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Mehler–Heine Formula: a Generalization in the Context of Spherical Functions

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Abstract. Using the notion of group contraction, we obtain the spherical functions of the strong Gelfand pair $(\mathrm{M}(n), \mathrm{SO}(n))$ as an appropriate limit of spherical functions of the strong Gelfand pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ and also of the strong Gelfand pair $(\mathrm{SO}_0(n,1), \mathrm{SO}(n))$.

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1. Introduction and motivation

The classic Mehler–Heine formula, introduced by Heine in 1861 and by Mehler in 1868 (who was motivated by the problem of knowing the distribution of electricity on spherical domains [15]), states that the Bessel function J_0 is a limit of Legendre polynomials P_N of order N in the following sense

$$\lim_{N\to\infty} P_N\Big(\cos(\frac{z}{N})\Big) = J_0(z),$$

where the limit is uniform over z in an arbitrary bounded domain in the complex plane. Observe that the functions on the left hand side are the spherical functions of the Gelfand pair $(\mathrm{SO}(3), \mathrm{SO}(2))$ and the function on the right hand side is a spherical function of the Gelfand pair $(\mathrm{SO}(2) \ltimes \mathbb{R}^2, \mathrm{SO}(2))$ (for a reference see, for e.g., [21]). There is a generalization of this formula involving other classical special functions as follows

$$\lim_{N\to\infty} \frac{P_N^{\alpha,\beta}\left(\cos(\frac{z}{N})\right)}{N^\alpha} = \frac{J_\alpha(z)}{\left(\frac{z}{2}\right)^\alpha} ,$$

where $P_N^{\alpha,\beta}$ are the Jacobi polynomials and J_α is the Bessel function of first kind of order α (cf. [19]). If $\alpha = \beta = \frac{n-2}{2}$, on the left side we have the Gegenbauer polynomials that are orthogonal polynomials that correspond to the spherical functions associated with the Gelfand pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ and on the right side the function

$$\frac{J_{\frac{n-2}{2}}(z)}{\left(\frac{z}{2}\right)^{\frac{n-2}{2}}}$$

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is a spherical function associated with the Gelfand pair $(\mathrm{SO}(n)\ltimes\mathbb{R}^n, \mathrm{SO}(n))$ (without normalization). We will denote by $\mathrm{M}(n) := \mathrm{SO}(n) \ltimes \mathbb{R}^n$ the connected component of the identity of the n-dimensional euclidean motion group.

In this article we obtain the spherical functions (scalar and matrix-valued) of the strong Gelfand pair $(\mathrm{M}(n), \mathrm{SO}(n))$ as an appropriate limit of spherical functions (scalar and matrix-valued) of the strong Gelfand pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ and then as an appropriate limit of spherical functions of the strong Gelfand pair $(\mathrm{SO}_0(n,1), \mathrm{SO}(n))$, where $\mathrm{SO}_0(n,1)$ is connected component of the identity of the Lorentz group. The non-scalar spherical functions associated to the first Gelfand pair were calculated in [4] and [18], the ones corresponding to the second pair were studied in [16, 20] and for the third pair can be found in [2] and in [24] for the scalar case. We will need the notion of group contraction introduced by Inönü and Wigner in [12]. For our purpose the results given by Dooley and Rice in the papers [7] and [8] will be extremely useful. Their results allow to show how to approximate matrix coefficients of irreducible representations of $\mathrm{M}(n)$ by a sequence of matrix coefficients of irreducible representations of $\mathrm{SO}(n+1)$ (see [3]).

The case that involves the compact group $\mathrm{SO}(n+1)$ is more difficult than the case with the non-compact group $\mathrm{SO}_0(n,1)$. Indeed, only the last section will be devoted to gain an asymptotic formula involving the spherical functions of $(\mathrm{SO}_0(n,1), \mathrm{SO}(n))$. Moreover, we can treat this case from a much more global optic, we will work with Cartan motions groups that arise from non-compact semisimple groups.

For the first part of this work we will follow the same writing structure as the paper [7] of Dooley and Rice and our main result is the Theorem 3.7 that states the following:

Let (τ, V_τ) be an irreducible unitary representation of $\mathrm{SO}(n)$ and let $\Phi^{\tau,\mathrm{M}(n)}$ be a spherical function of type τ of the strong Gelfand pair $(\mathrm{M}(n), \mathrm{SO}(n))$. There exists a sequence $\{\Phi_\ell^{\tau,\mathrm{SO}(n+1)}\}_{\ell\in\mathbb{Z}_{\geq 0}}$ of spherical functions of type τ of the strong Gelfand pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ and a contraction $\{D_\ell\}_{\ell\in\mathbb{Z}_{\geq 0}}$ of $\mathrm{SO}(n+1)$ to $\mathrm{M}(n)$ such that

$$\lim_{\ell\to\infty} \Phi_\ell^{\tau,\mathrm{SO}(n+1)} \circ D_\ell = \Phi^{\tau,\mathrm{M}(n)},$$

where the convergence is pointwise on V_τ and uniform on compact sets of $\mathrm{M}(n)$.

In the last section we obtain an analogous result changing $\mathrm{SO}(n+1)$ by $\mathrm{SO}_0(n,1)$.

2. Preliminaries

2.1. Spherical functions

Let (G, K, τ) be a triple where G is a locally compact Hausdorff unimodular topological group (or just a Lie group), K be a compact subgroup of G and (τ, V_τ) be an irreducible unitary representation of K of dimension d_τ. We denote by χ_τ the character associated to τ, by $\mathrm{End}(V_\tau)$ the group of endomorphisms of the vector space V_τ and by $\widehat{G}$ (respectively, $\widehat{K}$) the set of equivalence classes of irreducible unitary representations of G (respectively, of K). We assume that for each $\pi \in \widehat{G}$, the multiplicity $m(\tau, \pi)$ of τ in $\pi_{|_K}$ is at most one. In these cases the triple (G, K, τ) is said *commutative* because the convolution algebra of $\mathrm{End}(V_\tau)$-valued integrable functions on G such that are bi-τ-equivariant (i.e., $f(k_1 g k_2) = \tau(k_2)^{-1} f(g) \tau(k_1)^{-1}$ for all $g \in G$ and for all $k_2, k_2 \in K$) turns out to be commutative. When τ is the trivial representation we have the notion of *Gelfand pair*. It is said that (G,K) is a *strong Gelfand pair* if (G,K,τ) is a commutative triple for every $\tau \in \widehat{K}$.

Let $\widehat{G}(\tau)$ be the set of those representations $\pi \in \widehat{G}$ which contain τ upon restriction to K. For $\pi \in \widehat{G}(\tau)$, let $\mathcal{H}_\pi$ be the Hilbert space where π acts and let $\mathcal{H}_\pi(\tau)$ be the subspace of vectors which transforms under K according to τ. Since $m(\tau,\pi)=1$, $\mathcal{H}_\pi(\tau)$ can be identified with V_τ. Let $P_\pi^\tau : \mathcal{H}_\pi \longrightarrow \mathcal{H}_\pi(\tau)$ be the orthogonal projection (see, e.g., [22, Proposition 5.3.7] and [2, Section 3]) given by

$$P_\pi^\tau = d_\tau \pi_{|K}(\overline{\chi_\tau}) = d_\tau \int_K \chi_\tau(k^{-1})\pi(k)dk. \tag{1}$$

Definition 2.1. Let $\pi \in \widehat{G}$. The function

$$\Phi_\pi^\tau(g) := P_\pi^\tau \circ \pi(g) \circ P_\pi^\tau \qquad (\forall g \in G)$$

is called a *spherical function of type* τ.

Remark 2.2. (i) Observe that the spherical functions depend only on the classes of equivalence of irreducible unitary representations of G. That is, if π_1 y π_2 are two equivalent irreducible unitary representations of G with intertwining operator $A : \mathcal{H}_{\pi_1} \longrightarrow \mathcal{H}_{\pi_2}$ (i.e., $A\circ\pi_1(g)\circ A^{-1} = \pi_2(g)$ for all $g \in G$), then $A\circ P_{\pi_1}^\tau \circ A^{-1} = P_{\pi_2}^\tau$ and so

$$A \circ \Phi_{\pi_1}^\tau(g) \circ A^{-1} \;=\; \Phi_{\pi_2}^\tau(g) \qquad \forall g \in G.$$

As a result, $\Phi_{\pi_1}^\tau(g)$ and $\Phi_{\pi_2}^\tau(g)$ are conjugated by the same isomorphism A for all $g \in G$.

(ii) Apart from that, as we say before, given $\pi \in \widehat{G}$ such that $\tau \subset \pi$ as a K-module and $m(\tau,\pi)=1$, the vector space $\mathcal{H}_\pi(\tau)$ is isomorphic to V_τ. If $T : \mathcal{H}_\pi(\tau) \longrightarrow V_\tau$ is the isomorphism between them, we will not make distinctions between $\Phi_\pi^\tau(g) \in \mathrm{End}(\mathcal{H}_\pi(\tau))$ and $T \circ \Phi_\pi^\tau(g) \circ T^{-1} \in \mathrm{End}(V_\tau)$. ■

In this work we consider the strong Gelfand pairs $(\mathrm{M}(n), \mathrm{SO}(n))$, $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ and $(\mathrm{SO}_0(n,1), \mathrm{SO}(n))$. For a reference see for e.g [4, 18] for the first pair, [16, 20] for the second pair and [2] for the third pair.

The natural action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ will be denote by

$$\mathrm{SO}(n) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad (k,x) \mapsto k \cdot x.$$

From now on we will denote by K the group isomorphic to $\mathrm{SO}(n)$ which is, depending on the context, a subgroup of $\mathrm{SO}(n+1)$ or a subgroup of $\mathrm{M}(n)$. In the first case it must be identified with $\{g \in \mathrm{SO}(n+1) | \; g\cdot e_1 = e_1\}$ (where e_1 is the canonical vector $(1,0,...,0) \in \mathbb{R}^{n+1}$) and in the second with $\mathrm{SO}(n) \times \{0\}$.

2.2. The representation theory of $\mathrm{SO}(N)$

Let N be an arbitrary natural number. The Lie algebra $\mathfrak{so}(N)$ of $\mathrm{SO}(N)$ is the space of antisymmetric matrices of order N. Its complexification $\mathfrak{so}(N,\mathbb{C})$ is the space of complex such matrices. Let M be the integral part of $N/2$. For a maximal torus T of $\mathrm{SO}(2M)$ we consider

$$\left\{ \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) & & & \\ -\sin(\theta_1) & \cos(\theta_1) & & & \\ & & \ddots & & \\ & & & \cos(\theta_M) & \sin(\theta_M) \\ & & & -\sin(\theta_M) & \cos(\theta_M) \end{pmatrix} \mid \theta_1, ..., \theta_M \in \mathbb{R} \right\}$$

and for $\mathrm{SO}(2M+1)$ the same but with a one in the right bottom corner. In what follows we describe the basic notions of the root system of $\mathfrak{so}(N,\mathbb{C})$, following [9, 13], in order to fix the notation.

Let $\mathfrak{t}$ denote the Lie algebra of T. A Cartan subalgebra $\mathfrak{h}$ of the complex Lie algebra $\mathfrak{so}(N,\mathbb{C})$ is given by the complexification of $\mathfrak{t}$.
If N is even we consider $\{H_1, ..., H_M\}$ the following basis of $\mathfrak{h}$ as a $\mathbb{C}$-vector space

$$\left\{ H_1 := \begin{pmatrix} 0 & i & & & \\ -i & 0 & & & \\ & & \ddots & & \\ & & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix}, \; ... \; , H_M := \begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & \ddots & & \\ & & & 0 & i \\ & & & -i & 0 \end{pmatrix} \right\}$$

(where $i = \sqrt{-1}$) and if N is odd we consider the same but with a zero in the right bottom corner. This basis is *orthogonal* with respect to the Killing form B, that is

$$B(H_i, H_j) = 0 \quad \forall i \neq j \quad \text{and}$$

$$B(H_i, H_i) = \begin{cases} 4(M-1) & \text{(if } N \text{ is even)} \\ 4(M-1)+2 & \text{(if } N \text{ is odd).} \end{cases}$$

Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$ and let $\{L_1, ..., L_M\}$ be the dual basis of $\{H_1, ..., H_M\}$ (that is, $L_i(H_j) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta). To each irreducible representation of $\mathfrak{so}(N,\mathbb{C})$ corresponds its highest weight $\lambda = \sum_{i=1}^{M} \lambda_i L_i$, where λ_i are all integers or all half integers satisfying

(i) $\lambda_1 \geq \lambda_2 \geq \; ... \; \geq \lambda_{M-1} \geq |\lambda_M|$ if N is even or

(ii) $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_M \geq 0$ if N is odd.

Thus, we can associate each irreducible representation of $\mathfrak{so}(N,\mathbb{C})$ with an M-tuple $(\lambda_1, ..., \lambda_M)$ fulfilling the mentioned conditions. We call such tuple a *partition.*

We recall a well-known formula regarding the decomposition of a representation of $\mathfrak{so}(N,\mathbb{C})$ under its restriction to $\mathfrak{so}(N-1,\mathbb{C})$ (cf. [9, (25.34) and (25.35)]).

Case odd to even: Let ρ_λ be the irreducible representation of $\mathfrak{so}(2M+1,\mathbb{C})$ that is in correspondence to the partition $\lambda = (\lambda_1, ..., \lambda_M)$ where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_M \geq 0$.

Then
$$(\rho_\lambda)_{|_{\mathfrak{so}(2M,\mathbb{C})}} = \bigoplus_{\overline{\lambda}} \rho_{\overline{\lambda}} \tag{2}$$

where the sum runs over all the partitions $\overline{\lambda} = (\overline{\lambda_1}, ..., \overline{\lambda_M})$ that satisfy

$$\lambda_1 \geq \overline{\lambda_1} \geq \lambda_2 \geq \overline{\lambda_2} \geq ... \geq \overline{\lambda_{M-1}} \geq \lambda_M \geq |\overline{\lambda_M}|,$$

with the λ_i and $\overline{\lambda_i}$ simultaneously all integers or all half integers.

Case even to odd: Let ρ_λ be the irreducible representation of $\mathfrak{so}(2M)$ that is in correspondence to the partition $\lambda = (\lambda_1, ..., \lambda_M)$ where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{M-1} \geq |\lambda_M|)$.

Then
$$(\rho_\lambda)_{|\mathfrak{so}(2M-1,\mathbb{C})} = \bigoplus_{\overline{\lambda}} \rho_{\overline{\lambda}} \tag{3}$$
where the sum runs over all the partitions $\overline{\lambda} = (\overline{\lambda_1}, ..., \overline{\lambda_{M-1}})$ that satisfy
$$\lambda_1 \geq \overline{\lambda_1} \geq \lambda_2 \geq \overline{\lambda_2} \geq \ ... \ \geq \overline{\lambda_{M-1}} \geq |\lambda_M|,$$
with the λ_i and $\overline{\lambda_i}$ simultaneously all integers or all half integers.

Finally, we recall that each irreducible representation of the group $\mathrm{SO}(N)$ corresponds to a partition $(\lambda_1, ..., \lambda_M)$ (with the properties (i) or (ii) depending on the parity of its dimension) where λ_i are all integers.

2.2.1. The Borel-Weil-Bott Theorem

The Borel-Weil-Bott theorem provides a concrete model for the irreducible representations of the rotation group, since it is a compact Lie group. Let T be a maximal torus of $\mathrm{SO}(N)$. Given a character χ of T, $\mathrm{SO}(N)$ acts on the space of holomorphic sections of the line bundle $G \times_\chi \mathbb{C}$ by the left regular representation. This representation is either zero or irreducible, moreover, it is irreducible when χ is dominant integral. The theorem asserts that each irreducible representation of $\mathrm{SO}(N)$ arises from this way for a unique character χ of the maximal torus T. (For a reference see [22, Section 6.3] and [7, Section 1].)

Remark 2.3. The holomorphic sections of the line bundle $G \times_\chi \mathbb{C}$ may be identified with C^∞ functions on $\mathrm{SO}(N)$ satisfying the following two conditions:

(i) $f(gt) = \overline{\chi(t)} f(g) \quad \forall t \in \mathrm{T}$ and $g \in \mathrm{SO}(N)$ and

(ii) for each $X \in \eta^+$, $\quad Xf(g) := \frac{d}{ds}_{|s=0} f(g \ \exp(sX)) = 0 \quad \forall g \in \mathrm{SO}(N)$.

With this identification, the representation of $\mathrm{SO}(N)$ is given by the left regular action, i.e., $L_g(f)(x) := f(g^{-1}x) \quad \forall g \in \mathrm{SO}(N)$.

2.2.2. A special character and a special function

For the case $N = n + 1$ we will introduce a character γ and a function ψ that will play an important role later on. Let m be the integral part of $(n+1)/2$ and let $\mathbb{T}^m$ denote a maximal torus of $\mathrm{SO}(n + 1)$ as above.

Let $\gamma : \mathbb{T}^m \longrightarrow \mathbb{C}$ be the projection onto the first factor, i.e., $\gamma(e^{i\theta_1}, ..., e^{i\theta_m}) = e^{i\theta_1}$. The irreducible representation of $\mathrm{SO}(n + 1)$ associated with γ (through the Borel-Weil-Bott theorem) is equivalent to the standard representation [7, Lemma 1]. Moreover, for each $\ell \in \mathbb{N}$, the irreducible representation of $\mathrm{SO}(n+1)$ associated with the ℓ-th power of γ (i.e. $\gamma^\ell(e^{i\theta_1}, ..., e^{i\theta_m}) = e^{i\ell\theta_1}$) has $(\ell, 0, ..., 0)$, that is, the one that can be realized on the space of harmonic homogeneous polynomials of degree ℓ on $\mathbb{R}^{n+1}$ with complex coefficients.

In the standard representation of $\mathrm{SO}(n + 1)$, the trivial representation of $\mathrm{SO}(n)$ appears as a $\mathrm{SO}(n)$-submodule. As a consequence, we can take a $\mathrm{SO}(n)$-fixed vector for the standard representation, i.e., a function $\psi : \mathrm{SO}(n+1) \longrightarrow \mathbb{C}$ as in the Remark

2.3 satisfying $\psi(k^{-1}g) = \psi(g)$ for all $k \in \mathrm{SO}(n)$ and $g \in \mathrm{SO}(n+1)$. Moreover, we can choose ψ such that $\psi(k) = 1$ for all $k \in \mathrm{SO}(n)$.

2.3. The representation theory of $\mathrm{M}(n)$

We will follow Mackey's orbital analysis to describe the irreducible representations of $\mathrm{M}(n)$ (for a reference see [14, Section 14] and [7, Section 2]). The orbits of the natural action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ are the spheres of radius $R > 0$ and the origin set point $\{0\}$ (which is a fixed point for the whole group $\mathrm{SO}(n)$). The irreducible representations corresponding to the trivial orbit $\{0\}$ are one-dimensional, parametrized by $\lambda \in \mathbb{R}^n$ and explicitly they are

$$(k, x) \mapsto e^{i\langle \lambda, x\rangle} \qquad \forall (k, x) \in \mathrm{M}(n), \tag{4}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product on $\mathbb{R}^n$. Since these representations have zero Plancherel measure we are not interested in them. (They will not provide spherical functions appearing in the inversion formula for the spherical Fourier transform.)

The irreducible representations that arises from the non-trivial orbits will be more interesting for us. One must fix a point Re_1 on the sphere of radius $R > 0$ (where $e_1 := (1, 0, ..., 0) \in \mathbb{R}^n$), take its stabilizer

$$K_{Re_1} := \{k \in \mathrm{SO}(n) |\ k \cdot Re_1 = Re_1\},$$

the character $$\chi_R(x) := e^{iR\langle x, e_1\rangle} \qquad (\forall x \in \mathbb{R}^n).$$

and a representation $\sigma \in \widehat{\mathrm{SO}(n-1)}$ (note that K_{Re_1} is isomorphic to $SO(n-1)$). Finally, inducing the representation $\sigma \otimes \chi_R$ from $K_{Re_1} \ltimes \mathbb{R}^n$ to $\mathrm{M}(n)$ one obtains an irreducible representation $\omega_{\sigma,R}$ of $\mathrm{M}(n)$.

It can be seen (using the Borel-Weil-Bott model for $\sigma \in \widehat{\mathrm{SO}(n-1)}$) that this representation can be realized on a subspace of scalar-valued square integrable functions on $\mathrm{SO}(n)$. This space consists of the functions $f \in L^2(\mathrm{SO}(n))$ that satisfy the following two conditions,

(i) if $\mathbb{T}^{m-1}$ denotes the maximal torus of $K_{Re_1} \simeq \mathrm{SO}(n-1)$, then

$$f(kt) = \chi_\sigma(t^{-1}) f(k) \qquad \forall k \in \mathrm{SO}(n) \text{ and } \forall t \in \mathbb{T}^{m-1},$$

where χ_σ is the character associated with σ;

(ii) for each $k \in \mathrm{SO}(n)$, the function

$$\mathrm{SO}(n-1) \longrightarrow \mathbb{C}, \quad \tilde{k} \mapsto f(k\tilde{k})$$

satisfies condition (ii) from Remark 2.3 (with $N = n - 1$).

We denote this space as $\mathcal{H}_{\sigma,R}$. The irreducible representation $\omega_{\sigma,R}$ acts on $\mathcal{H}_{\sigma,R}$ in the following way, let $f \in \mathcal{H}_{\sigma,R}$ and let $(k, x) \in \mathrm{SO}(n) \times \mathbb{R}^n$, then

$$(\omega_{\sigma,R}(k, x)(f))(h) = e^{iR\langle h^{-1}\cdot x, e_1\rangle} f(k^{-1}h) \qquad (\forall h \in \mathrm{SO}(n)). \tag{5}$$

It is known that all the irreducible representations of $\mathrm{M}(n)$ are equivalent to the ones given by (4) or to the ones given by (5).

2.4. Contraction

The notion of *contraction*, at the Lie algebra level, was introduced Inönü and Wigner in [12]. We recall its definition at the Lie group level (cf. [17, p. 211]).

Definition 2.4. If G and H are two connected Lie groups of the same dimension, we say that the family $\{D_\alpha\}$ of infinitely differentiable maps $D_\alpha : H \longrightarrow G$, mapping the identity e_H to the identity e_G of G, defines a *contraction of G to H* if given any relatively compact open neighborhood V of e_H

(i) there is $\alpha_V \in \mathbb{N}$ such that for $\alpha > \alpha_V$, $(D_\alpha)_{|V}$ is a diffeomorphism,

(ii) if W is such that $W^2 \subset V$ and $\alpha > \alpha_V$, then $D_\alpha(W)^2 \subset D_\alpha(V)$ and

(iii) for $h_1, h_2 \in W$, $\lim_{\alpha\to\infty} D_\alpha^{-1}\left(D_\alpha(h_1)D_\alpha(h_2)^{-1}\right) = h_1h_2^{-1}$ uniformly on $V \times V$.

In particular, for $G = \mathrm{SO}(n+1)$ and $H = \mathrm{M}(n)$ we consider the following family of contraction maps $\{D_\alpha\}_{\alpha\in\mathbb{R}_{>0}}$,

$$D_\alpha : \mathrm{M}(n) \longrightarrow \mathrm{SO}(n+1), \quad D_\alpha(k, x) := \exp\left(\frac{x}{\alpha}\right) k, \tag{6}$$

where exp denotes the exponential map $\mathfrak{so}(n+1) \longrightarrow \mathrm{SO}(n+1)$ and we identified (as vector spaces) $\mathbb{R}^n$ with the complement of $\mathfrak{so}(n)$ in $\mathfrak{so}(n+1)$, which is invariant under the adjoint action of K. (Note that we are using the so called Cartan decomposition.) Writing

$$D_\alpha(k_1, x_1)D_\alpha(k_2, x_2) = \exp\left(\frac{1}{\alpha}x_1\right) k_1 \ \exp\left(\frac{1}{\alpha}x_2\right) k_2$$

$$= \exp\left(\frac{1}{\alpha}x_1\right)\left[k_1 \ \exp\left(\frac{1}{\alpha}x_2\right) k_1^{-1}\right] k_1\, k_2 = \exp\left(\frac{1}{\alpha}x_1\right) \ \exp\left(\mathrm{Ad}(k_1)\frac{1}{\alpha}x_2\right) k_2$$

(where Ad denotes the adjoint representation of $\mathrm{SO}(n)$) and using the Bake-Campbell-Hausdorff formula we can find at the limit of $\alpha \to \infty$ property (iii) for all $(k_1, x_1), (k_2, x_2) \in \mathrm{M}(n)$.

2.5. The contracting sequence of an irreducible representation of $\mathrm{M}(n)$

In this section we summarize the results proved by Dooley and Rice in [7, Sections 3 and 4] that will be frequently used in the sequel.

Let $R \in \mathbb{R}_{>0}$, let $\sigma \in \widehat{\mathrm{SO}(n-1)}$ corresponding to the partition $(\sigma_1, ..., \sigma_{m-1})$ and let $\omega_{\sigma,R} \in \widehat{\mathrm{M}(n)}$ be the irreducible unitary representation given by (5). Finally, let γ be the character given in Section 2.2.2. The following definition will be very important.

Definition 2.5. [7, Definition 4] The sequence $\{\gamma^\ell\chi_\sigma\}_{\ell=1}^\infty$ of characters of $\mathbb{T}^m$ defines, for $\ell \geq \sigma_1$, a sequence $\{\rho_{\sigma,\ell}\}_\ell$ of irreducible unitary representations of $\mathrm{SO}(n+1)$ (as in Section 2.2) and it is called the *contracting sequence* associated with $\omega_{\sigma,R}$. For each non-negative integer $\ell \geq \sigma_1$, we denote by $\mathcal{H}_{\sigma,\ell}$ the space given by Remark 2.3, which is a model for $\rho_{\sigma,\ell}$.

We will use the following results proved by Dooley and Rice.

Lemma 2.6. [7, Lemma 5]

(i) *For each $\ell \in \mathbb{N}$, the multiplication by the function ψ (given in Section* 2.2.2*) defines a linear map from $\mathcal{H}_{\sigma,\ell}$ to $\mathcal{H}_{\sigma,\ell+1}$.*

(ii) *If* $\tilde{f} \in \mathcal{H}_{\sigma,\ell}$*, then the restrictions of* $\tilde{f}$ *and* $\psi\tilde{f} \in \mathcal{H}_{\sigma,\ell+1}$ *to* $\mathrm{SO}(n)$ *are the same (since* $\psi_{|\mathrm{SO}(n)} \equiv 1$*).*

(iii) *The spaces* $\{\mathcal{H}_{\sigma,\ell}{}_{|\mathrm{SO}(n)}\}_{\ell\in\mathbb{N}}$ *of restrictions to* $\mathrm{SO}(n)$ *form an increasing sequence of subspaces of* $\mathcal{H}_{\sigma,R}$*.*

Theorem 2.7. [7, Theorem 1 and Corollary 1] *Let* ψ^ℓ *denote the* ℓ*-th power of* ψ *(i.e,* $\psi^\ell = \psi \circ ... \circ \psi$*). Let* B *be a compact subset of* $\mathbb{R}^n$*. For an arbitrary function* $\tilde{f} \in \mathcal{H}_{\sigma,\ell_0}$*, it follows that*

(i) *for all* $s \in \mathrm{SO}(n)$*,*
$$\lim_{\ell\to\infty} \left(\rho_{\sigma,\ell_0+\ell}(D_{\ell/R}(k,x))(\psi^\ell\tilde{f})\right)(s) = \left(\omega_{\sigma,R}(k,x)(\tilde{f}_{|\mathrm{SO}(n)})\right)(s) \tag{7}$$
uniformly for $(k,x) \in \mathrm{SO}(n) \times B$*;*

(ii) *and also,*
$$\lim_{\ell\to\infty} \left\|\rho_{\sigma,\ell_0+\ell}(D_{\ell/R}(k,x))(\psi^\ell\tilde{f}) - \omega_{\sigma,R}(k,x)(\tilde{f}_{|\mathrm{SO}(n)})\right\|_{L^2(\mathrm{SO}(n))} = 0 \tag{8}$$
uniformly for $(k,x) \in \mathrm{SO}(n) \times B$*.*

Corollary 2.8. [7, Corollary 2] *The increasing union* $\bigcup_{\ell=1}^{\infty}\left(\mathcal{H}_{\sigma,\ell}{}_{|\mathrm{SO}(n)}\right)$ *is dense in* $\mathcal{H}_{\sigma,R}$ *with respect to the* $L^2(\mathrm{SO}(n))$*-norm.*

3. The approximation theorem

The aim of this section is to prove that the spherical functions of type τ corresponding to the strong Gelfand pair $(\mathrm{SO}(n) \ltimes \mathbb{R}^n, \mathrm{SO}(n))$ can be obtained as an appropriate limit of spherical functions of type τ associated to the strong Gelfand pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$.

Let (τ, V_τ) be an arbitrary irreducible unitary representation of K. We now take $(\omega_{\sigma,R}, \mathcal{H}_{\sigma,R}) \in \widehat{\mathrm{M}(n)}$ such that $\tau \subset \omega_{\sigma,R}$ as a K-module, that is, τ appears in the decomposition of $\omega_{\sigma,R}$ into irreducible representations as a K-module. According to Section 2.3
$$\omega_{\sigma,R} = \mathrm{Ind}_{\mathrm{SO}(n-1)\ltimes\mathbb{R}^n}^{\mathrm{SO}(n)\ltimes\mathbb{R}^n}(\sigma \otimes \chi_R).$$

By the Frobenius reciprocity theorem, the representation $\tau \subset \omega_{\sigma,R}$ as an $\mathrm{SO}(n)$-module if and only if $\sigma \subset \tau$ as an $\mathrm{SO}(n-1)$-module. Moreover,
$$m(\tau, \omega_{\sigma,R}) = m(\sigma, \tau). \tag{9}$$

We denote by $(\tau_1, ..., \tau_m)$ the partition associated to τ if $n = 2m$ and $(\tau_1, ..., \tau_{m-1})$ if $n = 2m-1$.

Remark 3.1. Let $(\sigma_1, ..., \sigma_{m-1})$ be the partition corresponding to the representation $\sigma \in \widehat{\mathrm{SO}(n-1)}$ and assume $\tau \subset \omega_{\sigma,R}$. From (9) and the branching formulas given in Section 2.2 we have the following:

(i) If $n = 2m$, from (3) we have that
$$\tau_1 \geq \sigma_1 \geq \tau_2 \geq \sigma_2 \geq \ ... \ \geq \tau_{m-1} \geq \sigma_{m-1} \geq |\tau_m|. \tag{10}$$

(ii) If $n = 2m-1$, from (2)

$$\tau_1 \geq \sigma_1 \geq \tau_2 \geq \sigma_2 \geq \ \dots \ \geq \sigma_{m-2} \geq \tau_{m-1} \geq |\sigma_{m-1}|. \tag{11}$$

Let $\mathcal{H}_{\sigma,R}(\tau)$ be the τ-isotypic component of $\mathcal{H}_{\sigma,R}$. We fix $\Phi^{\tau,\mathrm{M}(n)}_{\omega_{\sigma,R}}$ the spherical function of type τ of $(\mathrm{M}(n), \mathrm{SO}(n))$ associated to the representation $\omega_{\sigma,R}$ of $\mathrm{M}(n)$ (see Definition 2.1).

We will consider a family of irreducible unitary representations of $\mathrm{SO}(n+1)$ that is a contracting sequence associated to $\omega_{\sigma,R}$. Let χ_σ denote the character associated to σ. The special orthogonal group $\mathrm{SO}(n-1)$ can be embedded in $\mathrm{SO}(n+1)$ by starting with a 2×2 identity block in the top left hand corner. Let $\mathbb{T}^m$ be the maximal torus of $\mathrm{SO}(n+1)$ and $\mathbb{T}^{m-1}$ be the maximal torus of $\mathrm{SO}(n-1)$ that are of the form given in Section 2.2. That is, if $n+1$ is even

$$\mathbb{T}^m \supset \mathbb{T}^{m-1} = \left\{ \begin{pmatrix} 1 & 0 & & & & & \\ 0 & 1 & & & & & \\ & & \cos(\theta_2) & \sin(\theta_2) & & & \\ & & -\sin(\theta_2) & \cos(\theta_2) & & & \\ & & & & \ddots & & \\ & & & & & \cos(\theta_m) & \sin(\theta_m) \\ & & & & & -\sin(\theta_m) & \cos(\theta_m) \end{pmatrix} \mid \theta_2, \dots, \theta_m \in \mathbb{R} \right\}.$$

When $n+1$ is odd, the picture is similar, but the maximal torus have an additional 1 in the bottom right hand corner. We consider a Cartan subalgebra of $\mathfrak{so}(n+1,\mathbb{C})$ generated by $\{H_1, H_2, \dots, H_m\}$ as in Section 2.2. By the relations of orthogonality with respect to the Killing form, we can take $\{H_2, \dots, H_m\}$ to be a basis of a Cartan subalgebra of $\mathfrak{so}(n-1,\mathbb{C})$ embedded in $\mathfrak{so}(n+1,\mathbb{C})$.

For each non-negative integer ℓ, let $(\rho_{\sigma,\ell}, \mathcal{H}_{\sigma,\ell})$ be the representation of $\mathrm{SO}(n+1)$ constructed as in Section 2.2 from the character $\gamma^\ell \chi_\sigma$ of $\mathbb{T}^m$. It is easy to see that the corresponding partition is $(\ell, \sigma_1, \dots, \sigma_{m-1})$. If $\ell \in \mathbb{N}$ is such that $\ell < \sigma_1$, the representation $\rho_{\sigma,\ell}$ is trivial and if $\ell \geq \sigma_1$, the representation $\rho_{\sigma,\ell}$ is irreducible.

We remind the reader that sometimes to reduce the notation we use K for $\mathrm{SO}(n)$.

Lemma 3.2. *If τ appears in the decomposition into irreducible representations of $\omega_{\sigma,R}$ as a K-module, then τ appears in the decomposition of $\rho_{\sigma,\ell}$ as a K-module for all $\ell \geq \tau_1$.*

Proof. We will apply the results given in Section 2.2 to our case recalling (9).
(i): Let $n+1 = 2m+1$. Since $\rho_{\sigma,\ell}$ corresponds to the partition $(\ell, \sigma_1, \dots, \sigma_{m-1})$, it follows from (10) and (2) that τ appears in the decomposition of $\rho_{\sigma,\ell}$ as a K-module if $\ell \geq \tau_1$.

(ii): Let $n+1 = 2m$. If $\ell \geq \tau_1$, it follows from (11) and (3) that τ appears in the decomposition of $\rho_{\sigma,\ell}$ as a K-module. ■

Remark 3.3. Note that, from (5), the representation $\omega_{\sigma,R}$ restricted to K acts on $\mathcal{H}_{\sigma,R}$ as the left regular action, i.e., for each $k \in K$,

$$(\omega_{\sigma,R}(k,0)f)(k_0) = (L_k(f))(k_0) = f(k^{-1}k_0) \quad \forall k_0 \in K \quad \text{and} \quad \forall f \in \mathcal{H}_{\sigma,R}.$$

Apart from that, for each $\ell \in \mathbb{Z}_{\geq 0}$, ${\mathcal{H}_{\sigma,\ell}}_{|K}$ is a K-submodule of $\mathcal{H}_{\sigma,R}$. Thus, the restriction operator given by

$$Res_\ell(\tilde{f}) := \tilde{f}_{|K} \qquad \forall \tilde{f} \in \mathcal{H}_{\sigma,\ell}$$

intertwines $\mathcal{H}_{\sigma,\ell}$ and $\mathcal{H}_{\sigma,R}$ as K-modules.

Lemma 3.4. *If $f \in \mathcal{H}_{\sigma,R}(\tau)$, then there exists $\ell' \in \mathbb{N}$ such that $f \in {\mathcal{H}_{\sigma,\ell}}_{|K}$ for all $\ell \geq \ell'$. Moreover, let $\ell_0 := \max\{\tau_1, \ell'\}$, then there exists a unique $\tilde{f} \in \mathcal{H}_{\sigma,\ell_0}(\tau)$ such that $f = \tilde{f}_{|K}$.*

Proof. The space $\mathcal{H}_{\sigma,R}(\tau) \simeq V_\tau$ is an invariant factor in the decomposition of $\mathcal{H}_{\sigma,R}$ as a K-module. From Section 2.5, each ${\mathcal{H}_{\sigma,\ell}}_{|K}$ is a subspace of $\mathcal{H}_{\sigma,R}$, moreover, $\bigcup_{\ell=1}^{\infty}\left({\mathcal{H}_{\sigma,\ell}}_{|\mathrm{SO}(n)}\right)$ is dense in $\mathcal{H}_{\sigma,R}$. Since the dimension of V_τ is finite, there exists $\ell' \in \mathbb{N}$ such that $\mathcal{H}_{\sigma,R}(\tau)$ is contained in ${\mathcal{H}_{\sigma,\ell'}}_{|K}$. Furthermore, as ${\mathcal{H}_{\sigma,\ell}}_{|K} \subset {\mathcal{H}_{\sigma,\ell+1}}_{|K}$ for all $\ell \in \mathbb{N}$, it follows that $\mathcal{H}_{\sigma,R}(\tau) \subset {\mathcal{H}_{\sigma,\ell}}_{|K}$ for all $\ell \geq \ell'$.

Apart from that, since the decomposition of $\rho_{\sigma,\ell}$ as a K-module is multiplicity free (for all $\ell \in \mathbb{N}$), then the operator Res_ℓ is a linear isomorphism that maps the irreducible component $\mathcal{H}_{\sigma,\ell}(\tau)$ into the irreducible component ${\mathcal{H}_{\sigma,\ell}}_{|K}(\tau)$, for all $\ell \geq \tau_1$. Finally, if f is an arbitrary function in $\mathcal{H}_{\sigma,R}(\tau)$, there is a unique $\tilde{f} \in \mathcal{H}_{\sigma,\ell_0}(\tau)$ such that $\tilde{f}(k) = f(k)$ for all $k \in K$. ■

Lemma 3.5. *Let $\ell_0 \in \mathbb{N}$ as in Lemma 3.4 and let $\tilde{f} \in \mathcal{H}_{\sigma,\ell_0}(\tau)$. It follows that $\psi^\ell \tilde{f} \in \mathcal{H}_{\sigma,\ell_0+\ell}(\tau)$ for all $\ell \in \mathbb{N}$.*

Proof. From Section 2.5 if $\tilde{f} \in \mathcal{H}_{\sigma,\ell_0}$, then $\psi^\ell \tilde{f} \in \mathcal{H}_{\sigma,\ell_0+\ell}$ for all $\ell \in \mathbb{N}$. Also, since ψ is a K-invariant function (i.e, $\psi(k^{-1}g) = \psi(g)$ for all $k \in K$ and $g \in \mathrm{SO}(n+1)$), then the multiplication by ψ^ℓ is an intertwining operator between $(\rho_{\sigma,\ell_0}, \mathcal{H}_{\sigma,\ell_0})$ and $(\rho_{\sigma,\ell_0+\ell}, \mathcal{H}_{\sigma,\ell_0+\ell})$ as K-modules. Since the decomposition of $\rho_{\sigma,\ell}$ as a K-module is multiplicity free (for all $\ell \in \mathbb{N}$), the multiplication by ψ^ℓ maps irreducible component to irreducible component, that is, maps $\tilde{f} \in \mathcal{H}_{\sigma,\ell_0}(\tau)$ into $\psi^\ell \tilde{f} \in \mathcal{H}_{\sigma,\ell_0+\ell}(\tau)$. ■

Let ℓ_0 as in Lemma 3.4. We consider the family

$$\{\Phi^{\tau,\mathrm{SO}(n+1)}_{\rho_{\sigma,\ell_0+\ell}}\}_{\ell \in \mathbb{Z}_{\geq 0}} \tag{12}$$

of spherical functions of type τ of the strong Gelfand pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ associated with the representations $\rho_{\sigma,\ell_0+\ell}$.

With all the previous notation we enunciate the following result.

Theorem 3.6. *Let $\tau \in \widehat{\mathrm{SO}(n)}$ and $(\omega_{\sigma,R}, \mathcal{H}_{\sigma,R}) \in \widehat{\mathrm{M}(n)}$ such that $\sigma \subset \tau$ as a $\mathrm{SO}(n-1)$-module. Let $\Phi^{\tau,\mathrm{M}(n)}_{\omega_{\sigma,R}}$ be the spherical function of type τ of $(\mathrm{M}(n), \mathrm{SO}(n))$ corresponding to $\omega_{\sigma,R}$. Then the family $\{\Phi^{\tau,\mathrm{SO}(n+1)}_{\rho_{\sigma,\ell}}\}_{\ell \in \mathbb{Z}_{\geq 0}}$ of spherical functions of type τ of $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ satisfying:*

For each $f \in \mathcal{H}_{\sigma,R}(\tau)$ there exists a unique $\tilde{f} \in \mathcal{H}_{\sigma,\ell_0}(\tau)$ such that for every compact subset B of $\mathbb{R}^n$ we have

(i) $\displaystyle\lim_{\ell \to \infty}\left(\Phi^{\tau,\mathrm{SO}(n+1)}_{\rho_{\sigma,\ell_0+\ell}}(D_{\ell/R}(k,x))(\psi^\ell \tilde{f})\right)(s) = \left(\Phi^{\tau,\mathrm{M}(n)}_{\omega_{\sigma,R}}(k,x)(f)\right)(s)$

for all $s \in \mathrm{SO}(n)$ and

(ii) $$\lim_{\ell\to\infty}\left\|\left(\Phi^{\tau,\mathrm{SO}(n+1)}_{\rho_{\sigma,\ell_0+\ell}}(D_{\ell/R}(k,x))(\psi^\ell\tilde f)\right)_{|\mathrm{SO}(n)}-\Phi^{\tau,\mathrm{M}(n)}_{\omega_{\sigma,R}}(k,x)(f)\right\|_{L^2(\mathrm{SO}(n))}=0$$
where the convergence is uniform for $(k,x)\in \mathrm{SO}(n)\times B$*.*

Proof. Let $\tilde f$ be given by Lemma 3.4.

First of all, note that $\Phi^{\tau,\mathrm{SO}(n+1)}_{\rho_{\sigma,\ell_0+\ell}}(g)\in \mathrm{End}(\mathcal{H}_{\sigma,\ell_0+\ell}(\tau))$ and $\psi^\ell\tilde f\in\mathcal{H}_{\sigma,\ell_0+\ell}(\tau)$ from Lemma 3.5.

Since the convergence in (7) and (8) is uniform for $(k,x)\in \mathrm{SO}(n)\times B$, we may take a convolution (over K) with $d_\tau\overline{\chi_\tau}$ and obtain that for all $s\in K$,

$$\lim_{\ell\to\infty}\left(d_\tau\overline{\chi_\tau}*\left(\rho_{\sigma,\ell_0+\ell}(D_{\ell/R}(k,x))(\psi^\ell\tilde f)_{|K}\right)\right)(s)=\left(d_\tau\overline{\chi_\tau}*\omega_{\sigma,R}(k,x)(f)\right)(s)$$

and also,

$$\lim_{\ell\to\infty}||d_\tau\overline{\chi_\tau}*_K(\rho_{\sigma,\ell_0+\ell}(D_{\ell/R}(k,x))(\psi^\ell\tilde f))-d_\tau\overline{\chi_\tau}*(\omega_{\sigma,R}(k,x)(f))||_{L^2(\mathrm{SO}(n))}=0.$$

Since it is obvious that

$$d_\tau\overline{\chi_\tau}*\left(\rho_{\sigma,\ell_0+\ell}(g)(\psi^\ell\tilde f)_{|K}\right)=\left(d_\tau\overline{\chi_\tau}*_K\rho_{\sigma,\ell_0+\ell}(g)(\psi^\ell\tilde f)\right)_{|K}\qquad\forall g\in\mathrm{SO}(n+1),$$

then for each $g\in\mathrm{SO}(n+1)$ and for all $s\in K$,

$$\begin{aligned}\left(d_\tau\overline{\chi_\tau}*_K\rho_{\sigma,\ell_0+\ell}(g)(\psi^\ell\tilde f)\right)(s)&=d_\tau\int_K\overline{\chi_\tau}(k)\left(\rho_{\sigma,\ell_0+\ell}(g)(\psi^\ell\tilde f)\right)(k^{-1}s)dk\\&=d_\tau\int_K\overline{\chi_\tau}(k)L_k\left(\rho_{\sigma,\ell_0+l}(g)(\psi^\ell\tilde f)\right)(s)dk\\&=d_\tau\int_K\overline{\chi_\tau}(k)\rho_{\sigma,\ell_0+\ell}(k)\left(\rho_{\sigma,\ell_0+\ell}(g)(\psi^\ell\tilde f)\right)(s)dk\\&=P^\tau_{\rho_{\sigma,\ell_0+\ell}}\left(\rho_{\sigma,\ell_0+\ell}(g)(\psi^\ell\tilde f)\right)(s)=\left(\Phi^{\tau,\mathrm{SO}(n+1)}_{\rho_{\sigma,\ell_0+\ell}}(g)(\psi^\ell\tilde f)\right)(s).\end{aligned}$$

Similarly, for each $(k,x)\in\mathrm{M}(n)$ and for all $s\in K$,

$$\left(d_\tau\overline{\chi_\tau}*\omega_{\sigma,R}(k,x)(f)\right)(s)=P^\tau_{\omega_{\sigma,R}}\left(\omega_{\sigma,R}(k,x)(f)\right)(s)=\left(\Phi^{\tau,\mathrm{M}(n)}_{\omega_{\sigma,R}}(k,x)(f)\right)(s).\qquad\blacksquare$$

We would like to end this paper, as in the last remark given by Dooley and Rice in [7], saying that the harmonic analysis (scalar-valued and vector or matrix-valued) on $\mathrm{M}(n)$ can be obtained as a limit (in an appropriate sense) of the harmonic analysis on $\mathrm{SO}(n+1)$. Indeed, consider for each $\ell\in\mathbb{Z}_{\geq 0}$ the map

$$Res_{\ell_0+\ell}:\mathcal{H}_{\sigma,\ell_0+\ell}(\tau)\longrightarrow\mathcal{H}_{\sigma,R}(\tau),\quad h\longmapsto h_{|K}$$

and the map $$\mathcal{H}_{\sigma,R}(\tau)\longrightarrow\mathcal{H}_{\sigma,\ell_0+\ell}(\tau),\quad f\longmapsto\psi^\ell\tilde f,$$

where $\tilde f$ is as in Lemma 3.4. These two maps are inverses of one another.

From the previous theorem for each $f\in\mathcal{H}_{\sigma,R}(\tau)$ it follows that

$$\lim_{\ell\to\infty}\left\|\left[Res_{\ell_0+\ell}\circ\Phi^{\tau,\mathrm{SO}(n+1)}_{\rho_{\sigma,\ell_0+\ell}}(D_{\ell/R}(\cdot))\circ(Res_{\ell_0+\ell})^{-1}-\Phi^{\tau,\mathrm{M}(n)}_{\omega_{\sigma,R}}(\cdot)\right](f)\right\|_{L^2(\mathrm{SO}(n))}=0,\quad(13)$$

where the convergence is uniform on compact sets of $\mathrm{M}(n)$. As we saw in the Remark 2.2, for all $\ell \in \mathbb{Z}_{\geq 0}$, the functions $\Phi^{\tau,\mathrm{SO}(n+1)}_{\rho_{\sigma,\ell_0+\ell}}$ and $Res_{\ell_0+\ell} \circ [\Phi^{\tau,\mathrm{SO}(n+1)}_{\rho_{\sigma,\ell_0+\ell}}(\cdot)] \circ (Res_{\ell_0+\ell})^{-1}$ represent the same spherical function. Now, using the isomorphism $\mathcal{H}_{\sigma,R}(\tau) \simeq V_\tau$ and again the Remark 2.2, the limit given in (13) can be rewritten as

$$\lim_{\ell\to\infty} \left\| \left[\Phi^{\tau,\mathrm{SO}(n+1)}_{\rho_{\sigma,\ell_0+\ell}}(D_{\ell/R}(\cdot)) - \Phi^{\tau,\mathrm{M}(n)}_{\omega_{\sigma,R}}(\cdot) \right] (v) \right\|_{V_\tau} = 0 \qquad \text{for all } v \in V_\tau, \tag{14}$$

where $\|\cdot\|_{V_\tau}$ is a norm on the finite-dimensional vector space V_τ and the limit is uniform on compact sets of $\mathrm{M}(n)$.

Therefore we have proved the following theorem.

Theorem 3.7. *Let* $(\tau, V_\tau) \in \widehat{\mathrm{SO}(n)}$ *and let* $\Phi^{\tau,\mathrm{M}(n)}$ *be a spherical function of type* τ *of the strong Gelfand pair* $(\mathrm{M}(n), \mathrm{SO}(n))$*. There exists a sequence* $\{\Phi^{\tau,\mathrm{SO}(n+1)}_\ell\}_{\ell\in\mathbb{Z}_{\geq 0}}$ *of spherical functions of type* τ *of the strong Gelfand pair* $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ *with*

$$\lim_{\ell\to\infty} \Phi^{\tau,\mathrm{SO}(n+1)}_\ell(D_\ell(k,x)) = \Phi^{\tau,\mathrm{M}(n)}(k,x),$$

where the convergence is pointwise on V_τ *and it is uniform on compact sets of* $\mathrm{M}(n)$*, where the family* $\{D_\ell\}_{\ell\in\mathbb{Z}_{\geq 0}}$ *is from contractions of* $\mathrm{SO}(n+1)$ *to* $\mathrm{M}(n)$*.*

Remark 3.8. We emphasize that the above result is independent of the model chosen for the representations that parametrize the spherical functions.

Remark 3.9. The family of contractions $\{D_\ell\}_{\ell\in\mathbb{Z}_{\geq 0}}$ does not depend on the spherical function of $\mathrm{M}(n)$. The contraction mappings D_α as defined in Section 2.4 are defined without any reference to a given spherical function on $\mathrm{M}(n)$. They are the contraction maps introduced in [7, 8].

4. The approximation theorem in the dual case

In this paragraph we will consider first a general framework. Let G be connected Lie group with Lie algebra $\mathfrak{g}$ and K be a closed subgroup with Lie algebra $\mathfrak{k}$. The coset space G/K is called reductive if $\mathfrak{k}$ admits an $\mathrm{Ad}_G(K)$-invariant complement $\mathfrak{p}$ in $\mathfrak{g}$. In this case it can be form the semidirect product $K \ltimes \mathfrak{p}$ with respect to the adjoint action of K on $\mathfrak{p}$. We will restrict ourselves to the case where G is semisimple with finite center. In particular, let θ be an analytic involution on G such that (G,K) is a Riemannian symmetric pair, that is, K is contained in the fixed point set K_θ of the involution θ, it contains the connected component of the identity and $\mathrm{Ad}_G(K)$ is compact. The subalgebra $\mathfrak{k}$ is the $+1$ eigenspace of $d\theta_e$ and naturally we choose $\mathfrak{p}$ as the -1 eigenspace. Furthermore, we will just consider G non-compact. In this case K is compact and connected [11, p. 252] and $d\theta_e$ is a Cartan involution, so $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called a Cartan decomposition [11, p. 182]. The semidirect product $K \ltimes \mathfrak{p}$ is called the *Cartan motion group* associated to the pair (G,K).

The unitary dual $\widehat{K \ltimes \mathfrak{p}}$ can be described as the one given in Section 2.3. First one must fix a character of $\mathfrak{p}$. Any character of $\mathfrak{p}$ can be uniquely expressed as $e^{i\phi(x)}$ for a linear functional $\phi \in \mathfrak{p}^*$. Then one must consider

$$K_\phi := \left\{ k \in K |\ e^{i\phi(\mathrm{Ad}(k^{-1})x)} = e^{i\phi(x)}\ \forall x \in \mathfrak{p} \right\}$$

and $(\sigma, H_\sigma) \in \widehat{K_\phi}$. After that we get an irreducible unitary representation $\omega_{\sigma,\phi}$ of $K \ltimes \mathfrak{p}$ inducing $\sigma \otimes e^{i\phi(\cdot)}$ from $K_\phi \ltimes \mathfrak{p}$ to $K \ltimes \mathfrak{p}$. By definition $\omega_{\sigma,\phi}$ acts by left translations on a space of functions $f : K \ltimes \mathfrak{p} \longrightarrow H_\sigma$ satisfying

$$f(gxm) = e^{-i\phi(x)}\sigma(m)^{-1}f(g) \qquad \forall x \in \mathfrak{p},\ m \in K_\phi \text{ and } g \in K \ltimes \mathfrak{p}.$$

Consequently,

$$f(xk) = f(k\,\mathrm{Ad}(k^{-1})x) = e^{-i\phi(\mathrm{Ad}(k^{-1})x)}f(k) \qquad \forall x \in \mathfrak{p},\ k \in K,$$

so any such f is completely determined by its restriction to K. Therefore for the representation $\omega_{\sigma,\phi}$ we can consider only those functions whose restrictions to K lie on $L^2(K, H_\sigma)$. This space is the closed subspace $H_{\omega_{\sigma,\phi}}$ of $L^2(K, H_\sigma)$ given by

$$H_{\omega_{\sigma,\phi}} := \left\{ f \in L^2(K, H_\sigma) |\ f(km) = \sigma(m)^{-1}f(k)\ \forall m \in K_\phi, k \in K \right\}$$

and $\omega_{\sigma,\phi}$ acts on $H_{\omega_{\sigma,\phi}}$ by $\left(\omega_{\sigma,\phi}(k,x)f\right)(k_0) := e^{i\phi(\mathrm{Ad}(k_0^{-1})x)}f(k^{-1}k_0)$.
Every irreducible unitary representation of $K \ltimes \mathfrak{p}$ occurs in this way and two irreducible unitary representations ω_{σ_1,ϕ_1} and ω_{σ_2,ϕ_2} are unitarily equivalent if and only if

- ϕ_1 and ϕ_2 lie in the same coadjoint orbit of K and
- σ_1 and σ_2 are unitarily equivalent.

Because K is compact we can endow $\mathfrak{p}$ with an $\mathrm{Ad}(K)$-invariant inner product $\langle \cdot, \cdot \rangle$ (for example, the Killing form restricted to $\mathfrak{p}$) and via $\langle \cdot, \cdot \rangle$ we identify $\mathfrak{p}$ with p^* and the adjoint with the coadjoint action of K. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Every adjoint orbit of K in $\mathfrak{p}$ intersects $\mathfrak{a}$ ([11, p. 247]). Hence every irreducible unitary representation of $K \ltimes \mathfrak{p}$ has the form $\omega_{\sigma,\phi}$ with $\phi(x) = \langle H, x \rangle$ for some $H \in \mathfrak{a}$, that is, we are allowed to suppose $\phi \in \mathfrak{a}^*$. Therefore K_ϕ coincides with the stabilizer of H under the adjoint action of K. Let M be the centralizer of $\mathfrak{a}$ in K. We say that $\omega_{\sigma,\phi} \in \widehat{K \ltimes \mathfrak{p}}$ is *generic* if $K_\phi = M$. Since the set of non-generic irreducible unitary representations of $K \ltimes \mathfrak{p}$ has zero Plancherel measure, we shall be concerned with the generic cases. That is we will consider

$$\omega_{\sigma,\phi} = \mathrm{Ind}_{M \ltimes \mathfrak{p}}^{K \ltimes \mathfrak{p}}(\sigma \otimes e^{i\phi(\cdot)}) \qquad (\sigma \in \widehat{M}, \phi \in \mathfrak{a}^*). \tag{15}$$

On the other hand, let $G = KAN$ be the Iwasawa decomposition of G, where $A := \exp_G(\mathfrak{a})$. Let $(\sigma, H_\sigma) \in \widehat{M}$. Let $\gamma \in \mathfrak{a}^* \otimes \mathbb{C}$ such that $\gamma = \phi + i\nu$ where $\phi \in \mathfrak{a}^*$ and $\nu \in \mathfrak{a}^*$ is the particular linear map $\nu := \frac{1}{2}\sum_{r \in P^+} c_r r$ where P^+ is the set of positive restricted roots and c_r is the multiplicity of the root r. Let 1_N denote the trivial representation of N. A principal series representation $\rho_{\sigma,\phi}$ of G can be given by inducing $\gamma \otimes \sigma \otimes 1_N$ from MAN to $KAN = G$, that is,

$$\rho_{\sigma,\phi} = \mathrm{Ind}_{MAN}^{G}(\gamma \otimes \sigma \otimes 1_N) \qquad (\sigma \in \widehat{M}, \phi \in \mathfrak{a}^*). \tag{16}$$

As such, it is realised on a space of functions $F : G \longrightarrow H_\sigma$ satisfying

$$f(gman) = e^{-i\gamma(\log(a))}\sigma(m)^{-1}f(g) \qquad \forall g \in G,\ man \in MAN. \tag{17}$$

By the Iwasawa decomposition such functions are clearly determined by their restrictions to K. A principal series representation gives rise to a unitary representation when its representation space $H_{\rho_{\sigma,\phi}}$ consist of functions satisfying (17) and whose restrictions to K lie in $L^2(K,H_\sigma)$. These restrictions comprise the subspace of $L^2(K,H_\sigma)$ whose functions f satisfy

$$f(km)=\sigma(m)^{-1}f(k) \qquad \forall k\in K, m\in M.$$

Note that $H_{\omega_{\sigma,\phi}}$ coincides with $\left(H_{\rho_{\sigma,\phi}}\right)_{|K}$.

Given any generic irreducible unitary representation $\omega_{\sigma,\phi}$ of $K\ltimes\mathfrak{p}$, we can associate the sequence $\{\rho_{\sigma,\ell\phi}\}_{\ell=1}^{\infty}$ of unitary principal series representations of G. As in (6) we consider the contraction maps $\{D_\beta\}_{\beta\in\mathbb{R}_{>0}}$

$$D_\beta: K\ltimes\mathfrak{p}\longrightarrow G, \quad D_\beta(k,x):=\exp_G(\frac{1}{\beta}x)\ k. \tag{18}$$

As in Section 2.2.2 we consider the special function

$$s_\phi: G\longrightarrow\mathbb{C}, \quad s_\phi(kan):=e^{-i\phi(\log(a))}, \tag{19}$$

which is K-invariant and has value 1 on K. We have that, if $f\in H_{\rho_{\sigma,\ell\phi}}$, then $s_\phi(f)\in H_{\rho_{\sigma,(\ell+1)\phi}}$ and $s_\phi(f)$ has the same restriction to K as f. The following result, due to Dooley and Rice, shows how the sequence $\{\rho_{\sigma,\ell\phi}\}_{\ell=1}^{\infty}$ approximates $\omega_{\sigma,\phi}$.

Theorem 4.1. [8, Theorem 1 and Corollary (4.4)] *For all* $(k,x)\in K\ltimes\mathfrak{p}$ *and* $F\in H_{\rho_{\sigma,\phi}}$

$$\lim_{\ell\to\infty}\left\|\left(\rho_{\sigma,\ell\phi}(D_\ell(k,x))(s_\phi^\ell F)\right)_{|K}-\omega_{\sigma,\phi}(k,x)(F_{|K})\right\|_{L^2(K,H_\sigma)}=0. \tag{20}$$

Moreover, if F is a smooth function, the convergence is uniform on compact subsets of $K\ltimes\mathfrak{p}$.

Let $\tau\in\widehat{K}$. It follows from the Frobenius reciprocity theorem that $\tau\subset(\omega_{\sigma,\phi})_{|K}$ and that $\tau\subset(\rho_{\sigma,\ell\phi})_{|K}$ if and only if $\sigma\subset\tau_{|M}$. In particular,

$$m(\tau,\omega_{\sigma,\phi})=m(\sigma,\tau)=m(\tau,\rho_{\sigma,\phi}). \tag{21}$$

We fix $\omega_{\sigma,\phi}\in\widehat{K\ltimes\mathfrak{p}}$ such that τ is a K-submodule of $\omega_{\sigma,\phi}$.

Consider the restriction operator $Res_{\ell\phi}(F):=F_{|K}$ for all $F\in H_{\rho_{\sigma,\ell\phi}}$. Since the action of $\rho_{\sigma,\ell\phi}$ is by left translations it is obvious that $Res_{\ell\phi}$ intertwines $H_{\rho_{\sigma,\ell\phi}}$ and $H_{\omega_{\sigma,\phi}}$ as K-modules.

Moreover, $Res_{\ell\phi}$ sends $H_{\rho_{\sigma,\ell\phi}}(\tau)$ to $H_{\omega_{\sigma,\phi}}(\tau)$. Apart from that, observe that the multiplication by the function s_ϕ is an intertwining operator between $H_{\rho_{\sigma,\ell\phi}}$ and $H_{\rho_{\sigma,(\ell+1)\phi}}$ as K-modules (for all $\ell\in\mathbb{N}$).

Now, let $f\in H_{\omega_{\sigma,\phi}}$, we extend it to G by

$$F(g)=F(k_g a_g n_g):=e^{-i\gamma(\log(a_g))}f(k_g), \tag{22}$$

where $g=k_g a_g n_g$ with $k_g\in K$, $a_g\in A$ and $n_g\in N$ is the Iwasawa decomposition of $g\in G$. The inverse of the restriction map defined previously is $Res_{\ell\phi}^{-1}(f):=(s_\phi)^\ell F$ for all $f\in H_{\omega_{\sigma,\phi}}(\tau)$ where F is defined as (22).

With all this in mind, Theorem 4.1 can be rewritten in the following way: For all $(k,x)\in K\ltimes\mathfrak{p}$ and $f\in H_{\omega_{\sigma,\phi}}(\tau)$,

$$\lim_{\ell\to\infty}\left\|\left(Res_{\ell\phi}\circ\rho_{\sigma,\ell\phi}\left(D_\ell(k,x)\right)\circ Res_{\ell\phi}^{-1}-\omega_{\sigma,\phi}(k,x)\right)(f)\right\|_{L^2(K,H_\sigma)}=0. \tag{23}$$

Finally, by (1), the projections $P^{\tau}_{\omega_{\sigma,\phi}}$ and $P^{\tau}_{\rho_{\sigma,\ell\phi}}$ are given by the same formula, i.e., by the convolution on K with $d_\tau\overline{\chi_\tau}$. Moreover, they are continuous operators. Therefore from (23) we get the asymptotic formula

$$\lim_{\ell\to\infty}\left\|\left(P^{\tau}_{\rho_{\sigma,\ell\phi}}\circ Res_{\ell\phi}\circ\rho_{\sigma,\ell\phi}\left(D_\ell(k,x)\right)\circ Res^{-1}_{\ell\phi}-P^{\tau}_{\omega_{\sigma,\phi}}\circ\omega_{\sigma,\phi}(k,x)\right)(f)\right\|_{L^2(K,H_\sigma)}=0. \quad (24)$$

Proposition 4.2. *Let G be a connected, non-compact semisimple Lie group and K be a closed subgroup of G such that (G,K) is a Riemannian symmetric pair. Let $K\ltimes\mathfrak{p}$ be the Cartan motion group associated to (G,K) and let $\tau\in\widehat{K}$. The triple (G,K,τ) is commutative if and only if $(K\ltimes\mathfrak{p},K,\tau)$ is commutative. In particular, (G,K) is a strong Gelfand pair if and only if $(K\ltimes\mathfrak{p},K)$ is a strong Gelfand pair.*

Proof. The Plancherel measure of $K\ltimes\mathfrak{p}$ is concentrated on the set of generic irreducible unitary representations of $K\ltimes\mathfrak{p}$. Respectively, the Plancherel measure of G is concentrated on the set of principal series representations. From [5, Theorem 3], $(K\ltimes\mathfrak{p},K,\tau)$ is a commutative triple if and only if $m(\tau,\omega)\leq 1$ for all ω in the subset of $\widehat{K\ltimes\mathfrak{p}}$ which has non-zero Plancherel measure. (This result is bases on the ideas given in [1] for the case of a Gelfand pair). So we take arbitrary generic and principal series representations $\omega_{\sigma,\phi}\in\widehat{K\ltimes\mathfrak{p}}$ and $\rho_{\sigma,\phi}\in\widehat{G}$ as in (15) and (16) respectively, for $\sigma\in\widehat{M}$ and $\phi\in\mathfrak{a}^*$. By (21), $m(\tau,\omega_{\sigma,\phi})=m(\tau,\rho_{\sigma,\phi})$ and the conclusion of this proposition follows immediately. ■

Theorem 4.3. *Let G be a connected, non compact semisimple Lie group and K be a maximal compact subgroup of G such that (G,K) is a Riemannian symmetric pair. Let $K\ltimes\mathfrak{p}$ be the Cartan motion group associated to (G,K) and let $(\tau,V_\tau)\in\widehat{K}$ such that $(K\ltimes\mathfrak{p},K,\tau)$ is a commutative triple. Let $\Phi^{\tau}_{\omega_{\sigma,\phi}}:K\ltimes\mathfrak{p}\longrightarrow\mathrm{End}(V_\tau)$ be the spherical function of type τ corresponding to $\omega_{\sigma,\phi}$. Then there exists a family $\{\Phi^{\tau}_{\rho_{\sigma,\ell\phi}}\}_{\ell\in\mathbb{Z}_{\geq 0}}$ where $\Phi^{\tau}_{\rho_{\sigma,\ell\phi}}:G\longrightarrow\mathrm{End}(V_\tau)$ is a spherical function of type τ corresponding to $\rho_{\sigma,\ell\phi}$ and such that for each $(k,x)\in K\ltimes\mathfrak{p}$*

$$\lim_{\ell\to\infty}\Phi^{\tau}_{\omega_{\sigma,\ell\phi}}(D_\ell(k,x))=\Phi^{\tau}_{\rho_{\sigma,\ell\phi}}(k,x),$$

where the convergence is pointwise on V_τ.

Proof. From Proposition 4.2, (G,K,τ) is also a commutative triple and the proof follows from (24) and Remark 2.2. ■

In particular, if we consider $G=\mathrm{SO}_0(n,1)$ the Lorentz group and $K=\mathrm{SO}(n)$, then $K\ltimes\mathfrak{p}=\mathrm{M}(n)$. The pair $(\mathrm{SO}_0(n,1),\mathrm{SO}(n))$ is a strong Gelfand pair and analogously to the case $(SO(n+1),SO(n))$ we have the following result.

Corollary 4.4. *Let $(\tau,V_\tau)\in\widehat{\mathrm{SO}(n)}$ and let $\Phi^{\tau,\mathrm{M}(n)}$ be a spherical function of type τ of the strong Gelfand pair $(\mathrm{M}(n),\mathrm{SO}(n))$. There exists a sequence $\{\Phi^{\tau,\mathrm{SO}_0(n,1)}_\ell\}_{\ell\in\mathbb{Z}_{\geq 0}}$ of spherical functions of type τ of the strong Gelfand pair $(\mathrm{SO}_0(n,1),\mathrm{SO}(n))$ such that for all $(k,x)\in\mathrm{M}(n)$*

$$\lim_{\ell\to\infty}\Phi^{\tau,\mathrm{SO}_0(n,1)}_\ell(D_\ell(k,x))=\Phi^{\tau,\mathrm{M}(n)}(k,x),$$

where the convergence is pointwise on V_τ *and where* $\{D_\ell\}_{\ell\in\mathbb{Z}_{\geq 0}}$ *is a family of contraction maps between* $\mathrm{M}(n)$ *and* $\mathrm{SO}_0(n,1)$.

Both the referee and we guess it should be possible to prove analogous results for the contraction of (G,K) to $(K \ltimes \mathfrak{p}, K, \tau)$ for any Riemannian symmetric pair of the compact type (G,K). That is, to prove an analogous of Theorem 4.3 for the case of the compact type and obtain the Theorem 3.7 as a corollary similar to the Corollary 4.4 for the case of the non-compact type. We consider that this work might be more technical. However, we suggest that it will be possible to achieve it using the results of [6, 8].

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