseteq U_{p_i}$.

(2) for each $x \in X$ we have $\sum \big(\rho_i(x)\big)^2 = 1$.

Step 2: *The maps X_i.* Recall that in the chart $\varphi_{p_i}\colon U_{p_i} \to \mathbb{R}^d$ induces diffeomorphisms $TU_{p_i} \cong \mathbb{R}^d \times \mathbb{R}^d$ and $\mathfrak{X}(U_{p_i}) \to C^\infty(\mathbb{R}^d, \mathbb{R}^d), Y \mapsto \mathrm{pr}_2 \circ T\varphi_{p_i} \circ Y \circ \varphi_{p_i}^{-1}$. Using these identifications we define a map $X_{p_i}\colon TU_{p_i} \to \mathfrak{X}(U_{p_i})$ as the map which corresponds to $\mathbb{R}^d \times \mathbb{R}^d \to C^\infty(\mathbb{R}^d, \mathbb{R}^d), (q,v) \mapsto (x \mapsto v)$. Note that X_{p_i} is **not** smooth (not even continuous!). We will remedy this problem by using our variant

[6]A set is relatively compact if its closure is compact

partition of unity. However, let us record the following properties of X_{p_i} first

$$\begin{aligned} X_{p_i}(v)\big(\pi(v)\big) &= v \quad \forall v \in TU_{p_i} \text{ and} \\ \text{if } T\theta(v) = 0 \text{ then } T\theta\big(X_{p_i}(v)(x)\big) &= 0 \quad \forall v \in TU_{p_i}, x \in U_{p_i}. \end{aligned} \tag{12}$$

Define for $i \in I$ a mapping $X_i \colon TX \to \mathfrak{X}_c(X)$ via the formula

$$X_i(v)(q) := \begin{cases} \rho_i\big(\pi(v)\big)\rho_i(q)X_{p_i}(v)(q) & \text{if } \pi(v), q \in W_i \\ 0 & \text{otherwise} \end{cases}.$$

For each $v \in TM$ the map $X_i(v)$ is smooth with compact support in $\operatorname{supp}\rho_i$. In particular, X_i takes its values in $\mathfrak{X}_c(X)$ and is smooth by Lemma B.2, where we identify $\mathfrak{X}_c(U_i) \cong C^\infty_c(\mathbb{R}^d, \mathbb{R}^d)$ via φ_{p_i}. Further, X_i inherits the properties (12).

Step 3: *The map X_θ.* Set $X_\theta \colon TX \to \mathfrak{X}_c(X)$, $v \mapsto \sum_{i\in I} X_i(v)$. Since $X_i(v) \in \mathfrak{X}_c(X)$ and for a given v only finitely many of the X_i are non-zero, the map X_θ makes sense. To establish continuity of X_θ observe that on the closed set $\pi^{-1}(\overline{W}_i)$ the map X_θ is given by a finite sum of maps X_i (since every $\overline{W}_i$ is compact and the supports of the X_i form a locally finite cover). Thus $X_\theta|_{\pi^{-1}(\overline{W}_i)}$ is continuous and smooth on $\pi^{-1}(W_i)$ as a finite sum of such mappings (using that $\mathfrak{X}_c(X)$ is a locally convex space). As the family $\{\overline{W}_i\}_{i\in I}$ is locally finite, this proves that X_θ is continuous. Now X_θ is smooth as it restricts to a smooth map on each member of the open cover $\{\pi^{-1}(W_i)\}_{i\in I}$.
By definition of X_θ, properties (10), (11) follow from (12) (cf. [33, p. 29]). ∎

Lemma B.2. *Let M be a (possibly infinite-dimensional) manifold. Consider a smooth map $f \colon M \times X \to \mathbb{R}^n$ with $n \in \mathbb{N}$ such that f vanishes outside $M \times K$ for $K \subseteq X$ compact. Then $f^\wedge \colon M \to C^\infty_c(X, \mathbb{R}^n), m \mapsto f(m, \cdot)$ is smooth.*

Proof. Every map $f(m, \cdot) \colon X \to \mathbb{R}^n$ vanishes outside of K, whence $f^\wedge$ takes its image in $C^\infty_K(X, \mathbb{R}^n) := \big\{h \in C^\infty(X, \mathbb{R}^n) \mid \operatorname{supp} h \subseteq K\big\} \subseteq C^\infty_c(X, \mathbb{R}^n)$. Since $C^\infty_K(X, \mathbb{R}^n)$ is a closed subspace of $C^\infty_c(X, \mathbb{R}^n)$ with the fine very strong topology, it suffices to prove that $f^\wedge$ is smooth as a mapping into $C^\infty_K(X, \mathbb{R}^n)$. The subspace topology on $C^\infty_K(X, \mathbb{R}^n)$ coincides with the topology induced by the compact open C^∞ topology, by [16, Remark 4.5]. Choose a locally finite family $\{V_\alpha\}_{\alpha\in I}$ of relatively compact sets which cover X. Then [12, Proposition 8.13 (a) and (b)] asserts that the map

$$\Gamma \colon C^\infty_K(X, \mathbb{R}^n) \to \bigoplus_{\alpha\in I, V_\alpha\cap K\neq\emptyset} C^\infty(V_\alpha, \mathbb{R}^n)_{c.o.}, \quad f \mapsto (f|_{V_\alpha})_{\alpha\in I}$$

is a topological embedding with closed image, where the spaces on the right side are endowed with the compact open C^∞-topology. As K is compact, only finitely many $\alpha \in I$ satisfy $V_\alpha \cap K \neq \emptyset$. Now $\Gamma \circ f^\wedge \colon M \to \prod_{\alpha\in I, V_\alpha\cap K\neq\emptyset} C^\infty(V_\alpha, \mathbb{R}^n)$ is smooth if the components $M \to C^\infty(V_\alpha, \mathbb{R}^n)$ are smooth. By [2, Theorem A] this follows as $f|_{M\times V_\alpha} \colon M \times V_\alpha \to \mathbb{R}^n$ is smooth. ∎

Lemma B.3 (Extracted from [33, Proof of Theorem 5.1])**.** *Denote by $\mathcal{V} \subseteq TX$ the vertical subbundle given fibre-wise by* $\operatorname{Ker} T_p\theta$. *There exists a smooth horizontal* *distribution $\mathcal{H} \subseteq TX$ (i.e. a smooth subbundle such that $TX = \mathcal{V} \oplus \mathcal{H}$) and local additions η_X on X and η_Y on Y such that the following diagram commutes:*

$$\begin{array}{ccccc} TX = \mathcal{V} \oplus \mathcal{H} & \xleftarrow{\supseteq} & \Omega_X & \xrightarrow{\eta_X} & X \\ \downarrow{\scriptstyle 0 \oplus T\theta|_{\mathcal{H}}} & & & & \downarrow{\scriptstyle \theta} \\ TY & \xleftarrow{\supseteq} & \Omega_Y & \xrightarrow{\eta_Y} & Y \end{array} \tag{13}$$

Proof. As in [9, Lemma 2.1] we turn θ we choose a θ-transverse Riemannian metric G_t on X as follows: Choose an Ehresmann connection for $\mathcal{V}$, i.e. a smooth horizontal distribution $\mathcal{H}$ complementing $\mathcal{V}$, and declare $\mathcal{V}$ and $\mathcal{H}$ to be orthogonal. Choose a Riemannian metric G_Y on Y which induces a metric on $\mathcal{H}$ via pullback by $T\theta$. For $\mathcal{V}$ we choose now an arbitrary bundle metric $G_{\mathcal{V}}$. This yields a Riemannian metric G_t which is θ-transverse, thus (cf. [9, Section 2.1]) θ becomes a Riemannian submersion.

The Riemannian metrics G_t and G_Y give rise to Riemannian exponential maps $\exp_X\colon \Omega_X \to X$ and $\exp_Y\colon \Omega_Y \to Y$. Here Ω_X and Ω_Y are neighborhoods of the zero section such that $T\theta(\Omega_X) \subseteq \Omega_Y$ and $(\pi_X, \exp_X)$ and $(\pi_Y, \exp_Y)$ with π_X, π_Y the canonical bundle projections, induce diffeomorphisms onto a neighborhood of the diagonal. In other words $\exp_X$ and $\exp_Y$ induce local additions and we set $\eta_Y := \exp_Y$. The Riemannian submersion θ maps horizontal geodesics (i.e. geodesics with initial value in $\mathcal{H}$) to geodesics in Y, whence we obtain the formula.

$$\theta \circ \exp_X(v) = \eta_Y \circ T\theta(v) \quad v \in \Omega_X \cap \mathcal{H} \tag{14}$$

Now we modify $\exp_X$ to obtain a local addition for which (14) holds for each $v \in \Omega_X$:

Step 1: *Parallel transport in $\mathcal{V}$.* Denote for a $\gamma\colon [0,1] \to X$ smooth the parallel transport along γ with respect to the vertical distribution[7] by $P_\gamma^{t_0,t_1}\colon T_{\gamma(t_0)}X \to T_{\gamma(t_1)}X$. From the exponential law [30, Theorem 7.8 (d)], we see that the assignment $\alpha\colon TX \to C^\infty([0,1], X)$, $v \mapsto \alpha_v$ is smooth, where $\alpha_v(t) := \exp_X(tv)$. Now as parallel transport arises as the flow of an ordinary differential equation which depends smoothly on the parameters [22, 17.8 Theorem], we obtain a smooth map

$$P\colon TX = \mathcal{V} \oplus \mathcal{H} \to \mathcal{V} \subseteq TX, \quad (v_{\mathcal{V}}, v_{\mathcal{H}}) \mapsto P^{0,1}_{\alpha_{v_{\mathcal{H}}}}(v_{\mathcal{V}}).$$

Step 2: *A θ-adapted local addition.* Consider the smooth map $X_\theta\colon TX \to \mathfrak{X}_c(X)$ constructed in Lemma B.1. Then let $\mathrm{Fl}_1\colon \mathfrak{X}_c(X) \to \mathrm{Diff}_c(X)$ be the mapping which takes a vector field to its flow evaluated at time 1. Recall that Fl_1 is the Lie group exponential map ([21, 4.6] or cf. [29, Theorem 5.4.11] together with [13, Corollary 13.7]) for the Lie group $\mathrm{Diff}_c(X)$, whence it is smooth. The evaluation $\mathrm{ev}\colon C^\infty(X,X) \times X \to X$ is smooth by A.12. We then obtain a smooth map

$$\eta_1\colon TX \to X, \quad v \mapsto \mathrm{ev}\big(\mathrm{Fl}_1(X_\theta(v)), \pi_X(v)\big).$$

Since $T\theta(X_\theta(v)) = 0$ for every $v \in \mathcal{V}$, θ is constant on the integral curves of $X_\theta(v)$, i.e.

$$\theta \circ \eta_1|_{\mathcal{V}} = \theta \circ \pi_X|_{\mathcal{V}}. \tag{15}$$

[7] The parallel transport with respect to the bundle metric on $\mathcal{V}$ exists above any smooth curve (by choosing a metric on $\mathcal{V}$, cf. [22, 17.9 Theorem]). This yields a smooth transport staying in $\mathcal{V}$.

Further, we note that $X_\theta(0_x) = 0 \in \mathfrak{X}_c(X)$, whence $\eta_1(0_x) = x$ for every $x \in X$. Let us show that η_1 restricts to a local addition on some 0-neighborhood, i.e. we prove that $T_{0_x}(\pi_X, \eta_1)\colon T_{0_x}(TX) \to T(X \times X)$ has invertible differential for every $x \in X$. Since Fl_1 is the Lie group exponential, $T_0 \mathrm{Fl}_1 = \mathrm{id}_{\mathfrak{X}_c(X)}$ yields with (10)

$$\begin{aligned} T_{0_x}(\mathrm{ev} \circ (\mathrm{Fl}_1 \circ X_\theta, \pi_X)) &= T\mathrm{ev} \circ ((T_0 \mathrm{Fl}_1) \circ (T_{0_x} X_\theta), T_{0_x}\pi_X) = T\mathrm{ev} \circ (T_{0_x} X_\theta, T_{0_x}\pi_X) \\ &= T_{0_x}(\mathrm{ev} \circ (X_\theta, \pi_X)) = T_{0_x} \mathrm{id}_{TX}\,. \end{aligned}$$

Hence the derivative of (π_X, η_1) over 0_x is (up to identification)[8] given by the matrix $\begin{bmatrix} \mathrm{id} & 0 \\ \mathrm{id} & \mathrm{id} \end{bmatrix}$. By the inverse function theorem η_1 restricts near the zero-section to a local addition (which is even adapted to θ in the sense of [30, Definition 3.1]).

Step 3: *Construction of the map η_X.* Using η_1 we modify $\exp_X$ as follows:

$$\eta_X\colon \Omega_X \cap \mathcal{V} \oplus \mathcal{H} \to X, \quad (v_\mathcal{V}, v_\mathcal{H}) \mapsto \eta_1\big(P(v_\mathcal{V}, v_\mathcal{H})\big)$$

The map η_X is smooth and by (15) and (14) we have for $(v_\mathcal{V}, v_\mathcal{H}) \in \Omega_X$

$$\theta\eta_X(v_\mathcal{V}, v_\mathcal{H}) = \theta\big(\alpha_{v_\mathcal{H}}(1)\big) = \theta \circ \exp_X(v_\mathcal{H}) = \exp_Y\big(\theta(v_\mathcal{H})\big) = \eta_Y \circ T\theta(v_\mathcal{V}, v_\mathcal{H}).$$

Thus η_X satisfies (13).

Step 4: *η_X induces a local addition.* Recall that $\eta_X = \eta_1 \circ P$ and $P(0_x) = 0_x$ and $T_{0_x}(\pi_X, \eta_1)$ is invertible. Hence it suffices to prove that $T_{0_x}P\colon T_{0_x}(T\Omega_X) \to T_{0_X}(TX)$ is invertible for every $x \in X$. This question is local, and using bundle charts we may assume that X is an open submanifold of $\mathbb{R}^d$. Thus $TX \cong X \times \mathbb{R}^d$ and $T_{0_x}(TX) \cong \mathbb{R}^d \times \mathbb{R}^d$. In the following, we consider smooth curves

$$\beta\colon\,]-\varepsilon, \varepsilon[\to \Omega_X \subseteq TX = X \times \mathbb{R}^d, \quad s \mapsto (\beta_0(s), \beta_\mathcal{V}(s) + \beta_\mathcal{H}(s))$$

with $\beta(0) = 0_x$, where $(\beta_0, \beta_\mathcal{V})$ takes its image in $\mathcal{V}$ and $(\beta_0, \beta_\mathcal{H})$ takes its image in $\mathcal{H}$. The splitting $TX \cong \mathcal{V} \oplus \mathcal{H}$ induces a splitting $T_{0_x}(TX) \cong \mathbb{R}^d \times (\mathfrak{v} \oplus \mathfrak{h})$, where $\mathfrak{v}$ (resp. $\mathfrak{h}$) is spanned by all vectors tangent to $\mathcal{V}$ (resp. $\mathcal{H}$). We compute $T_{0_x}P\big(a, (v, h)\big) = T_{0_x}P\big(a, (v, 0)\big) + T_{0_x}P\big(0, (0, h)\big)$ in two steps:

Step 4a: *$T_{0_x}P$ on vectors tangent to $\mathcal{V}$.* Take a smooth curve $\beta = (\beta_0, \beta_\mathcal{V})$ such that $\frac{d}{ds}\big|_{s=0} \beta(s) = \big(x, 0, a, (v, 0)\big)$. Then $\alpha_{(\beta_0, 0)} = \exp_X \circ (\beta_0, t0) = \beta_0$ is independent of t and

$$\begin{aligned} TP\Big(x, 0, \big(a, (v, 0)\big)\Big) &= \frac{d}{ds}\bigg|_{s=0} \underbrace{P_{\alpha_{(\beta_0,0)}}\big(\beta_\mathcal{V}(s)\big)}_{\text{transport along constant path}} = \frac{d}{ds}\bigg|_{s=0} \big(\beta_0(s), \beta_\mathcal{V}(s)\big) \\ &= \frac{d}{ds}\bigg|_{s=0} \beta(s) = \big(x, 0, a, (v, 0)\big). \end{aligned}$$

[8]Due to [18, X §4, Theorem 4.3] the zero section $\mathbf{0}\colon X \to TX$ induces an isomorphism $\mathbf{0}^*(T^2X) \cong TX \oplus TX$ given in coordinates by $(x, 0, v, w) \mapsto ((x, v), (x, w))$. Further the diagonal map $\Delta\colon X \to X \times X$ induces $\Delta^*(T(X \times X)) \cong TX \oplus TX$ via the local formula $((x, x)(v, w)) \mapsto ((x, v), (x, w))$.

Step 4b: *$T_{0_x}P$ on vectors of the form $\big(0,(0,h)\big)$.* Choose $\beta = \big(x,\beta_{\mathcal{H}}\big)$ with $\frac{d}{ds}\big|_{s=0}\beta(s) = \big(x,0,0,(0,h)\big)$. Then

$$T_{0_x}P\big(0,\big(0,h\big)\big) = \frac{d}{ds}\bigg|_{s=0} P_{\alpha_{\beta(s)}}(0) = \frac{d}{ds}\bigg|_{s=0}\Big(\exp_X\big(\beta(s)\big),0\Big)$$
$$= T_{0_x}\exp_X\left(\frac{d}{ds}\bigg|_{s=0}\big(\beta(s)\big)\right) = \big(x,0,\big(0,(0,h)\big)\big)$$

Here we have used that the Riemannian exponential map $\exp_X$ satisfies $T_{0_x}\exp_X = \mathrm{id}_{T_{0_x}X}$, $\exp_X(t0_X) = x$ for all $x\in X, t\in[0,1]$ and $\beta(0) = 0_x$. Summing up, $T_{0_x}P$ is the identity, whence due to the inverse function theorem η_X is a local addition. ■

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Received September 6, 2019
and in final form September 19, 2019