The Strong Trotter Property for
Locally μ-Convex Lie Groups

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Abstract.   We show that an infinite dimensional Lie group in Milnor's sense has the strong Trotter property if it is locally μ-convex. This is a continuity condition imposed on the Lie group multiplication that generalizes the triangle inequality for locally convex vector spaces, and is equivalent to C^0-continuity of the evolution map on its domain. In particular, the result proven in this paper significantly extends the respective result obtained by Glöckner in the context of measurable regularity.

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1.   Introduction

Let G be an infinite dimensional Lie group in Milnor's sense, with exponential map $\exp\colon \mathfrak{g} \supseteq \mathrm{dom}[\exp] \to G$. We say that G has the *strong Trotter property* (cf. [3]) if for each $\mu \in C^1([0,1], G)$ with $\mu(0) = e$ and $\dot{\mu}(0) \in \mathrm{dom}[\exp]$, we have

$$\lim_n \mu(\tau/n)^n = \exp(\tau \cdot \dot{\mu}(0)) \qquad \forall\, \tau \in [0,\ell] \tag{1}$$

uniformly for each $\ell > 0$. As already shown in [3], this implies[1] the *strong commutator property*, and also the *Trotter-*, and the *commutator property* that are relevant, e.g., in representation theory of infinite dimensional Lie groups [9]. More importantly, Theorem I in [3] states that G has the strong Trotter property if it is R-regular. Now, R-regularity implies C^0-continuity of the evolution map, so that Theorem 1 in [5] shows that G is *locally μ-convex*. This condition has originally been introduced in [2], and states that to each continuous seminorm $\mathfrak{u}$ on the modeling space E of G, and to each chart $\Xi\colon G \supseteq \mathcal{U} \to \mathcal{V} \subseteq E$ of G around e with $\Xi(e) = 0$, there exists a continuous seminorm $\mathfrak{u} \leq \mathfrak{o}$ on E such that $\Xi^{-1}(X_1) \cdot \ldots \cdot \Xi^{-1}(X_n) \in \mathcal{U}$

and $\qquad (\mathfrak{u} \circ \Xi)\big(\Xi^{-1}(X_1) \cdot \ldots \cdot \Xi^{-1}(X_n)\big) \leq \mathfrak{o}(X_1) + \ldots + \mathfrak{o}(X_n) \qquad (2)$

holds for all $X_1, \ldots, X_n \in E$ with $\mathfrak{o}(X_1) + \ldots + \mathfrak{o}(X_n) \leq 1$.

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[1] Although in [3] $\mathrm{dom}[\exp] = \mathfrak{g}$ is presumed, the proofs of the mentioned implications just carry over to the situation considered in this paper – provided, of course, that the definitions given in [3] for the (strong) commutator-, and the Trotter property are adapted in the obvious way.

Evidently, this condition generalizes the triangle inequality for locally convex vector spaces; and, in general (without regularity presumptions on G) is equivalent to C^0-continuity of the evolution map on its domain, cf. Theorem 1 in [5]. In this paper, we show that local μ-convexity already suffices to ensure validity of (1), i.e., that we have the following theorem.

Theorem 1.1. *If G is locally μ-convex, then G has the strong Trotter property.*

In particular, this drops the presumptions made in [3] on the domain of the evolution map, as well as the completeness presumptions made in [3] on $\mathfrak{g}$.

2. Preliminaries

In this section, we fix the notations, and discuss the properties of the product integral (evolution map) that we shall need in Sect. 3 to prove Theorem 1.1. The proofs of the facts mentioned but not verified in this section can be found, e.g., in Sect. 3 and Sect. 4 in [5].

2.1. Lie groups.   In the following, G will denote an infinite dimensional Lie group in Milnor's sense [1, 4, 7, 8] that is modeled over the Hausdorff locally convex vector space E, with corresponding system of continuous seminorms $\mathfrak{P}$. We denote the Lie algebra of G by $\mathfrak{g}$, fix a chart

$$\Xi\colon G \supseteq \mathcal{U} \to \mathcal{V} \subseteq E$$

with $\mathcal{V}$ convex, $e \in \mathcal{U}$, and $\Xi(e) = 0$; and identify $\mathfrak{g} \cong E$ via $\mathrm{d}_e\Xi\colon \mathfrak{g} \to E$. Specifically, the latter condition means that we will write $\mathfrak{p}(X)$ instead of $(\mathfrak{p}\circ\mathrm{d}_e\Xi)(X)$ for each $\mathfrak{p} \in \mathfrak{P}$ and $X \in \mathfrak{g}$ in the following. We let $\mathrm{m}\colon G \times G \to G$ denote the Lie group multiplication, $\mathrm{R}_g := \mathrm{m}(\cdot, g)$ the right translation by $g \in G$, and $\mathrm{Ad}\colon G \times \mathfrak{g} \to \mathfrak{g}$ the adjoint action, i.e., we have

$$\mathrm{Ad}_g(X) := \mathrm{Ad}(g, X) := \mathrm{d}_e\mathbf{c}_g(X) \quad \text{with} \quad \mathbf{c}_g\colon G \ni h \mapsto g \cdot h \cdot g^{-1} \in G$$

for each $g \in G$ and $X \in \mathfrak{g}$.

2.2. The product integral.   Let $\mathfrak{K} := \{[r, r'] \subseteq \mathbb{R} \mid r < r'\}$ denote the set of all proper compact intervals in $\mathbb{R}$. The *right logarithmic derivative* is given by

$$\delta^r\colon C^1([r, r'], G) \to C^0([r, r'], \mathfrak{g}), \qquad \mu \mapsto \mathrm{d}_\mu\mathrm{R}_{\mu^{-1}}(\dot\mu)$$

for all $[r, r'] \in \mathfrak{K}$. We let $\mathfrak{D} := \bigsqcup_{[r,r']\in\mathfrak{K}} \mathfrak{D}_{[r,r']}$ with $\mathfrak{D}_{[r,r']} := \delta^r(C^1([r, r'], G))$ for each $[r, r'] \in \mathfrak{K}$, and define

$$C^1_*([r, r'], G) := \{\mu \in C^1([r, r'], G) \mid \mu(r) = e\}$$

as well as

$$\mathrm{Evol}\colon \mathfrak{D}_{[r,r']} \to C^1_*([r, r'], G), \qquad \delta^r(\mu) \mapsto \mu \cdot \mu(r)^{-1}$$

for all $[r, r'] \in \mathfrak{K}$. The *product integral* is given by

$$\textstyle\oint_s^t \phi := \mathrm{Evol}\big(\phi|_{[s,t]}\big)(t) \in G \qquad \forall\, [s, t] \subseteq \mathrm{dom}[\phi], \ \phi \in \mathfrak{D},$$

and we let $\oint \phi := \oint_r^{r'} \phi$ for $\phi \in \mathfrak{D}$ with $\mathrm{dom}[\phi] = [r, r']$.

Remark 2.1. Evidently, for $[r, r'] = [0, 1]$, $\oint \phi$ just equals the "small evolution map", usually denoted by evol in the literature. Moreover, $\oint \phi$ equals the Riemann integral for the case that $(G, \cdot) = (F, +)$ is the additive group of a Hausdorff locally convex vectors space F. The formulas (a)–(d) below then just generalize the respective formulas for the Riemann integral. ∎

We have the following elementary identities:

(a) $\oint_r^t \phi \cdot \oint_r^t \psi = \oint_r^t \phi + \mathrm{Ad}_{\oint_r^\bullet \phi}(\psi)$ $\qquad \forall\, \phi, \psi \in \mathfrak{D}_{[r,r']}, \ t \in [r, r'].$

(b) $\left[\oint_r^t \phi\right]^{-1}\left[\oint_r^t \psi\right] = \oint_r^t \mathrm{Ad}_{[\oint_r^\bullet \phi]^{-1}}(\psi - \phi)$ $\qquad \forall\, \phi, \psi \in \mathfrak{D}_{[r,r']}, \ t \in [r, r'].$

(c) For $r = t_0 < \ldots < t_n = r'$ and $\phi \in \mathfrak{D}_{[r,r']}$, we have

$$\oint_r^t \phi = \oint_{t_p}^t \phi \cdot \oint_{t_{p-1}}^{t_p} \phi \cdot \ldots \cdot \oint_r^{t_1} \phi \qquad \forall\, t \in (t_p, t_{p+1}], \ p = 0, \ldots, n-1.$$

(d) For $\varrho \colon [\ell, \ell'] \to [r, r']$ of class C^1, we have

$$\oint_r^{\varrho(\bullet)} \phi = \left[\oint_\ell^\bullet \dot{\varrho} \cdot (\phi \circ \varrho)\right] \cdot \left[\oint_r^{\varrho(\ell)} \phi\right] \qquad \forall\, \phi \in \mathfrak{D}_{[r,r']}.$$

Next, for $X \in \mathfrak{g}$, we write ϕ_X for the constant map $[0, 1] \ni t \mapsto X \in \mathfrak{g}$. If $\phi_X \in \mathfrak{D}_{[0,1]}$ holds, we define $\exp(X) := \oint \phi_X$. Evidently, we have $0 \in \mathrm{dom}[\exp]$, and it is straightforward from (d) that, cf. Appendix A.1

$$X \in \mathrm{dom}[\exp] \qquad \Longrightarrow \qquad \mathbb{R}_{\geq 0} \cdot X \subseteq \mathrm{dom}[\exp]. \tag{3}$$

Then, we let $\mathfrak{D}\mathrm{P}^0(\mathfrak{K}, \mathfrak{g}) := \bigsqcup_{[r,r'] \in \mathfrak{K}} \mathfrak{D}\mathrm{P}^0([r, r'], \mathfrak{g})$; where $\mathfrak{D}\mathrm{P}^0([r, r'], \mathfrak{g})$ (for $[r, r'] \in \mathfrak{K}$) denotes the set of all maps $\phi \colon [r, r'] \to \mathfrak{g}$ such that there exist $r = t_0 < \ldots < t_n = r'$, and $\phi[p] \in \mathfrak{D}_{[t_p, t_{p+1}]}$ with

$$\phi|_{(t_p, t_{p+1})} = \phi[p]|_{(t_p, t_{p+1})} \qquad \forall\, p = 0, \ldots, n-1.$$

In this situation, we define $\oint_r^r \phi := e$, and let

$$\oint_r^t \phi := \oint_{t_p}^t \phi[p] \cdot \oint_{t_{p-1}}^{t_p} \phi[p-1] \cdot \ldots \cdot \oint_r^{t_1} \phi[0] \qquad \forall\, t \in (t_p, t_{p+1}] \tag{4}$$

for $p = 0, \ldots, n-1$. A standard refinement argument in combination with (c) then shows that this is well defined (cf. Sect. 4.3 in [5]), i.e., independent of any choices we have made. It is furthermore not hard to see that for $\phi, \psi \in \mathfrak{D}\mathrm{P}^0([r, r'], \mathfrak{g})$, we have $\mathrm{Ad}_{[\oint_r^\bullet \phi]^{-1}}(\psi - \phi) \in \mathfrak{D}\mathrm{P}^0([r, r'], \mathfrak{g})$ with

$$\left[\oint_r^t \phi\right]^{-1}\left[\oint_r^t \psi\right] = \oint_r^t \mathrm{Ad}_{[\oint_r^\bullet \phi]^{-1}}(\psi - \phi) \qquad \forall\, t \in [r, r']. \tag{5}$$

2.3. Some estimates. We recall several facts (cf. Sect. 3.4.1 in [5]).

i) For each compact $\mathrm{C} \subseteq G$, and each $\mathfrak{q} \in \mathfrak{P}$, there exists some $\mathfrak{q} \leq \mathfrak{m} \in \mathfrak{P}$, as well as $O \subseteq G$ open with $\mathrm{C} \subseteq O$, such that

$$\mathfrak{q} \circ \mathrm{Ad}_g \leq \mathfrak{m} \qquad \forall\, g \in O.$$

ii) Assume we are given $\mu \in C^1([0,1], G)$, as well as $\ell > 0$ and $m \geq 1$ with $[0, \ell] \cdot [0, 1/m] \subseteq [0, 1]$. For $\tau \in [0, \ell]$, we define $\mu_\tau \colon [0, 1/m] \ni t \mapsto \mu(\tau \cdot t) \in G$. The chain rule yields $\delta^r(\mu_\tau)(t) = \tau \cdot \delta^r(\mu)(\tau \cdot t)$ for each $t \in [0, 1/m]$, so that

$$\alpha \colon [0, \ell] \times [0, 1/m] \to \mathfrak{g}, \qquad (\tau, t) \mapsto \delta^r(\mu_\tau)(t) \tag{6}$$

is continuous.

We say that $C^0([0, \ell], G) \supseteq \{\mu_n\}_{n \in \mathbb{N}} \to \mu \in C^0([0, \ell], G)$ converges uniformly for $\ell > 0$ if to each neighbourhood $U \subseteq G$ of e, there exists some $n_U \in \mathbb{N}$ with

$$\mu_n(t) \in U \cdot \mu(t) \cap \mu(t) \cdot U \qquad \forall\, n \geq n_U, \;\; t \in [0, \ell].$$

It is straightforward to see that

Lemma 2.2. *A sequence $C^0([0, \ell], G) \supseteq \{\mu_n\}_{n \in \mathbb{N}} \to \mu \in C^0([0, \ell], G)$ converges uniformly if and only if to each neighbourhood $V \subseteq G$ of e, there exists some $n_V \in \mathbb{N}$ with*

$$\mu_n(t) \in \mu(t) \cdot V \qquad \forall\, n \geq n_V, \;\; t \in [0, \ell].$$

Proof.   The proof is elementary, and can be found in Appendix A.2.   ∎

2.4. Continuity of the integral.   As already mentioned in the introduction, Theorem 1 in [5] shows that[2] local μ-convexity (2) is equivalent to continuity of the product integral on $\mathfrak{D} \cap C^k([r, r'], \mathfrak{g})$ for any $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$, and $[r, r'] \in \mathfrak{K}$ w.r.t. the C^0-topology, i.e., w.r.t. the seminorms

$$\mathfrak{p}_\infty(\phi) := \sup_{t \in [r, r']} \mathfrak{p}(\phi(t)) \qquad \forall\, \phi \in \mathfrak{D} \cap C^k([r, r'], \mathfrak{g}) \tag{7}$$

for $\mathfrak{p} \in \mathfrak{P}$. It was furthermore shown in [5] that local μ-convexity implies that the product integral is continuous at zero on $\mathfrak{D}\mathrm{P}^0(\mathfrak{K}, \mathfrak{g})$ w.r.t. the L^1-topology, i.e., that the following proposition holds, cf. Proposition 2 in [5].

Proposition 2.3.   *Assume that G is locally μ-convex. Then, to each $\mathfrak{p} \in \mathfrak{P}$, there exists some $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$, such that for each $\phi \in \mathfrak{D}\mathrm{P}^0(\mathfrak{K}, \mathfrak{g})$ we have*

$$\int_r^{r'} \mathfrak{q}(\phi(s)) \, \mathrm{d}s \leq 1 \qquad \Longrightarrow \qquad (\mathfrak{p} \circ \Xi)\big(\textstyle\oint_r^\bullet \phi\big) \leq \int_r^\bullet \mathfrak{q}(\phi(s)) \, \mathrm{d}s$$

with $r, r' \in \mathbb{R}$ such that $\mathrm{dom}[\phi] = [r, r']$.

Using (5), this generalizes as follows.

Lemma 2.4.   *Assume that G is locally μ-convex, and let $\mathrm{K} \subseteq G$ be compact. Then, to each $\mathfrak{p} \in \mathfrak{P}$, there exist $\mathfrak{p} \leq \mathfrak{m} \in \mathfrak{P}$ and $O \subseteq G$ open with $\mathrm{K} \subseteq O$, such that for each $[r, r'] \in \mathfrak{K}$, we have*

$$(\mathfrak{p} \circ \Xi)\big(\big[\textstyle\oint_r^\bullet \phi\big]^{-1} \big[\oint_r^\bullet \psi\big]\big) \leq \int_r^\bullet \mathfrak{m}(\psi(s) - \phi(s)) \, \mathrm{d}s$$

for all $\phi, \psi \in \mathfrak{D}\mathrm{P}^0([r, r'], \mathfrak{g})$ with $\mathrm{im}[\oint_r^\bullet \phi] \subseteq O$ and $\int_r^{r'} \mathfrak{m}(\psi(s) - \phi(s)) \, \mathrm{d}s \leq 1$.

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[2]Here, one additionally has to apply Lemma 15 in [5], together with the observation that this lemma also holds if the interval $[0, 1]$ is replaced by some fixed arbitrary interval $[r, r'] \in \mathfrak{K}$ there.

Proof. For $\mathfrak{p} \in \mathfrak{P}$ fixed, we choose $\mathfrak{p} \leq \mathfrak{q}$ as in Proposition 2.3. Since $\mathrm{C} := \mathrm{K}^{-1}$ is compact, by i), there exists some $\mathfrak{q} \leq \mathfrak{m} \in \mathfrak{P}$, as well as $O \subseteq G$ open with $\mathrm{K} \subseteq O$, such that

$$\mathfrak{q} \circ \mathrm{Ad}_{g^{-1}} \leq \mathfrak{m} \qquad \forall\, g \in O. \tag{8}$$

Let now $\phi, \psi \in \mathfrak{D}\mathrm{P}^0([r, r'], \mathfrak{g})$ with $\mathrm{im}[\oint_r^\bullet \phi] \subseteq O$ and $\int_r^{r'} \mathfrak{m}(\psi(s) - \phi(s))\,\mathrm{d}s \leq 1$ be given. We obtain from (8) that

$$\mathfrak{q}(\chi) \leq \mathfrak{m}(\psi - \phi) \quad \text{holds for} \quad \chi := \mathrm{Ad}_{[\oint_r^\bullet \phi]^{-1}}(\psi - \phi), \tag{9}$$

hence, $\int_r^{r'} \mathfrak{q}(\chi(s))\,\mathrm{d}s \leq \int_r^{r'} \mathfrak{m}(\psi(s) - \phi(s))\,\mathrm{d}s \leq 1$. Then, Proposition 2.3 shows

$$(\mathfrak{p} \circ \Xi)\big(\oint_r^t \chi\big) \leq \int_r^t \mathfrak{q}(\chi(s))\,\mathrm{d}s \overset{(9)}{\leq} \int_r^t \mathfrak{m}(\psi(s) - \phi(s))\,\mathrm{d}s \qquad \forall\, t \in [r, r'].$$

Since $\oint_r^t \chi$ equals the right hand side of (5), we obtain

$$(\mathfrak{p} \circ \Xi)\big([\oint_r^\bullet \phi]^{-1}[\oint_r^\bullet \psi]\big) \leq \int_r^\bullet \mathfrak{m}(\psi(s) - \phi(s))\,\mathrm{d}s,$$

which proves the claim. ∎

3. The Strong Trotter Property

We now prove Theorem 1.1. We start with the following observation.

Lemma 3.1. *Let G be locally μ-convex, and assume that $L \cdot \phi \subseteq \mathfrak{D}\mathrm{P}^0([0, 1], \mathfrak{g})$ holds for $L \in \mathfrak{K}$ and $\phi \colon [0, 1] \to \mathfrak{g}$. Then,*

$$\Phi \colon L \times [0, 1] \to G, \qquad (\tau, t) \mapsto \oint_0^t \tau \cdot \phi \tag{10}$$

is continuous, thus has compact image.

Proof. Let $\tau \in L$, $t \in [0, 1]$, and $h, h' \in [-1, 1]$ be such that $\tau + [0, 1] \cdot h \subseteq L$ and $t + [0, 1] \cdot h' \subseteq [0, 1]$ holds. Then,

$$\mathrm{B}_h := [\oint_0^t (\tau + h) \cdot \phi] \cdot [\oint_0^t \tau \cdot \phi]^{-1}$$
$$= [\oint_0^t \tau \cdot \phi] \cdot \underbrace{[\oint_0^t \tau \cdot \phi]^{-1} \cdot [\oint_0^t (\tau + h) \cdot \phi]}_{\mathrm{C}_h} \cdot [\oint_0^t \tau \cdot \phi]^{-1}$$

tends to e for $h \to 0$; because C_h tends to e for $h \to 0$, by Lemma 2.4 applied to $\mathrm{K} = \mathrm{im}[\oint_0^\bullet \tau \cdot \phi]$. We obtain from (c) that

$$\Phi(\tau + h, t + h') \cdot \Phi(\tau, t)^{-1} = \big[\underbrace{\oint_t^{t+h'} (\tau + h) \cdot \phi}_{\mathrm{A}_{h'}^+}\big] \quad \cdot\ \mathrm{B}_h \quad \text{holds for} \quad h' > 0,$$

$$\Phi(\tau + h, t + h') \cdot \Phi(\tau, t)^{-1} = \big[\underbrace{\oint_{t-|h'|}^{t} (\tau + h) \cdot \phi}_{\mathrm{A}_{h'}^-}\big]^{-1} \cdot\ \mathrm{B}_h \quad \text{holds for} \quad h' < 0.$$

Since the integrands are bounded, Proposition 2.3 shows that $\lim_{h' \to 0} \mathrm{A}_{h'}^\pm = e$ converges uniformly in h, from which the claim is clear. ∎

Combining Lemma 3.1 with Lemma 2.4, we obtain the following corollary.

Corollary 3.2.   *Let G be locally μ-convex, and assume $L \cdot \phi \subseteq \mathfrak{D}\mathrm{P}^0([0,1], \mathfrak{g})$ for $L \in \mathfrak{K}$ and $\phi \colon [0,1] \to \mathfrak{g}$. Then, to each $\mathfrak{p} \in \mathfrak{P}$, there exists some $\mathfrak{p} \leq \mathfrak{m} \in \mathfrak{P}$, such that*

$$(\mathfrak{p} \circ \Xi)\big([\textstyle\oint_0^\bullet \tau \cdot \phi]^{-1}\,[\textstyle\oint_0^\bullet \psi]\big) \leq \int_0^\bullet \mathfrak{m}(\psi(s) - \tau \cdot \phi(s))\, \mathrm{d}s$$

holds for each $\tau \in L$ and $\psi \in \mathfrak{D}\mathrm{P}^0([0,1], \mathfrak{g})$ with $\int_0^1 \mathfrak{m}(\psi(s) - \tau \cdot \phi(s))\, \mathrm{d}s \leq 1$.

Proof.   Let Φ be defined by (10). Since Lemma 3.1 shows that $\mathrm{K} := \mathrm{im}[\Phi]$ is compact, the claim is clear from Lemma 2.4.   ∎

We are ready for the proof of Theorem 1.1.

Proof of Theorem 1.1.   Let $\mu \in C^1([0,1], G)$ with $\mu(0) = e$ and $\dot{\mu}(0) \in \mathrm{dom}[\exp]$ be given. We fix $\ell > 0$, let $X := \dot{\mu}(0)$, and choose $m \geq 1$ so large that $\ell/m \leq 1$ and $\mu([0, \ell/m]) \subseteq \mathrm{dom}[\Xi] = \mathcal{U}$ holds. We obtain from (3) that $\{X_\tau := \tau \cdot X \mid \tau \in [0, \ell]\} \subseteq \mathrm{dom}[\exp]$ holds, hence

$$[0, \ell] \cdot \phi_X \subseteq \mathfrak{D}\mathrm{P}^0([0,1], \mathfrak{g}). \tag{11}$$

The hypotheses of Corollary 3.2 are thus fulfilled for $L = [0, \ell]$ and $\phi = \phi_X$ there. We proceed as follows.

- For $\tau \in [0, \ell]$ and $n \geq m$, we define

$$\chi_{\tau, n} := \delta^r(\mu_\tau)|_{[0, 1/n]} \qquad \text{for} \qquad \mu_\tau \colon [0, 1/m] \ni t \mapsto \mu(\tau \cdot t),$$

  and let $t_p := p/n$ for $p = 0, \ldots, n$. We furthermore let

$$\phi_{\tau, n}[p] \colon [t_p, t_{p+1}] \ni t \mapsto \chi_{\tau, n}(t - t_p) \qquad \forall\, p = 0, \ldots, n - 1,$$

  and define $\phi_{\tau, n} \in \mathfrak{D}\mathrm{P}^0([0,1], \mathfrak{g})$ by putting $\chi_{\tau, n}$ n-times in a row, i.e., we let

$$\phi_{\tau, n}|_{[t_p, t_{p+1})} := \phi_{\tau, n}[p]|_{[t_p, t_{p+1})} \qquad \forall\, p = 0, \ldots, n - 2$$

  as well as $\phi_{\tau, n}|_{[t_{n-1}, t_n]} := \phi_{\tau, n}[n - 1]$.

- For $\tau \in [0, \ell]$, $n \geq m$, and $0 \leq p \leq n - 1$, we apply (d) to

$$\varrho_p \colon [t_p, t_{p+1}] \ni t \mapsto t - t_p \in [0, 1/n],$$

  and obtain

$$\textstyle\oint \phi_{\tau, n}[p] = \oint \chi_{\tau, n} \circ \varrho_p = \oint \dot{\varrho}_p \cdot \chi_{\tau, n} \circ \varrho_p \overset{(d)}{=} \oint \chi_{\tau, n} = \mu_\tau(1/n) = \mu(\tau/n). \tag{12}$$

  Then, (4) provides us with

$$\textstyle\oint \phi_{\tau, n} \overset{(4)}{=} \oint \phi_{\tau, n}[n - 1] \cdot \ldots \cdot \oint \phi_{\tau, n}[0] \overset{(12)}{=} \mu(\tau/n)^n. \tag{13}$$

- For each $\tau \in [0, \ell]$, $n \geq m$, and $\mathfrak{m} \in \mathfrak{P}$, we have

$$\mathfrak{m}_\infty(\phi_{\tau,n} - \phi_{X_\tau}) \overset{(7)}{=} \sup_{t \in [0,1/n]} \mathfrak{m}(\chi_{\tau,n}(t) - \tau \cdot X)$$

with $\chi_{\tau,n}(0) = \delta^r(\mu_\tau)(0) = \dot{\mu}_\tau(0) = \tau \cdot \dot{\mu}(0) = \tau \cdot X$.

It follows from continuity of (6) in ii) that to each $\mathfrak{m} \in \mathfrak{P}$ and $0 < \varepsilon \leq 1$, there exists some $n_{\mathfrak{m},\varepsilon} \geq m$ with

$$\mathfrak{m}_\infty(\phi_{\tau,n} - \phi_{X_\tau}) < \varepsilon \qquad\qquad \forall \, \tau \in [0, \ell], \ \ n \geq n_{\mathfrak{m},\varepsilon}$$

implying $\qquad \int_0^1 \mathfrak{m}(\phi_{\tau,n}(s) - \tau \cdot \phi_X(s)) \, \mathrm{d}s < \varepsilon \qquad \forall \, \tau \in [0, \ell], \ \ n \geq n_{\mathfrak{m},\varepsilon}.$ \hfill (14)

Let now $\mathfrak{p} \in \mathfrak{P}$ and $0 < \varepsilon \leq 1$ be fixed. We choose $\mathfrak{p} \leq \mathfrak{m} \in \mathfrak{P}$ as in Corollary 3.2 for $L = [0, \ell]$ and $\phi = \phi_X$ there (recall that $[0, \ell] \cdot \phi_X \subseteq \mathfrak{D}\mathrm{P}^0([0, 1], \mathfrak{g})$ holds by (11)), and let $n_{\mathfrak{m},\varepsilon} \geq m$ be as in (14). Since

$$\oint_0^t \tau \cdot \phi_X \overset{(d)}{=} \exp(t \cdot \tau \cdot X) \qquad\qquad \forall \, t \in [0, 1], \ \ \tau \in [0, \ell]$$

holds, we obtain from (14) and Corollary 3.2 that

$$(\mathfrak{p} \circ \Xi)\big(\exp(t \cdot \tau \cdot X)^{-1} \cdot \oint_0^t \phi_{\tau,n} \big) \leq \int_0^t \mathfrak{m}(\phi_{\tau,n}(s) - \tau \cdot \phi_X(s)) \, \mathrm{d}s \overset{(14)}{<} \varepsilon$$

for each $t \in [0, 1]$, $\tau \in [0, \ell]$, and $n \geq n_{\mathfrak{m},\varepsilon}$. It is thus clear that to each open neighbourhood $V \subseteq G$ of e, there exists some $n_V \geq m$ with

$$\mu(\tau/n)^n \overset{(13)}{=} \oint_0^1 \phi_{\tau,n} \in \exp(\tau \cdot \dot{\mu}(0)) \cdot V \qquad\qquad \forall \, n \geq n_V, \ \ \tau \in [0, \ell].$$

The claim now follows from Lemma 2.2. \hfill ∎

Acknowledgements

A. Appendix

A.1. Appendix

Proof of Implication (3). It follows, e.g., from Lemma 11 in [5] that for each $n \geq 1$, the constant map $\phi_X^n \colon [0, n] \ni t \mapsto X \in \mathfrak{g}$ is in $\mathfrak{D}_{[0,n]}$. Let $0 < s \leq 1$ and $n \geq 1$ be fixed. Then, (d) applied to $\varrho \colon [0, 1] \ni t \mapsto s \cdot n \cdot t \in [0, s \cdot n]$ gives

$$\oint_0^{s \cdot n} \phi_X^n \overset{(d)}{=} \oint_0^1 \phi_{s \cdot n \cdot X} = \exp(s \cdot n \cdot X) \qquad \Longrightarrow \qquad s \cdot n \cdot X \in \mathrm{dom}[\exp].$$

Since $\mathbb{R}_{>0} = \bigcup_{n \geq 1}(0, n]$ holds, the claim follows. \hfill ∎

A.2. Appendix

Proof of Lemma 2.2. The one direction is evident. For the other direction, let $U \subseteq G$ be a fixed neighbourhood of e. To establish the proof, it suffices to show that there exists a neighbourhood $V \subseteq U$ of e with

$$\mu(t) \cdot V \subseteq U \cdot \mu(t) \qquad \forall\, t \in [0, \ell]. \tag{15}$$

Since $\mathrm{im}[\mu]$ is compact, Theorem 4.9 in [6] provides us with an identity neighbourhood $V \subseteq G$, such that $\mu(t) \cdot V \cdot \mu(t)^{-1} \subseteq U$ holds for each $t \in [0, \ell]$, which is equivalent to (15). ∎

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