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Locally Compact Groups with Compact Open Subgroups Having Open Chabauty Spaces

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Abstract. Let G be a locally compact group. We denote by $\mathcal{SUB}(G)$ the space of closed subgroups of G equipped with the *Chabauty topology*; this is a compact space. The topological space $\mathcal{SUB}(G)$ is called the *Chabauty space* of G. For a closed subgroup H of G the subspace $\{L \in \mathcal{SUB}(G) \mid L \subseteq H\}$ of $\mathcal{SUB}(G)$ is homeomorphic to the Chabauty space $\mathcal{SUB}(H)$ of H and so $\mathcal{SUB}(H)$ is a compact subspace of $\mathcal{SUB}(G)$. The paper discusses the scope of validity of an assertion having appeared recently in the book of Herfort-Hofmann-Russo about the openness of the subspace $\mathcal{SUB}(H)$ in $\mathcal{SUB}(G)$. We study the class $\mathfrak{X}$ of locally compact groups G such that the subspace $\mathcal{SUB}(H)$ is open in $\mathcal{SUB}(G)$ for any compact open subgroup H of G. We show that a locally compact abelian group A is in $\mathfrak{X}$ if and only if A contains a compact open subgroup U such that A/U is a finite direct sum of subgroups each of which is either cyclic or is a Prüfer group isomorphic to $\mathbb{Z}(p^\infty)$.

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1. Introduction and background

Let G be a locally compact group with identity element e. We denote by $\mathcal{SUB}(G)$ the space of closed subgroups of G equipped with the *Chabauty topology*; this is a compact space. In this space, each closed subgroup H of G has a neighborhood base consisting of sets

$$\mathcal{U}_G(H; K, W) \stackrel{\text{def}}{=} \{L \in \mathcal{SUB}(G) \mid L \cap K \subseteq WH \text{ and } H \cap K \subseteq WL\}, \qquad (1)$$

where K ranges through the set $\mathcal{K}(G)$ of all compact subsets of G and W through the filter $\mathcal{U}(e)$ of all neighborhoods of the identity (The statement "U is a neighborhood of a point x" is used in the Bourbaki sense throughout this paper; i.e., U is any subset which contains an open subset containing x). In particular, the trivial subgroup $E = \{e\}$ has a neighborhood base consisting of sets

$$\mathcal{U}_G(E; K, W) = \{L \in \mathcal{SUB}(G) \mid L \cap K \subseteq W\}, \qquad (2)$$

$K \in \mathcal{K}(G)$ and $W \in \mathcal{U}(e)$. If H is a closed subgroup of the group G, then $\mathcal{SUB}(H)$ is homeomorphic to the subspace $\{L \in \mathcal{SUB}(G) \mid L \subseteq H\}$ of $\mathcal{SUB}(G)$ (see Proposition 1 in [16]).

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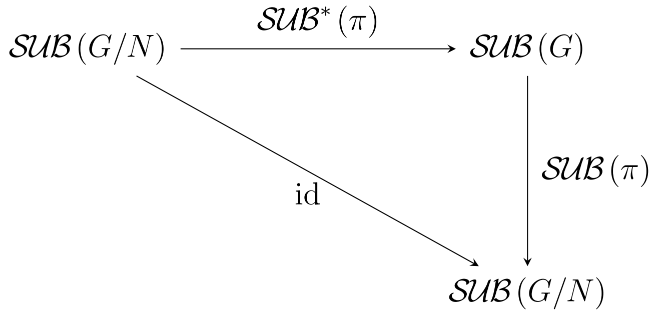
Let N be a normal closed subgroup of a locally compact topological group G and let $\pi\colon G \longrightarrow G/N$ be the canonical projection. By Corollary 2.7 in [10] the following map

$$\mathcal{SUB}^*(\pi) : \mathcal{SUB}(G/N) \to \mathcal{SUB}(G), \quad L \mapsto \pi^{-1}(L) \tag{3}$$

is continuous. If moreover N is compact, π is proper (Lemma 32.8 in [17]) and therefore the map

$$\mathcal{SUB}(\pi) : \mathcal{SUB}(G) \to \mathcal{SUB}(G/N), \quad H \mapsto \pi(H) \tag{4}$$

is continuous (Corollary 2.4 in [10]). Furthermore, the following diagram is commutative.



That is, the subspace $\mathcal{SUB}(G/N)$ is a retract of $\mathcal{SUB}(G)$.

The identity component of G, denoted by G_0, is the connected component of the identity in G. We note that G_0 is compact if and only if G contains a compact open subgroup (see Corollary 2.E.7(2) in [5]).

The following proposition appears in [12] as Proposition 1.22 (i):

Proposition 1.1. *Let G be a locally compact group such that G_0 is compact. Then for each compact open subgroup U of G, the subset $\mathcal{SUB}(U)$ of closed subgroups of U is a compact open retract of $\mathcal{SUB}(G)$.*

We shall show that the assertion of openness of $\mathcal{SUB}(U)$ is false in the stated generality by presenting appropriate counterexamples and discuss circumstances in which Proposition 1.1 remains correct, for instance for all compact groups G.

2. The openness of the Chabauty space of an open compact subgroup

The following example gives a counterexample to Proposition 1.1.

Example 2.1 (The group $\mathbb{Z}$ of integers)**.** Let $G = \mathbb{Z}$ be the group of all integers and let

$$X_{\mathbb{Z}} \overset{\text{def}}{=} \left\{\tfrac{1}{n} \mid n \in \mathbb{N} \setminus \{0\}\right\} \cup \{0\}. \tag{5}$$

As the mapping $\phi_{\mathbb{Z}}\colon X_{\mathbb{Z}} \to \mathcal{SUB}(\mathbb{Z})$ defined by

$$\phi_{\mathbb{Z}}(t) = \begin{cases} n\,\mathbb{Z} & \text{if } \; t = \frac{1}{n}, \\ \{0\} & \text{if } \; t = 0 \end{cases}$$

is a homeomorphism (see Proposition 1.6 in [9]), $\mathcal{SUB}(\{0\}) = \{\{0\}\}$ is not open in $\mathcal{SUB}(\mathbb{Z})$.

Let $\mathfrak{X}$ be the class of locally compact groups G with compact identity component and such that for any compact open subgroup U of G the subset $\mathcal{SUB}(U)$ is open in $\mathcal{SUB}(G)$.

Proposition 2.2. *The class $\mathfrak{X}$ contains all compact groups.*

Proof. Let U be an open subgroup of a compact group G. It is easy to see that

$$\begin{aligned} \mathcal{SUB}(U) &= \{L \in \mathcal{SUB}(G) \mid L \subseteq U\} \\ &= \{L \in \mathcal{SUB}(G) \mid L \cap (G \setminus U) = \varnothing\} \end{aligned}$$

and so $\mathcal{SUB}(U)$ is open in $\mathcal{SUB}(G)$ since the set of all closed subgroups which have empty intersection with the compact subset $G \setminus U$ is an open subset in the Chabauty space $\mathcal{SUB}(G)$. ∎

Lemma 2.3 (Stability properties of the class $\mathfrak{X}$)**.** *Let G be a locally compact group and N a closed subgroup of G.*

(1) *If N is open in G, then $G \in \mathfrak{X} \Longrightarrow N \in \mathfrak{X}$.*

(2) *If N is normal and compact, then $G \in \mathfrak{X} \Longleftrightarrow G/N \in \mathfrak{X}$.*

Proof. (1) is trivial. (2) Let $\pi\colon G \longrightarrow G/N$ be the canonical projection. Suppose that $G \in \mathfrak{X}$ and let L be a compact open subgroup of G/N. We shall prove that $\mathcal{SUB}(L)$ is open in $\mathcal{SUB}(G/N)$. Since π is proper, $H \overset{\text{def}}{=} \pi^{-1}(L)$ is a compact open subgroup of G and therefore $\mathcal{SUB}(H)$ is open in $\mathcal{SUB}(G)$. Then $\mathcal{SUB}^*(\pi)^{-1}(\mathcal{SUB}(H))$ is open in $\mathcal{SUB}(G/N)$ (see (3)). On the other hand, since H contains N, $\mathcal{SUB}^*(\pi)^{-1}(\mathcal{SUB}(H)) = \mathcal{SUB}(L)$ and so $\mathcal{SUB}(L)$ is open in $\mathcal{SUB}(G/N)$. Thus $G/N \in \mathfrak{X}$.

Conversely, let H be a compact open subgroup of G. As $\pi(H)$ is open in G/N, $\mathcal{SUB}(\pi(H))$ is open in $\mathcal{SUB}(G/N)$ and so $\mathcal{SUB}(HN) = \mathcal{SUB}(\pi)^{-1}(\mathcal{SUB}(\pi(H)))$ is open in $\mathcal{SUB}(G)$. On the other hand, as H is a compact open subgroup of the compact group HN, by Proposition 2.2, $\mathcal{SUB}(H)$ is open in $\mathcal{SUB}(HN)$ and hence in $\mathcal{SUB}(G)$. Consequently the group G belongs to the class $\mathfrak{X}$. ∎

The next theorem reduces the study of the class $\mathfrak{X}$ to the case of totally disconnected locally compact groups.

Theorem 2.4. *For a locally compact group G the following conditions are equivalent:*

(1) $G \in \mathfrak{X}$.

(2) *G_0 is compact and $G/G_0 \in \mathfrak{X}$.*

Proof. This follows from the definition of the class $\mathfrak{X}$ and Lemma 2.3. ∎

For a totally disconnected locally compact group G, the set of compact open subgroups of G is a basis of identity neighborhoods in G (van Dantzig's Theorem, Theorem 7.7 in [13]).

Proposition 2.5. *Let H be a closed subgroup of a totally disconnected locally compact group G. If $\mathcal{SUB}(H)$ is open in $\mathcal{SUB}(G)$ then H is open in G.*

Proof. Since $\mathcal{SUB}(H)$ is a neighborhood of the trivial subgroup E, there exist a compact subset K of G and a compact open subgroup U of G such that $\mathcal{U}_G(E; K, U) \subseteq \mathcal{SUB}(H)$. By (2), $U \in \mathcal{U}_G(E; K, U)$ and so $U \subseteq H$. Thus H is open. ∎

Remark 2.6. Proposition 2.5 can also be deduced from Theorem 4 in [14]. If G is a totally disconnected locally compact group then the function

$$m\colon G \to \mathcal{SUB}(G),\ m(g) = \overline{\langle g \rangle}$$

is continuous ([14]). If H is a closed subgroup of G and $g \notin H$, then $m(g) \notin \mathcal{SUB}(H)$, that is $m^{-1}(\mathcal{SUB}(H)) = H$. So if $\mathcal{SUB}(H)$ is open in $\mathcal{SUB}(G)$, then H is open in G.

As a consequence of the above proposition we get

Corollary 2.7. *Let H be a closed subgroup of a locally compact group G such that $G_0 \subseteq H$. If $\mathcal{SUB}(H)$ is open in $\mathcal{SUB}(G)$ then H is open in G.*

Proof. The map $\mathcal{SUB}^*(\pi) : \mathcal{SUB}(G/G_0) \to \mathcal{SUB}(G)$ induced by the canonical projection $\pi\colon G \to G/G_0$ is continuous and so $\mathcal{SUB}^*(\pi)^{-1}(\mathcal{SUB}(H))$ is open in $\mathcal{SUB}(G/G_0)$. On the other hand since $G_0 \subseteq H$, $\mathcal{SUB}^*(\pi)^{-1}(\mathcal{SUB}(H)) = \mathcal{SUB}(\pi(H))$. By Proposition 2.5 $\pi(H)$ is open in G/G_0 and therefore H is open in G. ∎

The assumption that H contains the identity component G_0 is essential in Corollary 2.7, by considering the following remark (see the proof of Lemma 4 in [15]).

Remark 2.8 (Protasov-Tsybenko)**.** If G is a compact Lie group then the trivial subgroup is isolated in $\mathcal{SUB}(G)$, i.e., $\mathcal{SUB}(E)$ is open in $\mathcal{SUB}(G)$. However E is not open in G if G is not discrete.

3. The case of locally compact abelian groups

A set C of nonidentity elements in an abelian group A is called a set of *cogenerators* if, every non-trivial subgroup of A contains an element of C. An abelian group is *finitely cogenerated* if it has a finite set of cogenerators ([7], page 145). A finitely cogenerated group is necessarily discrete (Theorem 2.2 in [1]). In the following we give an equivalent formulation of finitely cogenerated groups in terms of Chabauty topology.

Lemma 3.1. *For a discrete abelian group A the following two conditions are equivalent:*

(1) *A is finitely cogenerated.*

(2) *The trivial subgroup E is isolated in $\mathcal{SUB}(A)$.*

Proof. (1) $\Rightarrow$ (2): Let C be a finite subset of nonidentity elements in A such that every nontrivial subgroup of A contains an element of C. Then

$$\mathcal{U}_G(E; C, E) = \{L \in \mathcal{SUB}(G) \mid L \cap C \subseteq E\} = \{E\}$$

and so E is isolated in $\mathcal{SUB}(A)$.

(2) $\Rightarrow$ (1): It is easy to see that if C is a finite subset of A with $\mathcal{U}_G(E; C, E) = \{E\}$ then C is a set of cogenerators for A and so A is finitely cogenerated. ■

A group is *cocyclic* if it is isomorphic to $\mathbb{Z}(p^k)$ for some prime p and for some $k \in \mathbb{N} \cup \{\infty\}$ (see Theorem 3.3 of [7]), where $\mathbb{Z}(p^k)$ is the cyclic group of order p^k, if $k \in \mathbb{N}$, and $\mathbb{Z}(p^\infty)$ is the Prüfer p-group. A characterization of finitely cogenerated discrete abelian groups is given in [18], Theorem 4.4, page 184 (see also [7], Theorem 5.3, page 146).

Theorem 3.2. *For a discrete abelian group A, the following conditions are equivalent:*

(1) *A is finitely cogenerated.*

(2) *A is a torsion group of finite rank.*

(3) *A is a direct sum of a finite number of cocyclic groups.*

(4) *The subgroups of A satisfy the minimum condition.*

Remark 3.3. We observe that from the equivalence of (1) and (4) it results that quotient groups of finitely cogenerated groups are finitely cogenerated (Theorem 4.6(ii) in [18]).

Proposition 3.4 (Characterization of discrete abelian $\mathfrak{X}$-groups)**.** *For a discrete abelian group A, the following conditions are equivalent:*

(1) *A is finitely cogenerated.*

(2) *The trivial subgroup $\{e\}$ is isolated in $\mathcal{SUB}(A)$.*

(3) *$A \in \mathfrak{X}$.*

Proof. (1) $\Leftrightarrow$ (2): Follows from Lemma 3.1. (3) $\Rightarrow$ (2): Trivial. (2) $\Rightarrow$ (3): Let N be a finite subgroup of A. In view of 3.3 the group A/N is finitely cogenerated and so the trivial subgroup $E = \{\pi(e)\}$ is open in $\mathcal{SUB}(A/N)$. On the other hand the canonical projection $\pi\colon A \to A/N$ induces a continuous map

$$\mathcal{SUB}(\pi) : \mathcal{SUB}(A) \to \mathcal{SUB}(A/N).$$

Then $\mathcal{SUB}(N) = \mathcal{SUB}(\pi)^{-1}(E)$ is open in $\mathcal{SUB}(A)$. ■

As a consequence we get the following example.

Example 3.5. For an infinite set X, $\mathbb{Z}(2)^{(X)} \notin \mathfrak{X}$.

Remark 3.6. Let G be a locally compact group.

(a) If $(H_i)_{i \in I}$ is a convergent net of closed subgroups of G then its limit contains $\bigcap_{i \in I} H_i$.

(b) In view of (a), for a closed subgroup H of G the subset $\{L \in \mathcal{SUB}(G) \mid H \subseteq L\}$ is closed in $\mathcal{SUB}(G)$.

Example 3.7 ($\mathbb{Q}_p \in \mathfrak{X}$)**.** For the group $\mathbb{Q}_p$ of p-adic numbers we have

$$\mathcal{SUB}(\mathbb{Q}_p) = \left\{\{0\}, \mathbb{Q}_p, p^k\mathbb{Z}_p \mid k \in \mathbb{Z}\right\}$$

with

$$\{0\} = \bigcap_{k\geq 0} p^k\mathbb{Z}_p = \lim_{k\geq 0} p^k\mathbb{Z}_p, \qquad \mathbb{Q}_p = \bigcup_{k\leq 0} p^k\mathbb{Z}_p = \lim_{k\leq 0} p^k\mathbb{Z}_p$$

in $\mathcal{SUB}(\mathbb{Q}_p)$. Moreover, $\left\{p^k\mathbb{Z}_p \mid k \in \mathbb{Z}\right\}$ is the set of all compact open subgroups of $\mathbb{Q}_p$. We have the following:

(A) As the closed subgroups of $\mathbb{Q}_p$ are totally ordered by inclusion, in view of Remark 3.6(b) the subset $\mathcal{SUB}\left(p^k\mathbb{Z}_p\right)$ is open in $\mathcal{SUB}(\mathbb{Q}_p)$ for any $k \in \mathbb{Z}$ and so $\mathbb{Q}_p$ belongs to the class $\mathfrak{X}$.

(B) By [6, Examples 5.1, page 870], $\mathcal{SUB}(\mathbb{Z}_p)$ is homeomorphic to $X_{\mathbb{Z}}$ (see (5)) and therefore any compact open subgroup of $\mathbb{Q}_p$ is isolated in $\mathcal{SUB}(\mathbb{Q}_p)$.

We prove the following more general result.

Theorem 3.8 (Characterization of LCA $\mathfrak{X}$-groups)**.** *For a locally compact abelian group A the following conditions are equivalent:*

(a) $A \in \mathfrak{X}$.

(b) *A contains a compact open subgroup U such that A/U is finitely cogenerated (i.e., a finite direct sum of subgroups each of which is either cyclic or is a Prüfer group isomorphic to $\mathbb{Z}(p^\infty)$).*

(c) *A contains a compact open subgroup U such that A/U is a direct sum of finitely many Prüfer groups.*

Proof. (a) ⇒ (b): Since A_0 is compact, A contains a compact open subgroup U. By Lemma 2.3(2) $A/U \in \mathfrak{X}$ and so, by Proposition 3.4, it is finitely cogenerated.
(b) ⇒ (a): This follows from Lemma 2.3(2) and Proposition 3.4.
(b) ⇔ (c) is trivial. ∎

Remark 3.9. Combining Lemma 2.3(2) and Theorem 3.8, we deduce that a locally compact $[FD]^-$-group G (i.e., the commutator subgroup $\mathrm{D}(G)$ is relatively compact) is in $\mathfrak{X}$ if and only if it contains a compact open subgroup U such that $G/U\overline{\mathrm{D}(G)}$ is finitely cogenerated.

Remark 3.10. If a locally compact abelian group A belongs to the class $\mathfrak{X}$ then for any compact open subgroup U we have A/U is finitely cogenerated.

Remark 3.11. As $\mathbb{Q}_p/\mathbb{Z}_p$ and $\mathbb{Z}(p^\infty)$ are isomorphic as discrete abelian groups, assertion (A) in Example 3.7 is a consequence of Theorem 3.8.

Remark 3.12. Let us observe that the implication (2) ⇒ (3) in Proposition 3.4 remains valid for nondiscrete locally compact abelian groups. In fact, if the trivial subgroup E is isolated in $\mathcal{SUB}(A)$ then, by Lemma 5.1 in [2], A is isomorphic to $(\mathbb{R}/\mathbb{Z})^k \times D$, where D is finitely cogenerated. Thus in view of Theorem 3.8 $A \in \mathfrak{X}$. However, the group of p-adic integers $\mathbb{Z}_p$ shows that the implication (3) ⇒ (2) in Proposition 3.4 is not valid for nondiscrete locally compact abelian groups.

A topological space is called *perfect* if it has no isolated point. We end this paper with the following in which we show that in a discrete uncountable abelian group A the subspace $\mathcal{SUB}(U)$ is not open in $\mathcal{SUB}(A)$ for every finite subgroup U.

Proposition 3.13. *Let A be a discrete abelian group. Then*

(a) *If $\mathcal{SUB}(A)$ is not perfect then A is countable.*

(b) *If A is uncountable, then $\mathcal{SUB}(U)$ is not open in $\mathcal{SUB}(G)$ for any finite subgroup U of A.*

Proof. (a): Let N be an isolated point in $\mathcal{SUB}(A)$ and let $\mathcal{A} = A/N$. As the subset of finitely generated subgroups of A is dense in $\mathcal{SUB}(A)$ ([11, Proposition 2.6]), N is finitely generated and hence it is countable. On the other hand, the continuity of the mapping (see (3))

$$\mathcal{SUB}^*(\pi) : \mathcal{SUB}(\mathcal{A}) \to \mathcal{SUB}(A)\,,\; H \mapsto \pi^{-1}(H)$$

where $\pi\colon A \to \mathcal{A}$ is the canonical projection, implies that the trivial subgroup $E = \{\pi(e)\}$ is isolated in $\mathcal{SUB}(\mathcal{A})$. Then, by Lemma 3.1, $\mathcal{A}$ is finitely cogenerated and so it is countable (Theorem 3.2). Consequently A is countable.

(b): Since $\mathcal{SUB}(A)$ is a Hausdorff space then, by (a), any finite subset of $\mathcal{SUB}(A)$ is not open. ■

Remark 3.14. We note that the assertion given in Proposition 3.13(a) appears in [4, Proposition A(2)].

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