Smooth Duality and Co-Contra Correspondence

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Communicated by K.-H. Neeb

Abstract. The aim of this paper is to explain how to get a complex of smooth representations out of the dual vector space to a smooth representation of a p-adic Lie group, in natural characteristic. The construction does not depend on any finiteness/admissibility assumptions. Imposing such an assumption, one obtains an involutive duality on the derived category of complexes of smooth modules with admissible cohomology modules. The paper can serve as an introduction to the results about representations of locally profinite groups contained in the author's monograph on semi-infinite homological algebra [L. Positselski: *Homological Algebra of Semimodules and Semi-contramodules: Semi-infinite Homological Algebra of Associative Algebraic Structures*, Monografie Matematyczne IMPAN, 70, Birkhäuser/Springer, Basel (2010)].

Mathematics Subject Classification: 22E50, 22D35, 20E18, 16T15, 18E30.

Key Words: Locally profinite groups, p-adic Lie groups, modular representation theory, coalgebras and semialgebras, comodules and contramodules, admissible representations, derived categories, involutive triangulated duality.

1. Introduction

1.1. Dualizing modules produces modules. If M is an B-A-bimodule and V is a left B-module, then the group of all B-linear maps $\mathrm{Hom}_B(M,V)$ is naturally a left A-module. When the ring A is isomorphic to its opposite ring A^{op} (say, A is commutative or endowed with a Hopf algebra antipode), this means that we started from a right A-module and came back to a right A-module.

The situation gets more complicated when one starts from a module of a particular class, like a torsion, discrete, or smooth module, and wants to obtain a module from the same class after the dualization. If $\mathcal{M}$ is a torsion abelian group, then the Pontryagin dual group $\mathcal{M}^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, \mathbb{Q}/\mathbb{Z})$ is no longer torsion, generally speaking. There is, however, a covariant derived functor producing a *two-term complex* of torsion abelian groups out of the group $\mathcal{M}^\vee$ (see, e. g., [14, Sections 1.4–1.6] and the introduction to [17]). This construction does not depend on the topology on the group $\mathcal{M}^\vee$, but only on its abelian group structure.

The general philosophy is that dualizing comodules produces contramodules, and one can get back from a contramodule to a complex of comodules using the *de-*

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rived comodule-contramodule correspondence constructions [11, 12]. Various kinds of torsion, discrete, or smooth modules are viewed as species of comodules, in one or another sense [14, Sections 0.1–0.2]. Oversimplifying matters a bit, one can say that for every abelian category of torsion, discrete, or smooth modules there is a much less familiar, but no less interesting, related abelian category of contramodules. To be more careful, one has to distinguish between left and right co/contramodules.

In the simplest example of the coalgebra $\mathcal{C}$ dual to the algebra $\mathcal{C}^\vee = k[[t]]$ of formal power series in one variable over a field k, the $\mathcal{C}$-comodules are just the $k[t]$-modules with a locally nilpotent action of t. The $\mathcal{C}$-contramodules also form a full subcategory in $k[t]$-modules; it is described in [11, Section A.1.1] or [14, Sections 1.3 and 1.5–1.6].

For any coalgebra $\mathcal{C}$ over a field k, the dual vector space $\mathcal{N}^\vee$ to a right $\mathcal{C}$-comodule $\mathcal{N}$ is a left $\mathcal{C}$-contramodule. So is the vector space $\mathrm{Hom}_k(\mathcal{N}, V)$ for an arbitrary k-vector space V. Applying the co-contra correspondence, one obtains a nonpositively cohomologically graded complex of left $\mathcal{C}$-comodules $\mathbb{L}\Phi_\mathcal{C}(\mathrm{Hom}_k(\mathcal{N}, V))$ out of $\mathrm{Hom}_k(\mathcal{N}, V)$; in particular, the complex $\mathbb{L}\Phi_\mathcal{C}(\mathcal{N}^\vee)$. In the case of $\mathcal{C}^\vee = k[[t]]$, this will be a two-term complex; in the case of $\mathcal{C}^\vee = k[[t_1, \ldots, t_n]]$, an $(n+1)$-term complex. When $n = \infty$, it will be sometimes an acyclic complex! (See [11, Sections 0.2.6–0.2.7].)

1.2.   In this paper, we are interested in representations of locally profinite groups G. To a profinite group H and a field k, one assigns the coalgebra $\mathcal{C} = k(H)$ of locally constant k-valued functions on H. Discrete H-modules over k are then the same thing as $\mathcal{C}$-comodules. To a locally profinite group G with a compact open subgroup $H \subset G$, one assigns a $\mathcal{C}$-*semialgebra* $\boldsymbol{\mathcal{S}} = k(G)$ of compactly supported locally constant functions on G. Smooth G-modules over k are the same thing as $\boldsymbol{\mathcal{S}}$-*semimodules* [11, Section E.1], [14, Example 2.6].

More specifically, a $\mathcal{C}$-semialgebra is an associative algebra object in the tensor category of $\mathcal{C}$-$\mathcal{C}$-bicomodules with respect to the *cotensor product* operation $\square_\mathcal{C}$. In other words, a semialgebra $\boldsymbol{\mathcal{S}}$ over $\mathcal{C}$ is a $\mathcal{C}$-$\mathcal{C}$-bicomodule endowed with a *semimultiplication* morphism $\boldsymbol{\mathcal{S}} \square_\mathcal{C} \boldsymbol{\mathcal{S}} \longrightarrow \boldsymbol{\mathcal{S}}$ and a *semiunit* morphism $\mathcal{C} \longrightarrow \boldsymbol{\mathcal{S}}$ satisfying the conventional axioms. In particular, given a locally profinite group G with a compact open subgroup H, one can decompose the multiplication map $G \times G \longrightarrow G$ into the composition $G \times G \longrightarrow G \times_H G \longrightarrow G$, where $G \times_H G$ is the quotient of $G \times G$ by the equivalence relation $(g'h, g'') \sim (g', hg'')$, where g', $g'' \in G$ and $h \in H$. Then the pullback of compactly supported locally constant functions with respect to the quotient map $G \times G \longrightarrow G \times_H G$ identifies $k(G \times_H G)$ with $\boldsymbol{\mathcal{S}} \square_\mathcal{C} \boldsymbol{\mathcal{S}} \subset \boldsymbol{\mathcal{S}} \otimes_k \boldsymbol{\mathcal{S}}$, and the pushforward of such functions with respect to the multiplication map $G \times_H G \longrightarrow G$ provides the semimultiplication morphism $\boldsymbol{\mathcal{S}} \square_\mathcal{C} \boldsymbol{\mathcal{S}} \longrightarrow \boldsymbol{\mathcal{S}}$.

1.3.   The opposite category to the category of k-vector spaces is identified with the category of *linearly compact* or *pseudo-compact* topological vector spaces. The identification is provided by the functor assigning to a vector space V is dual vector space $V^\vee$; the inverse functor assigns to a topological vector space W the vector space of all continuous linear functions $W \longrightarrow k$. Similarly, the opposite category to the category comod–$\mathcal{C}$ of right comodules over a coalgebra $\mathcal{C}$ is identified with the category $\mathcal{C}^\vee$–pscomp of *pseudo-compact left modules* over the *pseudo-compact*

algebra $\mathcal{C}^\vee$. In particular, when $\mathcal{C} = k(H)$ for a profinite group H, one has

$$\mathcal{C}^\vee = k[[H]] = \varprojlim_{U \subset H} k[H/U],$$

where the projective limit of the group algebras $k[H/U]$ is taken over all the open normal subgroups $U \subset H$.

One can also consider the conventional category of left modules $\mathcal{C}^\vee$–mod over the algebra $\mathcal{C}^\vee$ (viewed as an abstract k-algebra with the topology forgotten). Then there is the forgetful (or in other terms, the dualization) functor

$$(\mathsf{comod}\text{–}\mathcal{C})^{\mathsf{op}} \simeq \mathcal{C}^\vee\text{–}\mathsf{pscomp} \longrightarrow \mathcal{C}^\vee\text{–}\mathsf{mod},$$

which can be also constructed as the composition

$$(\mathsf{comod}\text{–}\mathcal{C})^{\mathsf{op}} = (\mathsf{discr}\text{–}\mathcal{C}^\vee)^{\mathsf{op}} \longrightarrow (\mathsf{mod}\text{–}\mathcal{C}^\vee)^{\mathsf{op}} \longrightarrow \mathcal{C}^\vee\text{–}\mathsf{mod}$$

of the fully faithful embedding of comod–$\mathcal{C}$ as the full subcategory of discrete right modules in mod–$\mathcal{C}^\vee$ and the dualization functor $N \longmapsto N^\vee \colon (\mathsf{mod}\text{–}\mathcal{C}^\vee)^{\mathsf{op}} \longrightarrow \mathcal{C}^\vee$–$\mathsf{mod}$.

The category of left $\mathcal{C}$-contramodules $\mathcal{C}$–contra stands "in between" the category of pseudo-compact left $\mathcal{C}^\vee$-modules $\mathcal{C}^\vee$–pscomp and the category of abstract left $\mathcal{C}^\vee$-modules $\mathcal{C}^\vee$–mod. In other words, the forgetful functor $\mathcal{C}^\vee$–$\mathsf{pscomp} \longrightarrow \mathcal{C}^\vee$–$\mathsf{mod}$ factorizes naturally into the composition

$$\mathcal{C}^\vee\text{–}\mathsf{pscomp} \longrightarrow \mathcal{C}\text{–}\mathsf{contra} \longrightarrow \mathcal{C}^\vee\text{–}\mathsf{mod}.$$

All the three categories $\mathcal{C}^\vee$–pscomp, $\mathcal{C}$–contra, and $\mathcal{C}^\vee$–mod are abelian, and both the natural (forgetful) functors between them are exact.

A $\mathcal{C}$-contramodule does not remember the topology of a pseudo-compact $\mathcal{C}^\vee$-module; still it carries more structure than that of an abstract $\mathcal{C}^\vee$-module. This intermediate structure is that of *infinite summation operations* with zero-converging families of coefficients in $\mathcal{C}^\vee$ [14, Sections 1.1, 2.1, and 2.3]. Sometimes (e. g., when $\mathcal{C}^\vee$ is a quotient algebra of the algebra of formal power series in a finite set of variables), this structure can be uniquely recovered from the $\mathcal{C}^\vee$-module structure, so $\mathcal{C}$–contra is a full subcategory in $\mathcal{C}^\vee$–mod [11, Remark A.1.1], [16, Theorem B.1.1], [13, Theorem C.5.1], [14, Sections 1.6 and 2.2]. Generally speaking, it can not and it is not.

1.4. There are enough injective objects in the abelian category $\mathcal{C}$–comod and enough projective objects in the abelian category $\mathcal{C}$–contra. Moreover, the additive categories of injective left $\mathcal{C}$-comodules and projective left $\mathcal{C}$-contramodules are naturally equivalent [14, Section 1.2], [11, Section 0.2.6]. Generally speaking, this allows to construct a natural triangulated equivalence between the *coderived category* of left $\mathcal{C}$-comodules and the *contraderived category* of left $\mathcal{C}$-contramodules [11, Section 0.2.6], [12, Sections 5.1–5.2],

$$\mathbb{R}\Psi_\mathcal{C} \colon \mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}) \colon \mathbb{L}\Phi_\mathcal{C}. \tag{1}$$

This equivalence of exotic derived categories can sometimes assign an acyclic complex to an irreducible co/contramodule [11, Section 0.2.7].

When the coalgebra $\mathcal{C}$ has finite homological dimension n, there is no difference between the co/contraderived and the conventional derived categories, that is

$$\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–comod}) = \mathsf{D}(\mathcal{C}\text{–comod}) \quad \text{and} \quad \mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–contra}) = \mathsf{D}(\mathcal{C}\text{–contra})$$

(see [12, Section 4.5] and [11, Remark 2.1]). Hence the triangulated equivalence

$$\mathbb{R}\Psi_{\mathcal{C}} : \mathsf{D}(\mathcal{C}\text{–comod}) \simeq \mathsf{D}(\mathcal{C}\text{–contra}) : \mathbb{L}\Phi_{\mathcal{C}}. \tag{2}$$

The left derived functor $\mathbb{L}\Phi_{\mathcal{C}}$ takes a $\mathcal{C}$-contramodule to an (at most) $(n+1)$-term complex of $\mathcal{C}$-comodules, and the right derived functor $\mathbb{R}\Psi_{\mathcal{C}}$ takes a $\mathcal{C}$-comodule to an $(n+1)$-term complex of $\mathcal{C}$-contramodules.

1.5.   Given a k-coalgebra $\mathcal{C}$ and a $\mathcal{C}$-semialgebra $\mathcal{S}$, for any right $\mathcal{S}$-semimodule $\mathcal{N}$ and a k-vector space V the vector space $\operatorname{Hom}_k(\mathcal{N}, V)$ has a natural structure of left $\mathcal{S}$-*semicontramodule* [11, Sections 0.3.2 and 0.3.5], [14, Section 2.6]. In particular, given a locally profinite group G and a field k, for any smooth G-module $\mathcal{N}$ over k the vector space $\operatorname{Hom}_k(\mathcal{N}, V)$ has a natural structure of G-*contramodule* over k.

The G-contramodules over k are the same thing as the $\mathcal{S}$-semicontramodules for the semialgebra $\mathcal{S} = k(G)$ over the coalgebra $\mathcal{C} = k(H)$. One has to be careful: the semialgebra $\mathcal{S}$ depends on the choice of a compact open subgroup $H \subset G$ (because the coalgebra $\mathcal{C}$ does), but the notion of a G-contramodule over k does not depend on this choice [11, Section E.1], [14, Example 2.6]. One has to choose a compact open subgroup in G if one wants to interpret smooth G-modules as semimodules and G-contramodules as semicontramodules over some semialgebra.

In particular, the H-contramodules over k are the same thing as the $\mathcal{C}$-contramodules for the coalgebra $\mathcal{C} = k(H)$.

1.6.   For any $\mathcal{C}$-semialgebra $\mathcal{S}$ (satisfying certain adjustness conditions which always hold for semialgebras arising from locally profinite groups), there is a natural triangulated equivalence between certain exotic derived categories of the abelian categories of left $\mathcal{S}$-semimodules and left $\mathcal{S}$-semicontramodules. These are called the *semiderived categories* and denoted by $\mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–simod})$ and $\mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–sicntr})$; the natural equivalence between them is called the *derived semimodule-semicontramodule correspondence* [11, Sections 0.3.7 and 6.3] (cf. [14, Section 3.5]).

The triangulated equivalence $\mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–simod}) \simeq \mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–sicntr})$ forms a commutative diagram with the triangulated equivalence (1) and the forgetful functors from semi-(contra)-modules to co(ntra)-modules,

$$
\begin{array}{ccc}
\mathbb{R}\Psi_{\mathcal{S}} : \mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–simod}) & \!\!=\!\!\!=\!\!\!=\!\! & \mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–sicntr}) : \mathbb{L}\Phi_{\mathcal{S}} \\
\downarrow & & \downarrow \\
\mathbb{R}\Psi_{\mathcal{C}} : \mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–comod}) & \!\!=\!\!\!=\!\!\!=\!\! & \mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–contra}) : \mathbb{L}\Phi_{\mathcal{C}}
\end{array}
\tag{3}
$$

The functor $\mathbb{R}\Psi_{\mathcal{S}}$ is the right derived functor of the functor

$$\Psi_{\mathcal{S}} : \mathcal{S}\text{–simod} \longrightarrow \mathcal{S}\text{–sicntr}, \qquad \Psi_{\mathcal{S}}(\mathcal{M}) = \operatorname{Hom}_{\mathcal{S}}(\mathcal{S}, \mathcal{M})$$

of homomorphisms in the category of left $\boldsymbol{S}$-semimodules. The functor $\mathbb{L}\Phi_{\boldsymbol{S}}$ is the left derived functor of the functor

$$\Phi_{\boldsymbol{S}} \colon \boldsymbol{S}\text{–sicntr} \longrightarrow \boldsymbol{S}\text{–simod},$$

which is constructed using the operation of *contratensor product* of right $\boldsymbol{S}$-semimodules and left $\boldsymbol{S}$-semicontramodules,

$$\Phi_{\boldsymbol{S}}(\mathfrak{P}) = \boldsymbol{S} \odot_{\boldsymbol{S}} \mathfrak{P}.$$

The contratensor product operation is in some sense dual to the functor $\mathrm{Hom}^{\boldsymbol{S}}$ of homomorphisms in the category of left $\boldsymbol{S}$-semicontramodules: for any right $\boldsymbol{S}$-semimodule $\mathcal{N}$, left $\boldsymbol{S}$-semicontramodule $\mathfrak{P}$, and k-vector space V one has

$$\mathrm{Hom}_k(\mathcal{N} \odot_{\boldsymbol{S}} \mathfrak{P},\, V) \simeq \mathrm{Hom}^{\boldsymbol{S}}(\mathfrak{P}, \mathrm{Hom}_k(\mathcal{N}, V)).$$

It follows that the functor $\Phi_{\boldsymbol{S}}$ is left adjoint to the functor $\Psi_{\boldsymbol{S}}$.

1.7. In particular, when the coalgebra $\mathcal{C}$ has finite homological dimension n, there is no difference between the semiderived and the conventional derived categories of $\boldsymbol{S}$-semi(contra)modules,

$$\mathsf{D}^{\mathsf{si}}(\boldsymbol{S}\text{–simod}) = \mathsf{D}(\boldsymbol{S}\text{–simod}) \quad \text{and} \quad \mathsf{D}^{\mathsf{si}}(\boldsymbol{S}\text{–sicntr}) = \mathsf{D}(\boldsymbol{S}\text{–sicntr}).$$

Hence the commutative diagram of triangulated equivalences and triangulated forgetful functors

$$
\begin{array}{ccc}
\mathbb{R}\Psi_{\boldsymbol{S}} \colon \mathsf{D}(\boldsymbol{S}\text{–simod}) & =\!=\!=\!= & \mathsf{D}(\boldsymbol{S}\text{–sicntr}) : \mathbb{L}\Phi_{\boldsymbol{S}} \\
\downarrow & & \downarrow \\
\mathbb{R}\Psi_{\mathcal{C}} \colon \mathsf{D}(\mathcal{C}\text{–comod}) & =\!=\!=\!= & \mathsf{D}(\mathcal{C}\text{–contra}) : \mathbb{L}\Phi_{\mathcal{C}}
\end{array}
\tag{4}
$$

The left derived functor $\mathbb{L}\Phi_{\boldsymbol{S}}$ takes an $\boldsymbol{S}$-semicontramodule to an $(n+1)$-term complex of $\boldsymbol{S}$-semimodules, and the right derived functor $\mathbb{R}\Psi_{\boldsymbol{S}}$ takes an $\boldsymbol{S}$-semimodule to an $(n+1)$-term complex of $\boldsymbol{S}$-semicontramodules.

Let us emphasize that no homological dimension condition on the semialgebra $\boldsymbol{S}$ is imposed here, but only on the coalgebra $\mathcal{C}$.

1.8. Given a $\mathcal{C}$-semialgebra $\boldsymbol{S}$ and a k-vector space V, we compose the dualization functor from Section 1.5

$$\mathrm{Hom}(-, V) \colon \mathsf{D}^{\mathsf{si}}(\mathsf{simod}\text{–}\boldsymbol{S})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{si}}(\boldsymbol{S}\text{–sicntr})$$

with the derived semimodule-semicontramodule correspondence functor (3)

$$\mathbb{L}\Phi_{\boldsymbol{S}} \colon \mathsf{D}^{\mathsf{si}}(\boldsymbol{S}\text{–sicntr}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\boldsymbol{S}\text{–simod}),$$

obtaining a contravariant triangulated functor

$$\Delta_{\boldsymbol{S}}^V \colon \mathsf{D}^{\mathsf{si}}(\mathsf{simod}\text{–}\boldsymbol{S})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{si}}(\boldsymbol{S}\text{–simod}).
\tag{5}$$

Similarly one constructs a triangulated functor in the opposite direction

$$\Delta_{\boldsymbol{\mathcal{S}}^{\mathrm{op}}}^{V} : \mathsf{D}^{\mathsf{si}}(\boldsymbol{\mathcal{S}}\text{--simod})^{\mathrm{op}} \longrightarrow \mathsf{D}^{\mathsf{si}}(\text{simod--}\boldsymbol{\mathcal{S}}).$$

Consider the simplest (and most interesting) case when $V = k$. In what sense and under what restrictions can one expect the two functors $\Delta_{\boldsymbol{\mathcal{S}}}^{k}$ and $\Delta_{\boldsymbol{\mathcal{S}}^{\mathrm{op}}}^{k}$ to be mutually inverse?

The answer is, first of all, that this *cannot* be literally true in this form. It is instructive to consider the trivial case when $\boldsymbol{\mathcal{S}} = \mathcal{C} = k$ is the ground field. Then $\boldsymbol{\mathcal{S}}$--simod = simod--$\boldsymbol{\mathcal{S}}$ is the category of k-vector spaces, $\mathsf{D}^{\mathsf{si}}(\boldsymbol{\mathcal{S}}\text{--simod}) = \mathsf{D}^{\mathsf{si}}(\text{simod--}\boldsymbol{\mathcal{S}})$ is the derived category of k-vector spaces, and the functor Δ_k^k takes a complex of k-vector spaces $U^\bullet$ to the complex $U^{\bullet\vee} = \mathrm{Hom}_k(U^\bullet, k)$. Obviously, unless all the cohomology spaces of the complex $U^\bullet$ are finite-dimensional, $U^\bullet$ is *not* isomorphic to $U^{\bullet\vee\vee}$. So some finiteness conditions have to be imposed at this point.

What can one say without imposing any finiteness conditions? The answer is that the contravariant functors $\Delta_{\boldsymbol{\mathcal{S}}}^{V}$ and $\Delta_{\boldsymbol{\mathcal{S}}^{\mathrm{op}}}^{V}$ are *right adjoint to each other*. In other words, for any two objects $\boldsymbol{\mathcal{M}}^\bullet \in \mathsf{D}^{\mathsf{si}}(\boldsymbol{\mathcal{S}}\text{--simod})$ and $\boldsymbol{\mathcal{N}}^\bullet \in \mathsf{D}^{\mathsf{si}}(\text{simod--}\boldsymbol{\mathcal{S}})$ there is a natural adjunction isomorphism of k-vector spaces of morphisms

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{si}}(\boldsymbol{\mathcal{S}}\text{--simod})}(\boldsymbol{\mathcal{M}}^\bullet, \Delta_{\boldsymbol{\mathcal{S}}}^{V}\boldsymbol{\mathcal{N}}^\bullet) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{si}}(\text{simod--}\boldsymbol{\mathcal{S}})}(\boldsymbol{\mathcal{N}}^\bullet, \Delta_{\boldsymbol{\mathcal{S}}^{\mathrm{op}}}^{V}\boldsymbol{\mathcal{M}}^\bullet)$$

induced by the natural adjunction morphisms $\boldsymbol{\mathcal{M}}^\bullet \longrightarrow \Delta_{\boldsymbol{\mathcal{S}}}^{V}\Delta_{\boldsymbol{\mathcal{S}}^{\mathrm{op}}}^{V}(\boldsymbol{\mathcal{M}}^\bullet)$ in the category $\mathsf{D}^{\mathsf{si}}(\boldsymbol{\mathcal{S}}\text{--simod})$ and $\boldsymbol{\mathcal{N}}^\bullet \longrightarrow \Delta_{\boldsymbol{\mathcal{S}}^{\mathrm{op}}}^{V}\Delta_{\boldsymbol{\mathcal{S}}}^{V}(\boldsymbol{\mathcal{N}}^\bullet)$ in $\mathsf{D}^{\mathsf{si}}(\text{simod--}\boldsymbol{\mathcal{S}})$.

1.9. A general discussion of finiteness and quasi-finiteness conditions on coalgebras, comodules, and contramodules can be found in the paper [18, Section 2], the preprint [19, Section 2], and the references therein. The most general and convenient condition for our purposes is that the coalgebra $\mathcal{C}$ should be *left* and *right quasi-cocoherent* [19].

In this case, the full subcategories of *quasi-finitely copresented left $\mathcal{C}$-comodules* $\mathcal{C}$--comod$_{\mathsf{qfc}} \subset \mathcal{C}$--comod, quasi-finitely copresented right $\mathcal{C}$-comodules comod$_{\mathsf{qfc}}$--$\mathcal{C} \subset$ comod--$\mathcal{C}$, and *quasi-finitely presented left $\mathcal{C}$-contramodules* $\mathcal{C}$--contra$_{\mathsf{qfp}} \subset \mathcal{C}$--contra are closed under the kernels, cokernels, and extensions in $\mathcal{C}$--comod, comod--$\mathcal{C}$, and $\mathcal{C}$--contra, respectively. So $\mathcal{C}$--comod$_{\mathsf{qfc}}$, comod$_{\mathsf{qfc}}$--$\mathcal{C}$, and $\mathcal{C}$--contra$_{\mathsf{qfp}}$ are abelian categories. The functor assigning to a right $\mathcal{C}$-comodule $\mathcal{N}$ its dual left $\mathcal{C}$-contramodule $\mathfrak{P} = \mathcal{N}^\vee = \mathrm{Hom}_k(\mathcal{N}, k)$ restricts to an anti-equivalence of abelian categories

$$(\text{comod}_{\mathsf{qfc}}\text{--}\mathcal{C})^{\mathsf{op}} \simeq \mathcal{C}\text{--contra}_{\mathsf{qfp}},$$

(so any quasi-finitely presented left $\mathcal{C}$-contramodule acquires a natural pseudo-compact left $\mathcal{C}^\vee$-module structure); while the equivalence between the additive categories of injective left $\mathcal{C}$-comodules and projective left $\mathcal{C}$-contramodules $\mathcal{C}$--comod$^{\mathsf{inj}} \simeq \mathcal{C}$--contra$^{\mathsf{proj}}$ restricts to an equivalence between the categories of quasi-finitely cogenerated injective left $\mathcal{C}$-comodules and quasi-finitely generated projective left $\mathcal{C}$-contramodules,

$$\mathcal{C}\text{--comod}_{\mathsf{qfc}}^{\mathsf{inj}} \simeq \mathcal{C}\text{--contra}_{\mathsf{qfp}}^{\mathsf{proj}}.$$

Now let us assume that the coalgebra $\mathcal{C}$ is not only left and right quasi-cocoherent, but also has finite homological dimension. Then the derived equivalence (2) restricts

to an equivalence between the full subcategories $\mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–comod}_{\mathsf{qfc}}) \subset \mathsf{D}(\mathcal{C}\text{–comod})$ and $\mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–contra}_{\mathsf{qfp}}) \subset \mathsf{D}(\mathcal{C}\text{–contra})$. So one obtains triangulated equivalences

$$\mathsf{D}^{\mathsf{b}}(\mathsf{comod}_{\mathsf{qfc}}\text{–}\mathcal{C})^{\mathsf{op}} \simeq \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–contra}_{\mathsf{qfp}}) \simeq \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–comod}_{\mathsf{qfc}}), \qquad (6)$$

where the composition $\mathsf{D}^{\mathsf{b}}(\mathsf{comod}_{\mathsf{qfc}}\text{–}\mathcal{C}) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–comod}_{\mathsf{qfc}})$ is the restriction of the functor $\Delta_{\mathcal{C}}^{k}$. Thus the restrictions of the adjoint functors $\Delta_{\mathcal{C}}^{k}$ and $\Delta_{\mathcal{C}^{\mathsf{op}}}^{k}$ to the full subcategories $\mathsf{D}^{\mathsf{b}}(\mathsf{comod}_{\mathsf{qfc}}\text{–}\mathcal{C}) \subset \mathsf{D}(\mathsf{comod}\text{–}\mathcal{C})$ and $\mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–comod}_{\mathsf{qfc}}) \subset \mathsf{D}(\mathcal{C}\text{–comod})$ are mutually inverse anti-equivalences.

1.10.   Finally, let $\mathcal{S}$ be a semialgebra over a coalgebra $\mathcal{C}$. Consider the full subcategories $\mathcal{S}\text{–simod}_{\mathcal{C}\text{-qfc}} \subset \mathcal{S}\text{–simod}$ and $\mathsf{simod}\text{–}\mathcal{S}_{\mathsf{qfc}\text{-}\mathcal{C}} \subset \mathsf{simod}\text{–}\mathcal{S}$ consisting of all the left/right $\mathcal{S}$-semimodules that are quasi-finitely copresented *as $\mathcal{C}$-comodules*. Similarly, consider the full subcategory $\mathcal{S}\text{–sicntr}_{\mathcal{C}\text{-qfp}} \subset \mathcal{S}\text{–sicntr}$ consisting of al the left $\mathcal{S}$-semicontramodules that are quasi-finitely presented *as $\mathcal{C}$-contramodules*. Then the functor assigning to a right $\mathcal{S}$-semimodule $\mathcal{N}$ its dual left $\mathcal{S}$-semicontramodule $\mathfrak{P} = \mathcal{N}^{\vee}$ restricts to an anti-equivalence of categories

$$(\mathsf{simod}\text{–}\mathcal{S}_{\mathsf{qfc}\text{-}\mathcal{C}})^{\mathsf{op}} \simeq \mathcal{S}\text{–sicntr}_{\mathcal{C}\text{-qfp}}$$

(so there is a natural pseudo-compact topology on any $\mathcal{S}$-semicontramodule that is quasi-finitely presented as a $\mathcal{C}$-contramodule). The two equivalent categories here are abelian whenever the coalgebra $\mathcal{C}$ is right quasi-cocoherent.

Now suppose that the coalgebra $\mathcal{C}$ is left and right quasi-cocoherent of finite homological dimension. Consider the full subcategories $\mathsf{D}_{\mathcal{C}\text{-qfc}}^{\mathsf{b}}(\mathcal{S}\text{–simod}) \subset \mathsf{D}(\mathcal{S}\text{–simod})$ and $\mathsf{D}_{\mathsf{qfc}\text{-}\mathcal{C}}^{\mathsf{b}}(\mathsf{simod}\text{–}\mathcal{S}) \subset \mathsf{D}(\mathsf{simod}\text{–}\mathcal{S})$ in the derived categories of left and right $\mathcal{S}$-semimodules formed by the bounded complexes of semimodules with the *cohomology semimodules* that are *quasi-finitely copresented as $\mathcal{C}$-comodules*, i. e., the underlying $\mathcal{C}$-comodules of the cohomology $\mathcal{S}$-semimodules are quasi-finitely copresented. Similarly, consider the full subcategory $\mathsf{D}_{\mathcal{C}\text{-qfp}}^{\mathsf{b}}(\mathcal{S}\text{–sicntr}) \subset \mathsf{D}(\mathcal{S}\text{–sicntr})$ in the derived category of left $\mathcal{S}$-semicontramodules formed by the bounded complexes of semicontramodules with the *cohomology semicontramodules* that are *quasi-finitely presented as $\mathcal{C}$-contramodules*, i. e., the underlying $\mathcal{C}$-contramodules of the cohomology $\mathcal{S}$-semicontramodules are quasi-finitely presented.

Then it follows from the discussion in Section 1.9 that the derived equivalence in the upper line of the diagram (4) restricts to an equivalence between the full subcategories $\mathsf{D}_{\mathcal{C}\text{-qfc}}^{\mathsf{b}}(\mathcal{S}\text{–simod}) \subset \mathsf{D}(\mathcal{S}\text{–simod})$ and $\mathsf{D}_{\mathcal{C}\text{-qfp}}^{\mathsf{b}}(\mathcal{S}\text{–sicntr}) \subset \mathsf{D}(\mathcal{S}\text{–sicntr})$. Hence one obtains a commutative diagram of triangulated equivalences and triangulated forgetful functors

$$
\begin{array}{ccccc}
\mathsf{D}_{\mathsf{qfc}\text{-}\mathcal{C}}^{\mathsf{b}}(\mathsf{simod}\text{–}\mathcal{S})^{\mathsf{op}} & = & \mathsf{D}_{\mathcal{C}\text{-qfp}}^{\mathsf{b}}(\mathcal{S}\text{–sicntr}) & = & \mathsf{D}_{\mathcal{C}\text{-qfc}}^{\mathsf{b}}(\mathcal{S}\text{–simod}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{D}^{\mathsf{b}}(\mathsf{comod}_{\mathsf{qfc}}\text{–}\mathcal{C})^{\mathsf{op}} & = & \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–contra}_{\mathsf{qfp}}) & = & \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–comod}_{\mathsf{qfc}})
\end{array}
\qquad (7)
$$

where the composition $\mathsf{D}_{\mathsf{qfc}\text{-}\mathcal{C}}^{\mathsf{b}}(\mathsf{simod}\text{–}\mathcal{S}) \longrightarrow \mathsf{D}_{\mathcal{C}\text{-qfc}}^{\mathsf{b}}(\mathcal{S}\text{–simod})$ is the restriction of the functor $\Delta_{\mathcal{S}}^{k}$. In other words, the restrictions of the adjoint functors $\Delta_{\mathcal{S}}^{k}$ and $\Delta_{\mathcal{S}^{\mathsf{op}}}^{k}$

to the full subcategories $\mathsf{D}^{\mathsf{b}}_{\mathsf{qfc}\text{-}\mathcal{C}}(\mathsf{simod}\text{--}\mathcal{S}) \subset \mathsf{D}(\mathsf{simod}\text{--}\mathcal{S})$ and $\mathsf{D}^{\mathsf{b}}_{\mathcal{C}\text{-qfc}}(\mathcal{S}\text{--}\mathsf{simod}) \subset \mathsf{D}(\mathcal{S}\text{--}\mathsf{simod})$ are mutually inverse anti-equivalences.

1.11. Specializing (1) to the case of the coalgebra $\mathcal{C} = k(H)$ for a profinite group H, we obtain a natural triangulated equivalence between the coderived category of discrete H-modules and the contraderived category of H-contramodules over k,

$$\mathbb{R}\Psi_H \colon \mathsf{D}^{\mathsf{co}}(H\text{--}\mathsf{discr}_k) \simeq \mathsf{D}^{\mathsf{ctr}}(H\text{--}\mathsf{contra}_k) \colon \mathbb{L}\Phi_H. \qquad (8)$$

When the profinite group H has finite k-cohomological dimension (that is, the homological dimension of the abelian category of discrete H-modules over k is finite), this reduces to an equivalence between the conventional derived categories (2)

$$\mathbb{R}\Psi_H \colon \mathsf{D}(H\text{--}\mathsf{discr}_k) \simeq \mathsf{D}(H\text{--}\mathsf{contra}_k) \colon \mathbb{L}\Phi_H. \qquad (9)$$

In the case of the semialgebra $\mathcal{S} = k(G)$ over the coalgebra $\mathcal{C} = k(H)$ corresponding to a locally profinite group G with a compact open subgroup H, we obtain a natural triangulated equivalence between the semiderived categories of the abelian categories of smooth G-modules and G-contramodules over k,

$$\mathbb{R}_H\Psi_G \colon \mathsf{D}^{\mathsf{si}}_H(G\text{--}\mathsf{smooth}_k) \simeq \mathsf{D}^{\mathsf{si}}_H(G\text{--}\mathsf{contra}_k) \colon \mathbb{L}_H\Phi_G. \qquad (10)$$

The subindices H are necessary, because the semiderived categories depend on the choice of a compact open subgroup $H \subset G$, even though the abelian categories of smooth G-modules and G-contramodules do not. Moreover, the triangulated equivalences (8) and (10) form a commutative diagram with the forgetful functors (remembering only the action of H and forgetting the action of the rest of G) (3)

$$(11)$$

$$\begin{array}{ccc}
\mathbb{R}_H\Psi_G \colon \mathsf{D}^{\mathsf{si}}_H(G\text{--}\mathsf{smooth}_k) & =\!\!=\!\!= & \mathsf{D}^{\mathsf{si}}_H(G\text{--}\mathsf{contra}_k) \colon \mathbb{L}_H\Phi_G \\
\downarrow & & \downarrow \\
\mathbb{R}\Psi_H \colon \mathsf{D}^{\mathsf{co}}(H\text{--}\mathsf{discr}_k) & =\!\!=\!\!= & \mathsf{D}^{\mathsf{ctr}}(H\text{--}\mathsf{contra}_k) \colon \mathbb{L}\Phi_H
\end{array}$$

1.12. The situation simplifies considerably when a locally profinite group G has a compact open subgroup H of finite k-cohomological dimension. Notice that every open subgroup $H' \subset H$ then also has finite k-cohomological dimension; so the group G has a base of neighborhoods of zero formed by compact open subgroups of finite k-cohomological dimension.

Furthermore, the H-semiderived categories of smooth G-modules and G-contramodules over k coincide with the conventional derived categories when H has finite k-cohomological dimension,

$$\mathsf{D}^{\mathsf{si}}_H(G\text{--}\mathsf{smooth}_k) = \mathsf{D}(G\text{--}\mathsf{smooth}_k) \quad \text{and} \quad \mathsf{D}^{\mathsf{si}}_H(G\text{--}\mathsf{contra}_k) = \mathsf{D}(G\text{--}\mathsf{contra}_k).$$

Moreover, the derived functors $\mathbb{R}_H\Psi_G$ and $\mathbb{L}_H\Phi_G$ do not depend on the choice of a compact open subgroup $H \subset G$ of finite k-cohomological dimension. So we come to

the commutative diagram of triangulated equivalences and forgetful functors (4)

$$\mathbb{R}\Psi_G\colon \mathsf{D}(G\text{–}\mathsf{smooth}_k) \;=\!=\!=\; \mathsf{D}(G\text{–}\mathsf{contra}_k) \colon \mathbb{L}\Phi_G$$

$$\mathbb{R}\Psi_H\colon \mathsf{D}(H\text{–}\mathsf{discr}_k) \;=\!=\!=\; \mathsf{D}(H\text{–}\mathsf{contra}_k) \colon \mathbb{L}\Phi_H \tag{12}$$

where the upper line does not depend on the choice of H.

If the locally profinite group G has a compact open subgroup H of k-cohomological dimension n, then the homological dimension of the functors $\mathbb{R}\Psi_G$ and $\mathbb{L}\Phi_G$ does not exceed n. In other words, the functor $\mathbb{L}\Phi_G$ takes a G-contramodule over k to a nonpositively cohomologically graded $(n+1)$-term complex of smooth G-modules over k, while the functor $\mathbb{R}\Psi_G$ takes a smooth G-module over k to a nonnegatively cohomologically graded $(n+1)$-term complex of G-contramodules over k.

More generally, for any locally profinite group G admitting a compact open subgroup of finite k-cohomological dimension, and for any open subgroup $G' \subset G$, one has a commutative diagram of triangulated equivalences and forgetful functors

$$\mathbb{R}\Psi_G\colon \mathsf{D}(G\text{–}\mathsf{smooth}_k) \;=\!=\!=\; \mathsf{D}(G\text{–}\mathsf{contra}_k) \colon \mathbb{L}\Phi_G$$

$$\mathbb{R}\Psi_{G'}\colon \mathsf{D}(G'\text{–}\mathsf{smooth}_k) \;=\!=\!=\; \mathsf{D}(G'\text{–}\mathsf{contra}_k) \colon \mathbb{L}\Phi_{G'} \tag{13}$$

where the triangulated forgetful functors are induced by the exact forgetful functors between the abelian categories

$$G\text{–}\mathsf{smooth}_k \longrightarrow G'\text{–}\mathsf{smooth}_k \quad \text{and} \quad G\text{–}\mathsf{contra}_k \longrightarrow G'\text{–}\mathsf{contra}_k.$$

1.13. In fact, the situation considered in Section 1.12 stands in the intersection of several derived covariant duality theories.

Let k be a field and G be a locally profinite group having a compact open subgroup $H \subset G$ of finite k-cohomological dimension n. Let $\boldsymbol{\mathcal{S}} = k(G)$ be the related semialgebra over the coalgebra $\mathcal{C} = k(H)$. Then the $\boldsymbol{\mathcal{S}}$-$\boldsymbol{\mathcal{S}}$-bisemimodule $\boldsymbol{\mathcal{S}}$, viewed as a one-term complex, is a dedualizing complex for the pair of semialgebras $(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}})$ [18, Examples 4.2], so [18, Theorem 4.3] applies.

Furthermore, in the same assumptions, the smooth G-module $\boldsymbol{\mathcal{S}} = k(G)$ is an n-tilting object in the Grothendieck abelian category $\mathsf{A} = G\text{–}\mathsf{smooth}_k$ in the sense of, e. g., [9, Section 6] and [3, Definition 2.1 and Theorem 2.2]. The tilting heart B is equivalent to the abelian category of G-contramodules $G\text{–}\mathsf{contra}_k$. Either approach (see also Section 4.1 below) allows to construct the triangulated equivalences

$$\mathsf{D}^\star(G\text{–}\mathsf{smooth}_k) \simeq \mathsf{D}^\star(G\text{–}\mathsf{contra}_k) \tag{14}$$

for all the bounded or unbounded, conventional or absolute derived categories with the symbols $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, or abs. We refer to the paper [20, Corollary 5.6 and Example 10.10] for the details.

1.14. Let G be a locally profinite group, $H \subset G$ be a compact open subgroup, and k be a field. As usually, we say that a smooth G-module $\mathcal{M}$ over k is *admissible* if, for every compact open subgroup $F \subset G$, the subspace of F-invariant elements $\mathcal{M}^F \subset \mathcal{M}$ is finite-dimensional. Equivalently, this means that the underlying smooth H-module of the smooth G-module $\mathcal{M}$ is quasi-finitely cogenerated as a $k(H)$-comodule. Hence the latter property of a smooth G-module $\mathcal{M}$ over k does not depend on the choice of a compact open subgroup $H \subset G$.

Dual-analogously, for any G-contramodule $\mathfrak{P}$ over k and a compact open subgroup $F \subset G$ we consider the maximal quotient contramodule of $\mathfrak{P}$ on which the contraaction of F is trivial, and denote it by $\mathfrak{P}_F$. This is the contramodule version of the space of coinvariants of F in $\mathfrak{P}$. We say that a G-contramodule $\mathfrak{P}$ over k is *contraadmissible* if, for every compact open subgroup $F \subset G$, the vector space $\mathfrak{P}_F$ is finite-dimensional. Equivalently, this means that the underlying H-contramodule of the G-contramodule $\mathfrak{P}$ is quasi-finitely generated as a $k(H)$-contramodule. It follows that the latter property of a G-contramodule $\mathfrak{P}$ over k does not depend on the choice of a compact open subgroup $H \subset G$.

A smooth G-module $\mathcal{M}$ over k is said to be *admissibly copresented* if its underlying smooth H-module is quasi-finitely copresented as a $k(H)$-comodule, i. e., in other words, the underlying smooth H-module of $\mathcal{M}$ is the kernel of a morphism between two admissible injective smooth H-modules over k. One can check that this property of a smooth G-module over k does not depend on the choice of a compact open subgroup $H \subset G$. We denote the full subcategory of admissibly copresented smooth G-modules by $k(G)$–$\mathsf{simod}_{k(H)\text{-qfc}} = G$–$\mathsf{smooth}_{k,\mathsf{acp}} \subset G$–$\mathsf{smooth}_k$.

Similarly, a G-contramodule $\mathfrak{P}$ over k is said to be *contraadmissibly presented* if its underlying H-contramodule is quasi-finitely presented as a $k(H)$-contramodule, i. e., the underlying H-contramodule of $\mathfrak{P}$ is the cokernel of a morphism between two contraadmissible projective H-contramodules over k. Once again, one can check that this property of a G-contramodule over k does not depend on the choice of a compact open subgroup $H \subset G$. We denote the full subcategory of contraadmissibly presented G-contramodules by $k(G)$–$\mathsf{sicntr}_{k(H)\text{-qfp}} = G$–$\mathsf{contra}_{k,\mathsf{caap}} \subset G$–$\mathsf{contra}_k$.

The functor assigning to a smooth G-module $\mathcal{M}$ over k its dual G-semicontramodule $\mathfrak{P} = \mathcal{M}^\vee = \mathrm{Hom}_k(\mathcal{M}, k)$ restricts to an anti-equivalence of categories

$$(G\text{–}\mathsf{smooth}_{k,\mathsf{acp}})^{\mathsf{op}} \simeq G\text{–}\mathsf{contra}_{k,\mathsf{caap}}.$$

The two equivalent categories here are abelian whenever the coalgebra $\mathcal{C} = k(H)$ is quasi-cocoherent. The latter property does not depend on the choice of a compact open subgroup H in a given locally profinite group G.

1.15. Let G be a locally profinite group and k be a field. Suppose that the coalgebra $\mathcal{C} = k(H)$ is quasi-cocoherent for some (equivalently, for all) compact open subgroup $H \subset G$. Suppose further that there exists a compact open subgroup $H \subset G$ of finite k-cohomological dimension.

As in Section 1.10, we consider the full subcategory $\mathsf{D}^\mathsf{b}_\mathsf{acp}(G\text{–}\mathsf{smooth}_k) \subset \mathsf{D}(G\text{–}\mathsf{smooth}_k)$ in the derived category of smooth G-modules over k formed by the bounded complexes of smooth G-modules *with admissibly copresented cohomology G-modules.* Similarly, we consider the full subcategory $\mathsf{D}^\mathsf{b}_\mathsf{caap}(G\text{–}\mathsf{contra}_k) \subset \mathsf{D}(G\text{–}\mathsf{contra}_k)$ in the derived category of G-contramodules over k formed by the bounded complexes

of G-contramodules *with contraadmissibly presented cohomology G-contramodules.* Then the commutative diagram of triangulated equivalences and triangulated forgetful functors (7) takes the form

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{--smooth}_k)^{\mathsf{op}} \quad\Longequal\quad \mathsf{D}^{\mathsf{b}}_{\mathsf{caap}}(G\text{--contra}_k) \quad\Longequal\quad \mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{--smooth}_k)$$

$$\Big\downarrow \qquad\qquad\qquad \Big\downarrow \qquad\qquad\qquad \Big\downarrow \qquad\qquad (15)$$

$$\mathsf{D}^{\mathsf{b}}(H\text{--discr}_{k,\mathsf{acp}})^{\mathsf{op}} \quad\Longequal\quad \mathsf{D}^{\mathsf{b}}(H\text{--contra}_{k,\mathsf{caap}}) \quad\Longequal\quad \mathsf{D}^{\mathsf{b}}(H\text{--discr}_{k,\mathsf{acp}})$$

where $H \subset G$ is any chosen compact open subgroup and $H\text{--discr}_{k,\mathsf{acp}}$ is a notation for the abelian category of admissibly copresented discrete/smooth H-modules over k. The composition $\mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{--smooth}_k)^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{--smooth}_k)$ is an involutive auto-anti-equivalence of the triangulated category $\mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{--smooth}_k)$ which can be obtained as the restriction of the functor $\Delta^k_{\boldsymbol{\mathcal{S}}} = \Delta^k_G = \Delta^k_{\boldsymbol{\mathcal{S}}^{\mathsf{op}}}$, where $\boldsymbol{\mathcal{S}} = k(G) \simeq \boldsymbol{\mathcal{S}}^{\mathsf{op}}$, to the full subcategory $\mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{--smooth}_k) \subset \mathsf{D}(G\text{--smooth}_k)$.

Here the upper line of the diagram (15) does not depend on the choice of a compact open subgroup $H \subset G$, because the upper line of the diagram (12) does not depend on it. More generally, for any locally profinite group G admitting a compact open subgroup H of finite k-cohomological dimension with a quasi-cocoherent coalgebra $k(H)$, and for any open subgroup $G' \subset G$, we have a commutative diagram of triangulated equivalences and forgetful functors (cf. (13))

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{--smooth}_k)^{\mathsf{op}} \quad\Longequal\quad \mathsf{D}^{\mathsf{b}}_{\mathsf{caap}}(G\text{--contra}_k) \quad\Longequal\quad \mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{--smooth}_k)$$

$$\Big\downarrow \qquad\qquad\qquad \Big\downarrow \qquad\qquad\qquad \Big\downarrow \qquad\qquad (16)$$

$$\mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G'\text{--smooth}_k)^{\mathsf{op}} \quad\Longequal\quad \mathsf{D}^{\mathsf{b}}_{\mathsf{caap}}(G'\text{--contra}_k) \quad\Longequal\quad \mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G'\text{--smooth}_k)$$

1.16. Let H be a profinite group of the proorder not divisible by the characteristic of a field k. (In particular, H can be an arbitrary profinite group and k a field of characteristic 0.) Then the k-cohomological dimension of H is zero, $n = 0$. In other words, the coalgebra $\mathcal{C} = k(H)$ is cosemisimple.

Denote by I_α the irreducible discrete representations of H over k. Then an arbitrary smooth representation of H over k has the form

$$\mathcal{M} = \bigoplus_\alpha V_\alpha \otimes_k I_\alpha,$$

where V_α are some k-vector spaces. At the same time, an arbitrary H-contramodule over k has the form [11, Lemma A.2.2]

$$\mathfrak{P} = \prod_\alpha V_\alpha \otimes_k I_\alpha.$$

The derived equivalence $\mathsf{D}(H\text{--discr}_k) \simeq \mathsf{D}(H\text{--contra}_k)$ from the diagram (12) is induced by the equivalence of (semisimple) abelian categories

$$\Psi_H \colon H\text{--discr}_k \simeq H\text{--contra}_k : \Phi_H$$

taking the discrete H-module $\bigoplus_\alpha V_\alpha \otimes I_\alpha$ to the H-contramodule $\prod_\alpha V_\alpha \otimes I_\alpha$ and back. The functor Φ_H can be also described as taking an H-contramodule $\mathfrak{P}$ over k to its H-submodule $\mathcal{M} \subset \mathfrak{P}$ formed by all the H-discrete vectors (i. e., vectors whose stabilizers are open in H).

Furthermore, any cosemisimple coalgebra is left and right co-Noetherian [18, Section 2], hence quasi-co-Noetherian, and therefore quasi-cocoherent [19, Section 2]. In the situation at hand, the full subcategory of all admissibly copresented (= admissible) discrete H-modules $H\text{–discr}_{k,\mathsf{acp}} \subset H\text{–discr}_k$ consists of all the representations $\mathcal{M} = \bigoplus_\alpha V_\alpha \otimes_k I_\alpha$ such that V_α is a finite-dimensional k-vector space for every α. Similarly, the full subcategory of all contraadmissibly presented (= contraadmissible) H-contramodules $H\text{–contra}_{k,\mathsf{caap}} \subset H\text{–contra}_k$ consists of all the H-contramodules $\mathfrak{P} = \prod_\alpha V_\alpha \otimes_k I_\alpha$ such that all the vector spaces V_α are finite-dimensional.

Now let G be a locally profinite group containing a compact open subgroup $H \subset G$ of the proorder not divisible by $\operatorname{char} k$. Then the assertions of Sections 1.12–1.13 are applicable. In particular, the smooth G-module $\mathcal{S}$ is a projective generator of the abelian category $G\text{–smooth}_k$. Moreover, the derived equivalence $\mathsf{D}(G\text{–smooth}_k) \simeq \mathsf{D}(G\text{–contra}_k)$ from the diagram (12) is induced by an equivalence of abelian categories

$$\Psi_G \colon G\text{–smooth}_k \simeq G\text{–contra}_k \colon \Phi_G.$$

The functor Φ_G takes a G-contramodule $\mathfrak{P}$ over k to its submodule $\mathcal{M} = \Phi_G(\mathfrak{P}) \subset \mathfrak{P}$ of all G-smooth vectors. The functor Ψ_G can be described by the rule

$$\mathfrak{P} = \Psi_G(\mathcal{M}) = \operatorname{Hom}_{k[G]}(\mathcal{S}, \mathcal{M}),$$

where $\mathcal{S} = k(G)$ is the smooth G-G-bimodule of all compactly supported locally constant functions $G \longrightarrow k$.

Restricting the triangulated equivalences and triangulated forgetful functors from the diagram (15) to the full subcategories of one-term complexes, one obtains a diagram of abelian category equivalences and forgetful functors

$$
\begin{array}{ccccc}
G\text{–smooth}^{\mathsf{op}}_{k,\mathsf{acp}} & = & G\text{–contra}_{k,\mathsf{caap}} & = & G\text{–smooth}_{k,\mathsf{acp}} \\
\downarrow & & \downarrow & & \downarrow \\
H\text{–discr}^{\mathsf{op}}_{k,\mathsf{acp}} & = & H\text{–contra}_{k,\mathsf{caap}} & = & H\text{–discr}_{k,\mathsf{acp}}
\end{array}
\tag{17}
$$

1.17. Let G be a p-adic Lie group. If $\operatorname{char} k \neq p$ (the "nonnatural characteristic" case), then G contains a compact open subgroup H of the proorder not divisible by $\operatorname{char} k$, so the assertions of Section 1.16 apply.

In this paper, we are mostly interested in the case $\operatorname{char} k = p$ (the "natural characteristic" case). According to [1, Definition 4.1, Theorem 4.5, and Theorem 8.32] and [23, Corollaire (1)] (see also [8, Theorem 3.1 and the discussion in the preceding paragraph]), G contains an open subgroup H that is a pro-p-group of finite cohomological dimension. Hence the assertions of Sections 1.12–1.13 apply to H and G.

Furthermore, the pro-p-group H is finitely generated. As we show in this paper, it follows that the the forgetful functor

$$H\text{--contra}_k = \mathcal{C}\text{--contra} \longrightarrow \mathcal{C}^\vee\text{--mod}$$

from the category of $\mathcal{C}$-contramodules to the category of abstract $\mathcal{C}^\vee$-modules is fully faithful. So is the forgetful functor

$$G\text{--contra}_k \longrightarrow G\text{--mod}_k$$

from the category of G-contramodules over k to the category of abstract G-modules over k.

It also follows that the derived functor $\mathbb{L}\Phi_G$ in the diagram (12) is simply the left derived functor of the functor of tensor product

$$\Phi_G \colon G\text{--contra}_k \longrightarrow G\text{--smooth}_k, \qquad \Phi_G(\mathfrak{P}) = \mathcal{S} \otimes_{k[G]} \mathfrak{P}.$$

Alternatively, the functor Φ_G can be described as

$$\Phi_G(\mathfrak{P}) = \mathcal{S} \otimes_{\mathfrak{T}} \mathfrak{P},$$

where $\mathfrak{T} = \mathrm{Hom}_{k[G]}(\mathcal{S}, \mathcal{S})^{\mathrm{op}}$ is the opposite ring to the ring of endomorphisms of the left G-module $\mathcal{S}$ over k. The functor $\mathbb{L}\Phi_G \colon \mathsf{D}(G\text{--contra}_k) \longrightarrow \mathsf{D}(G\text{--smooth}_k)$ agrees with the conventional derived tensor product functor $\mathcal{S} \otimes_{\mathfrak{T}}^{\mathbb{L}} -$.

Similarly, the derived functor $\mathbb{R}\Psi_G$ is simply the right derived functor of

$$\Psi_G \colon G\text{--smooth}_k \longrightarrow G\text{--contra}_k, \qquad \Psi_G(\mathcal{M}) = \mathrm{Hom}_{k[G]}(\mathcal{S}, \mathcal{M}) = \mathrm{Hom}_{\mathfrak{T}^{\mathrm{op}}}(\mathcal{S}, \mathcal{M}).$$

The functor $\mathbb{R}\Psi_G \colon \mathsf{D}(G\text{--smooth}_k) \longrightarrow \mathsf{D}(G\text{--contra}_k)$ agrees with the conventional $\mathbb{R}\,\mathrm{Hom}_{\mathfrak{T}^{\mathrm{op}}}(\mathcal{S}, -)$.

Finally, the ring $k[[H]] = k(H)^\vee$ is left and right Noetherian [1, Corollary 7.25 and Exercise 7.6], hence the coalgebra $\mathcal{C} = k(H)$ is left and right Artinian and consequently co-Noetherian [18, Section 2]. It follows that $\mathcal{C}$ is quasi-co-Noetherian and quasi-cocoherent [19, Section 2]. A discrete H-module is admissibly copresented if and only if it is admissible, and if and only if the related $\mathcal{C}$-comodule is Artinian. An H-contramodule is contraadmissibly presented if and only if it is contraadmissible, and if and only if the related $\mathcal{C}$-contramodule is co-Artinian [18, Section 2].

Thus the right self-adjoint contravariant triangulated functor

$$\Delta_G^k = \Delta_{k(G)}^k \colon \mathsf{D}(G\text{--smooth}_k)^{\mathrm{op}} \longrightarrow \mathsf{D}(G\text{--smooth}_k)$$

restricts to an involutive triangulated auto-anti-equivalence of the full subcategory $\mathsf{D}_{\mathsf{acp}}^{\mathsf{b}}(G\text{--smooth}_k) \subset \mathsf{D}(G\text{--smooth}_k)$ of bounded complexes of smooth G-modules with admissible cohomology G-modules, and we have the commutative diagrams (15–16) of triangulated equivalences and triangulated forgetful functors, as in Section 1.15.

1.18. Before we finish this introduction, let us mention a result demonstrating that the semiderived category $\mathsf{D}_H^{\mathsf{si}}(G\text{--smooth}_k)$ is a natural triangulated category to assign to a locally profinite group G with a compact open pro-p subgroup H of a more general kind than p-adic Lie groups. The question becomes nontrivial when the pro-p-group H has infinite cohomological dimension (and char $k = p$).

For any coalgebra $\mathcal{C}$ over a field k, the coderived category of left $\mathcal{C}$-comodules $\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–comod})$ is compactly generated by its full subcategory of finite complexes of finite-dimensional comodules. This full subcategory is equivalent to the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–comod}_{\mathsf{fd}})$ of the abelian category of finite-dimensional left $\mathcal{C}$-comodules [12, Sections 3.11 and/or 4.6 and 5.5].

In particular, let H be a profinite group and k be a field. Denote by I_α the irreducible discrete representations of H over k. Then the objects $I_\alpha \in H\text{–discr}_k \subset \mathsf{D}^{\mathsf{co}}(H\text{–discr}_k)$ form a set of compact generators of the coderived category of discrete representations $\mathsf{D}^{\mathsf{co}}(H\text{–discr}_k)$. When H is a pro-p-group and $\operatorname{char} k = p$, there is a unique irreducible discrete representation of H over k, namely, the trivial representation $I_0 = k$. So I_0 is a single compact generator of the coderived category $\mathsf{D}^{\mathsf{co}}(H\text{–discr}_k)$.

For a semialgebra $\mathcal{S}$ over $\mathcal{C}$, one can consider the induction functor

$$\operatorname{ind}_{\mathcal{C}}^{\mathcal{S}} \colon \mathcal{C}\text{–comod} \longrightarrow \mathcal{S}\text{–simod}, \qquad \operatorname{ind}_{\mathcal{C}}^{\mathcal{S}}(\mathcal{M}) = \mathcal{S} \mathbin{\square}_{\mathcal{C}} \mathcal{M}.$$

The functor $\operatorname{ind}_{\mathcal{C}}^{\mathcal{S}}$ is left adjoint to the forgetful functor $\mathcal{S}\text{–simod} \longrightarrow \mathcal{C}\text{–comod}$. The functor $\operatorname{ind}_{\mathcal{C}}^{\mathcal{S}}$ is exact, so it induces a triangulated functor

$$\operatorname{ind}_{\mathcal{C}}^{\mathcal{S}} \colon \mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–comod}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–simod}).$$

left adjoint to the forgetful functor $\mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–simod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–comod})$. The triangulated forgetful functor preserves coproducts, so the triangulated induction functor takes compact objects to compact objects.

Denote by I_α the irreducible left $\mathcal{C}$-comodules. Then the objects $\operatorname{ind}_{\mathcal{C}}^{\mathcal{S}}(I_\alpha) \in \mathcal{S}\text{–simod} \subset \mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–simod})$ are compact generators of the triangulated category $\mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–simod})$. This follows from the fact that the triangulated forgetful functor $\mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{–simod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–comod})$ is, by the definition of the semiderived category, conservative, i. e., it takes nonzero objects to nonzero objects.

Now let G be a locally profinite group with a compact open subgroup $H \subset G$. Specializing to the case $\mathcal{C} = k(H)$ and $\mathcal{S} = k(G)$, we obtain the functor of compactly supported smooth induction

$$\operatorname{ind}_H^G \colon H\text{–discr}_k \longrightarrow G\text{–smooth}_k$$

left adjoint to the forgetful functor $G\text{–smooth}_k \longrightarrow H\text{–discr}_k$. Let I_α be the irreducible discrete representations of H over k. Then the objects $\operatorname{ind}_H^G(I_\alpha) \in G\text{–smooth}_k \subset \mathsf{D}_H^{\mathsf{si}}(G\text{–smooth}_k)$ are compact generators of the semiderived category $\mathsf{D}_H^{\mathsf{si}}(G\text{–smooth}_k)$. When H is a pro-p-group and $\operatorname{char} k = p$, the smooth G-module $\operatorname{ind}_H^G(I_0)$ is a single compact generator of the category $\mathsf{D}_H^{\mathsf{si}}(G\text{–smooth}_k)$.

Now we can define the Hecke DG-algebra. The full subcategories of bounded below complexes in $\mathsf{D}(G\text{–smooth}_k)$, $\mathsf{D}_H^{\mathsf{si}}(G\text{–smooth}_k)$, and $\mathsf{D}^{\mathsf{co}}(G\text{–smooth}_k)$ are all equivalent, so it does not matter in (a DG-enhancement of) which of these triangulated categories such a DG-algebra is computed. As in [22, Section 3], it suffices to pick a right injective resolution $\mathcal{J}^\bullet$ of the object $\operatorname{ind}_H^G(I_0)$ in the abelian category $G\text{–smooth}_k$, and set $A^\bullet = \operatorname{Hom}_{k[G]}(\mathcal{J}^\bullet, \mathcal{J}^\bullet)^{\mathrm{op}}$.

The same arguments as in [22] provide a triangulated equivalence

$$\mathsf{D}_H^{\mathsf{si}}(G\text{–smooth}_k) \simeq \mathsf{D}(A^\bullet\text{–mod}) \tag{18}$$

between the H-semiderived category of smooth G-modules over k and the derived category of left DG-modules over the Hecke DG-algebra $A^\bullet$. This is a generalization of the Schneider derived equivalence [22, Theorem 9] to the case of an arbitrary locally profinite group G with a compact open pro-p subgroup $H \subset G$.

1.19. In conclusion, let us mention that many results of this paper can be extended easily to the case of a discrete commutative coefficient ring k of finite homological dimension (in place of a field). Indeed, the exposition in [11, Sections E.1–E.3] is given in precisely this generality. In order not to intimidate the reader, we prefer to work over a field in this paper. For the same reason, we do not go into a discussion of the absolute derived categories.

2. Discrete H-modules and H-contramodules

2.1. Comodules and contramodules. Let k be a field. We will denote the category of (discrete or abstract) k-vector spaces by k–vect. A complete, separated topological k-vector space W is said to be *pseudo-compact* if its open vector subspaces of finite codimension form a base of neighborhoods of zero in it. In particular, a finite-dimensional topological vector space is pseudo-compact if and only if it is discrete, and a pseudo-compact topological vector space is discrete if and only if it is finite-dimensional. The category of pseudo-compact topological vector spaces and continuous linear maps between them will be denoted by k–pscomp.

For any discrete k-vector space V, the dual vector space $V^\vee = \mathrm{Hom}_k(V, k)$ carries a natural pseudo-compact topology in which the annihilators of finite-dimensional subspaces in V form a base of neighborhoods of zero. Conversely, to any pseudo-compact k-vector space W one assigns the discrete vector space V of all continuous linear functions $W \longrightarrow k$ (where the topology on k is discrete). This correspondence defines an anti-equivalence between the categories of discrete and pseudo-compact k-vector spaces,

$$(k\text{–vect})^{\mathsf{op}} \simeq k\text{–pscomp}. \tag{19}$$

The *completed tensor product* of two pseudo-compact vector spaces W_1 and W_2 is defined as the projective limit

$$W_1 \mathbin{\widehat{\otimes}}_k W_2 = \varprojlim_{U_1, U_2} W_1/U_1 \otimes_k W_2/U_2,$$

where U_1 ranges over the open vector subspaces in W_1 and U_2 over the open vector subspaces in W_2. The k-vector space $W_1 \mathbin{\widehat{\otimes}}_k W_2$ is endowed with the topology

of projective limit of discrete finite-dimensional vector spaces $W_1/U_1 \otimes_k W_2/U_2$, making $W_1 \widehat{\otimes}_k W_2$ a pseudo-compact topological vector space. The anti-equivalence of categories (19) takes the tensor product of discrete k-vector spaces to the completed tensor product of pseudo-compact k-vector spaces; so it is an anti-equivalence of tensor categories.

A (coassociative and counital) *coalgebra* $\mathcal{C}$ over k is k-vector space endowed with a *comultiplication* map $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and a *counit* map $\varepsilon\colon \mathcal{C} \longrightarrow k$ satisfying the conventional coassociativity and counitality axioms. A *left $\mathcal{C}$-comodule* $\mathcal{M}$ is a k-vector space endowed with a *left coaction* map $\nu_{\mathcal{M}}\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$, and a *right $\mathcal{C}$-comodule* $\mathcal{N}$ is a k-vector space endowed with a *right coaction* map $\nu_{\mathcal{N}}\colon \mathcal{N} \longrightarrow \mathcal{N} \otimes_k \mathcal{C}$. The coassociativity and counitality axioms have to be satisfied in both cases. We refer to [24] or [14] for the details.

The *Sweedler notation* [24] for the comultiplication in a coalgebra $\mathcal{C}$ has the form

$$\mu(c) = \sum_i c_i^{(1)} \otimes c_i^{(2)} = c^{(1)} \otimes c^{(2)},$$

where c, $c_i^{(1)}$, $c_i^{(2)} \in \mathcal{C}$, while $c^{(1)} \otimes c^{(2)}$ is a simplified symbolic form of the notation. Similarly, the Sweedler notation for the left coaction map $\nu_{\mathcal{M}}$ is

$$\nu_{\mathcal{M}}(x) = \sum_i x_i^{(-1)} \otimes x_i^{(0)} = x^{(-1)} \otimes x^{(0)},$$

where x, $x_i^{(0)} \in \mathcal{M}$ and $x_i^{(-1)} \in \mathcal{C}$; and the notation for the right coaction map $\nu_{\mathcal{N}}$ is

$$\nu_{\mathcal{N}}(y) = \sum_i y_i^{(0)} \otimes y_i^{(1)} = y^{(0)} \otimes y^{(1)},$$

where y, $y_i^{(0)} \in \mathcal{N}$ and $y_i^{(1)} \in \mathcal{C}$. In every one of these cases, the index i actually ranges over a finite set; and this index is often omitted, together with the summation sign over i, for the sake of brevity.

The anti-equivalence of tensor categories (19) transforms coalgebra objects in the category k–vect into algebra objects in the category k–pscomp and back. In other words, for any coassociative, counital coalgebra $\mathcal{C}$ over k, the dual pseudo-compact vector space $\mathcal{C}^\vee$ acquires an associative, unital *pseudo-compact algebra* structure given by a pseudo-compact multiplication map $m = \mu^\vee\colon \mathcal{C}^\vee \widehat{\otimes}_k \mathcal{C}^\vee \longrightarrow \mathcal{C}^\vee$ and a unit map $e = \varepsilon^\vee\colon k \longrightarrow \mathcal{C}^\vee$. The datum of a map m is equivalent to the datum of a conventional multiplication $\mathcal{C}^\vee \times \mathcal{C}^\vee \longrightarrow \mathcal{C}^\vee$ that is continuous as a function of two variables in the pseudo-compact topology, k-linear, associative, and unital with the unit element $e(1) \in \mathcal{C}$. In other words, the pseudo-compact algebra $\mathcal{C}^\vee$ has an underlying structure of a discrete or nontopological associative algebra (as well as an underlying structure of a pseudo-compact vector space).

There still remains a choice between the two opposite multiplications on the pseudo-compact vector space $\mathcal{C}^\vee$: which one of the two opposite pseudo-compact algebras is to be denoted by $\mathcal{C}^\vee$, and which one by $\mathcal{C}^{\vee\mathrm{op}}$? (See the discussion in [14, Section 1.4].) As in [14], we make the slightly nonstandard choice of defining the multiplication in $\mathcal{C}^\vee$ by the formula

$$\langle fg, c \rangle = \langle f, c^{(2)} \rangle \langle g, c^{(1)} \rangle, \qquad f, g \in \mathcal{C}^\vee, \ c \in \mathcal{C},$$

where $\langle \, , \, \rangle$ denotes the natural pairing $\mathcal{C}^\vee \times \mathcal{C} \longrightarrow k$.

Then, for any coalgebra $\mathcal{C}$ over k, the anti-equivalence of tensor categories (19) transforms left $\mathcal{C}$-comodules $\mathcal{M}$ into pseudo-compact right $\mathcal{C}^\vee$-modules $\mathcal{M}^\vee$ and right $\mathcal{C}$-comodules $\mathcal{N}$ into pseudo-compact left $\mathcal{C}^\vee$-modules $\mathcal{N}^\vee$. So the anti-equivalence of tensor categories (19) induces an anti-equivalence between the categories of right $\mathcal{C}$-comodules and pseudo-compact left $\mathcal{C}^\vee$-modules,

$$(\mathsf{comod}\text{--}\mathcal{C})^{\mathsf{op}} \simeq \mathcal{C}^\vee\text{--}\mathsf{pscomp}, \tag{20}$$

and similarly on the other side. What is important for us, though, is that the right $\mathcal{C}^\vee$-module $\mathcal{M}^\vee$ can be obtained by applying the duality functor $\mathrm{Hom}_k(-,k)$ to a natural left $\mathcal{C}^\vee$-module structure on $\mathcal{M}$. The natural left $\mathcal{C}^\vee$-module structure on a left $\mathcal{C}$-comodule $\mathcal{M}$ is defined by the composion

$$\mathcal{C}^\vee \otimes_k \mathcal{M} \longrightarrow \mathcal{C}^\vee \otimes_k \mathcal{C} \otimes_k \mathcal{M} \longrightarrow \mathcal{M}$$

of the map $\mathcal{C}^\vee \otimes_k \mathcal{M} \longrightarrow \mathcal{C}^\vee \otimes_k \mathcal{C} \otimes_k \mathcal{M}$ induced by the coaction map $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ and the map $\mathcal{C}^\vee \otimes_k \mathcal{C} \otimes_k \mathcal{M} \longrightarrow \mathcal{M}$ induced by the pairing $\mathcal{C}^\vee \times \mathcal{C} \longrightarrow k$. The natural right $\mathcal{C}^\vee$-module structure on a right $\mathcal{C}$-comodule $\mathcal{N}$ is constructed similarly.

So we have two functors

$$\mathcal{C}\text{--}\mathsf{comod} \longrightarrow \mathcal{C}^\vee\text{--}\mathsf{mod}, \tag{21}$$

$$\mathsf{comod}\text{--}\mathcal{C} \longrightarrow \mathsf{mod}\text{--}\mathcal{C}^\vee. \tag{22}$$

These functors are always fully faithful, and identify the category of (left or right) $\mathcal{C}$-comodules with the category of *discrete* (left or right) $\mathcal{C}^\vee$-modules with respect to the pseudo-compact topology on $\mathcal{C}^\vee$, i. e., the $\mathcal{C}^\vee$-modules every vector in which has an open annihilator in $\mathcal{C}^\vee$ [24, Section 2.1].

A *left $\mathcal{C}$-contramodule* $\mathfrak{P}$ is a k-vector space endowed with a *left contraaction* map $\pi_{\mathfrak{P}}\colon \mathrm{Hom}_k(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$. The map $\pi_{\mathfrak{P}}$ must satisfy the *contraassociativity* and *contraunitality* axioms: namely, the two maps

$$\mathrm{Hom}_k(\mathcal{C}, \mathrm{Hom}_k(\mathcal{C},\mathfrak{P})) = \mathrm{Hom}_k(\mathcal{C} \otimes_k \mathcal{C},\, \mathfrak{P}) \rightrightarrows \mathrm{Hom}_k(\mathcal{C},\mathfrak{P}),$$

one of them induced by the comultiplication map μ and the other one by the contraaction map $\pi_{\mathfrak{P}}$, should have equal compositions with the contraaction map $\pi_{\mathfrak{P}}$,

$$\mathrm{Hom}_k(\mathcal{C}, \mathrm{Hom}_k(\mathcal{C},\mathfrak{P})) = \mathrm{Hom}_k(\mathcal{C} \otimes_k \mathcal{C},\, \mathfrak{P}) \rightrightarrows \mathrm{Hom}_k(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P},$$

and the composition of the map $\mathfrak{P} \longrightarrow \mathrm{Hom}_k(\mathcal{C},\mathfrak{P})$ induced by the counit map ε with the contraaction map $\pi_{\mathfrak{P}}$ should be equal to the identity map $\mathrm{id}_{\mathfrak{P}}$,

$$\mathfrak{P} \longrightarrow \mathrm{Hom}_k(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}.$$

Here the identification $\mathrm{Hom}_k(V, \mathrm{Hom}_k(U,W)) \simeq \mathrm{Hom}_k(U \otimes_k V,\, W)$ is presumed in the contraassociativity axiom (using $\mathrm{Hom}_k(U, \mathrm{Hom}_k(V,W)) \simeq \mathrm{Hom}_k(U \otimes_k V,\, W)$ would lead to the definition of a *right $\mathcal{C}$-contramodule*).

Given a right $\mathcal{C}$-comodule $\mathcal{N}$ and a k-vector space V, the vector space $\mathfrak{P} = \mathrm{Hom}_k(\mathcal{N}, V)$ has a natural left $\mathcal{C}$-contramodule structure with the contraaction map $\pi_{\mathfrak{P}}\colon \mathrm{Hom}_k(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$ constructed as the composition

$$\mathrm{Hom}_k(\mathcal{C}, \mathrm{Hom}_k(\mathcal{N}, V)) \simeq \mathrm{Hom}_k(\mathcal{N} \otimes_k \mathcal{C},\, V) \longrightarrow \mathrm{Hom}_k(\mathcal{N}, V) \tag{23}$$

of the natural isomorphism of Hom spaces and the map induced by the right coaction map $\nu_{\mathcal{N}}\colon \mathcal{N} \longrightarrow \mathcal{N} \otimes_k \mathcal{C}$.

The category of left $\mathcal{C}$-comodules $\mathcal{C}$–comod is abelian with exact functors of infinite direct sum and an injective cogenerator $\mathcal{C}$, while the category of left $\mathcal{C}$-contramodules $\mathcal{C}$–contra is abelian with exact functors of infinite product and a projective generator $\mathcal{C}^\vee$. More generally, a *cofree* left $\mathcal{C}$-comodule is a $\mathcal{C}$-comodule of the form $\mathcal{C} \otimes_k V$, and a *free* left $\mathcal{C}$-contramodule is a $\mathcal{C}$-contramodule of the form $\mathrm{Hom}_k(\mathcal{C}, V)$, where V is a k-vector space. For any left $\mathcal{C}$-comodule $\mathcal{M}$, left $\mathcal{C}$-comodule morphisms $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k V$ correspond bijectively to k-linear maps $\mathcal{M} \longrightarrow V$; and similarly, for any left $\mathcal{C}$-contramodule $\mathfrak{P}$, left $\mathcal{C}$-contramodule morphisms $\mathrm{Hom}_k(\mathcal{C}, V) \longrightarrow \mathfrak{P}$ correspond bijectively to k-linear maps $V \longrightarrow \mathfrak{P}$. The injective objects of $\mathcal{C}$–comod are precisely the direct summands of the cofree left $\mathcal{C}$-comodules, and the projective objects of $\mathcal{C}$–contra are precisely the direct summands of free left $\mathcal{C}$-contramodules [14, Section 1.2].

Every left $\mathcal{C}$-contramodule $\mathfrak{P}$ has a natural underlying structure of a left $\mathcal{C}^\vee$-module. The left action of $\mathcal{C}^\vee$ in $\mathfrak{P}$ is given by the composition

$$\mathcal{C}^\vee \otimes_k \mathfrak{P} \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$$

of the natural inclusion $\mathcal{C}^\vee \otimes_k \mathfrak{P} \hookrightarrow \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P})$ (whose image is the vector subspace of all linear operators $\mathcal{C} \longrightarrow \mathfrak{P}$ of finite rank in the vector space of arbitrary such operators) and the left contraaction map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$. So there is a faithful forgetful functor

$$\mathcal{C}\text{–contra} \longrightarrow \mathcal{C}^\vee\text{–mod}. \tag{24}$$

Following the above discussion, for any right $\mathcal{C}$-comodule $\mathcal{N}$ the vector space $\mathcal{N}^\vee = \mathrm{Hom}_k(\mathcal{N}, k)$ has a natural left $\mathcal{C}$-contramodule structure. On the other hand, $\mathcal{N}^\vee$ has a natural pseudo-compact topology and a structure of a pseudo-compact left $\mathcal{C}^\vee$-module. Moreover, all the pseudo-compact left $\mathcal{C}^\vee$-modules come from right $\mathcal{C}$-comodules $\mathcal{N}$ in this way (see (20)). Thus the functor of forgetting the pseudo-compact topology of a pseudo-compact left $\mathcal{C}^\vee$-module (assigning to a pseudo-compact left $\mathcal{C}^\vee$-module its underlying nontopological left $\mathcal{C}^\vee$-module) factorizes naturally through the category of left $\mathcal{C}$-contramodules,

$$\mathcal{C}^\vee\text{–pscomp} \longrightarrow \mathcal{C}\text{–contra} \longrightarrow \mathcal{C}^\vee\text{–mod}. \tag{25}$$

Let us say a few words about what the functor $\mathcal{C}^\vee$–pscomp $\longrightarrow \mathcal{C}$–contra does. It forgets the topology of a pseudo-compact $\mathcal{C}^\vee$-module while keeping a remnant of it in the form of the *infinite summation operations* with zero-convergent families of elements of $\mathcal{C}^\vee$ as the coefficients. We refer to [14, Sections 1.3, 2.1, and 2.3] for a discussion of the contramodule infinite summation operations. Subsequently, the functor $\mathcal{C}$–contra $\longrightarrow \mathcal{C}^\vee$–mod forgets the infinite summation operations, too, keeping only the conventional action of the k-algebra $\mathcal{C}^\vee$ in a module over it. (See also the discussion in Section 1.3 of the introduction.)

Both the functors in (25) are faithful, but neither one of them is full, generally speaking. In other words, the action of these functors on morphisms is injective, but not, in general, surjective. Furthermore, neither one of the two functors is surjective on the isomorphism classes of objects (not even of objects finite-dimensional over k). We refer to [11, Section A.1.2] for a detailed discussion.

The following trivial example is helpful to keep in mind: when $\mathcal{C} = k$ is the trivial coalgebra, one has $\mathcal{C}^\vee$–pscomp $= k$–pscomp and $\mathcal{C}$–contra $= k$–vect $= \mathcal{C}^\vee$–mod. So

the functor $\mathcal{C}$–contra $\longrightarrow \mathcal{C}^\vee$–mod is an equivalence of categories in this case, but the functor $\mathcal{C}^\vee$–pscomp $\longrightarrow \mathcal{C}$–contra is very far from being an equivalence. More generally, the functor $\mathcal{C}^\vee$–pscomp $\longrightarrow \mathcal{C}$–contra is *never* an equivalence of categories for a nonzero coalgebra $\mathcal{C}$, while the functor $\mathcal{C}$–contra $\longrightarrow \mathcal{C}^\vee$–mod is an equivalence for any finite-dimensional coalgebra $\mathcal{C}$, as well as in some more interesting cases covered by Theorem 2.1 below.

2.2. The full-and-faithfulness theorem.   A coalgebra $\mathcal{C}$ is called *conilpotent* if it has a unique irreducible (say, left) comodule, whose dimension over k is equal to 1. Alternatively, a (coassociative) coalgebra without counit $\mathcal{D}$ is called conilpotent if for every element $x \in \mathcal{D}$ there exists an integer $m \geq 0$ such that x is annihilated by the iterated comultiplication map $\mathcal{D} \longrightarrow \mathcal{D}^{\otimes m+1}$. A *coaugmented* coalgebra $\mathcal{C}$ is a (coassociative and counital) coalgebra endowed with a morphism of coalgebras (the coaugmentation) $\gamma \colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $\mathcal{C}$ is called conilpotent if the coalgebra without counit $\mathcal{C}_+ = \mathcal{C}/\gamma(k)$ is conilpotent [21]. This definition is equivalent to the previous one [24, Section 9.1]. A conilpotent (counital) coalgebra $\mathcal{C}$ always has a unique coaugmentation.

The *cohomology algebra $H^*(\mathcal{C})$* of a coaugmented coalgebra $\mathcal{C}$ can be defined as the Yoneda Ext-algebra $\mathrm{Ext}^*_{\mathcal{C}}(k, k)$ computed in the abelian category of left $\mathcal{C}$-comodules $\mathcal{C}$–comod. More explicitly, one has, in particular, $H^1(\mathcal{C}) = \ker(\mathcal{C}_+ \to \mathcal{C}_+ \otimes_k \mathcal{C}_+)$; this vector space can be interpreted as the *space of cogenerators* of the conilpotent coalgebra $\mathcal{C}$. We refer to [21], [10], and [15, Section 5] for further details.

Theorem 2.1.   *Let $\mathcal{C}$ be a conilpotent coalgebra such that the k-vector space $H^1(\mathcal{C})$ is finite-dimensional. Then the forgetful functor $\mathcal{C}$–contra $\longrightarrow \mathcal{C}^\vee$–mod is fully faithful.*
Moreover, for any dense subring R in the topological ring $\mathcal{C}^\vee$, the forgetful functor $\mathcal{C}$–contra $\longrightarrow R$–mod is fully faithful.

Proof.   Given two left $\mathcal{C}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ and a left R-module morphism $f \colon \mathfrak{P} \longrightarrow \mathfrak{Q}$, we have to show that f is a $\mathcal{C}$-contramodule morphism. Composing f with the contraaction morphism $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ and replacing $\mathfrak{P}$ with $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P})$, we can assume that $\mathfrak{P}$ is a free left $\mathcal{C}$-contramodule, $\mathfrak{P} = \mathrm{Hom}_k(\mathcal{C}, V)$ for some k-vector space V. Then the composition

$$V \longrightarrow \mathrm{Hom}_k(\mathcal{C}, V) \xrightarrow{\ f\ } \mathfrak{Q} \tag{26}$$

of the k-linear map $V \longrightarrow \mathrm{Hom}_k(\mathcal{C}, V)$ induced by the counit $\varepsilon \colon \mathcal{C} \longrightarrow k$ with the R-module morphism $f \colon \mathfrak{P} = \mathrm{Hom}_k(\mathcal{C}, V) \longrightarrow \mathfrak{Q}$ extends uniquely to a left $\mathcal{C}$-contramodule morphism $f' \colon \mathrm{Hom}_k(\mathcal{C}, V) \longrightarrow \mathfrak{Q}$. Replacing f with $f - f'$, we can assume that the composition (26) vanishes. Then we have to show that $f = 0$. Furthermore, replacing $\mathfrak{Q}$ with its $\mathcal{C}$-subcontramodule generated by $f(\mathfrak{P})$, we can assume that $\mathfrak{Q}$ has no proper subcontramodules containing $f(\mathfrak{P})$.

For any left $\mathcal{C}$-contramodule $\mathfrak{L}$, denote by $\mathfrak{L}^+ \subset \mathfrak{L}$ the image of the composition of maps $\mathrm{Hom}_k(\mathcal{C}_+, \mathfrak{L}) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \mathfrak{L}) \longrightarrow \mathfrak{L}$. Then $\mathfrak{L}^+$ is a $\mathcal{C}$-subcontramodule in $\mathfrak{L}$ and $\mathfrak{L}/\mathfrak{L}^+$ is the maximal quotient contramodule of $\mathfrak{L}$ with a trivial $\mathcal{C}$-contramodule structure (the latter notion being defined in terms of the coaugmentation of $\mathcal{C}$). According to the contramodule Nakayama lemma [11, Lemma A.2.1] (cf. [14, Lemma 2.1], [16, Lemma 1.3.1], [13, Lemma D.1.2]), $\mathfrak{L}/\mathfrak{L}^+ \neq 0$ whenever $\mathfrak{L} \neq 0$.

Let us first discuss the case when $R = \mathcal{C}^\vee$. Let $\mathcal{C}_+ \longrightarrow U$ be a k-linear map from $\mathcal{C}_+$ to a finite-dimensional k-vector space U such that the composition $H^1(\mathcal{C}) \longrightarrow \mathcal{C}_+ \longrightarrow U$ is injective. Since $\mathcal{C}$ is conilpotent, the composition $\mathcal{C}_+ \longrightarrow \mathcal{C} \otimes_k \mathcal{C}_+ \longrightarrow \mathcal{C} \otimes_k U$ is then also an injective left $\mathcal{C}$-comodule morphism [15, Lemma 5.1]. Hence the induced map $\operatorname{Hom}_k(\mathcal{C} \otimes_k U, \mathfrak{L}) \longrightarrow \operatorname{Hom}_k(\mathcal{C}_+, \mathfrak{L})$ is surjective, and it follows from the contraassociativity axiom that $\mathfrak{L}^+ \subset \mathfrak{L}$ is the image of the composition of the contraaction maps $\operatorname{Hom}_k(U, \operatorname{Hom}_k(\mathcal{C}, \mathfrak{L})) \longrightarrow \operatorname{Hom}_k(U, \mathfrak{L}) \longrightarrow \mathfrak{L}$. Thus $\mathfrak{L}^+$ is the image of the contraaction map $\operatorname{Hom}_k(U, \mathfrak{L}) \longrightarrow \mathfrak{L}$.

The vector space U being finite-dimensional, we have $\operatorname{Hom}_k(U, \mathfrak{L}) \simeq U^\vee \otimes_k \mathfrak{L}$. The map $U^\vee \otimes_k \mathfrak{L} \longrightarrow \mathfrak{L}$ can be obtained as the composition $U^\vee \otimes_k \mathfrak{L} \longrightarrow \mathcal{C}^\vee \otimes_k \mathfrak{L} \longrightarrow \mathfrak{L}$ of the map induced by the natural linear map $U^\vee \longrightarrow \mathcal{C}^\vee$ and the $\mathcal{C}^\vee$-action map. It follows that any left $\mathcal{C}^\vee$-module morphism $\mathfrak{P} \longrightarrow \mathfrak{Q}$ between two left $\mathcal{C}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ takes $\mathfrak{P}^+ \subset \mathfrak{P}$ into $\mathfrak{Q}^+ \subset \mathfrak{Q}$. We have shown that $f(\mathfrak{P}^+) \subset \mathfrak{Q}^+$.

Now we have $\mathfrak{P} = \operatorname{Hom}_k(\mathcal{C}, V)$, hence $\mathfrak{P}/\mathfrak{P}^+ = V$. By our assumption, the induced map $\mathfrak{P}/\mathfrak{P}^+ \longrightarrow \mathfrak{Q}/\mathfrak{Q}^+$ vanishes. Therefore, $f(\mathfrak{P}) \subset \mathfrak{Q}^+$. By another assumption of ours, it follows that $\mathfrak{Q}^+ = \mathfrak{Q}$. Applying the contramodule Nakayama lemma, we can conclude that $\mathfrak{Q} = 0$.

More generally, let R be a dense subring in the pseudo-compact algebra $\mathcal{C}^\vee$. The discrete k-vector space $H^1(\mathcal{C})$ is dual to the pseudo-compact quotient k-vector space of the augmentation ideal $\mathfrak{I} = \mathcal{C}_+^\vee \subset \mathcal{C}^\vee$ by the closure $\mathfrak{I}_2$ of the ideal $\mathfrak{I}^2 = (\mathcal{C}_+^\vee)^2$ in $\mathcal{C}^\vee$, that is $H^1(\mathcal{C})^\vee \simeq \mathfrak{I}/\mathfrak{I}_2$. Since $H^1(\mathcal{C})$ is finite-dimensional, it follows that $\mathfrak{I}/\mathfrak{I}_2$ is also finite-dimensional and discrete, and $\mathfrak{I}_2$ is open in $\mathfrak{I}$. (We will see below that $\mathfrak{I}^2$ is in fact an open ideal in $\mathcal{C}^\vee$, so $\mathfrak{I}_2 = \mathfrak{I}^2$.) Since R is dense in $\mathcal{C}^\vee$ (and consequently $R \cap \mathfrak{I}$ is dense in $\mathfrak{I}$, as the unit element of $\mathcal{C}^\vee$ belongs to R by the definition of a subring), there exists a finite-dimensional subspace $U^\vee \subset R \cap \mathfrak{I}$ such that the projection map $U^\vee \longrightarrow \mathfrak{I}/\mathfrak{I}_2$ is surjective.

Dualizing the composition of maps $U^\vee \longrightarrow R \cap \mathfrak{I} \longrightarrow \mathfrak{I} = \mathcal{C}_+^\vee$, we obtain a k-linear map $\mathcal{C}_+ \longrightarrow U$ from $\mathcal{C}_+$ to a finite-dimensional k-vector space U such that the composition $H^1(\mathcal{C}) \longrightarrow \mathcal{C}_+ \longrightarrow U$ is injective. Since $\mathcal{C}$ is conilpotent, the composition $\mathcal{C}_+ \longrightarrow \mathcal{C} \otimes_k \mathcal{C}_+ \longrightarrow \mathcal{C} \otimes_k U$ is then also an injective left $\mathcal{C}$-comodule morphism [15, Lemma 5.1]. Arguing as above, we see that for any left $\mathcal{C}$-contramodule $\mathfrak{L}$ the subcontramodule $\mathfrak{L}^+ \subset \mathfrak{L}$ is the image of the composition $U^\vee \otimes_k \mathfrak{L} \longrightarrow \mathcal{C}^\vee \otimes_k \mathfrak{L} \longrightarrow \mathfrak{L}$ of the map induced by the embedding $U^\vee \longrightarrow \mathcal{C}^\vee$ and the $\mathcal{C}^\vee$-action map.

The image of the map $U^\vee \longrightarrow \mathcal{C}^\vee$ lies in $R \subset \mathcal{C}^\vee$. So the map $U^\vee \otimes_k \mathfrak{L} \longrightarrow \mathfrak{L}$ can be also obtained as the composition $U^\vee \otimes_k \mathfrak{L} \longrightarrow R \otimes_k \mathfrak{L} \longrightarrow \mathfrak{L}$ of the map induced by the embedding $U^\vee \longrightarrow R$ and the left R-action map. It follows that any left R-module morphism $\mathfrak{P} \longrightarrow \mathfrak{Q}$ between two left $\mathcal{C}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ takes $\mathfrak{P}^+ \subset \mathfrak{P}$ into $\mathfrak{Q}^+ \subset \mathfrak{Q}$. Once again, we have shown that $f(\mathfrak{P}^+) \subset \mathfrak{Q}^+$, and the argument finishes as above.

Finally, let us explain why $\mathfrak{I}_2 = \mathfrak{I}^2 = \mathfrak{I}U^\vee \subset \mathcal{C}^\vee$ (notice that, the vector space $U^\vee$ being finite-dimensional, the ideal $\mathfrak{I}U^\vee \subset \mathcal{C}^\vee$ is clearly closed as the image of a continious linear map between pseudo-compact vector spaces). Equivalently, this means that the composition $\mathcal{C}_+ \longrightarrow \mathcal{C}_+ \otimes_k \mathcal{C}_+ \longrightarrow \mathcal{C}_+ \otimes_k U$ has the same kernel as the comultiplication map $\mathcal{C}_+ \longrightarrow \mathcal{C}_+ \otimes_k \mathcal{C}_+$. The kernel of the composition $\mathcal{C}_+ \longrightarrow \mathcal{C}_+ \otimes_k \mathcal{C}_+ \longrightarrow \mathcal{C}_+ \otimes_k \mathcal{C} \otimes_k U$ is equal to the kernel of the map $\mathcal{C}_+ \longrightarrow \mathcal{C}_+ \otimes_k \mathcal{C}_+$, as the map $\mathcal{C}_+ \longrightarrow \mathcal{C} \otimes_k U$ is injective as we have seen.

The composition $\mathcal{C}_+ \longrightarrow \mathcal{C}_+ \otimes_k U \longrightarrow \mathcal{C}_+ \otimes_k \mathcal{C} \otimes_k U$ provides the same map, so the kernel of $\mathcal{C}_+ \longrightarrow \mathcal{C}_+ \otimes_k U$ is contained in that of $\mathcal{C}_+ \longrightarrow \mathcal{C}_+ \otimes_k \mathcal{C}_+$. It follows that $\mathfrak{I}^n = \mathfrak{I}(U^\vee)^{n-1}$ is a closed ideal in $\mathcal{C}^\vee$ for every $n \geq 2$. ∎

2.3. Cotensor product and contratensor product. The *cotensor product* $\mathcal{N} \,\square_\mathcal{C}\, \mathcal{M}$ of a right $\mathcal{C}$-comodule $\mathcal{N}$ and a left $\mathcal{C}$-comodule $\mathcal{M}$ is a k-vector space constructed as the kernel of the difference of the two maps

$$\mathcal{N} \otimes_k \mathcal{M} \rightrightarrows \mathcal{N} \otimes_k \mathcal{C} \otimes_k \mathcal{M},$$

one of which is induced by the right coaction map $\mathcal{N} \longrightarrow \mathcal{N} \otimes_k \mathcal{C}$ and the other one by the left coaction map $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$. The equivalences of categories (19) and (20) transform the functor of cotensor product of $\mathcal{C}$-comodules into the functor of completed tensor product of pseudo-compact $\mathcal{C}^\vee$-modules.

For any pseudo-compact right $\mathcal{C}^\vee$-module $\mathcal{M}^\vee$ and any pseudo-compact left $\mathcal{C}^\vee$-module $\mathcal{N}^\vee$, the *completed tensor product* $\mathcal{M}^\vee \,\widehat{\otimes}_{\mathcal{C}^\vee}\, \mathcal{N}^\vee$ is a pseudo-compact k-vector space constructed as the cokernel of the difference of two maps

$$\mathcal{M}^\vee \,\widehat{\otimes}_k\, \mathcal{C}^\vee \,\widehat{\otimes}_k\, \mathcal{N}^\vee \rightrightarrows \mathcal{M}^\vee \,\widehat{\otimes}_k\, \mathcal{N}^\vee,$$

one of which is induced by the right action map $\mathcal{M}^\vee \,\widehat{\otimes}_k\, \mathcal{C}^\vee \longrightarrow \mathcal{M}^\vee$ and the other one by the left action map $\mathcal{C}^\vee \,\widehat{\otimes}_k\, \mathcal{N}^\vee \longrightarrow \mathcal{N}^\vee$. For any right $\mathcal{C}$-comodule $\mathcal{N}$ and a left $\mathcal{C}$-comodule $\mathcal{M}$, there is a natural isomorphism of pseudo-compact vector spaces

$$(\mathcal{N} \,\square_\mathcal{C}\, \mathcal{M})^\vee \simeq \mathcal{M}^\vee \,\widehat{\otimes}_{\mathcal{C}^\vee}\, \mathcal{N}^\vee.$$

The functor of cotensor product of $\mathcal{C}$-comodules is left exact. The functor of completed tensor product of pseudo-compact $\mathcal{C}^\vee$-modules is right exact.

The *contratensor product* $\mathcal{N} \odot_\mathcal{C} \mathfrak{P}$ of a right $\mathcal{C}$-comodule $\mathcal{N}$ and a left $\mathcal{C}$-contramodule $\mathfrak{P}$ is a k-vector space constructed as the cokernel of the difference of two natural maps

$$\mathcal{N} \otimes_k \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \rightrightarrows \mathcal{N} \otimes_k \mathfrak{P}. \tag{27}$$

Here the first map is simply induced by the left contraaction map $\pi_\mathfrak{P} \colon \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$, while the second map is the composition $\mathcal{N} \otimes_k \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathcal{N} \otimes_k \mathcal{C} \otimes_k \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathcal{N} \otimes_k \mathfrak{P}$ of the map induced by the right coaction map $\nu_\mathcal{N} \colon \mathcal{N} \longrightarrow \mathcal{N} \otimes_k \mathcal{C}$ and the map induced by the evaluation map $\mathcal{C} \otimes_k \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$.

For any right $\mathcal{C}$-comodule $\mathcal{N}$, left $\mathcal{C}$-contramodule $\mathfrak{P}$, and k-vector space V there is a natural isomorphism of k-vector spaces

$$\mathrm{Hom}_k(\mathcal{N} \odot_\mathcal{C} \mathfrak{P}, \, V) \simeq \mathrm{Hom}^\mathcal{C}(\mathfrak{P}, \mathrm{Hom}_k(\mathcal{N}, V)),$$

where we denote by $\mathrm{Hom}^\mathcal{C}$ the space of morphisms in the category of left $\mathcal{C}$-contramodules $\mathcal{C}$–contra [14, Section 3.1].

The functor of contratensor product of $\mathcal{C}$-comodules and $\mathcal{C}$-contramodules is right exact. Let us emphasize that the functor of completed tensor product is dual to the functor of cotensor product, as mentioned above. There is *no* simple connection between the completed tensor product and the contratensor product.

There is an important, natural pair of adjoint functors between the categories of left $\mathcal{C}$-comodules and left $\mathcal{C}$-contramodules,

$$\Psi_\mathcal{C} \colon \mathcal{C}\text{–comod} \rightleftarrows \mathcal{C}\text{–contra} \colon \Phi_\mathcal{C}, \tag{28}$$

defined by the rules $\Phi_{\mathcal{C}}(\mathfrak{P}) = \mathcal{C} \odot_{\mathcal{C}} \mathfrak{P}$ and $\Psi_{\mathcal{C}}(\mathcal{M}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$, where $\mathfrak{P}$ is a left $\mathcal{C}$-contramodule, $\mathcal{M}$ is a left $\mathcal{C}$-comodule, and $\mathrm{Hom}_{\mathcal{C}}$ denotes the space of morphisms in the category of left $\mathcal{C}$-comodules $\mathcal{C}$–comod [14, Section 3.1].

Here the construction of the vector space $\mathcal{C} \odot_{\mathcal{C}} \mathfrak{P}$ uses the right $\mathcal{C}$-comodule structure on the coalgebra $\mathcal{C}$, while the left $\mathcal{C}$-comodule structure on $\mathcal{C}$ induces a left $\mathcal{C}$-comodule structure on the contratensor product $\mathcal{C} \odot_{\mathcal{C}} \mathfrak{P}$. Similarly, the construction of the vector space $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$ uses the left $\mathcal{C}$-comodule structure on $\mathcal{C}$, while the right $\mathcal{C}$-comodule structure on $\mathcal{C}$ induces a left $\mathcal{C}$-contramodule structure on $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$ (see (23) for the construction). The pair of adjoint functors $\Psi_{\mathcal{C}}$ and $\Phi_{\mathcal{C}}$ restricts to an equivalence between the full subcategories of injective left $\mathcal{C}$-comodules and projective left $\mathcal{C}$-contramodules,

$$\mathcal{C}\text{–comod}^{\mathsf{inj}} \simeq \mathcal{C}\text{–contra}^{\mathsf{proj}}. \tag{29}$$

The equivalence of categories (29) takes the cofree left $\mathcal{C}$-comodule $\mathcal{C} \otimes_k V$ to the free left $\mathcal{C}$-contramodule $\mathrm{Hom}_k(\mathcal{C}, V)$ for any k-vector space V [14, Section 1.2], [11, Section 0.2.6].

Returning to the discussion of the comparison of contramodules with pseudo-compact modules in Section 2.1, one has to say that, generally speaking, of course, the $\mathcal{C}$-contramodule structure on $\Psi_{\mathcal{C}}(\mathcal{M})$ does *not* underlie any pseudo-compact topology. In fact, for the trivial coalgebra $\mathcal{C} = k$, one has $\mathcal{C}$–comod $= k$–vect $= \mathcal{C}$–contra, and both $\Phi_{\mathcal{C}}$ and $\Psi_{\mathcal{C}}$ are simply the identity functors k–vect $\longrightarrow k$–vect.

However, there is one notable particular case when a natural structure of a pseudo-compact left $\mathcal{C}^{\vee}$-module can be defined on the vector space $\Psi_{\mathcal{C}}(\mathcal{M})$, and the left $\mathcal{C}$-contramodule structure of $\Psi_{\mathcal{C}}(\mathcal{M})$ comes from this pseudo-compact module structure. This is the case for *finitely cogenerated* injective left $\mathcal{C}$-comodules $\mathcal{M}$ (i. e., the direct summands of cofree left $\mathcal{C}$-comodules $\mathcal{C} \otimes_k V$ with finite-dimensional vector spaces of cogenerators V) and, more generally, for *quasi-finitely cogenerated* left $\mathcal{C}$-comodules $\mathcal{M}$. We refer to the discussion in Section 1.9 of the introduction, and to the details in Sections 5.1–5.2 below.

For any coalgebra $\mathcal{C}$, a right $\mathcal{C}$-comodule $\mathcal{N}$, and a left $\mathcal{C}$-contramodule $\mathfrak{P}$ there is a natural surjective map from the tensor product to the contratensor product

$$\mathcal{N} \otimes_{\mathcal{C}^{\vee}} \mathfrak{P} \longrightarrow \mathcal{N} \odot_{\mathcal{C}} \mathfrak{P}. \tag{30}$$

Here the tensor product in the left-hand side is the abstract (uncompleted) tensor product of abstract (nontopological) $\mathcal{C}^{\vee}$-modules whose $\mathcal{C}^{\vee}$-module structures arise from the comodule and contramodule structures via the forgetful functors (22) and (24). In order to construct the surjective map (30), it suffices to observe that there is a pair of commutative diagrams

$$
\begin{array}{ccc}
\mathcal{N} \otimes_k \mathcal{C}^{\vee} \otimes_k \mathfrak{P} & \overset{\rightrightarrows}{\longrightarrow} & \mathcal{N} \otimes_k \mathfrak{P} \\
\downarrow & & \| \\
\mathcal{N} \otimes_k \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) & \overset{\rightrightarrows}{\longrightarrow} & \mathcal{N} \otimes_k \mathfrak{P}
\end{array}
$$

where the tensor product $\mathcal{N} \otimes_{\mathcal{C}^{\vee}} \mathfrak{P}$ is the cokernel of the difference of the two maps in the upper horizontal line and the contratensor product $\mathcal{N} \odot_{\mathcal{C}} \mathfrak{P}$ is the cokernel of the difference of the two arrows in the lower horizontal line.

Corollary 2.2. *Let $\mathcal{C}$ be a conilpotent coalgebra such that the k-vector space $H^1(\mathcal{C})$ is finite-dimensional. Then for any right $\mathcal{C}$-comodule $\mathcal{N}$ and any left $\mathcal{C}$-contramodule $\mathfrak{P}$ the natural map $\mathcal{N} \otimes_{\mathcal{C}^\vee} \mathfrak{P} \longrightarrow \mathcal{N} \odot_{\mathcal{C}} \mathfrak{P}$ is an isomorphism. Moreover, for any dense subring R in the topological ring $\mathcal{C}^\vee$, the natural map $\mathcal{N} \otimes_R \mathfrak{P} \longrightarrow \mathcal{N} \odot_{\mathcal{C}} \mathfrak{P}$ is an isomorphism.*

Proof. Applying the functor $\mathrm{Hom}_k(-, V)$ to the map in question, we obtain the map

$$\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, \mathrm{Hom}_k(\mathcal{N}, V)) \simeq \mathrm{Hom}_k(\mathcal{N} \odot_{\mathcal{C}} \mathfrak{P}, V)$$
$$\longrightarrow \mathrm{Hom}_k(\mathcal{N} \otimes_R \mathfrak{P}, V) \simeq \mathrm{Hom}_R(\mathfrak{P}, \mathrm{Hom}_k(\mathcal{N}, V)),$$

which is an isomorphism by Theorem 2.1. ∎

2.4. Cohomomorphisms. Yet another relevant tensor operation involving co-modules and contramodules is the vector space of *cohomomorphisms*, or Cohom. It can be thought of as answering the following question.

In the previous section, we have seen how to construct the dual vector space to the cotensor product $(\mathcal{N} \square_{\mathcal{C}} \mathcal{M})^\vee$ in terms of the pseudo-compact $\mathcal{C}^\vee$-modules $\mathcal{N}^\vee$ and $\mathcal{M}^\vee$. In this section we will see how to construct the vector space $(\mathcal{N} \square_{\mathcal{C}} \mathcal{M})^\vee$ in terms of the left $\mathcal{C}$-comodule $\mathcal{M}$ and the left $\mathcal{C}$-contramodule $\mathcal{N}^\vee$.

One difference is that we will only obtain $(\mathcal{N} \square_{\mathcal{C}} \mathcal{M})^\vee$ as an abstract vector space, without the pseudo-compact topology on it. On the other hand, the same construction will also produce the vector space $\mathrm{Hom}_k(\mathcal{N} \square_{\mathcal{C}} \mathcal{M}, V)$ from the left $\mathcal{C}$-comodule $\mathcal{M}$ and the left $\mathcal{C}$-contramodule $\mathrm{Hom}_k(\mathcal{N}, V)$ for any vector space V.

Let $\mathcal{M}$ be a left $\mathcal{C}$-comodule and $\mathfrak{P}$ be a left $\mathcal{C}$-contramodule. The k-vector space of *cohomomorphisms* $\mathrm{Cohom}_{\mathcal{C}}(\mathcal{M}, \mathfrak{P})$ is constructed as the cokernel of the difference of the two maps

$$\mathrm{Hom}_k(\mathcal{C} \otimes_k \mathcal{M}, \mathfrak{P}) = \mathrm{Hom}_k(\mathcal{M}, \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P})) \rightrightarrows \mathrm{Hom}_k(\mathcal{M}, \mathfrak{P}),$$

one of which is induced by the left coaction map $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ and the other one by the left contraaction map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$.

For any right $\mathcal{C}$-comodule $\mathcal{N}$, left $\mathcal{C}$-comodule $\mathcal{M}$, and k-vector space V there is a natural isomorphism of k-vector spaces [14, Section 2.6]

$$\mathrm{Hom}_k(\mathcal{N} \square_{\mathcal{C}} \mathcal{M}, V) \simeq \mathrm{Cohom}_k(\mathcal{M}, \mathrm{Hom}_k(\mathcal{N}, V)). \tag{31}$$

Let us emphasize that $\mathrm{Cohom}_{\mathcal{C}}$ is *not* the functor Hom in any category, if only because its two arguments are objects from two different categories: $\mathcal{M}$ is a left $\mathcal{C}$-comodule, and $\mathfrak{P}$ is a left $\mathcal{C}$-contramodule. Furthermore, the functor $\mathrm{Cohom}_{\mathcal{C}}$ is right exact, while the functor Hom in any abelian category is left exact.

2.5. Injective and projective objects. Our next aim is to show that, under certain assumptions, all injective $\mathcal{C}$-comodules are injective $\mathcal{C}^\vee$-modules and all projective $\mathcal{C}$-contramodules are flat $\mathcal{C}^\vee$-modules. It will be convenient to increase the generality slightly and use the language centered around the topological ring $\mathfrak{R} = \mathcal{C}^\vee$ rather than the coalgebra $\mathcal{C}$.

Let $\mathfrak{R}$ be an associative ring and $\mathfrak{I} \subset \mathfrak{R}$ be a two-sided ideal such that $\mathfrak{R}$ is separated and complete in the $\mathfrak{I}$-adic topology, that is $\mathfrak{R} = \varprojlim_n \mathfrak{R}/\mathfrak{I}^n$. Consider the associated graded ring $\mathrm{gr}_{\mathfrak{I}}\mathfrak{R} = \bigoplus_{n=0}^\infty \mathfrak{I}^n/\mathfrak{I}^{n+1}$ and the ideal $\mathrm{gr}_{\mathfrak{I}}\mathfrak{I} = \bigoplus_{n=1}^\infty \mathfrak{I}^n/\mathfrak{I}^{n+1} \subset \mathrm{gr}_{\mathfrak{I}}\mathfrak{R}$. The following assertion is a version of the Artin–Rees lemma.

Lemma 2.3. *Assume that the ring $\mathrm{gr}_{\mathfrak{J}}\mathfrak{R}$ is right Noetherian and the ideal $\mathrm{gr}_{\mathfrak{J}}\mathfrak{J} \subset \mathrm{gr}_{\mathfrak{J}}\mathfrak{R}$ is generated by (a finite set of) central elements in $\mathrm{gr}_{\mathfrak{J}}\mathfrak{R}$. Let M be a finitely generated right $\mathfrak{R}$-module with an $\mathfrak{R}$-submodule $N \subset M$. Then there exists an integer $m \geq 0$ such that $N \cap M\mathfrak{J}^{n+m} = (N \cap M\mathfrak{J}^m)\mathfrak{J}^n$ for all $n \geq 0$.*

Proof. As in [5, Lemma 13.2], it suffices to show that the Rees ring $\mathfrak{R}(\mathfrak{J}) = \bigoplus_{n=0}^{\infty} \mathfrak{J}^n$ is right Noetherian (as a graded ring). Indeed, $M(\mathfrak{J}) = \bigoplus_{n=0}^{\infty} M\mathfrak{J}^n$ is a finitely generated graded $\mathfrak{R}(\mathfrak{J})$-module, and $\bigoplus_{n=0}^{\infty} N \cap M\mathfrak{J}^n$ is a graded $\mathfrak{R}(\mathfrak{J})$-sub-module in $M(\mathfrak{J})$. It remains to choose $m \geq 0$ such that this submodule is generated by some (finite) set of elements of the degree $\leq m$.

Consider the ideal $\mathfrak{J} = \mathfrak{J} \oplus \mathfrak{J} \oplus \mathfrak{J}^2 \oplus \mathfrak{J}^3 \subset \mathfrak{R}(\mathfrak{J})$, so that the quotient ring is $\mathfrak{R}(\mathfrak{J})/\mathfrak{J} = \mathfrak{R}/\mathfrak{J}$. The ring $\mathfrak{R}(\mathfrak{J})$ is separated and complete (as a graded ring) in the $\mathfrak{J}$-adic topology, that is $\mathfrak{R}(\mathfrak{J}) = \varprojlim_n \mathfrak{R}(\mathfrak{J})/\mathfrak{J}^n$ in the category of graded abelian groups. The associated graded ring $\mathrm{gr}_{\mathfrak{J}}\mathfrak{R}(\mathfrak{J}) = \bigoplus_{n=0}^{\infty} \mathfrak{J}^n/\mathfrak{J}^{n+1}$ is isomorphic to the Rees ring of the graded ring $\mathrm{gr}_{\mathfrak{J}}\mathfrak{R}$ (endowed with the decreasing filtration associated with the grading), $\mathrm{gr}_{\mathfrak{J}}\mathfrak{R}(\mathfrak{J}) \simeq \mathrm{gr}_{\mathfrak{J}}\mathfrak{R}(\mathrm{gr}_{\mathfrak{J}}\mathfrak{J})$.

Hence the (bi)graded ring $\mathrm{gr}_{\mathfrak{J}}\mathfrak{R}(\mathfrak{J})$ is a quotient ring of the polynomial ring in a finite number of variables over the right Noetherian ring $\mathrm{gr}_{\mathfrak{J}}\mathfrak{R}$, so it is right Noetherian, as in [5, Theorems 1.9 and 13.3]. It remains to deduce the assertion that the graded ring $\mathfrak{R}(\mathfrak{J})$ is right Noetherian. This is a standard argument: given a homogeneous right ideal $\mathfrak{K} \subset \mathfrak{R}(\mathfrak{J})$, one chooses a finite set of bihomogeneous generators of the right ideal $\mathrm{gr}_{\mathfrak{J}}\mathfrak{K} \subset \mathrm{gr}_{\mathfrak{J}}\mathfrak{R}(\mathfrak{J})$ and lifts them to homogeneous elements in $\mathfrak{K}$, obtaining a finite set of generators of the right ideal $\mathfrak{K} \subset \mathfrak{R}(\mathfrak{J})$. ∎

Denote by $\mathsf{discr}\text{–}\mathfrak{R} \subset \mathsf{mod}\text{–}\mathfrak{R}$ the full subcategory of discrete right $\mathfrak{R}$-modules (i. e., right $\mathfrak{R}$-modules $\mathcal{N}$ such that the annihilator of every element $x \in \mathcal{N}$ is a open right ideal in $\mathfrak{R}$). The category $\mathsf{discr}\text{–}\mathfrak{R}$ is a Grothendieck abelian category, so it has enough injective objects.

The definition of a left $\mathfrak{R}$-contramodule can be found in [16, Section 1.2], [14, Section 2.1], not repeated here, as all we need is the construction of *free* left $\mathfrak{R}$-contramodules. Given a set X and a ring R, denote by $R[X]$ the free left R-module generated by the set X. Then the free left $\mathfrak{R}$-contramodule generated by X is

$$\mathfrak{R}[[X]] = \varprojlim_n (\mathfrak{R}/\mathfrak{J}^n)[X].$$

In other words, $\mathfrak{R}[[X]]$ is the set of all maps of sets $f \colon X \longrightarrow \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, which means that for every $n \geq 1$ one has $f(x) \in \mathfrak{J}^n$ for all but a finite subset of elements $x \in X$.

When $\mathcal{C}$ is a conilpotent coalgebra with finite-dimensional space of cogenerators $H^1(\mathcal{C})$, and $\mathfrak{R} = \mathcal{C}^{\vee}$ is the dual algebra, the pseudo-compact topology of $\mathcal{C}^{\vee}$ coincides with the adic topology for the augmentation ideal $\mathfrak{J} = \mathcal{C}_+^{\vee} \subset \mathcal{C}^{\vee}$. Hence the above definition of a free $\mathfrak{R}$-contramodule agrees with the definition of a free $\mathcal{C}$-contramodule (cf. [16, Section 1.10], [14, Section 2.3]).

Proposition 2.4. *Assume that the ring $\mathrm{gr}_{\mathfrak{J}}\mathfrak{R}$ is right Noetherian and the ideal $\mathrm{gr}_{\mathfrak{J}}\mathfrak{J} \subset \mathrm{gr}_{\mathfrak{J}}\mathfrak{R}$ is generated by (a finite set of) central elements in $\mathrm{gr}_{\mathfrak{J}}\mathfrak{R}$. Then*
(a) *every injective object of $\mathsf{discr}\text{–}\mathfrak{R}$ is also an injective object of $\mathsf{mod}\text{–}\mathfrak{R}$;*
(b) *for every set X, the left $\mathfrak{R}$-module $\mathfrak{R}[[X]]$ is flat.*

Proof.   This is a version of [16, Proposition B.9.1] and [14, Proposition 2.2.2]. Notice first of all that the ring $\mathfrak{R}$ is right Noetherian by [1, Proposition 7.27] (cf. the related argument in the proof of Lemma 2.3 above). Part (a): let $\mathcal{J}$ be an injective object of discr–$\mathfrak{R}$. In order to show that $\mathcal{J}$ is an injective right $\mathfrak{R}$-module, it suffices to check that, for any finitely generated right $\mathfrak{R}$-module M with an $\mathfrak{R}$-submodule $N \subset M$, any $\mathfrak{R}$-module morphism $f\colon N \longrightarrow \mathcal{J}$ can be extended to an $\mathfrak{R}$-module morphism $M \longrightarrow \mathcal{J}$. Since the $\mathfrak{R}$-module N is finitely generated and the $\mathfrak{R}$-module $\mathcal{J}$ is discrete, there exists $n \geq 0$ such that $f(N\mathfrak{I}^n) = f(N)\mathfrak{I}^n = 0$ in $\mathcal{J}$. By Lemma 2.3, there exists $m \geq 0$ such that $N \cap M\mathfrak{I}^m \subset N\mathfrak{I}^n$. Now $N/(N \cap M\mathfrak{I}^m)$ is a submodule in a discrete right $\mathfrak{R}$-module $M/M\mathfrak{I}^m$, so the $\mathfrak{R}$-module morphism $N/(N \cap M\mathfrak{I}^m) \longrightarrow N/N\mathfrak{I}^n \longrightarrow \mathcal{J}$ can be extended to an $\mathfrak{R}$-module morphism $M/M\mathfrak{I}^m \longrightarrow \mathcal{J}$.

Part (b): it suffices to show that the functor $M \longmapsto M \otimes_{\mathfrak{R}} \mathfrak{R}[[X]]$ is exact on the abelian category of finitely generated right $\mathfrak{R}$-modules. For any such M, consider the natural morphism of abelian groups

$$M \otimes_{\mathfrak{R}} \mathfrak{R}[[X]] \longrightarrow \varprojlim_n M \otimes_{\mathfrak{R}} \mathfrak{R}/\mathfrak{I}^n[X]. \tag{32}$$

For any short exact sequence of finitely generated right $\mathfrak{R}$-modules $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$, there is a short exact sequence of $\mathfrak{R}/\mathfrak{I}^n$-modules

$$0 \longrightarrow K/(K \cap L\mathfrak{I}^n) \longrightarrow L/L\mathfrak{I}^n \longrightarrow M/M\mathfrak{I}^n \longrightarrow 0,$$

which remains exact after applying the functor $- \otimes_{\mathfrak{R}/\mathfrak{I}^n} \mathfrak{R}/\mathfrak{I}^n[X]$. By Lemma 2.3, the projective systems $K/(K \cap L\mathfrak{I}^n)$ and $K/K\mathfrak{I}^n$ are cofinal, so

$$\varprojlim_n K/(K \cap L\mathfrak{I}^n) \otimes_{\mathfrak{R}/\mathfrak{I}^n} \mathfrak{R}/\mathfrak{I}^n[X] = \varprojlim_n K/K\mathfrak{I}^n \otimes_{\mathfrak{R}/\mathfrak{I}^n} \mathfrak{R}/\mathfrak{I}^n[X].$$

Since $K/(K \cap L\mathfrak{I}^n) \otimes_{\mathfrak{R}/\mathfrak{I}^n} \mathfrak{R}/\mathfrak{I}^n[X]$ is a projective system of surjective morphisms, passing to the projective limit produces a short exact sequence

$$0 \longrightarrow \varprojlim_n K \otimes_{\mathfrak{R}} \mathfrak{R}/\mathfrak{I}^n[X] \longrightarrow \varprojlim_n L \otimes_{\mathfrak{R}} \mathfrak{R}/\mathfrak{I}^n[X] \longrightarrow \varprojlim_n M \otimes_{\mathfrak{R}} \mathfrak{R}/\mathfrak{I}^n[X] \longrightarrow 0.$$

Hence the functor in the right-hand side of the morphism (32) is exact.

The functor in the left-hand side is right exact, and the morphism is an isomorphism when M is a finitely generated *free* right $\mathfrak{R}$-module. It follows that the morphism (32) is an isomorphism for every finitely generated right $\mathfrak{R}$-module M. Thus the left-hand side of (32) is also an exact functor of M. ■

2.6. Profinite groups.   Let H be a profinite group and k be a field. An H-*module over* k is a k-vector space in which H acts by k-linear automorphisms. An H-module $\mathcal{M}$ over k is called *discrete* if the stabilizer of every element $x \in \mathcal{M}$ is an open subgroup in H. The abelian category of H-modules over k is denoted by H–mod_k and the abelian category of discrete H-modules over k is denoted by H–$\mathsf{discr}_k \subset H$–$\mathsf{mod}_k$.

Let X denote a profinite set (or in other words, a compact totally disconnected topological space). For any k-vector space V, we denote by $V(X)$ the k-vector space of all locally constant functions $X \longrightarrow V$. We also denote by $V[[X]]$ the k-vector space of all finitely additive V-valued measures defined on all the open-closed subsets in X. Then there are natural isomorphisms

$$V(X) \simeq V \otimes_k k(X), \qquad V[[X]] \simeq \mathrm{Hom}_k(k(X), V),$$

and
$$V(H) = \varinjlim_{U \subset H} V(H/U), \qquad V[[H]] = \varprojlim_{U \subset H} V[H/U],$$

where the limits are taken over all the (if one wishes, normal) open subgroups $U \subset H$, and $V[H/U] = V(H/U)$ denotes the vector space of all functions $H/U \longrightarrow V$. For any two profinite sets X and Y, there are natural isomorphisms

$$k(X \times Y) \simeq k(X) \otimes_k k(Y), \qquad V[[X \times Y]] \simeq V[[X]][[Y]].$$

An H-*contramodule over* k is a k-vector space $\mathfrak{P}$ endowed with a k-linear H-*contra-action* map $\pi_{\mathfrak{P}} \colon \mathfrak{P}[[H]] \longrightarrow \mathfrak{P}$ satisfying the following two axioms. Firstly, the two maps $\mathfrak{P}[[H]][[H]] \simeq \mathfrak{P}[[H \times H]] \rightrightarrows \mathfrak{P}[[H]]$, one of them provided by the pushforward of measures with respect to the multiplication map $H \times H \longrightarrow H$ and the other one induced by the contraaction map $\pi_{\mathfrak{P}}$, should have equal compositions with the contraaction map $\pi_{\mathfrak{P}}$,

$$\mathfrak{P}[[H]][[H]] \simeq \mathfrak{P}[[H \times H]] \rightrightarrows \mathfrak{P}[[H]] \longrightarrow \mathfrak{P}.$$

Secondly, the composition of the map $\mathfrak{P} \longrightarrow \mathfrak{P}[[H]]$ assigning to a vector $x \in \mathfrak{P}$ the point measure concentrated at the unit element $e \in H$ with the value x and the contraaction map $\pi_{\mathfrak{P}}$ should be equal to the identity map $\mathrm{id}_{\mathfrak{P}}$,

$$\mathfrak{P} \longrightarrow \mathfrak{P}[[H]] \longrightarrow \mathfrak{P}.$$

We refer to [14, Section 1.8] for a discussion of the intuition behind this concept.

H-contramodules over k form an abelian category $H\text{--}\mathsf{contra}_k$. Given an H-contra-module $\mathfrak{P}$ over k, a vector $x \in \mathfrak{P}$, and an element $h \in H$, one can consider the point measure at $h^{-1} \in H$ with the value $x \in \mathfrak{P}$. Applying the contraaction map $\pi_{\mathfrak{P}}$ to this measure, one obtains an element denoted by $h(x) \in \mathfrak{P}$. This construction defines the underlying H-module structure on an H-contramodule $\mathfrak{P}$, providing an exact and faithful forgetful functor $H\text{--}\mathsf{contra}_k \longrightarrow H\text{--}\mathsf{mod}_k$.

The k-vector space $\mathcal{C} = k(H)$ has a natural coalgebra structure with the comultiplication map $\mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ provided by the pullback of functions with respect to the multiplication map $H \times H \longrightarrow H$ and the counit map $\mathcal{C} \longrightarrow k$ similarly induced by the unit map $\{*\} \longrightarrow H$. The dual algebra $\mathcal{C}^\vee = k[[H]]$ is the projective limit of the group algebras $\varprojlim_{U \subset H} k[H/U]$, as above.

The datum of a discrete action of H on a k-vector space $\mathcal{M}$ is equivalent to that of a (left or right) $\mathcal{C}$-comodule structure on $\mathcal{M}$. Analogously, the datum of an H-contramodule structure on a k-vector space $\mathfrak{P}$ is equivalent to that of a (left or right) $\mathcal{C}$-contramodule structure. So there are natural isomorphisms of categories $H\text{--}\mathsf{discr}_k = k(H)\text{--}\mathsf{comod} = \mathsf{comod}\text{--}k(H)$ and $H\text{--}\mathsf{contra}_k = k(H)\text{--}\mathsf{contra}$ (where the left and right $k(H)$-comodules are identified by means of the involutive anti-automorphism of the coalgebra $k(H)$ induced by the inverse element map $H \longrightarrow H$). As usually, we denote by $k[H]$ the group algebra of the group H (viewed as an abstract group with the topology forgotten). So we have natural isomorphisms of categories $H\text{--}\mathsf{mod}_k \simeq k[H]\text{--}\mathsf{mod} \simeq \mathsf{mod}\text{--}k[H]$. There is a natural injective homomorphism of k-algebras $k[H] \longrightarrow k[[H]] = \mathcal{C}^\vee$ inducing the embedding functor $H\text{--}\mathsf{discr}_k \longrightarrow H\text{--}\mathsf{mod}_k$ and the forgetful functor $H\text{--}\mathsf{contra}_k \longrightarrow H\text{--}\mathsf{mod}_k$.

The profinite group cohomology algebra $H^*(H, k)$ is naturally isomorphic to the cohomology algebra $H^*(\mathcal{C})$ of the coalgebra $\mathcal{C} = k(H)$ [10, Section 4.2]. One can show this by interpreting both the cohomology algebras in question as the Yoneda Ext-algebras in the related categories (of $\mathcal{C}$-comodules = discrete H-modules over k), or simply noticing that the complex of continuous (i. e., locally constant) cochains computing the profinite group cohomology coincides with the cobar-complex [21, Section 1.1], [10, Section 2.1] computing the cohomology of $\mathcal{C}$.

Now let us assume that H is a pro-p-group and k is a field of characteristic p. Then the minimal number of generators of the profinite group H can be computed as the dimension of the k-vector space $H^1(H, k) = H^1(\mathcal{C})$.

Corollary 2.5. *Let H be a finitely generated pro-p-group and k be a field of characteristic p. Then the forgetful functor H–$\mathsf{contra}_k \longrightarrow H$–$\mathsf{mod}_k$ is fully faithful. Moreover, for any dense subgroup $H' \subset H$, the forgetful functor H–$\mathsf{contra}_k \longrightarrow H'$–$\mathsf{mod}_k$ is fully faithful.*

Proof. Follows from Theorem 2.1, as $k[H']$ is a dense subring in $k[[H]]$. ∎

Corollary 2.6. *Let H be a finitely generated pro-p-group and k be a field of characteristic p. Then for any discrete H-module $\mathcal{N}$ and any H-contramodule $\mathfrak{P}$ over k the natural map $\mathcal{N} \otimes_{k[H]} \mathfrak{P} \longrightarrow \mathcal{N} \odot_{k(H)} \mathfrak{P}$ is an isomorphism. Moreover, for any dense subgroup $H' \subset H$, the natural map $\mathcal{N} \otimes_{k[H']} \mathfrak{P} \longrightarrow \mathcal{N} \odot_{k(H)} \mathfrak{P}$ is an isomorphism.*

Proof. Follows from Corollary 2.2. ∎

For the definition of a *uniform pro-p-group*, we refer to [1, Definition 4.1].

Corollary 2.7. *Let H be a uniform pro-p-group and k be a field of characteristic p. Then*
(a) *every injective object of H–discr_k is also an injective object of $k[[H]]$–mod;*
(b) *every projective object of H–contra_k is a flat $k[[H]]$-module.*

Proof. Set $\mathfrak{R} = k[[H]]$, and let $\mathfrak{I} \subset \mathfrak{R}$ be the augmentation ideal. According to [1, Theorem 7.24], the graded ring $\mathrm{gr}_{\mathfrak{I}}\mathfrak{R}$ is a commutative polynomial ring in a finite number of variables over k. So Proposition 2.4 applies. ∎

Remark 2.8. For the benefit of a reader having no prior experience with contramodules, let us revisit once again the concepts that we have defined and discussed so far. Returning to the discussion in Sections 1.3 and 2.1, for any coalgebra $\mathcal{C}$ over a field k, the forgetful functor $\mathcal{C}^\vee$–$\mathsf{pscomp} \longrightarrow \mathcal{C}^\vee$–$\mathsf{mod}$ from the category of pseudo-compact left $\mathcal{C}^\vee$-modules to the category of abstract left $\mathcal{C}^\vee$-modules factorizes naturally through the category of left $\mathcal{C}$-contramodules $\mathcal{C}$–contra. In particular, for any profinite group H the forgetful functor H–$\mathsf{pscomp}_k \longrightarrow H$–$\mathsf{mod}_k$ from the category of pseudo-compact H-modules to the category of abstract H-modules over k factorizes naturally through the category of H-contramodules H–contra_k,

$$H\text{–}\mathsf{pscomp}_k \longrightarrow H\text{–}\mathsf{contra}_k \longrightarrow H\text{–}\mathsf{mod}_k.$$

What is the difference between pseudo-compact modules and contramodules? This question is best answered by specializing to the case of the trivial group $H = \{e\}$

(corresponding to the trivial coalgebra $\mathcal{C} = k$). In this case, the category of pseudo-compact H-modules is just the category of pseudo-compact k-vector spaces, while the category of H-contramodules is the category of discrete k-vector spaces. So the functor $H\text{–pscomp}_k \longrightarrow H\text{–mod}_k$ forgets the topology of a pseudo-compact vector space, leaving only the discrete vector space structure; while the functor $H\text{–contra}_k \longrightarrow H\text{–mod}_k$ is an isomorphism of categories in this case.

What is the difference between contramodules and abstract modules? As it was mentioned above, for any coalgebra $\mathcal{C}$ over k, a $\mathcal{C}$-contramodule structure on a k-vector space $\mathfrak{P}$ is defined by a k-linear map $\operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$, while a $\mathcal{C}^\vee$-module structure is given by a k-linear map $\mathcal{C}^\vee \otimes_k \mathfrak{P} \longrightarrow \mathfrak{P}$. The forgetful functor $\mathcal{C}\text{–contra} \longrightarrow \mathcal{C}^\vee\text{–mod}$ takes the restriction of the contraaction map $\operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ to the subspace of all k-linear maps of finite rank $\mathcal{C}^\vee \otimes_k \mathfrak{P} \subset \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P})$. In particular, for any profinite group H, an H-contramodule structure on a k-vector space $\mathfrak{P}$ is defined by a k-linear map $\mathfrak{P}[[H]] \longrightarrow \mathfrak{P}$ from the space $\mathfrak{P}[[H]]$ of finitely additive $\mathfrak{P}$-valued measures defined on open-closed subsets of H. The forgetful functor $H\text{–contra}_k \longrightarrow H\text{–mod}_k$ takes the restriction of the contraaction map $\mathfrak{P}[[H]] \longrightarrow \mathfrak{P}$ to the subspace $\mathfrak{P}[H] \subset \mathfrak{P}[[H]]$ of all measures supported in a finite set of points in H.

What is the contratensor product of comodules and contramodules? Once again, considering the case of $H = \{e\}$ and $\mathcal{C} = k$ is instuctive. In this case, both the discrete H-modules and the H-contramodules over k are just the discrete k-vector spaces, and their contratensor product $\odot_{k(H)}$ is just the conventional tensor product $\otimes_k$ of discrete vector spaces over k. There are no pseudo-compact vector spaces here at all, and no topological completion is involved.

So, what is the difference between the tensor product $\mathcal{N} \otimes_{k[H]} \mathfrak{P}$ and the contratensor product $\mathcal{N} \odot_{k(H)} \mathfrak{P}$? It is that the contratensor product is a *smaller* vector space than the tensor product; indeed, generally speaking, $\mathcal{N} \odot_{k(H)} \mathfrak{P}$ is a (discrete) quotient vector space of a (discrete) k-vector space $\mathcal{N} \otimes_{k[H]} \mathfrak{P}$.

Let us recall that the contratensor product $\mathcal{N} \odot_{\mathcal{C}} \mathfrak{P}$ of a right comodule $\mathcal{N}$ and a left contramodule $\mathfrak{P}$ over a coalgebra $\mathcal{C}$ is defined as the quotient space of $\mathcal{N} \otimes_k \mathfrak{P}$ by the image of a natural k-linear map coming from the vector space $\mathcal{N} \otimes_k \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P})$, while the tensor product $\mathcal{N} \otimes_{\mathcal{C}^\vee} \mathfrak{P}$ of the $\mathcal{C}^\vee$-modules $\mathcal{N}$ and $\mathfrak{P}$ is defined as the quotient space of $\mathcal{N} \otimes_k \mathfrak{P}$ by the image of a natural k-linear map coming from the vector space $\mathcal{N} \otimes_k \mathcal{C}^\vee \otimes_k \mathfrak{P}$. Now, of course, $\mathcal{N} \otimes_k \mathcal{C}^\vee \otimes_k \mathfrak{P}$ is a subspace in $\mathcal{N} \otimes_k \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P})$, and the latter map is the restriction of the former one onto this subspace. Informally speaking, as the structure of a $\mathcal{C}$-contramodule on $\mathfrak{P}$ is richer than that of a $\mathcal{C}^\vee$-module, one can use it to construct a deeper (discrete) quotient of the k-tensor product space $\mathcal{N} \otimes_k \mathfrak{P}$ than the structure of $\mathcal{C}^\vee$-module on $\mathfrak{P}$ allows. Similarly, when $\mathcal{C} = k(H)$ for a profinite group H (so $\mathcal{C}^\vee = k[[H]]$), the tensor product $\mathcal{N} \otimes_{k[[H]]} \mathfrak{P}$, generally speaking, is a quotient space of the tensor product $\mathcal{N} \otimes_{k[H]} \mathfrak{P}$. Furthermore, when $H' \subset H$ is a dense subgroup, the tensor product $\mathcal{N} \otimes_{k[H]} \mathfrak{P}$ is, generally speaking, a quotient space of the tensor product $\mathcal{N} \otimes_{k[H']} \mathfrak{P}$. Thus, for a discrete H-module $\mathcal{N}$ and an H-contramodule $\mathfrak{P}$ over k, we have a sequence of surjective maps of discrete k-vector spaces

$$\mathcal{N} \otimes_{k[H']} \mathfrak{P} \longrightarrow \mathcal{N} \otimes_{k[H]} \mathfrak{P} \longrightarrow \mathcal{N} \otimes_{k[[H]]} \mathfrak{P} \longrightarrow \mathcal{N} \odot_{k(H)} \mathfrak{P}.$$

Corollary 2.6 claims that, when H is a finitely generated pro-p-group and k is a field of characteristic p, all these maps are isomorphisms.

3. Smooth G-modules and G-contramodules

3.1. Contramodules over a locally profinite group.   Let G be a locally profinite group and k be a field. A *G-module* over k is a k-vector space endowed with an action of G (viewed as an abstract group). A G-module $\boldsymbol{\mathcal{M}}$ over k is called *smooth* if the stabilizer of every element $x \in \boldsymbol{\mathcal{M}}$ is an open subgroup in G. The abelian category of G-modules over k is denoted by $G\text{–}\mathsf{mod}_k$ and the abelian category of smooth G-modules is denoted by $G\text{–}\mathsf{smooth}_k \subset G\text{–}\mathsf{mod}_k$.

Both the infinite direct sums and infinite products exist in the abelian category $G\text{–}\mathsf{smooth}_k$. The embedding functor $G\text{–}\mathsf{smooth}_k \longrightarrow G\text{–}\mathsf{mod}_k$ preserves the infinite direct sums (but not the infinite products). In fact, the infinite product $\boldsymbol{\mathcal{M}}$ of a family of objects $\boldsymbol{\mathcal{M}}_\alpha \in G\text{–}\mathsf{smooth}_k$ is the submodule of all G-smooth vectors $\boldsymbol{\mathcal{M}} \subset M$ in their product $M = \prod_\alpha \boldsymbol{\mathcal{M}}_\alpha$ as objects in $G\text{–}\mathsf{mod}_k$.

In order to define contramodules over locally profinite groups, we will need the following extension of the discussion of finitely additive measures on profinite sets in Subsection 2.6 to the case of locally profinite sets.

Let X be a locally profinite set (that is a locally compact totally disconnected topological space). For any k-vector space V, we denote by $V(X)$ the k-vector space of all compactly supported locally constant functions $X \longrightarrow V$. We also denote by $V[[X]]$ the k-vector space of all compactly supported finitely additive V-valued measures defined on all the open-closed subsets in X. Then for any two locally profinite sets X and Y there are natural isomorphisms

$$V(X) \simeq V \otimes_k k(X), \qquad k(X \times Y) \simeq k(X) \otimes_k k(Y)$$

and natural injective maps

$$V[[X]] \otimes_k k[[Y]] \longrightarrow V[[X \times Y]] \longrightarrow V[[X]][[Y]].$$

In particular, there is a natural injective map $V \otimes_k k[[X]] \longrightarrow V[[X]]$. For any continuous map of locally profinite sets $X \longrightarrow Y$, there is a natural pushforward map $V[[X]] \longrightarrow V[[Y]]$ (cf. [11, Section E.1.1] and [14, Section 1.8]).

A *G-contramodule over k* is a k-vector space $\boldsymbol{\mathfrak{P}}$ endowed with a k-linear *G-contraaction* map $\pi_{\boldsymbol{\mathfrak{P}}} \colon \boldsymbol{\mathfrak{P}}[[G]] \longrightarrow \boldsymbol{\mathfrak{P}}$ satisfying the *contraassociativity* and *contraunitality* axioms. Specifically, the two maps $\boldsymbol{\mathfrak{P}}[[G \times G]] \rightrightarrows \boldsymbol{\mathfrak{P}}[[G]]$, one of them provided by the pushforward of measures with respect to the multiplication map $G \times G \longrightarrow G$, and the other one constructed as the composition of the natural injection $\boldsymbol{\mathfrak{P}}[[G \times G]] \longrightarrow \boldsymbol{\mathfrak{P}}[[G]][[G]]$ with the map $\boldsymbol{\mathfrak{P}}[[G]][[G]] \longrightarrow \boldsymbol{\mathfrak{P}}[[G]]$ induced by the contraaction map $\pi_{\boldsymbol{\mathfrak{P}}}$, should have equal compositions with the contraaction map $\pi_{\boldsymbol{\mathfrak{P}}}$,

$$\boldsymbol{\mathfrak{P}}[[G \times G]] \rightrightarrows \boldsymbol{\mathfrak{P}}[[G]] \longrightarrow \boldsymbol{\mathfrak{P}}.$$

Besides, the composition of the map $\boldsymbol{\mathfrak{P}} \longrightarrow \boldsymbol{\mathfrak{P}}[[G]]$ assigning to a vector $x \in \boldsymbol{\mathfrak{P}}$ the point measure on G concentrated at the unit element $e \in G$ and taking the value x on the neighborhoods of e with the contraaction map $\pi_{\boldsymbol{\mathfrak{P}}}$ should be equal to the identity map $\mathrm{id}_{\boldsymbol{\mathfrak{P}}}$,

$$\boldsymbol{\mathfrak{P}} \longrightarrow \boldsymbol{\mathfrak{P}}[[G]] \longrightarrow \boldsymbol{\mathfrak{P}}.$$

G-contramodules over k form an abelian category $G\text{–}\mathsf{contra}_k$. For every element $g \in G$, one can consider the point measure at g with the value $1 \in k$. This defines

an injective map $k[G] \longrightarrow k[[G]]$. The composition of k-linear maps $\mathfrak{P} \otimes_k k[G] \longrightarrow \mathfrak{P} \otimes_k k[[G]] \longrightarrow \mathfrak{P}[[G]] \longrightarrow \mathfrak{P}$ provides a map $\mathfrak{P} \otimes_k k[G] \longrightarrow \mathfrak{P}$ defining the underlying G-module structure on a G-contramodule $\mathfrak{P}$. Hence we obtain an exact and faithful forgetful functor G–$\mathsf{contra}_k \longrightarrow G$–$\mathsf{mod}_k$.

Both the infinite direct sums and infinite products exist in the abelian category G–contra_k. The forgetful functor G–$\mathsf{contra}_k \longrightarrow G$–$\mathsf{mod}_k$ preserves the infinite products (but not the infinite direct sums). We refer to [11, Section 3.1.2] and [14, Section 2.6] for the construction of the infinite direct sums in the categories of semi-contramodules over semialgebras, which includes the category of G-contramodules as a particular case [14, Example 2.6].

For any smooth G-module $\mathcal{N}$ over k and any k-vector space V, there is a natural structure of G-contramodule on the k-vector space $\mathrm{Hom}_k(\mathcal{N}, V)$. We refer to [11, Section E.1.4] for the construction of the contraaction map $\pi\colon \mathrm{Hom}_k(\mathcal{N}, V)[[G]] \longrightarrow \mathrm{Hom}_k(\mathcal{N}, V)$ and to [14, Section 1.8] for the discussion of its intuitive meaning as a certain integration operation.

Corollary 3.1. *Let G be a locally profinite group with a compact open subgroup $H \subset G$. Assume that H is a finitely generated pro-p-group and k is a field of characteristic p. Then the forgetful functor G–$\mathsf{contra}_k \longrightarrow G$–$\mathsf{mod}_k$ is fully faithful. Moreover, for any dense subgroup $G' \subset G$, the forgetful functor G–$\mathsf{contra}_k \longrightarrow G'$–$\mathsf{mod}_k$ is fully faithful.*

Proof. Let $\mathfrak{P}$ and $\mathfrak{Q}$ be two G-contramodules over k, and let $f\colon \mathfrak{P} \longrightarrow \mathfrak{Q}$ be a $k[G']$-module morphism. Set $H' = H \cap G'$. Then f is, in particular, a $k[H']$-module morphism. By Corollary 2.5, it follows that f is a morphism of H-contramodules over k. Finally, we notice that the composition

$$\mathfrak{P}[[H]] \otimes_k k[G'] \longrightarrow \mathfrak{P}[[H]] \otimes_k k[[G]] \longrightarrow \mathfrak{P}[[H \times G]] \longrightarrow \mathfrak{P}[[G]]$$

of the natural injective maps and the pushforward map is surjective. As f is a morphism of H-contramodules and a morphism of G'-modules, we can conclude that it is a morphism of G-contramodules over k. ∎

3.2. Contratensor product over a locally profinite group.

Let G be locally profinite group and k be a field. The *contratensor product* $\mathcal{N} \odot_{k,G} \mathfrak{P}$ of a smooth G-module $\mathcal{N}$ and a G-contramodule $\mathfrak{P}$ over k is a k-vector space constructed as the cokernel of the difference of two natural maps

$$\mathcal{N} \otimes_k \mathfrak{P}[[G]] \rightrightarrows \mathcal{N} \otimes_k \mathfrak{P}. \tag{33}$$

The first map is simply induced by the contraaction map $\pi_{\mathfrak{P}}\colon \mathfrak{P}[[G]] \longrightarrow \mathfrak{P}$. The second map is defined by the formula

$$x \otimes \mu \longmapsto \int_G xg^{-1} \otimes d\mu_g$$

for all $x \in \mathcal{N}$ and $\mu \in \mathfrak{P}[[G]]$. Here $g \longmapsto xg^{-1} = gx$ is a smooth $\mathcal{N}$-valued function on G, which can be integrated with the compactly supported $\mathfrak{P}$-valued measure μ on open-closed subsets of G, resulting in an element of $\mathcal{N} \otimes_k \mathfrak{P}$.

For any smooth G-module $\mathcal{N}$, G-contramodule $\mathfrak{P}$, and a vector space V over k there is a natural isomorphism of k-vector spaces

$$\mathrm{Hom}_k(\mathcal{N} \odot_{k,G} \mathfrak{P}, V) \simeq \mathrm{Hom}_k^G(\mathfrak{P}, \mathrm{Hom}_k(\mathcal{N}, V)), \tag{34}$$

where $\mathrm{Hom}_k^G(\mathfrak{P}, \mathfrak{Q})$ denotes the space of all morphisms $\mathfrak{P} \longrightarrow \mathfrak{Q}$ is the category G–contra_k. Clearly, there is also a natural surjective morphism

$$\mathcal{N} \otimes_{k[G]} \mathfrak{P} \longrightarrow \mathcal{N} \odot_{k,G} \mathfrak{P}.$$

When H is a profinite group, the contratensor product $\mathcal{N} \odot_{k,H} \mathfrak{P}$ of a discrete H-module $\mathcal{N}$ and an H-contramodule $\mathfrak{P}$ over k is nothing but their contratensor product $\mathcal{N} \odot_{k(H)} \mathfrak{P}$ as a right comodule and a left contramodule over the coalgebra $\mathcal{C} = k(H)$ (as defined in Sections 2.3 and 2.6). Indeed, one has $\mathfrak{P}[[H]] \simeq \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P})$, and this isomorphism identifies the pair of maps $\mathcal{N} \otimes_k \mathfrak{P}[[H]] \rightrightarrows \mathcal{N} \otimes_k \mathfrak{P}$ (33) with the pair of maps $\mathcal{N} \otimes_k \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \rightrightarrows \mathcal{N} \otimes_k \mathfrak{P}$ (27). The following result is a generalization of Corollary 2.6 to locally profinite groups.

Corollary 3.2.   *Let G be a locally profinite group with a compact open subgroup $H \subset G$. Assume that H is a finitely generated pro-p-group and k is a field of characteristic p. Then for any smooth G-module $\mathcal{N}$ and any G-contramodule $\mathfrak{P}$ over k the natural map $\mathcal{N} \otimes_{k[G]} \mathfrak{P} \longrightarrow \mathcal{N} \odot_{k,G} \mathfrak{P}$ is an isomorphism.*
Moreover, for any dense subgroup $G' \subset G$, the natural map $\mathcal{N} \otimes_{k[G']} \mathfrak{P} \longrightarrow \mathcal{N} \odot_{k,G} \mathfrak{P}$ is an isomorphism.

Proof.   This is deduced from Corollary 3.1 in the same way as Corollary 2.2 is deduced from Theorem 2.1. ∎

3.3. Smooth module-contramodule correspondence.   Let G' and G'' be two locally profinite groups, and let $\mathcal{K}$ be a smooth $(G' \times G'')$-module over k. Then the functor

$$\mathcal{K} \odot_{k,G''} - \colon G''\text{–}\mathsf{contra}_k \longrightarrow G'\text{–}\mathsf{smooth}_k$$

taking a G''-contramodule $\mathfrak{P}$ over k to the smooth G'-module $\mathcal{K} \odot_{k,G''} \mathfrak{P}$ is left adjoint to the functor

$$\mathrm{Hom}_{k[G']}(\mathcal{K}, -) \colon G'\text{–}\mathsf{smooth}_k \longrightarrow G''\text{–}\mathsf{contra}_k$$

taking a smooth G'-module $\mathcal{M}$ over k to the G''-contramodule $\mathrm{Hom}_{k[G']}(\mathcal{K}, \mathcal{M})$. The smooth $(G \times G)$-module $\mathcal{S} = k(G)$ of compactly supported locally constant k-valued functions on a locally profinite group G plays a central role. We denote the related pair of adjoint functors by

$$\Phi_G = \mathcal{S} \odot_{k,G} - \colon G\text{–}\mathsf{contra}_k \longrightarrow G\text{–}\mathsf{smooth}_k \tag{35}$$

and $\qquad\qquad \Psi_G = \mathrm{Hom}_{k[G]}(\mathcal{S}, -) \colon G\text{–}\mathsf{smooth}_k \longrightarrow G\text{–}\mathsf{contra}_k. \tag{36}$

As it was mentioned and briefly discussed in Sections 1.2 and 1.5 of the introduction, choosing a compact open subgroup $H \subset G$ endows the smooth $(G \times G)$-module $\mathcal{S}$ viewed as a discrete $(H \times H)$-module with the structure of a semiassociative, semiunital *semialgebra* over the coalgebra $\mathcal{C} = k(H)$. The category of

smooth G-modules over k is then identified with the category of (left or right) $\mathcal{S}$-*semimodules*, G–$\mathsf{smooth}_k = \mathcal{S}$–$\mathsf{simod} = \mathsf{simod}$–$\mathcal{S}$, and the category of G-contramodules over k is isomorphic to the category of (left or right) $\mathcal{S}$-*semicontramodules*, G–$\mathsf{contra}_k = \mathcal{S}$–$\mathsf{sicntr}$ [11, Sections E.1.2–E.1.3], [14, Example 2.6].

Given a smooth G-module $\mathcal{N}$ and a G-contramodule $\mathfrak{P}$ over k, their contratensor product $\mathcal{N} \odot_{k,G} \mathfrak{P}$ is nothing but their contratensor product $\mathcal{N} \odot_{\mathcal{S}} \mathfrak{P}$ as a right $\mathcal{S}$-semimodule and a left $\mathcal{S}$-semicontramodule, in the sense of [11, Sections 0.3.7 and 6.1.1]. In particular, the pair of adjoint functors Φ_G and Ψ_G is isomorphic to the pair of adjoint functors $\Phi_{\mathcal{S}}$ and $\Psi_{\mathcal{S}}$ of [11, Sections 0.3.7 and 6.1.4–6.2].

For a profinite group H and the related coalgebra $\mathcal{C} = k(H)$, the functors Φ_H and Ψ_H agree with the functors $\Phi_{\mathcal{C}}$ and $\Psi_{\mathcal{C}}$ which were defined and discussed in Section 2.3. This was essentially explained in Section 3.2. Let us emphasize that the the functors Φ and Ψ are *covariant*, and no passage to the dual vector space is involved in their construction.

In particular, according to the discussion in Section 1.16 of the introduction, when the proorder of the profinite group H is not divisible by the characteristic of k (e. g., char $k = 0$), the functor Φ_H assigns to an H-contramodule $\mathfrak{P}$ its smooth H-submodule of all H-smooth vectors in $\mathfrak{P}$. Furthermore, following the brief discussions in Sections 1.9 and 2.3, when a $k(H)$-comodule $\mathcal{M}$ is *quasi-finitely cogenerated* (or equivalently, in the more traditional terminology, a smooth H-module $\mathcal{M}$ is *admissible*), the H-contramodule $\Psi_H(\mathcal{M})$ is associated with a certain pseudo-compact $k[[H]]$-module (see Lemmas 5.1 and 5.9(a) for the details; cf. Proposition 5.11).

Example 3.3. Generally speaking, when the characteristic of k divides the proorder of the group H, the functor Φ_H is quite different from the functor of maximal smooth H-submodule. In particular, the functor Φ_H is right exact, while the functor of maximal smooth H-submodule is left exact.

The following example is instructive. Let $H = \mathbb{Z}_p$ be the additive group of p-adic integers and k be a field of characteristic p. Then the ring $k[[H]] = \varprojlim_{n \geq 1} k[z]/(z^{p^n} - 1)$ is isomorphic to the ring of formal power series $k[[x]]$ in the variable $x = z - 1$. The category of smooth H-modules over k is equivalent to the category of $k[[x]]$-modules with a locally nilpotent action of x; while the category of H-contramodules over k is equivalent to the category of $k[[x]]$-contramodules, which means $k[[x]]$-modules with an x-power infinite summation operation, as described in [14, Section 1.3] (see also [14, Sections 1.5–1.6] for a discussion of this category's properties).

The smooth H-module $k(H)$ corresponds to the Prüfer $k[[x]]$-module $k((x))/xk[[x]]$, and the functor Φ_H assigns to an H-contramodule $\mathfrak{P}$ over k the smooth H-module $k(H) \otimes_{k[[H]]} \mathfrak{P} = k((x))/xk[[x]] \otimes_{k[[x]]} \mathfrak{P}$. This is quite different from the functor of the maximal smooth H-submodule (= maximal $k[[x]]$-submodule with a locally nilpotent action of x). For example, the projective H-contramodules $\mathfrak{P}$ over k correspond to the projective (= free) $k[[x]]$-contramodules $V[[x]]$, where V ranges over the k-vector spaces. There are no vectors with a locally nilpotent action of x in $\mathfrak{P} = V[[x]]$, so the maximal smooth H-submodule in $\mathfrak{P}$ vanishes; while $\Phi(\mathfrak{P}) = k(H) \otimes_k V = k((x))/xk[[x]] \otimes_k V$ is a quite nonzero smooth H-module. ∎

Let $G' \subset G$ be an open subgroup. Then the functors Φ and Ψ related to the groups G and G' form commutative diagrams with the forgetful functors G–$\mathsf{smooth}_k \longrightarrow G'$–$\mathsf{smooth}_k$ and G–$\mathsf{contra}_k \longrightarrow G'$–$\mathsf{contra}_k$,

$$\Psi_G : G\text{–smooth}_k \longrightarrow G\text{–contra}_k \qquad \Phi_G : G\text{–contra}_k \longrightarrow G\text{–smooth}_k$$

$$\tag{37}$$

$$\Psi_{G'} : G'\text{–smooth}_k \longrightarrow G'\text{–contra}_k \qquad \Phi_{G'} : G'\text{–contra}_k \longrightarrow G'\text{–smooth}_k$$

Commutativity of the leftmost diagram follows immediately from the fact that the G-module $\boldsymbol{\mathcal{S}}_G = k(G)$ can be obtained by applying the compactly supported smooth induction functor to the G'-module $\boldsymbol{\mathcal{S}}_{G'} = k(G')$, that is $k(G) = \operatorname{ind}_{G'}^G k(G')$. Commutativity of the rightmost diagram can be obtained, e. g., as a particular case of the results of [11, Section 8.1.2].

3.4. Weakly compactly injective and weakly compactly projective objects.
Let $H_1 \subset H_2$ be an open subgroup in a profinite group. Then one can easily see that the forgetful functor $H_2\text{–discr}_k \longrightarrow H_1\text{–discr}_k$ preserves injectives (and in fact takes cofree $k(H_2)$-comodules to cofree $k(H_1)$-comodules). Similarly, the forgetful functor $H_2\text{–contra}_k \longrightarrow H_1\text{–contra}_k$ preserves projectives (and in fact takes free $k(H_2)$-contramodules to free $k(H_1)$-contramodules).

Given a locally profinite group G, let us call a smooth G-module $\boldsymbol{\mathcal{M}}$ over k *weakly compactly injective* if there exists a compact open subgroup $H \subset G$ such that $\boldsymbol{\mathcal{M}}$ is injective as an object of $H\text{–discr}_k$. Similarly, let us call a G-contramodule $\boldsymbol{\mathfrak{P}}$ over k *weakly compactly projective* if there exists a compact open subgroup $H \subset G$ such that $\boldsymbol{\mathfrak{P}}$ is projective as an object of $H\text{–contra}_k$.

Proposition 3.4. *For any locally profinite group G and a field k, the functors Ψ_G and Φ_G restrict to mutually inverse equivalences between the full subcategories of weakly compactly injective objects in $G\text{–smooth}_k$ and weakly compactly projective objects in $G\text{–contra}_k$.*

Proof. It suffices to show that for every compact open subgroup $H \subset G$ the functors Ψ_G and Φ_G restrict to mutually inverse equivalences between the full subcategory of smooth G-modules over k that are injective as discrete H-modules and the full subcategory of G-contramodules over k that are projective as H-contramodules. In view of commutativity of the diagrams (37) for $G' = H$, the question reduces to showing that the functors $\Psi_H = \Psi_{\mathcal{C}}$ and $\Phi_H = \Phi_{\mathcal{C}}$ (where $\mathcal{C} = k(H)$) restrict to mutually inverse equivalences between the full subcategories of injective objects in $H\text{–discr}_k = \mathcal{C}\text{–comod}$ and projective objects in $H\text{–contra}_k = \mathcal{C}\text{–contra}$. The latter is a standard result about comodules and contramodules over a coalgebra over a field [11, Sections 0.2.6 and 5.1.3], [14, Sections 1.2 and 3.4], [12, Sections 5.1–5.2]. Alternatively, one can apply directly the results of [11, Sections 0.3.7 and 6.2] (see also [14, Section 3.5]). ∎

A smooth G-module over k is called *semiprojective* if it is a direct summand of an (infinite) direct sum of copies of the smooth G-module $\boldsymbol{\mathcal{S}} = k(G)$, or in other words, if it is a direct summand of the smooth G-module $\boldsymbol{\mathcal{S}} \otimes_k V$ for some k-vector space V. A G-contramodule over k is called *semiinjective* if it is a direct summand of an (infinite) product of copies of the G-contramodule $\boldsymbol{\mathcal{S}}^\vee$ Pontryagin dual to the smooth G-module $\boldsymbol{\mathcal{S}} = k(G)$, or in other words, if it is a direct summand of the G-contramodule $\operatorname{Hom}_k(\boldsymbol{\mathcal{S}}, V)$ for some k-vector space V.

Semiprojective smooth G-modules over k are weakly compactly injective; in fact, they are injective objects of H–discr_k for every compact open subgroup $H \subset G$. Semiinjective G-contramodules over k are weakly compactly projective; in fact, they are projective objects of H–contra_k for every compact open subgroup $H \subset G$.

Proposition 3.5. (a) *There are enough injective objects in the abelian category G–smooth_k, and the forgetful functors G–$\mathsf{smooth}_k \longrightarrow H$–$\mathsf{discr}_k$ preserve injectives, so injectives in G–smooth_k are weakly compactly injective. The functors Ψ_G and Φ_G identify the full subcategory of injective objects in G–smooth_k with the full subcategory of semiinjective objects in G–contra_k.*
(b) *There are enough projective objects in the abelian category G–contra_k, and the forgetful functors G–$\mathsf{contra}_k \longrightarrow H$–$\mathsf{contra}_k$ preserve projectives, so projectives in G–contra_k are weakly compactly projective. The functors Φ_G and Ψ_G identify the full subcategory of projective objects in G–contra_k with the full subcategory of semiprojective objects in G–smooth_k.*

Proof. Follows from Proposition 3.4 and [14, Proposition 3.5] (see [18, Proposition 4.1] for a slightly more general result). ∎

3.5. The universal acting algebra. Introducing a k-algebra containing the group algebra $k[G]$ and acting in all the smooth G-modules over k is a natural thing to do (see, e. g, [8, Section 1]). In the context of the present paper, one may also wonder about a k-algebra containing $k[G]$ and acting in all the G-contramodules over k. We choose the approach of computing and working with the *universal* such k-algebra in both cases. It turns out that the two answers only differ by the passage to the opposite algebra.

Let us denote by $\mathfrak{T} = \mathrm{Hom}_{k[G]}(\mathcal{S}, \mathcal{S})^{\mathrm{op}}$ the opposite algebra to the k-algebra of endomorphisms of the smooth G-module $\mathcal{S}$ over k (with its left G-module structure). This means that there is a right action of $\mathfrak{T}$ in $\mathcal{S}$ commuting with the left action of G. The right action of G in $\mathcal{S}$ provides an injective homomorphism $k[G] \longrightarrow \mathfrak{T}$.

In particular, when $G = H$ is a profinite group, $\mathfrak{T} = \mathfrak{R} = \mathcal{C}^\vee = k[[H]]$ is the Pontryagin dual algebra to the coalgebra $\mathcal{C} = k(H)$.

Proposition 3.6. (a) *The k-algebra of endomorphisms of the forgetful functor G–$\mathsf{smooth}_k \longrightarrow k$–$\mathsf{mod}$ is naturally isomorphic to $\mathfrak{T}^{\mathrm{op}}$.*
(b) *The k-algebra of endomorphisms of the forgetful functor G–$\mathsf{contra}_k \longrightarrow k$–$\mathsf{mod}$ is naturally isomorphic to $\mathfrak{T}$.*

Proof. First of all, we have to construct a natural right action of $\mathfrak{T}$ in smooth G-modules and a natural left action of $\mathfrak{T}$ in G-contramodules over k. The most straightforward approach would be to compute the forgetful functor simod–$\mathcal{S} = G$–$\mathsf{smooth}_k \longrightarrow k$–$\mathsf{mod}$ as the functor of semitensor product $- \lozenge_{\mathcal{S}} \mathcal{S}$ with the left $\mathcal{S}$-semimodule $\mathcal{S}$ [11, Sections 0.3.2 and 1.4.1] and the forgetful functor $\mathcal{S}$–$\mathsf{sicntr} = G$–$\mathsf{contra}_k \longrightarrow k$–$\mathsf{mod}$ as the functor of semihomomorphisms $\mathrm{SemiHom}_{\mathcal{S}}(\mathcal{S}, -)$ from the left $\mathcal{S}$-semimodule $\mathcal{S}$ [11, Sections 0.3.5 and 3.4.1–3.4.2]. A more roundabout argument below is based on the constructions discussed above in this paper.

Part (a): let G–$\mathsf{smooth}_k^{\mathsf{inj}} \subset G$–$\mathsf{smooth}_k$ denote the full subcategory of injective smooth G-modules and G–$\mathsf{contra}_k^{\mathsf{siin}} \subset G$–$\mathsf{contra}_k$ denote the full subcategory of semiinjective G-contramodules over k.

Since there are enough injectives in G–smooth_k, the algebra of endomorphisms of the forgetful functor G–$\mathsf{smooth}_k \longrightarrow k$–$\mathsf{mod}$ is isomorphic to the algebra of endomorphisms of the restriction of this functor to the full subcategory of injective objects G–$\mathsf{smooth}_k^{\mathsf{inj}} \subset G$–$\mathsf{smooth}_k$. In view of the equivalence of categories G–$\mathsf{smooth}_k^{\mathsf{inj}} \simeq G$–$\mathsf{contra}_k^{\mathsf{siin}}$ from Proposition 3.5(a), the latter algebra is isomorphic to the algebra of endomorphisms of the functor of contratensor product $\boldsymbol{\mathcal{S}} \odot_{k,G} -\colon G$–$\mathsf{contra}_k^{\mathsf{siin}} \longrightarrow k$–$\mathsf{mod}$. The ring $\mathfrak{T}^{\mathrm{op}}$ of endomorphisms of the smooth G-module $\boldsymbol{\mathcal{S}}$ (with its right G-module structure) acts naturally by endomorphisms of this functor on the left.

Now consider the smooth G-module $\boldsymbol{\mathcal{S}}$ with its left G-module structure. The group G acts on the right by automorphisms of the object $\boldsymbol{\mathcal{S}} \in G$–$\mathsf{smooth}_k$. Hence any endomorphism of the forgetful functor G–$\mathsf{smooth}_k \longrightarrow k$–$\mathsf{mod}$ must act in the k-vector space $\boldsymbol{\mathcal{S}}$ by an operator commuting with the right action of G, i. e., by an operator coming from an element of $\mathfrak{T}^{\mathrm{op}}$. It remains to show that every nonzero endomorphism of the forgetful functor acts in $\boldsymbol{\mathcal{S}}$ by a nonzero operator. Indeed, in view of Proposition 3.5(b), $\boldsymbol{\mathcal{S}}$ is a projective generator of the exact category of weakly compactly injective smooth G-modules over k, which by Proposition 3.5(a) contains an injective cogenerator of the abelian category G–smooth_k.

Part (b): let G–$\mathsf{contra}_k^{\mathsf{proj}} \subset G$–$\mathsf{contra}_k$ denote the full subcategory of projective G-contramodules and G–$\mathsf{smooth}_k^{\mathsf{sipr}} \subset G$–$\mathsf{smooth}_k$ denote the full subcategory of semiprojective smooth G-modules over k. Since there are enough projectives in G–contra_k, the algebra of endomorphisms of the forgetful functor G–$\mathsf{contra}_k \longrightarrow k$–$\mathsf{mod}$ is isomorphic to the algebra of endomorphisms of the restriction of this functor to the full subcategory of projective objects G–$\mathsf{contra}_k^{\mathsf{proj}} \subset G$–$\mathsf{contra}_k$. In view of the equivalence of categories G–$\mathsf{contra}_k^{\mathsf{proj}} \simeq G$–$\mathsf{smooth}_k^{\mathsf{sipr}}$ from Proposition 3.5(b), the latter algebra is isomorphic to the algebra of endomorphisms of the corepresentable functor $\mathrm{Hom}_{k[G]}(\boldsymbol{\mathcal{S}}, -)\colon G$–$\mathsf{smooth}_k^{\mathsf{sipr}} \longrightarrow k$–$\mathsf{mod}$. The latter algebra is isomorphic to the opposite algebra to the algebra of endomophisms of the corepresenting object $\boldsymbol{\mathcal{S}}$ (with its left G-module structure), that is the algebra $\mathfrak{T}$. ∎

Let $\boldsymbol{\mathcal{N}}$ be a smooth G-module and $\boldsymbol{\mathfrak{P}}$ be a left G-contramodule over k. Then for every k-vector space V there is a natural morphism of k-vector spaces

$$\mathrm{Hom}_k(\boldsymbol{\mathcal{N}} \odot_{k,G} \boldsymbol{\mathfrak{P}},\, V) \simeq \mathrm{Hom}_k^G(\boldsymbol{\mathfrak{P}},\, \mathrm{Hom}_k(\boldsymbol{\mathcal{N}}, V))$$

$$\longrightarrow \mathrm{Hom}_{\mathfrak{T}}(\boldsymbol{\mathfrak{P}},\, \mathrm{Hom}_k(\boldsymbol{\mathcal{N}}, V)) \simeq \mathrm{Hom}_k(\boldsymbol{\mathcal{N}} \otimes_{\mathfrak{T}} \boldsymbol{\mathfrak{P}},\, V).$$

This k-linear map being functorial in V, it follows that there is a natural morphism of k-vector spaces

$$\boldsymbol{\mathcal{N}} \otimes_{\mathfrak{T}} \boldsymbol{\mathfrak{P}} \longrightarrow \boldsymbol{\mathcal{N}} \odot_{k,G} \boldsymbol{\mathfrak{P}},$$

which is surjective, because the map

$$\mathrm{Hom}_k^G(\boldsymbol{\mathfrak{P}},\, \mathrm{Hom}_k(\boldsymbol{\mathcal{N}}, V)) \longrightarrow \mathrm{Hom}_{\mathfrak{T}}(\boldsymbol{\mathfrak{P}},\, \mathrm{Hom}_k(\boldsymbol{\mathcal{N}}, V))$$

is injective (as the identity embedding of two subspaces in $\mathrm{Hom}_k(\boldsymbol{\mathfrak{P}}, \mathrm{Hom}_k(\boldsymbol{\mathcal{N}}, V))$).

Corollary 3.7. *Let G be a locally profinite group with a compact open subgroup $H \subset G$. Assume that H is a finitely generated pro-p-group and k is a field of characteristic p. Then the forgetful functor G–$\mathsf{contra}_k \longrightarrow \mathfrak{T}$–$\mathsf{mod}$ is fully faithful.*

Proof. This is a weaker version of Corollary 3.1, as $k[G]$ is a subalgebra in $\mathfrak{T}$. ∎

Corollary 3.8. *Let G be a locally profinite group with a compact open subgroup $H \subset G$. Assume that H is a finitely generated pro-p-group and k is a field of characteristic p. Then for any smooth G-module $\mathcal{N}$ and any G-contramodule $\mathfrak{P}$ over k the natural map $\mathcal{N} \otimes_{\mathfrak{T}} \mathfrak{P} \longrightarrow \mathcal{N} \odot_{k,G} \mathfrak{P}$ is an isomorphism.*

Proof.   Follows from Corollary 3.7 or Corollary 3.2. ∎

We denote by $\mathrm{Ext}_{\mathfrak{T}^{\mathrm{op}}}$ and $\mathrm{Tor}^{\mathfrak{T}}$ the conventional Ext and Tor functors over the ring $\mathfrak{T}^{\mathrm{op}}$ or $\mathfrak{T}$ (i. e., the derived functors of Hom and tensor product computed in the abelian categories of $\mathfrak{T}$-modules).

Corollary 3.9. *Let G be a p-adic Lie group and k a field of characteristic p. Then*
(a) *for any weakly compactly injective smooth G-module $\mathcal{M}$ over k one has $\mathrm{Ext}^i_{\mathfrak{T}^{\mathrm{op}}}(\mathcal{S}, \mathcal{M}) = 0$ for all $i > 0$;*
(b) *for any weakly compactly projective G-contramodule $\mathfrak{P}$ over k one has $\mathrm{Tor}^{\mathfrak{T}}_i(\mathcal{S}, \mathfrak{P}) = 0$ for all $i > 0$.*

Proof.   The group G has a base of neighborhoods of zero formed by open subgroups that are uniform pro-p-groups [1, Proposition 1.16, Theorem 4.2, and Theorem 8.32]. So we can fix such a subgroup $H \subset G$ for which $\mathcal{M}$ is an injective object in H–discr_k and $\mathfrak{P}$ is a projective object in H–contra_k. Set $\mathfrak{R} = k[[H]] = \mathcal{C}^{\vee} \simeq \mathfrak{R}^{\mathrm{op}}$; then $\mathfrak{R} = \mathrm{Hom}_{k[H]}(\mathcal{C}, \mathcal{C})$ is a subring in $\mathfrak{T} = \mathrm{Hom}_{k[G]}(\mathcal{S}, \mathcal{S})^{\mathrm{op}} = \mathrm{Hom}_{k[H]}(\mathcal{C}, \mathcal{S})$. Furthermore, $\mathfrak{T} = \Psi_G(\mathcal{S})$ is naturally a G-contramodule over k with its left $\mathfrak{T}$-module structure induced by its G-contramodule structure. The forgetful functor G–$\mathsf{contra}_k \longrightarrow H$–$\mathsf{contra}_k$ takes $\mathfrak{T}$ to a projective H-contramodule over k, since a semiprojective smooth G-module $\mathcal{S}$ over k is injective as a discrete H-module (cf. the proof of Proposition 3.4 and the subsequent discussion). By Corollary 2.7(b), it follows that $\mathfrak{T}$ is a flat left $\mathfrak{R}$-module. In view of Corollary 2.2, the commutative diagram (37), and Proposition 3.4 or Corollary 3.8, we have

$$\mathcal{C} \otimes_{\mathfrak{R}} \mathfrak{T} \overset{2.2}{=} \mathcal{C} \odot_{\mathcal{C}} \mathfrak{T} = \Phi_H(\mathfrak{T}) \overset{(37)}{=} \Phi_G(\mathfrak{T}) \overset{3.4 \text{ or } 3.8}{\simeq} \mathcal{S},$$

where the isomorphism $\Phi_{\mathcal{C}}(\mathfrak{T}) = \mathcal{C} \odot_{\mathcal{C}} \mathfrak{T} \simeq \mathcal{C} \odot_{k,H} \mathfrak{T} = \Phi_H(\mathfrak{T})$ holds according to the discussion in Sections 3.2–3.3.
Now we can compute

$$\mathbb{R}\mathrm{Hom}^i_{\mathfrak{T}^{\mathrm{op}}}(\mathcal{S}, \mathcal{M}) \simeq \mathbb{R}\mathrm{Hom}^i_{\mathfrak{T}^{\mathrm{op}}}(\mathfrak{T}^{\mathrm{op}} \otimes^{\mathbb{L}}_{\mathfrak{R}} \mathcal{C}, \mathcal{M}) \simeq \mathbb{R}\mathrm{Hom}^i_{\mathfrak{R}}(\mathcal{C}, \mathcal{M}) = 0, \quad i > 0$$

by Corollary 2.7(a), and

$$H_i(\mathcal{S} \otimes^{\mathbb{L}}_{\mathfrak{T}} \mathfrak{P}) = H_i((\mathcal{C} \otimes^{\mathbb{L}}_{\mathfrak{R}} \mathfrak{T}) \otimes^{\mathbb{L}}_{\mathfrak{T}} \mathfrak{P}) = H_i(\mathcal{C} \otimes^{\mathbb{L}}_{\mathfrak{R}} \mathfrak{P}) = 0, \quad i > 0$$

by Corollary 2.7(b). ∎

4. Derived equivalence and duality adjunction

4.1. Derived equivalence. Let H be a profinite group and k be a field. The *k-cohomological dimension of H* is conventionally defined as the supremum of the set of all integers i for which there exists a discrete H-module $\mathcal{M}$ over k such that

$H^i(H, \mathcal{M}) \neq 0$. Alternatively, the k-cohomological dimension of H can be defined as the homological dimension of the abelian category H–discr_k. The k-cohomological dimension of any open subgroup $H' \subset H$ does not exceed that of H; moreover, the k-cohomological dimensions of H and H' coincide if H contains no elements of finite order equal to char k [23].

For any coalgebra $\mathcal{C}$ over k, the homological dimensions of the abelian categories of left $\mathcal{C}$-comodules, right $\mathcal{C}$-comodules, and left $\mathcal{C}$-contramodules are equal to each other, as all of them are equal to the homological dimensions of the derived functors $\mathrm{Cotor}^{\mathcal{C}}_*$ and $\mathrm{Coext}^*_{\mathcal{C}}$ [11, Sections 0.2.2, 0.2.5, and 0.2.9], [12, Section 4.5], [16, Corollary 1.9.4]. In particular, the homological dimensions of the abelian categories H–discr_k and H–contra_k are equal to each other (and to the k-cohomological dimension of H).

Let us say that a locally profinite group G is *locally of finite k-cohomological dimension n* if it has a compact open subgroup $H \subset G$ of finite k-cohomological dimension n. In this case, compact open subgroups of k-cohomological dimension n form a base of neighborhoods of zero in G.

In particular, a p-adic Lie group G is locally of finite k-cohomological dimension for any field k. Specifically, in the unnatural characteristic char $k \neq p$ one has $n = 0$, and in the natural characteristic char $k = p$ the local k-cohomological dimension n is equal to the dimension of the group.

Theorem 4.1. *Let k be a field and G be a locally profinite group locally of finite k-cohomological dimension. Then for any derived category symbol $\star = \mathsf{b}, +, -,$ or $\varnothing$ there is a natural triangulated equivalence between the derived categories of smooth G-modules and G-contramodules over k,*

$$\mathbb{R}\Psi_G \colon \mathsf{D}^\star(G\text{–}\mathsf{smooth}_k) \simeq \mathsf{D}^\star(G\text{–}\mathsf{contra}_k) \colon \mathbb{L}\Phi_G \qquad (38)$$

provided by the derived functors of the adjoint functors (35) *and* (36).

Proof. Denote by G–$\mathsf{smooth}_k^{\mathsf{wcin}} \subset G$–$\mathsf{smooth}_k$ the full subcategory of weakly compactly injective smooth G-modules and by G–$\mathsf{contra}_k^{\mathsf{wcpr}} \subset G$–$\mathsf{contra}_k$ the full subcategory of weakly compactly projective G-contramodules over k.

The full subcategory G–$\mathsf{smooth}_k^{\mathsf{wcin}}$ is closed under the cokernels of monomorphisms, extensions, and direct summands in the abelian category G–smooth_k; and every object of G–smooth_k is a subobject of an object from G–$\mathsf{smooth}_k^{\mathsf{wcin}}$. In other words, G–$\mathsf{smooth}_k^{\mathsf{wcin}}$ is a *coresolving subcategory* in G–smooth_k. In particular, as a full subcategory closed under extensions, the category G–$\mathsf{smooth}_k^{\mathsf{wcin}}$ inherits a Quillen exact category structure from the abelian category G–smooth_k.

Similarly, the full subcategory G–$\mathsf{contra}_k^{\mathsf{wcpr}}$ is closed under the kernels of epimorphisms, extensions, and direct summands in the abelian category G–contra_k; and every object of G–contra_k is a quotient object of an object from G–$\mathsf{contra}_k^{\mathsf{wcpr}}$. In other words, G–$\mathsf{contra}_k^{\mathsf{wcpr}}$ is a *resolving subcategory* in G–contra_k. In particular, the full subcategory G–$\mathsf{contra}_k^{\mathsf{wcpr}}$ inherits an exact category structure from the ambient abelian category G–contra_k.

The result of Proposition 3.4 provides an equivalence between the two exact categories G–$\mathsf{smooth}_k^{\mathsf{wcin}}$ and G–$\mathsf{contra}_k^{\mathsf{wcpr}}$. All of these assertions do not yet depend on the locally finite k-cohomological dimension assumption.

When the group G is locally of finite k-cohomological dimension n, every smooth G-module over k has a finite right resolution of length $\leq n$ by modules from G–$\mathsf{smooth}_k^{\mathsf{wcin}}$, and every G-contramodule over k has a finite left resolution of length $\leq n$ by contramodules from G–$\mathsf{contra}_k^{\mathsf{wcpr}}$. In other words, one can say that the weakly compactly injective dimension of any smooth G-module over k does not exceed n, and the weakly compactly projective dimension of any G-contramodule over k does not exceed n.

Applying the result of [7, Proposition 13.2.6] or [13, Proposition A.5.6] and the assertion dual to it (see also [6, §§ I.5 and I.7]), we obtain triangulated equivalences

$$\mathsf{D}^\star(G\text{–}\mathsf{smooth}_k) \simeq \mathsf{D}^\star(G\text{–}\mathsf{smooth}_k^{\mathsf{wcin}}) \simeq \mathsf{D}^\star(G\text{–}\mathsf{contra}_k^{\mathsf{wcpr}}) \simeq \mathsf{D}^\star(G\text{–}\mathsf{contra}_k).$$

In other words, one can say that the (mutually inverse) derived functors $\mathbb{R}\Psi$ and $\mathbb{L}\Phi$ providing the desired triangulated equivalence are constructed using resolutions of compexes of smooth G-modules and complexes of G-contramodules by arbitrary complexes of weakly compactly injective smooth G-modules and weakly compactly projective G-contramodules over k.   ∎

In conclusion, let us restate the simplified descriptions of some elements of the picture of Theorem 4.1 provided by some results of Section 3.

The functor $\Psi_G\colon G$–$\mathsf{smooth}_k \longrightarrow G$–$\mathsf{contra}_k$ is simply $\Psi_G\colon \boldsymbol{\mathcal{M}} \longmapsto \mathrm{Hom}_{k[G]}(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{M}})$, where $\boldsymbol{\mathcal{S}} = k(G)$ is the smooth $G \times G$-module of compactly supported locally constant k-valued functions on G.

The functor $\Phi_G\colon G$–$\mathsf{contra}_k \longrightarrow G$–$\mathsf{smooth}_k$ is, generally speaking, defined as the contratensor product $\Phi_G\colon \mathfrak{P} \longmapsto \boldsymbol{\mathcal{S}} \odot_{k,G} \mathfrak{P}$. However, when the group G has a compact open subgroup H which is a finitely generated pro-p-group, and k is a field of characteristic p, the contratensor product does not differ from the conventional tensor product, $\Phi_G\colon \mathfrak{P} \longmapsto \boldsymbol{\mathcal{S}} \otimes_{k[G]} \mathfrak{P}$ (see Corollary 3.2). In particular, this is applicable to any p-adic Lie group G.

When G is a p-adic Lie group, the derived functor $\mathbb{R}\Psi_G$ can be computed as the conventional $\mathbb{R}\,\mathrm{Hom}$ over the ring $\mathfrak{T}^{\mathrm{op}}$,

$$\mathbb{R}\Psi_G\colon \boldsymbol{\mathcal{M}}^\bullet \longmapsto \mathbb{R}\,\mathrm{Hom}_{\mathfrak{T}^{\mathrm{op}}}(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{M}}^\bullet),$$

and the derived functor $\mathbb{L}\Phi_G$ can be computed as the conventional derived tensor product over the ring $\mathfrak{T}$,

$$\mathbb{L}\Phi_G\colon \mathfrak{P}^\bullet \longmapsto \boldsymbol{\mathcal{S}} \otimes_{\mathfrak{T}}^{\mathbb{L}} \mathfrak{P}^\bullet.$$

This follows essentially from Corollary 3.9.

4.2. Duality adjunction. Let k be a field and G be a locally profinite group locally of finite k-cohomological dimension. Let V be a fixed k-vector space. Then the contravariant exact functor $\mathrm{Hom}_k(-, V)\colon (G$–$\mathsf{smooth}_k)^{\mathrm{op}} \longrightarrow G$–$\mathsf{contra}_k$ induces a contravariant triangulated functor between the derived categories

$$\mathsf{D}^\star(G\text{–}\mathsf{smooth}_k)^{\mathrm{op}} \longrightarrow \mathsf{D}^\star(G\text{–}\mathsf{contra}). \tag{39}$$

Composing the contravariant triangulated functor (39) with the triangulated equivalence $\mathbb{L}\Phi_G\colon \mathsf{D}^\star(G$–$\mathsf{contra}_k) \longrightarrow \mathsf{D}^\star(G$–$\mathsf{smooth}_k)$ (38), we obtain a contravariant triangulated functor

$$\Delta_G^V\colon \mathsf{D}^\star(G\text{–}\mathsf{smooth}_k)^{\mathrm{op}} \longrightarrow \mathsf{D}^\star(G\text{–}\mathsf{smooth}_k). \tag{40}$$

The following result is a particular case of the discussion in Section 1.8 of the introduction.

Proposition 4.2. *The duality functor* (40) *is* right self-adjoint, *that is, for any two complexes of smooth G-modules $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet \in \mathsf{D}^\star(G\text{–smooth}_k)$, there is a natural isomorphism of Hom spaces*

$$\mathrm{Hom}_{\mathsf{D}^\star(G\text{–smooth}_k)}(\mathcal{M}^\bullet, \Delta_G^V(\mathcal{N}^\bullet)) \simeq \mathrm{Hom}_{\mathsf{D}^\star(G\text{–smooth}_k)}(\mathcal{N}^\bullet, \Delta_G^V(\mathcal{M}^\bullet)). \qquad (41)$$

Sketch of proof. The argument is based on the results of the book [11] and uses the formalism of semi-infinite homology and cohomology of locally profinite groups (which is a particular case of the semi-infinite homology and cohomology of semialgebras over coalgebras). Both side of the desired isomorphism (41) are computed as a certain semi-infinite (co)homology space.

We choose a compact open subgroup $H \subset G$ of finite k-cohomological dimension and consider the semialgebra $\mathcal{S} = k(G)$ over the coalgebra $\mathcal{C} = k(H)$ [11, Section E.1], [14, Example 2.6]. Then smooth G-modules over k are the same thing as $\mathcal{S}$-semimodules, and G-contramodules over k are the same thing as $\mathcal{S}$-semicontramodules. The involutive anti-authomorphisms of $\mathcal{C}$ and $\mathcal{S}$ induced by the inverse element maps $H \longrightarrow H$ and $G \longrightarrow G$ identify the left and right $\mathcal{S}$-semimodules.

For any complex of left semimodules $\mathcal{M}^\bullet$ and any complex of right semimodules $\mathcal{N}^\bullet$ over a semialgebra $\mathcal{S}$, the construction of [11, Section 2.7] produces a cohomologically graded k-vector space of semi-infinite homology

$$\mathrm{SemiTor}^{\mathcal{S}}(\mathcal{N}^\bullet, \mathcal{M}^\bullet) \in \mathsf{D}(k\text{–vect}).$$

In the situation at hand, when $\mathcal{S} = k(G)$ and both $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ are complexes of smooth G-modules over k, the functor $\mathrm{SemiTor}^{\mathcal{S}}$ can be thought of as a double-sided derived functor of the functor of (G, H)-*semiinvariants* $(\mathcal{N}^\bullet \otimes_k \mathcal{M}^\bullet)_{G,H}$ in the tensor product of complexes of smooth G-modules $\mathcal{N}^\bullet \otimes_k \mathcal{M}^\bullet$ [11, Section E.2]. In particular, one has

$$\mathrm{SemiTor}^{\mathcal{S}}(\mathcal{N}^\bullet, \mathcal{M}^\bullet) \simeq \mathrm{SemiTor}^{\mathcal{S}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet).$$

Furthermore, for any complex of left semimodules $\mathcal{M}^\bullet$ and any complex of left semicontramodules $\mathfrak{P}^\bullet$ over a semialgebra $\mathcal{S}$, the construction of [11, Section 4.7] produces a graded k-vector space of semi-infinite cohomology

$$\mathrm{SemiExt}_{\mathcal{S}}(\mathcal{M}^\bullet, \mathfrak{P}^\bullet) \in \mathsf{D}(k\text{–vect}).$$

Generally speaking, for a locally profinite group G and the related semialgebra $\mathcal{S} = k(G)$ over the coalgebra $\mathcal{C} = k(H)$, the functors $\mathrm{SemiTor}$ and $\mathrm{SemiExt}$ depend on the choice of a compact open subgroup $H \subset G$; but when G is locally of finite k-cohomological dimension and H is chosen to be of finite k-cohomological dimension, they don't. (We do not need to use this fact.)

According to [11, last formula of Section 4], for any complexes of smooth G-modules $\mathcal{N}^\bullet$ and $\mathcal{M}^\bullet$ we have

$$\mathrm{Hom}_k(\mathrm{SemiTor}^{\mathcal{S}}(\mathcal{N}^\bullet, \mathcal{M}^\bullet), V) \simeq \mathrm{SemiExt}_{\mathcal{S}}(\mathcal{M}^\bullet, \mathrm{Hom}_k(\mathcal{N}^\bullet, V)).$$

Furthermore, by [11, Corollary 6.6(a)], for any complex of smooth G-modules $\mathcal{M}^\bullet$ and any complex of G-contramodules $\mathfrak{P}^\bullet$ over k we have

$$\operatorname{SemiExt}_{\boldsymbol{\mathcal{S}}}(\mathcal{M}^\bullet, \mathfrak{P}^\bullet) \simeq \operatorname{Ext}_{\boldsymbol{\mathcal{S}}}(\mathcal{M}^\bullet, \mathbb{L}\Phi_{\boldsymbol{\mathcal{S}}}(\mathfrak{P}^\bullet)).$$

Combining the natural isomorphisms above, we get

$$\operatorname{Ext}_{\boldsymbol{\mathcal{S}}}(\mathcal{M}^\bullet, \mathbb{L}\Phi_{\boldsymbol{\mathcal{S}}}\operatorname{Hom}_k(\mathcal{N}^\bullet, V)) \simeq \operatorname{SemiExt}_{\boldsymbol{\mathcal{S}}}(\mathcal{M}^\bullet, \operatorname{Hom}_k(\mathcal{N}^\bullet, V))$$
$$\simeq \operatorname{Hom}_k(\operatorname{SemiTor}^{\boldsymbol{\mathcal{S}}}(\mathcal{N}^\bullet, \mathcal{M}^\bullet), V) \simeq \operatorname{Hom}_k(\operatorname{SemiTor}^{\boldsymbol{\mathcal{S}}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet), V)$$
$$\simeq \operatorname{SemiExt}_{\boldsymbol{\mathcal{S}}}(\mathcal{N}^\bullet, \operatorname{Hom}_k(\mathcal{M}^\bullet, V)) \simeq \operatorname{Ext}_{\boldsymbol{\mathcal{S}}}(\mathcal{N}^\bullet, \mathbb{L}\Phi_{\boldsymbol{\mathcal{S}}}\operatorname{Hom}_k(\mathcal{M}^\bullet, V)). \quad (42)$$

Passing to the degree-zero components in the isomorphism of graded vector spaces (42), we obtain the desired adjunction isomorphism (41). ∎

The self-adjunction of Proposition 4.2 can be also described in terms of the adjunction morphisms rather than adjunction isomorphisms. For any complex of smooth G-modules $\mathcal{M}^\bullet \in \mathsf{D}^\star(G\text{–smooth}_k)$ there is a natural adjunction morphism

$$\mathcal{M}^\bullet \longrightarrow \Delta_G^V \Delta_G^V(\mathcal{M}^\bullet) \quad (43)$$

in the derived category $\mathsf{D}^\star(G\text{–smooth}_k)$ which corresponds to the identity morphism $\Delta_G^V(\mathcal{M}^\bullet) \longrightarrow \Delta_G^V(\mathcal{M}^\bullet)$ under the adjunction isomorphism (41). The natural morphism (43) can be constructed as follows.

Let $H \subset G$ be a compact open subgroup of finite k-cohomological dimension. Up to an isomorphism in $\mathsf{D}^\star(G\text{–smooth}_k)$, we can assume $\mathcal{M}^\bullet$ to be a complex of smooth G-modules over k whose terms are injective as discrete H-modules. Then $\mathfrak{P}^\bullet = \Psi_G(\mathcal{M}^\bullet)$ and $\mathfrak{Q}^\bullet = \operatorname{Hom}_k(\mathcal{M}^\bullet, V)$ are complexes of G-contramodules over k whose terms are projective as H-contramodules over k. By Proposition 3.4, we have a natural isomorphism of complexes of smooth G-modules $\Phi_G(\mathfrak{P}^\bullet) \simeq \mathcal{M}^\bullet$. According to formula (34), we have a natural isomorphism of complexes of k-vector spaces $\operatorname{Hom}_k(\mathcal{M}^\bullet \odot_{k,G} \mathfrak{Q}^\bullet, V) \simeq \operatorname{Hom}_k^G(\mathfrak{Q}^\bullet, \operatorname{Hom}_k(\mathcal{M}^\bullet, V)) = \operatorname{Hom}_k^G(\mathfrak{Q}^\bullet, \mathfrak{Q}^\bullet)$. Hence the identity morphism of complexes of G-contramodules $\mathfrak{Q}^\bullet \longrightarrow \mathfrak{Q}^\bullet$ corresponds to a morphism of complexes of k-vector spaces

$$\mathcal{M}^\bullet \odot_{k,G} \mathfrak{Q}^\bullet \longrightarrow V. \quad (44)$$

Lemma 4.3. *For any locally profinite group G, any field k, and any two G-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ over k, there is a natural isomorphism of k-vector spaces*

$$\Phi_G(\mathfrak{P}) \odot_{k,G} \mathfrak{Q} \simeq \Phi_G(\mathfrak{Q}) \odot_{k,G} \mathfrak{P}.$$

Proof. We construct an isomorphism $(\boldsymbol{\mathcal{S}} \odot_{k,G} \mathfrak{P}) \odot_{k,G} \mathfrak{Q} \simeq (\boldsymbol{\mathcal{S}} \odot_{k,G} \mathfrak{Q}) \odot_{k,G} \mathfrak{P}$. Following the definition of the contratensor product in Section 3.2, both the vector spaces in question can be identified with the cokernel of a certain natural map

$$t\colon \boldsymbol{\mathcal{S}} \otimes_k \mathfrak{P}[[G]] \otimes_k \mathfrak{Q} \oplus \boldsymbol{\mathcal{S}} \otimes_k \mathfrak{P} \otimes_k \mathfrak{Q}[[G]] \longrightarrow \boldsymbol{\mathcal{S}} \otimes_k \mathfrak{P} \otimes_k \mathfrak{Q}.$$

Let us explain what the map t is. It has two components $t_1\colon \boldsymbol{\mathcal{S}} \otimes_k \mathfrak{P}[[G]] \otimes_k \mathfrak{Q} \longrightarrow \boldsymbol{\mathcal{S}} \otimes_k \mathfrak{P} \otimes_k \mathfrak{Q}$ and $t_2\colon \boldsymbol{\mathcal{S}} \otimes_k \mathfrak{P} \otimes_k \mathfrak{Q}[[G]] \longrightarrow \boldsymbol{\mathcal{S}} \otimes_k \mathfrak{P} \otimes_k \mathfrak{Q}$, each of which is the

difference of two natural maps, $t_i = r_i - s_i$. The maps r_1 and r_2 are induced by the G-contraaction maps $\pi_{\mathfrak{P}}$ and $\pi_{\mathfrak{Q}}$, respectively. The maps s_1 and s_2 are constructed using the smooth G-action in $\mathcal{S}$, as explained in Section 3.2. There are two such smooth actions, the left and the right one; and one uses, say, the right action to construct the map $\mathcal{S} \otimes_k \mathfrak{P}[[G]] \longrightarrow \mathcal{S} \otimes_k \mathfrak{P}$ inducing the map s_1, and the left action to construct the map $\mathcal{S} \otimes_k \mathfrak{Q}[[G]] \longrightarrow \mathcal{S} \otimes_k \mathfrak{Q}$ inducing the map s_2. ∎

Returning to the situation at hand, we now have

$$\mathcal{M}^{\bullet} \circledcirc_{k,G} \mathfrak{Q}^{\bullet} \simeq \Phi_G(\mathfrak{P}^{\bullet}) \circledcirc_{k,G} \mathfrak{Q}^{\bullet} \simeq \Phi_G(\mathfrak{Q}^{\bullet}) \circledcirc_{k,G} \mathfrak{P}^{\bullet}.$$

Set $\mathcal{N}^{\bullet} = \Phi_G(\mathfrak{Q}^{\bullet})$. We have constructed a morphism of complexes of k-vector spaces

$$\mathcal{N}^{\bullet} \circledcirc_{k,G} \mathfrak{P}^{\bullet} \longrightarrow V. \tag{45}$$

Once again, by (34), we have a natural isomorphism of complexes of k-vector spaces $\mathrm{Hom}_k(\mathcal{N}^{\bullet} \circledcirc_{k,G} \mathfrak{P}^{\bullet}, V) \simeq \mathrm{Hom}_k^G(\mathfrak{P}^{\bullet}, \mathrm{Hom}_k(\mathcal{N}^{\bullet}, V))$. In view of this isomorphism, the map (45) corresponds to a morphism of complexes of G-contramodules over k

$$\mathfrak{P}^{\bullet} \longrightarrow \mathrm{Hom}_k(\mathcal{N}^{\bullet}, V).$$

Applying the functor Φ_G, we obtain

$$\mathcal{M}^{\bullet} \simeq \Phi_G(\mathfrak{P}^{\bullet}) \longrightarrow \Phi_G \mathrm{Hom}_k(\mathcal{N}^{\bullet}, V) = \Phi_G \mathrm{Hom}_k(\Phi_G \mathrm{Hom}_k(\mathcal{M}^{\bullet}, V), V). \tag{46}$$

It remains to recall that $\mathfrak{Q}^{\bullet}$ is a complex of G-contramodules whose terms are projective as H-contramodules over k, hence $\mathcal{N}^{\bullet} = \Phi_G(\mathfrak{Q}^{\bullet})$ is a complex of smooth G-modules whose terms are injective as discrete H-modules over k, hence $\mathrm{Hom}_k(\mathcal{N}^{\bullet}, V)$ is also a complex of G-contramodules whose terms are projective as H-contramodules over k.

Thus the complexes $\mathrm{Hom}_k(\mathcal{M}^{\bullet}, V)$ and $\mathrm{Hom}_k(\Phi_G \mathrm{Hom}_k(\mathcal{M}^{\bullet}, V), V)$ are adjusted to the derived functor $\mathbb{L}\Phi_G$, that is, one can compute $\mathbb{L}\Phi_G$ for these complexes simply by applying the functor Φ_G.

The map (46) provides the desired self-adjunction morphism (43), and our construction is finished.

5. Admissibility conditions

5.1. Quasi-finite comodules and contramodules. Let $\mathcal{C}$ be a (coassociative, counital) coalgebra over a field k. The definitions of finitely cogenerated $\mathcal{C}$-comodules and finitely generated $\mathcal{C}$-contramodules are straightforward: a left $\mathcal{C}$-comodule $\mathcal{M}$ is said to be *finitely cogenerated* if it is a subcomodule of a cofree left $\mathcal{C}$-comodule $\mathcal{C} \otimes_k V$ with a finite-dimensional vector space of cogenerators V. Similarly, a left $\mathcal{C}$-contramodule $\mathfrak{P}$ is said to be *finitely generated* if it is a quotient contramodule of a free left $\mathcal{C}$-contramodule $\mathrm{Hom}_k(\mathcal{C}, V)$ with a finite-dimensional vector space of generators V. However, these finiteness conditions (discussed in detail in [18, Section 2]) are sometimes too restrictive, and *quasi-finiteness* conditions are preferable.

Let $\mathcal{C}$ and $\mathcal{E}$ be (coassociative, counital) coalgebras over a field k, and let $\mathcal{E} \longrightarrow \mathcal{C}$ be a coalgebra morphism. Then any left $\mathcal{E}$-comodule $\mathcal{N}$ can be also considered

as a left $\mathcal{C}$-comodule, with the $\mathcal{C}$-coaction map defined as the composition $\mathcal{N} \longrightarrow \mathcal{E} \otimes_k \mathcal{N} \longrightarrow \mathcal{C} \otimes_k \mathcal{N}$. So there is an exact, faithful functor of "corestriction of scalars" $\mathcal{E}$–comod $\longrightarrow$ $\mathcal{C}$–comod. Similarly, any left $\mathcal{E}$-contramodule $\mathfrak{Q}$ can be considered as a left $\mathcal{C}$-contramodule, with the $\mathcal{C}$-contraaction map provided by the composition $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{Q}) \longrightarrow \mathrm{Hom}_k(\mathcal{E}, \mathfrak{Q}) \longrightarrow \mathfrak{Q}$. So there is an exact, faithful functor of "contrarestriction of scalars" $\mathcal{E}$–contra $\longrightarrow$ $\mathcal{C}$–contra.

We are interested in the particular case when $\mathcal{E}$ is a subcoalgebra in $\mathcal{C}$. In this case, the functors $\mathcal{E}$–comod $\longrightarrow$ $\mathcal{C}$–comod and $\mathcal{E}$–contra $\longrightarrow$ $\mathcal{C}$–contra are fully faithful. We will say that a left $\mathcal{C}$-comodule $\mathcal{M}$ *is a left $\mathcal{E}$-comodule* if it belongs to the essential image of the functor $\mathcal{E}$–comod $\longrightarrow$ $\mathcal{C}$–comod. This means that the image of the coaction map $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ is contained in the subspace $\mathcal{E} \otimes_k \mathcal{M} \subset \mathcal{C} \otimes_k \mathcal{M}$. Similarly, a left $\mathcal{C}$-contramodule $\mathfrak{P}$ is said to *be a left $\mathcal{E}$-contramodule* if it belongs to the essential image of the functor $\mathcal{E}$–contra $\longrightarrow$ $\mathcal{C}$–contra. This means that the contraaction map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ factorizes through the surjection $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathrm{Hom}_k(\mathcal{E}, \mathfrak{P})$ induced by the inclusion $\mathcal{E} \longrightarrow \mathcal{C}$.

Any left $\mathcal{C}$-comodule $\mathcal{M}$ has a unique maximal subcomodule which is an $\mathcal{E}$-comodule. This subcomodule, denoted by $_{\mathcal{E}}\mathcal{M} \subset \mathcal{M}$, can be constructed as the full preimage of the subspace $\mathcal{E} \otimes_k \mathcal{M} \subset \mathcal{C} \otimes_k \mathcal{M}$ under the coaction map $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$. The functor $\mathcal{M} \longmapsto {}_{\mathcal{E}}\mathcal{M}$ is right adjoint to the functor of corestriction of scalars $\mathcal{E}$–comod $\longrightarrow$ $\mathcal{C}$–comod discussed above. Similarly, any left $\mathcal{C}$-contramodule $\mathfrak{P}$ has a unique maximal quotient contramodule which is an $\mathcal{E}$-contramodule. This quotient contramodule, denoted by $^{\mathcal{E}}\mathfrak{P}$, can be constructed as the cokernel of the composition $\mathrm{Hom}_k(\mathcal{C}/\mathcal{E}, \mathfrak{P}) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ of the injective map $\mathrm{Hom}_k(\mathcal{C}/\mathcal{E}, \mathfrak{P}) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P})$ induced by the natural surjection $\mathcal{C} \longrightarrow \mathcal{C}/\mathcal{E}$ with the contraaction map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$. The functor $\mathfrak{P} \longmapsto {}^{\mathcal{E}}\mathfrak{P}$ is left adjoint to the functor of contrarestriction of scalars $\mathcal{E}$–contra $\longrightarrow$ $\mathcal{C}$–contra discussed above. The maximal $\mathcal{E}$-subcomodule of a right $\mathcal{C}$-comodule $\mathcal{M}$ is denoted by $\mathcal{M}_{\mathcal{E}} \subset \mathcal{M}$.

A left $\mathcal{C}$-comodule $\mathcal{M}$ is said to be *quasi-finitely cogenerated* (or simply *quasi-finite*) if the vector space of $\mathcal{C}$-comodule homomorphisms $\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{M})$ is finite-dimensional for any finite-dimensional left $\mathcal{C}$-comodule $\mathcal{L}$. This definition goes back to the classical paper of Takeuchi [25]; later expositions can be found, e. g., in [4] and [19, Section 2]. A left $\mathcal{C}$-comodule $\mathcal{M}$ is quasi-finitely cogenerated if and only if the left $\mathcal{E}$-comodule $_{\mathcal{E}}\mathcal{M}$ is finite-dimensional for every finite-dimensional subcoalgebra $\mathcal{E} \subset \mathcal{C}$ [19, Lemma 2.1]. Similarly, a left $\mathcal{C}$-contramodule $\mathfrak{P}$ is said to be *quasi-finitely generated* (or just *quasi-finite*) if the vector space of $\mathcal{C}$-contramodule homomorphisms $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, \mathfrak{L})$ is finite-dimensional for any finite-dimensional left $\mathcal{C}$-contramodule $\mathfrak{L}$, or equivalently if the left $\mathcal{E}$-contramodule $^{\mathcal{E}}\mathfrak{P}$ is finite-dimensional for every finite-dimensional subcoalgebra $\mathcal{E} \subset \mathcal{C}$ [19, Lemma 2.5].

Lemma 5.1. *For any quasi-finitely cogenerated left comodule $\mathcal{M}$ over a coalgebra $\mathcal{C}$, there is a natural, functorially defined pseudo-compact left $\mathcal{C}^{\vee}$-module structure on the k-vector space $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$ such that the left $\mathcal{C}$-contramodule structure of $\Psi_{\mathcal{C}}(\mathcal{M}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$ discussed in Section 2.3 underlies this pseudo-compact left $\mathcal{C}^{\vee}$-module structure.*

Proof. In fact, for any coalgebra $\mathcal{D}$ and any $\mathcal{C}$-$\mathcal{D}$-bicomodule $\mathcal{K}$ the left $\mathcal{D}$-contramodule structure of the Hom space $\mathrm{Hom}_{\mathcal{C}}(\mathcal{K}, \mathcal{M})$ for a quasi-finitely cogenerated left $\mathcal{C}$-comodule $\mathcal{M}$ is associated with a certain functorially defined pseudo-compact left $\mathcal{D}^{\vee}$-module structure. This is essentially the result of [25, Section 1.7], where a

right $\mathcal{D}$-comodule denoted by $h_{\mathcal{C}\text{-}}(\mathcal{M}, \mathcal{K})$ is constructed so that $\mathrm{Hom}_{\mathcal{C}}(\mathcal{K}, \mathcal{M}) = h_{\mathcal{C}\text{-}}(\mathcal{M}, \mathcal{K})^{\vee}$. As no contramodules are mentioned in [25], we give a sketch of the argument in the situation at hand (with $\mathcal{K} = \mathcal{C} = \mathcal{D}$).

The coalgebra $\mathcal{C}$ is the union of its finite-dimensional subcoalgebras $\mathcal{E}$, that is $\mathcal{C} = \varinjlim_{\mathcal{E}} \mathcal{E}$. Hence $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{M}) = \varprojlim_{\mathcal{E}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{M})$. The left $\mathcal{C}$-contramodule structure on $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$ (induced by the right $\mathcal{C}$-comodule structure on $\mathcal{C}$) is the projective limit of the left $\mathcal{C}$-contramodule structures on $\mathrm{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{M})$, which in turn arise from left $\mathcal{E}$-contramodule structures (induced by the right $\mathcal{E}$-comodule structure on $\mathcal{E}$). By assumption, the k-vector space $\mathfrak{Q} = \mathrm{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{M})$ is finite-dimensional. Any finite-dimensional left contramodule $\mathfrak{Q}$ over a finite-dimensional coalgebra $\mathcal{E}$ is dual to a finite-dimensional right $\mathcal{E}$-comodule $\mathfrak{Q}^{\vee}$; so the formula

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{M}) = \varprojlim_{\mathcal{E}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{M}) = (\varinjlim_{\mathcal{E}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{M})^{\vee})^{\vee}$$

represents $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$ as the dual vector space to a certain right $\mathcal{C}$-comodule $\mathcal{N} = \varinjlim_{\mathcal{E}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{E}, \mathcal{M})^{\vee}$. For any right $\mathcal{C}$-comodule $\mathcal{N}$, the dual vector space $\mathcal{N}^{\vee}$ is a pseudo-compact left $\mathcal{C}^{\vee}$-module. In other words, any finite-dimensional contramodule $\mathfrak{Q}$ over a finite-dimensional coalgebra $\mathcal{E}$ is naturally a pseudo-compact $\mathcal{E}^{\vee}$-module, and the projective limit of pseudo-compact modules is a pseudo-compact module. ∎

Lemma 5.2. *A right comodule $\mathcal{N}$ over a coalgebra $\mathcal{C}$ is quasi-finitely cogenerated if and only if the left contramodule $\mathcal{N}^{\vee} = \mathrm{Hom}_k(\mathcal{N}, k)$ over $\mathcal{C}$ is quasi-finitely generated.*

Proof. For any subcoalgebra $\mathcal{E} \subset \mathcal{C}$ and any k-vector space V, there is a natural isomorphism of left $\mathcal{E}$-contramodules

$$^{\mathcal{E}}\mathrm{Hom}_k(\mathcal{N}, V) \simeq \mathrm{Hom}_k(\mathcal{N}_{\mathcal{E}}, V),$$

as one can see, e. g., from the constructions of the $\mathcal{E}$-comodule $_{\mathcal{E}}\mathcal{M}$ and the $\mathcal{E}$-contramodule $^{\mathcal{E}}\mathfrak{P}$ above. In particular, the left $\mathcal{E}$-contramodule $^{\mathcal{E}}(\mathcal{N}^{\vee}) \simeq (\mathcal{N}_{\mathcal{E}})^{\vee}$ is finite-dimensional if and only if the right $\mathcal{E}$-comodule $\mathcal{N}_{\mathcal{E}}$ is. ∎

5.2. Quasi-finitely copresented comodules and quasi-finitely presented contramodules. Let $\mathcal{C}$ be a coalgebra over a field k. It is well-known that any quasi-finitely cogenerated $\mathcal{C}$-comodule is a subcomodule of a quasi-finitely cogenerated injective $\mathcal{C}$-comodule (see, e. g., [19, Lemma 2.2(b)]). Similarly, any quasi-finitely generated $\mathcal{C}$-contramodule is a quotient contramodule of a quasi-finitely generated projective $\mathcal{C}$-contramodule [19, Lemma 2.6(b)].

A $\mathcal{C}$-comodule is said to be *quasi-finitely copresented* if it is the kernel of a morphism between two quasi-finitely cogenerated injective $\mathcal{C}$-comodules. A $\mathcal{C}$-contramodule is said to be *quasi-finitely presented* if it is the cokernel of a morphism between two quasi-finitely generated projective $\mathcal{C}$-contramodules. We denote the full subcategory of quasi-finitely copresented left $\mathcal{C}$-comodules by $\mathcal{C}\text{–}\mathsf{comod}_{\mathsf{qfc}} \subset \mathcal{C}\text{–}\mathsf{comod}$, the full subcategory of quasi-finitely copresented right $\mathcal{C}$-comodules by $\mathsf{comod}_{\mathsf{qfc}}\text{–}\mathcal{C} \subset \mathsf{comod}\text{–}\mathcal{C}$, and the full subcategory of quasi-finitely presented left $\mathcal{C}$-contramodules by $\mathcal{C}\text{–}\mathsf{contra}_{\mathsf{qfp}} \subset \mathcal{C}\text{–}\mathsf{contra}$.

Obviously, an injective $\mathcal{C}$-comodule is quasi-finitely cogenerated if and only if it is quasi-finitely copresented; and a projective $\mathcal{C}$-contramodule is quasi-finitely generated if and only if it is quasi-finitely presented.

Proposition 5.3.   *The equivalence between the additive categories of injective left*
$\mathcal{C}$-comodules and projective left $\mathcal{C}$-contramodules (28–29)

$$\Psi_{\mathcal{C}} \colon \mathcal{C}\text{–comod}^{\mathsf{inj}} \simeq \mathcal{C}\text{–contra}^{\mathsf{proj}} : \Phi_{\mathcal{C}}$$

restricts to an equivalence between the full subcategories of quasi-finitely cogenerated
injective left $\mathcal{C}$-comodules and quasi-finitely generated projective left $\mathcal{C}$-contramodules,

$$\mathcal{C}\text{–comod}^{\mathsf{inj}}_{\mathsf{qfc}} \simeq \mathcal{C}\text{–contra}^{\mathsf{proj}}_{\mathsf{qfp}}.$$

Proof.   For any subcoalgebra $\mathcal{E} \subset \mathcal{C}$, there is a commutative diagram of additive
functors and equivalences [11, Section 7.1.4], [12, Section 5.4]

$$
\begin{array}{ccc}
\Psi_{\mathcal{C}} \colon \mathcal{C}\text{–comod} & === & \mathcal{C}\text{–contra} : \Phi_{\mathcal{C}} \\
\downarrow & & \downarrow \\
\Psi_{\mathcal{E}} \colon \mathcal{E}\text{–comod} & === & \mathcal{E}\text{–contra} : \Phi_{\mathcal{E}}
\end{array}
$$

where the leftmost vertical functor takes a left $\mathcal{C}$-comodule $\mathcal{M}$ to the left $\mathcal{E}$-comodule
$_{\mathcal{E}}\mathcal{M}$, while the rightmost vertical functor takes a left $\mathcal{C}$-contramodule $\mathfrak{P}$ to the left
$\mathcal{E}$-contramodule $^{\mathcal{E}}\mathfrak{P}$. It remains to observe that over a finite-dimensional coalge-
bra $\mathcal{E}$ the functors $\Psi_{\mathcal{E}} = \mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, -)$ and $\Phi_{\mathcal{E}} = \mathcal{E} \odot_{\mathcal{E}} -$ take finite-dimensional
$\mathcal{E}$-comodules to finite-dimensional $\mathcal{E}$-contramodules and vice versa.   ∎

Proposition 5.4.   *The dualization functor*

$$\mathcal{N} \longmapsto \mathcal{N}^{\vee} = \mathrm{Hom}_{k}(\mathcal{N}, k) \colon \mathsf{comod}\text{–}\mathcal{C} \longrightarrow \mathcal{C}\text{–contra}$$

restricts to an anti-equivalence between the full subcategories of quasi-finitely co-
generated injective right $\mathcal{C}$-comodules and quasi-finitely generated projective left
$\mathcal{C}$-contramodules,

$$(\mathsf{comod}^{\mathsf{inj}}_{\mathsf{qfc}}\text{–}\mathcal{C})^{\mathsf{op}} \simeq \mathcal{C}\text{–contra}^{\mathsf{proj}}_{\mathsf{qfp}}.$$

Moreover, the same functor restricts to an anti-equivalence between the full subcat-
egories of quasi-finitely copresented right $\mathcal{C}$-comodules and quasi-finitely presented
left $\mathcal{C}$-contramodules,

$$(\mathsf{comod}_{\mathsf{qfc}}\text{–}\mathcal{C})^{\mathsf{op}} \simeq \mathcal{C}\text{–contra}_{\mathsf{qfp}}.$$

Thus any quasi-finitely presented left $\mathcal{C}$-contramodule carries a natural, functorially
defined structure of a pseudo-compact left $\mathcal{C}^{\vee}$-module.

Proof.   This is [19, Proposition 2.8(a)]. Essentially, one first proves the first
assertion and then deduces the second one by passing to the (co)kernels of morphisms.
The particular case for finitely (co)presented co/contramodules in place of the quasi-
finitely (co)presented ones can be found in [18, Proposition 2.9(a)].   ∎

5.3. Coextension and contraextension of scalars.   Let $\mathcal{C} \longrightarrow \mathcal{D}$ be a mor-
phism of coalgebras. Then, according to the discussion in Section 5.1, any left
$\mathcal{C}$-comodule can be considered as a left $\mathcal{D}$-comodule and any left $\mathcal{C}$-contramodule
can be considered as a left $\mathcal{D}$-contramodule. So there are exact, faithful functors

of corestriction of scalars $\mathcal{C}$–comod $\longrightarrow$ $\mathcal{D}$–comod and contrarestriction of scalars $\mathcal{C}$–contra $\longrightarrow$ $\mathcal{D}$–contra.

The functor of corestriction of scalars $\mathcal{C}$–comod $\longrightarrow$ $\mathcal{D}$–comod has a right adjoint functor of "coextension of scalars" [12, Section 4.8], [11, Section 7.1.2] assigning to a left $\mathcal{D}$-comodule $\mathcal{N}$ the left $\mathcal{C}$-comodule ${}_\mathcal{C}\mathcal{N}$, which can be constructed as the cotensor product ${}_\mathcal{C}\mathcal{N} = \mathcal{C} \,\square_\mathcal{D}\, \mathcal{N}$ (see the discussion of cotensor products in Section 2.3). The coextension of scalars can be also constructed as the unique left exact functor $\mathcal{D}$–comod $\longrightarrow$ $\mathcal{C}$–comod taking the cofree left $\mathcal{D}$-comodule $\mathcal{D} \otimes_k V$ to the cofree left $\mathcal{C}$-comodule $\mathcal{C} \otimes_k V$ for every k-vector space V.

Similarly, the functor of contrarestriction of scalars $\mathcal{C}$–contra $\longrightarrow$ $\mathcal{D}$–contra has a left adjoint functor of "contraextension of scalars" [12, Section 4.8], [11, Section 7.1.2] assigning to a left $\mathcal{D}$-contramodule $\mathfrak{Q}$ the left $\mathcal{C}$-contramodule ${}^\mathcal{C}\mathfrak{Q}$, which can be constructed as the vector space of cohomomorphisms ${}^\mathcal{C}\mathfrak{Q} = \mathrm{Cohom}_\mathcal{D}(\mathcal{C}, \mathfrak{Q})$ (see Section 2.4) with the left $\mathcal{C}$-contramodule structure induced by the right $\mathcal{C}$-comodule structure on $\mathcal{C}$. The contraextension of scalars can be also constructed as the unique right exact functor $\mathcal{D}$–contra $\longrightarrow$ $\mathcal{C}$–contra taking the free left $\mathcal{D}$-contramodule $\mathrm{Hom}_k(\mathcal{D}, V)$ to the free left $\mathcal{C}$-contramodule $\mathrm{Hom}_k(\mathcal{C}, V)$ for every k-vector space V. The following two lemmas will be useful in Section 5.5.

Lemma 5.5. *Let $\mathcal{C} \longrightarrow \mathcal{D}$ be a morphism of coalgebras over a field k. Then*
(a) *the functor of coextension of scalars $\mathcal{D}$–comod $\longrightarrow$ $\mathcal{C}$–comod takes quasi-finitely cogenerated $\mathcal{D}$-comodules to quasi-finitely cogenerated $\mathcal{C}$-comodules;*
(b) *the functor of coextension of scalars $\mathcal{D}$–comod $\longrightarrow$ $\mathcal{C}$–comod takes quasi-finitely copresented $\mathcal{D}$-comodules to quasi-finitely copresented $\mathcal{C}$-comodules;*
(c) *the functor of contraextension of scalars $\mathcal{D}$–contra $\longrightarrow$ $\mathcal{C}$–contra takes quasi-finitely generated $\mathcal{D}$-contramodules to quasi-finitely generated $\mathcal{C}$-contramodules;*
(d) *the functor of contraextension of scalars $\mathcal{D}$–contra $\longrightarrow$ $\mathcal{C}$–contra takes quasi-finitely presented $\mathcal{D}$-contramodules to quasi-finitely presented $\mathcal{C}$-contramodules.*

Proof. Part (a): let $\mathcal{N}$ be a quasi-finitely cogenerated left $\mathcal{D}$-comodule and $\mathcal{L}$ be a finite-dimensional left $\mathcal{C}$-comodule. Then $\mathcal{L}$ is also a finite-dimensional left $\mathcal{D}$-comodule, so the k-vector space $\mathrm{Hom}_\mathcal{D}(\mathcal{L}, \mathcal{N})$ is finite-dimensional. Now the adjunction isomorphism $\mathrm{Hom}_\mathcal{C}(\mathcal{L}, {}_\mathcal{C}\mathcal{N}) \simeq \mathrm{Hom}_\mathcal{D}(\mathcal{L}, \mathcal{N})$ shows that the vector space $\mathrm{Hom}_\mathcal{C}(\mathcal{L}, {}_\mathcal{C}\mathcal{N})$ is finite-dimensional.

Part (b): we observe that the functor of coextension of scalars $\mathcal{N} \longmapsto {}_\mathcal{C}\mathcal{N}$ is right adjoint to an exact functor, hence it is left exact and takes injective objects to injective objects. Now let $\mathcal{N}$ be a quasi-finitely copresented left $\mathcal{D}$-comodule; so $\mathcal{N}$ is the kernel of a morphism of quasi-finitely cogenerated injective left $\mathcal{D}$-comodules $\mathcal{I} \longrightarrow \mathcal{J}$. Then ${}_\mathcal{C}\mathcal{N}$ is the kernel of the induced morphism of injective left $\mathcal{C}$-comodules ${}_\mathcal{C}\mathcal{I} \longrightarrow {}_\mathcal{C}\mathcal{J}$. By part (a), the $\mathcal{C}$-comodules ${}_\mathcal{C}\mathcal{I}$ and ${}_\mathcal{C}\mathcal{J}$ are also quasi-finitely cogenerated.

The proofs of parts (c) and (d) are dual-analogous to (a) and (b), respectively. ∎

Lemma 5.6. (a) *For any coalgebra morphism $\mathcal{C} \longrightarrow \mathcal{D}$ and any left $\mathcal{C}$-comodule $\mathcal{M}$, there is a natural exact sequence of left $\mathcal{C}$-comodules*

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{C} \,\square_\mathcal{D}\, \mathcal{M} \longrightarrow \mathcal{C} \,\square_\mathcal{D}\, \mathcal{C} \,\square_\mathcal{D}\, \mathcal{M}. \qquad (47)$$

Here the left $\mathcal{C}$-comodule $\mathcal{C} \,\square_\mathcal{D}\, \mathcal{M}$ can be obtained by applying to $\mathcal{M}$ the composition $\mathcal{C}$–comod $\longrightarrow$ $\mathcal{D}$–comod $\longrightarrow$ $\mathcal{C}$–comod of the functors of corestriction and coextension

of scalars. The left $\mathcal{C}$-comodule $\mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{M}$ is obtained by applying the same composition of functors twice to the left $\mathcal{C}$-comodule $\mathcal{M}$.

(b) *For any coalgebra morphism $\mathcal{C} \longrightarrow \mathcal{D}$ and any left $\mathcal{C}$-contramodule $\mathfrak{P}$, there is a natural exact sequence of left $\mathcal{C}$-contramodules*

$$\mathrm{Cohom}_{\mathcal{D}}(\mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{C},\, \mathfrak{P}) \simeq \mathrm{Cohom}_{\mathcal{D}}(\mathcal{C},\, \mathrm{Cohom}_{\mathcal{D}}(\mathcal{C}, \mathfrak{P}))$$
$$\longrightarrow \mathrm{Cohom}_{\mathcal{D}}(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P} \longrightarrow 0.$$

Here the left $\mathcal{C}$-contramodules $\mathrm{Cohom}_{\mathcal{D}}(\mathcal{C}, \mathfrak{P})$ and $\mathrm{Cohom}_{\mathcal{D}}(\mathcal{C}, \mathrm{Cohom}_{\mathcal{D}}(\mathcal{C}, \mathfrak{P}))$ can be obtained by applying to $\mathfrak{P}$ the composition of the functors of contrarestriction and contraextension of scalars $\mathcal{C}$–contra $\longrightarrow \mathcal{D}$–contra $\longrightarrow \mathcal{C}$–contra (one or two times).

Proof.   Part (a): in the sequence (47), the map $\mathcal{M} \longrightarrow \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{M}$ comes from the left $\mathcal{C}$-coaction map $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$, whose image is contained in the subspace $\mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{M} \subset \mathcal{C} \otimes_k \mathcal{M}$. In fact, the left coaction $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ is an injective map whose image is equal to $\mathcal{C} \,\square_{\mathcal{C}}\, \mathcal{M} \subset \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{M} \subset \mathcal{C} \otimes_k \mathcal{M}$.

In particular, the comultiplication $\mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ is an injective map whose image is equal to $\mathcal{C} \,\square_{\mathcal{C}}\, \mathcal{C} \subset \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{C} \subset \mathcal{C} \otimes_k \mathcal{C}$; so there is also a map $\mathcal{C} \longrightarrow \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{C}$. The map $\mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{M} \longrightarrow \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{M}$ in the sequence (47) is the difference of two maps $\mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{M} \rightrightarrows \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{M}$, one of which is induced by the map $\mathcal{C} \longrightarrow \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{C}$ and the other one by the map $\mathcal{M} \longrightarrow \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{M}$.

The assertion that the composition of the two maps in (47) vanishes is a restatement of the coassociativity equation for the left $\mathcal{C}$-coaction in $\mathcal{M}$. The assertion that the sequence (47) is exact simply means that $\mathcal{M}$ is the kernel of the map $\mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{M} \longrightarrow \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{M}$. Now we have $\mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{M} \subset \mathcal{C} \otimes_k \mathcal{M}$ and $\mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{M} \subset \mathcal{C} \otimes_k \mathcal{C} \otimes_k \mathcal{M}$, so it suffices to check that $\mathcal{M}$ is the kernel of the difference of two maps $\mathcal{C} \otimes_k \mathcal{M} \rightrightarrows \mathcal{C} \otimes_k \mathcal{C} \otimes_k \mathcal{M}$ induced by the coaction and comultiplication maps. The latter assertion is a restatement of the natural isomorphism $\mathcal{M} \simeq \mathcal{C} \,\square_{\mathcal{C}}\, \mathcal{M}$ [2, Corollary 2.2], [11, Section 1.2.1].

Part (b) is dual-analogous and uses the natural isomorphism $\mathfrak{P} \simeq \mathrm{Cohom}_{\mathcal{C}}(\mathcal{C}, \mathfrak{P})$ [11, Section 3.2.1]. We omit the details. ∎

5.4. Quasi-contra-Noetherian coalgebras.   For any coalgebra $\mathcal{C}$, the class of all quasi-finitely copresented right $\mathcal{C}$-comodules is closed under the kernels and extensions in comod–$\mathcal{C}$ [19, Lemma 2.4(a-b)]. Similarly, the class of all quasi-finitely presented left $\mathcal{C}$-contramodules is closed under the cokernels and extensions in $\mathcal{C}$–contra [19, Lemma 2.7(a-b)]. A coalgebra $\mathcal{C}$ is called *right quasi-cocoherent* if the class of all quasi-finitely copresented right $\mathcal{C}$-comodules is closed under cokernels in comod–$\mathcal{C}$, or equivalently (in view of Proposition 5.4), the class of quasi-finitely presented left $\mathcal{C}$-contramodules is closed under kernels in $\mathcal{C}$–contra. Over a right quasi-cocoherent coalgebra $\mathcal{C}$, the categories of quasi-finitely copresented right comodules and quasi-finitely presented left contramodules are abelian [19, Section 2].

Due to Propositions 5.3 and 5.4, left and right quasi-cocoherent coalgebras and semi-algebras over them are quite suitable for developing an involutive triangulated duality theory (see the discussion in Sections 1.9–1.10 of the introduction). However, the notions of quasi-finitely copresented comodules and quasi-finitely presented contramodules are somewhat complicated. To simplify matters a bit, we discuss certain more restrictive assumptions in this section.

A coalgebra $\mathcal{C}$ is said to be *right co-Noetherian* if every quotient comodule of a finitely cogenerated right $\mathcal{C}$-comodule is finitely cogenerated, or equivalently, every quotient comodule of the right $\mathcal{C}$-comodule $\mathcal{C}$ is finitely cogenerated [26, Theorem 3]. Finitely cogenerated right comodules over a right co-Noetherian coalgebra form an abelian category. We will say that a coalgebra $\mathcal{C}$ is *left contra-Noetherian* if every subcontramodule of a finitely generated left $\mathcal{C}$-contramodule is finitely generated, or equivalently, every subcontramodule of the left $\mathcal{C}$-contramodule $\mathcal{C}^\vee$ is finitely cogenerated. Finitely generated left contramodules over a left contra-Noetherian coalgebra form an abelian category. Any left contra-Noetherian coalgebra $\mathcal{C}$ is right co-Noetherian [18, Lemma 2.10(c)].

A coalgebra $\mathcal{C}$ is said to be *right quasi-co-Noetherian* if every quotient comodule of a quasi-finitely cogenerated right $\mathcal{C}$-comodule is quasi-finitely cogenerated, or equivalently, every quotient comodule of the right $\mathcal{C}$-comodule $\mathcal{C}$ is quasi-finitely cogenerated [4], [19, Proposition 2.3]. All quasi-finitely cogenerated right comodules over a right quasi-co-Noetherian coalgebra $\mathcal{C}$ are quasi-finitely copresented, and any such a coalgebra $\mathcal{C}$ is right quasi-cocoherent; so quasi-finitely cogenerated right comodules over it form an abelian category [4], [19, Section 2].

We will say that a coalgebra $\mathcal{C}$ is *left quasi-contra-Noetherian* if every subcontramodule of a quasi-finitely generated left $\mathcal{C}$-contramodule is quasi-finitely generated, or equivalently (in view of [19, Lemma 2.6(b)]), every subcontramodule of a quasi-finitely generated projective left $\mathcal{C}$-contramodule is quasi-finitely generated. All quasi-finitely generated left contramodules over a left quasi-contra-Noetherian coalgebra $\mathcal{C}$ are quasi-finitely presented, and any such a coalgebra $\mathcal{C}$ is right quasi-cocoherent; so quasi-finitely generated left contramodules over it form an abelian category. In fact, the following lemma holds.

Lemma 5.7. *Any left quasi-contra-Noetherian coalgebra is right quasi-co-Noetherian.*

Proof. Let $\mathcal{C}$ be a left quasi-contra-Noetherian coalgebra over k. Let $\mathcal{N}$ be a quasi-finitely cogenerated right $\mathcal{C}$-comodule and $\mathcal{L}$ be a quotient comodule of $\mathcal{N}$. Then $\mathcal{N}^\vee$ is a left $\mathcal{C}$-contramodule and $\mathcal{L}^\vee$ is a subcontramodule of $\mathcal{N}^\vee$. By Lemma 5.2, the left $\mathcal{C}$-contramodule $\mathcal{N}^\vee$ is quasi-finitely generated. Since $\mathcal{C}$ is left quasi-contra-Noetherian, it follows that the left $\mathcal{C}$-contramodule $\mathcal{L}^\vee$ is quasi-finitely generated. Applying Lemma 5.2 again, we can conclude that the right $\mathcal{C}$-comodule $\mathcal{L}$ is quasi-finitely cogenerated. Thus the coalgebra $\mathcal{C}$ is right quasi-co-Noetherian. ∎

A (coassicoative, counital) coalgebra $\mathcal{C}$ over a field k is said to be *cosemisimple* if the abelian category of left $\mathcal{C}$-comodules is semisimple, or equivalently, the abelian category of right $\mathcal{C}$-comodules is semisimple, or equivalently, the abelian category of left $\mathcal{C}$-contramodules is semisimple [12, Section 4.5]. A coalgebra is cosemisimple if and only if it is the sum of its cosimple subcoalgebras (where a coalgebra is called *cosimple* if it has no nonzero proper subcoalgebras) [12, Section 4.3]. Any coalgebra $\mathcal{C}$ has a unique maximal cosemisimple subcoalgebra $\mathcal{C}^{ss}$, which can be constructed as the (direct) sum of all cosimple subcoalgebras of $\mathcal{C}$. We refer to [24, Chapters 1–2 and 9] and [11, Section A.2] for the background material.

Examples 5.8. (1) Any cosemisimple coalgebra $\mathcal{C}$ is left and right quasi-contra-Noetherian. Indeed, any subcontramodule of a quasi-finitely generated $\mathcal{C}$-contramodule $\mathfrak{P}$ is at the same time a quotient contramodule of $\mathfrak{P}$, hence also quasi-finitely

generated (as the class of quasi-finitely generated contramodules over any coalgebra is closed under quotients).

(2) Let $\mathcal{C}$ be a coalgebra whose maximal cosemisimple subcoalgebra $\mathcal{C}^{\mathrm{ss}}$ is finite-dimensional (e. g., $\mathcal{C}$ is conilpotent, which means that $\mathcal{C}^{\mathrm{ss}} = k$; see the discussion in Section 2.2). Then any quasi-finitely cogenerated $\mathcal{C}$-comodule is finitely cogenerated [18, Lemma 2.2(e)], and similarly, any quasi-finitely generated $\mathcal{C}$-contramodule is finitely generated [18, Lemma 2.5(e)]. So the quasi-finiteness conditions on $\mathcal{C}$-comodules and $\mathcal{C}$-contramodules are equivalent to the similarly named finiteness conditions.

(3) Let $\mathcal{C}$ be a coalgebra whose dual algebra $\mathcal{C}^{\vee}$ is left Noetherian (as an abstract associative ring). Then the coalgebra $\mathcal{C}^{\mathrm{ss}}$ is finite-dimensional [18, Section 2], so (2) applies. Furthermore, by [18, Lemma 2.10(b)], the coalgebra $\mathcal{C}$ is *right Artinian* in the sense of [18, Section 2], and it follows that $\mathcal{C}$ is left contra-Noetherian [18, Lemmas 2.6(a) and 2.10(a)]. In view of the discussion in (2), we can conclude that the coalgebra $\mathcal{C}$ is also left quasi-contra-Noetherian.

5.5. Admissible smooth modules and contraadmissible contramodules.
Let G be locally profinite group, $H \subset G$ be a compact open subgroup, and k be a field. A smooth G-module $\mathcal{M}$ over k is said to be *admissible* if, for any (compact) open subgroup $F \subset G$, the k-vector subspace of F-invariant elements $\mathcal{M}^F \subset \mathcal{M}$ is finite-dimensional. Clearly, a smooth G-module is admissible if and only if it is admissible as a smooth H-module.

Here is the dual-analogous definition for G-contramodules over k. Given a contramodule $\mathfrak{P}$ over a (locally) profinite group F, its vector space of *contrainvariants* $\mathfrak{P}_F$ is defined as the (unique) maximal quotient F-contramodule of $\mathfrak{P}$ with the trivial contraaction of F. Here the trivial contraaction of F in a vector space U is the map $U[[F]] \longrightarrow U$ taking a U-valued measure μ on F to the vector $\mu(F) \in U$ which the measure μ assigns to the whole topological space F. This rule defines a F-contramodule structure on an arbitrary k-vector space U (cf. the definition of a G-contramodule in Section 3.1), called the *trivial F-contramodule structure*.

The functor of contrainvariants $\mathfrak{P} \longmapsto \mathfrak{P}_F \colon F\text{–}\mathsf{contra}_k \longrightarrow k\text{–}\mathsf{vect}$ is left adjoint to the functor assigning to any k-vector space U the vector space U endowed with the trivial F-contramodule structure. Explicitly, the vector space $\mathfrak{P}_F$ is the cokernel of the restriction $\pi_{\mathfrak{P}}|_{\mathfrak{P}[[F]]_0} \colon \mathfrak{P}[[F]]_0 \longrightarrow \mathfrak{P}$, where $\mathfrak{P}[[F]]_0 \subset \mathfrak{P}[[F]]$ is the subspace consisting of all the measures μ for which the measure of the whole space F vanishes, $\mu(F) = 0$ (and $\pi_{\mathfrak{P}} \colon \mathfrak{P}[[F]] \longrightarrow \mathfrak{P}$ is the contraaction map).

We will say that a G-contramodule $\mathfrak{P}$ over k is *contraadmissible* if, for any (compact) open subgroup $F \subset G$, the k-vector space of F-contrainvariants $\mathfrak{P}_F$ is finite-dimensional. Clearly, it suffices to check this condition for "small enough" open subgroups $F \subset G$, as for any open subgroups $F' \subset F'' \subset G$ the vector space $\mathfrak{P}_{F''}$ is a quotient space of $\mathfrak{P}_{F'}$. So a G-contramodule $\mathfrak{P}$ is contraadmissible if and only if it is contraadmissible as an H-contramodule.

We recall that for any profinite group H and any field k, the inverse element map $H \longrightarrow H$ induces an isomorphism $\mathcal{C} \simeq \mathcal{C}^{\mathrm{op}}$ between the coalgebra $\mathcal{C} = k(H)$ and its opposite coalgebra. So there is no difference between left and right $\mathcal{C}$-comodules.

Lemma 5.9. *Let H be a profinite group and k be a field. Then*
(a) *a discrete H-module over k is admissible if and only if it is a quasi-finitely cogenerated comodule over the coalgebra $k(H)$;*
(b) *an H-contramodule over k is contraadmissible if and only if it is a quasi-finitely generated contramodule over the coalgebra $k(H)$.*

Proof. For any open normal subgroup $F \subset H$, the coalgebra $\mathcal{E} = k(H/F)$ is a finite-dimensional subcoalgebra in $\mathcal{C} = k(H)$. Conversely, for any finite-dimensional subcoalgebra $\mathcal{E} \subset k(H)$ there exists an open normal subgroup $F \subset H$ such that $\mathcal{E} \subset k(H/F) \subset k(H)$. It remains to observe that for the subcoalgebra $\mathcal{E} = k(H/F)$, for any discrete H-module $\mathcal{M}$ over k, and for any H-contramodule $\mathfrak{P}$ over k, one has $_\mathcal{E}\mathcal{M} = \mathcal{M}^F$ and $^\mathcal{E}\mathfrak{P} = \mathfrak{P}_F$. In other words, the maximal $k(H/F)$-subcomodule of the $k(H)$-comodule $\mathcal{M}$ is the subspace of F-invariants $\mathcal{M}^F \subset \mathcal{M}$, and the maximal quotient $k(H/F)$-contramodule of the $k(H)$-contramodule $\mathfrak{P}$ is the quotient space of F-contrainvariants $\mathfrak{P} \twoheadrightarrow \mathfrak{P}_F$. ∎

Let H be a profinite group and k be a field. We will say that a discrete H-module $\mathcal{M}$ over k is *admissibly copresented* if it is quasi-finitely copresented as a $k(H)$-comodule. In view of Lemma 5.9(a), this means that $\mathcal{M}$ is the kernel of a morphism between two admissible injective discrete H-modules over k. Similarly, we will say that an H-contramodule $\mathfrak{P}$ over k is *contraadmissibly presented* if it is quasi-finitely presented as a $k(H)$-contramodule. In view of Lemma 5.9(b), this means that $\mathfrak{P}$ is the cokernel of a morphism between two contraadmissible projective H-contramodules over k.

Proposition 5.10. *Let H be a profinite group, $F \subset H$ be an open subgroup, and k be a field. Then*
(a) *a discrete H-module over k is admissibly copresented if and only if it is admissibly copresented as a discrete F-module;*
(b) *an H-contramodule over k is contraadmissibly presented if and only if it is contraadmissibly presented as a F-contramodule.*

Proof. Consider two coalgebras $\mathcal{C} = k(H)$ and $\mathcal{D} = k(F)$. Then there is a natural surjective coalgebra morphism $\mathcal{C} \longrightarrow \mathcal{D}$ assigning to a locally constant function $H \longrightarrow k$ its restriction to F. A key observation is that $\mathcal{C}$ is an injective (in fact, cofree) left $\mathcal{D}$-comodule (as well as right $\mathcal{D}$-comodule). Here, in the language of group representations, considering a $\mathcal{C}$-comodule as a $\mathcal{D}$-comodule means restricting the group action from H to F; and similarly with the contramodules.

Since $\mathcal{C}$ is an injective $\mathcal{D}$-comodule, any injective $\mathcal{C}$-comodule is also injective as a $\mathcal{D}$-comodule, and any projective $\mathcal{C}$-contramodule is also projective as a $\mathcal{D}$-contramodule. Furthermore, in view of Lemma 5.9 and the above discusssion of (contra)admissibility, any quasi-finitely cogenerated $\mathcal{C}$-comodule is also a quasi-finitely cogenerated $\mathcal{D}$-comodule, and any quasi-finitely generated $\mathcal{C}$-contramodule is also a quasi-finitely generated $\mathcal{D}$-contramodule.

Now let $\mathcal{M}$ be a quasi-finitely copresented left $\mathcal{C}$-comodule; so $\mathcal{M}$ is the kernel of a morphism of quasi-finitely cogenerated injective $\mathcal{C}$-comodules $\mathcal{I} \longrightarrow \mathcal{J}$. Then the left $\mathcal{D}$-comodule $\mathcal{M}$ is the kernel of the morphism of quasi-finitely cogenerated injective $\mathcal{D}$-comodules $\mathcal{I} \longrightarrow \mathcal{J}$. Hence $\mathcal{M}$ is a quasi-finitely copresented left $\mathcal{D}$-comodule. This proves the "only if" implication in part (a); and the proof of the "only if" implication in part (b) is similar.

To prove the "if" implication in part (a), consider the functor of coextension of scalars assigning to a left $\mathcal{D}$-comodule $\mathcal{N}$ the left $\mathcal{C}$-comodule $_\mathcal{C}\mathcal{N} = \mathcal{C} \,\square_\mathcal{D}\, \mathcal{N}$, as discussed in Section 5.3. In the language of smooth representations (in the situation at hand with $\mathcal{C} = k(H)$ and $\mathcal{D} = k(F)$), the coextension of scalars is the functor of induced representation from F to H, that is $_\mathcal{C}\mathcal{N} = \mathrm{ind}_F^H\,\mathcal{N}$.

Let $\mathcal{M}$ be a left $\mathcal{C}$-comodule that is quasi-finitely copresented as a left $\mathcal{D}$-comodule. Then, by Lemma 5.5(b), the left $\mathcal{C}$-comodule $_\mathcal{C}\mathcal{M} = \mathcal{C} \,\square_\mathcal{D}\, \mathcal{M}$ is quasi-finitely copresented. According to the "only if" implication, which we have already proved, it follows that $\mathcal{C} \,\square_\mathcal{D}\, \mathcal{M}$ is also quasi-finitely copresented as a $\mathcal{D}$-comodule. Applying Lemma 5.5(b) again, we see that the left $\mathcal{C}$-comodule $_\mathcal{C}(_\mathcal{C}\mathcal{M}) = \mathcal{C} \,\square_\mathcal{D}\, \mathcal{C} \,\square_\mathcal{D}\, \mathcal{M}$ is quasi-finitely copresented, too. By Lemma 5.6(a), we can conclude that $\mathcal{M}$ is the kernel of a morphism between two quasi-finitely copresented left $\mathcal{C}$-comodules. It remains to recall that the class of quasi-finitely copresented comodules over any coalgebra $\mathcal{C}$ is closed under kernels [19, Lemma 2.4(a)].

The proof of the "if" implication in part (b) is dual-analogous and based on Lemmas 5.5(d) and 5.6(b). ∎

Let G be a locally profinite group, $H \subset G$ be a compact open subgroup, and k be a field. We will say that a smooth G-module $\boldsymbol{\mathcal{M}}$ over k is *admissibly copresented* if it is admissibly copresented as a discrete H-module over k. According to Proposition 5.10(a), this property of a smooth G-module $\boldsymbol{\mathcal{M}}$ does not depend on the choice of a compact open subgroup H in the locally profinite group G. The full subcategory of admissibly copresented smooth G-modules will be denoted by $G\text{–smooth}_{k,\mathsf{acp}} \subset G\text{–smooth}_k$.

Similarly, we will say that a G-contramodule $\boldsymbol{\mathfrak{P}}$ over k is *contraadmissibly presented* if it is contraadmissibly presented as an H-contramodule over k. According to Proposition 5.10(b), this property of a G-contramodule $\boldsymbol{\mathfrak{P}}$ does not depend on the choice of a compact open subgroup $H \subset G$. The full subcategory of contraadmissibly presented G-contramodules will be denoted by $G\text{–contra}_{k,\mathsf{caap}} \subset G\text{–smooth}_k$.

Proposition 5.11. *The dualization functor*

$$\boldsymbol{\mathcal{M}} \longmapsto \boldsymbol{\mathcal{M}}^\vee = \mathrm{Hom}_k(\boldsymbol{\mathcal{M}}, k)\colon G\text{–smooth}_k \longrightarrow G\text{–contra}_k$$

restricts to an anti-equivalence between the full subcategories of admissibly copresented smooth G-modules and contraadmissibly presented G-contramodules over k,

$$(G\text{–smooth}_{k,\mathsf{acp}})^{\mathsf{op}} \simeq G\text{–contra}_{k,\mathsf{caap}}.$$

Sketch of proof. Proposition 5.4, applied to the case of the coalgebra $\mathcal{C} = k(H)$, tells that the dualization functor

$$\mathcal{M} \longmapsto \mathcal{M}^\vee = \mathrm{Hom}_k(\mathcal{M}, k)\colon H\text{–discr}_k \longrightarrow H\text{–contra}_k$$

restricts to an anti-equivalence between the full subcategories of admissibly copresented discrete H-modules and contraadmissibly presented H-contramodules over k,

$$(H\text{–discr}_{k,\mathsf{acp}})^{\mathsf{op}} \simeq H\text{–contra}_{k,\mathsf{caap}}.$$

The idea is to show that, given an admissibly copresented discrete H-module $\mathcal{M}$ and the dual contraadmissibly presented H-contramodule $\mathfrak{P} = \mathcal{M}^\vee$ over k, the

dualization functor $\mathcal{M} \longmapsto \mathfrak{P} = \mathcal{M}^\vee$ establishes a bijective correspondence between the ways to extend the discrete H-action in $\mathcal{M}$ to a smooth G-action and the ways to extend the H-contraaction in $\mathfrak{P}$ to a G-contraaction.

Following [11, Section E.1], given a discrete H-module $\mathcal{M}$ over k, extending the action of H in $\mathcal{M}$ to an action of the group G is equivalent to defining a structure of *right semimodule* over the $\mathcal{C}$-semialgebra $\boldsymbol{S} = k(G)$ on the right $\mathcal{C}$-comodule $\mathcal{M}$. This means specifying a *right semiaction* map $\mathcal{M} \,\square_\mathcal{C}\, \boldsymbol{S} \longrightarrow \mathcal{M}$, which should be a morphism of left $\mathcal{C}$-comodules satisfying the semiassociativity and semiunitality equations. Here $\mathcal{M} \,\square_\mathcal{C}\, \boldsymbol{S} = \mathrm{ind}_H^G \mathcal{M}$ is the smooth G-module induced from the smooth H-module $\mathcal{M}$. Similarly, given an H-contramodule $\mathfrak{P}$ over k, extending the contraaction of H in $\mathfrak{P}$ to a contraaction of G is equivalent to defining a structure of *left semicontramodule* over $\boldsymbol{S}$ on the left $\mathcal{C}$-contramodule $\mathfrak{P}$. This means specifying a *left semicontraaction* map $\mathfrak{P} \longrightarrow \mathrm{Cohom}_\mathcal{C}(\boldsymbol{S}, \mathfrak{P})$, which should be a morphism of left $\mathcal{C}$-contramodules satisfying the semicontraassociativity and semicontraunitality equations [14, Example 2.6]. See also the discussions in Sections 1.2 and 3.3 above. Now in the situation at hand, we have $\mathfrak{P} = \mathcal{M}^\vee$ and $\mathrm{Cohom}_\mathcal{C}(\boldsymbol{S}, \mathfrak{P}) = (\mathcal{M} \,\square_\mathcal{C}\, \boldsymbol{S})^\vee$ (see (31)). In order to prove the desired bijection between the $\boldsymbol{S}$-semimodule structures on $\mathcal{M}$ and the $\boldsymbol{S}$-semicontramodule structures on $\mathfrak{P}$, we use the result of [19, Proposition 2.8(b)], claiming, in particular, that for any coalgebra $\mathcal{C}$, and right $\mathcal{C}$-comodule $\mathcal{N}$, and any quasi-finitely cogenerated right $\mathcal{C}$-comodule $\mathcal{M}$, the dualization functor $\mathcal{N} \longmapsto \mathcal{N}^\vee$ induces an isomorphism between the Hom spaces in the categories of right $\mathcal{C}$-comodules and left $\mathcal{C}$-contramodules

$$\mathrm{Hom}_{\mathcal{C}^\mathrm{op}}(\mathcal{N}, \mathcal{M}) \simeq \mathrm{Hom}^\mathcal{C}(\mathcal{M}^\vee, \mathcal{N}^\vee).$$

Notice that the right $\mathcal{C}$-comodule $\mathcal{N} = \mathcal{M} \,\square_\mathcal{C}\, \boldsymbol{S}$ does not need to be quasi-finitely cogenerated, of course; but the right $\mathcal{C}$-comodule $\mathcal{M}$ is (quasi-finitely copresented, hence) quasi-finitely cogenerated by assumption, which is sufficient to make [19, Proposition 2.8(b)] applicable.

It is straightforward to check that a right $\mathcal{C}$-comodule map $\mathcal{M} \,\square_\mathcal{C}\, \boldsymbol{S} \longrightarrow \mathcal{M}$ satisfies the above-mentioned equations from the definition of a semimodule and only if the corresponding left $\mathcal{C}$-contramodule map $\mathfrak{P} \longrightarrow \mathrm{Cohom}_\mathcal{C}(\boldsymbol{S}, \mathfrak{P})$ satisfies the above-mentioned (dual-analogous) equations from the definition of a semicontramodule. This proves that our functor $(G\text{–smooth}_{k,\mathsf{acp}})^\mathsf{op} \longrightarrow G\text{–contra}_{k,\mathsf{caap}}$ is surjective on the isomorphism classes of objects. Checking that it is bijective on morphisms is another application of [19, Proposition 2.8(b)]. ∎

It follows from Proposition 5.11 that every contraadmissibly presented G-contramodule over k carries a natural, functorially defined pseudo-compact G-module structure (i. e., a pseudo-compact topology making it the pseudo-compact dual vector space to a smooth G-module).

5.6. Locally quasi-contra-Noetherian and locally quasi-cocoherent groups.
In the previous section we have shown that various quasi-finiteness/admissibility conditions on smooth modules and contramodules do not depend on the choice of a compact open subgroup H in a given locally profinite group G. In this section we prove similar results for local quasi-finiteness conditions on the group G itself (cf. the discussion in Section 1.14 of the introduction).

Lemma 5.12. *Let H be a profinite group, $F \subset H$ be an open subgroup, and k be a field. Consider the related surjective morphism $\mathcal{C} = k(H) \longrightarrow k(F) = \mathcal{D}$ of coalgebras over k. Then*

(a) *a $\mathcal{D}$-comodule $\mathcal{N}$ is quasi-finitely cogenerated if and only if the $\mathcal{C}$-comodule $_{\mathcal{C}}\mathcal{N}$ is quasi-finitely cogenerated;*

(b) *a $\mathcal{D}$-comodule $\mathcal{N}$ is quasi-finitely copresented if and only if the $\mathcal{D}$-comodule $_{\mathcal{C}}\mathcal{N}$ is quasi-finitely copresented;*

(c) *a $\mathcal{D}$-contramodule $\mathfrak{Q}$ is quasi-finitely generated if and only if the $\mathcal{C}$-contramodule $^{\mathcal{C}}\mathfrak{Q}$ is quasi-finitely generated;*

(d) *a $\mathcal{D}$-contramodule $\mathfrak{Q}$ is quasi-finitely presented if and only if the $\mathcal{C}$-contramodule $^{\mathcal{C}}\mathfrak{Q}$ is quasi-finitely presented.*

Proof. In all the four parts (a–d), the "only if" implication holds for any morphism of coalgebras $\mathcal{C} \longrightarrow \mathcal{D}$ by Lemma 5.5. Let us prove the "if".

The key observation is that the $\mathcal{D}$-$\mathcal{D}$-bicomodule $\mathcal{C}$ contains the $\mathcal{D}$-$\mathcal{D}$-bicomodule $\mathcal{D}$ as a direct summand (or in other words, the smooth $F \times F$-module $k(H)$ contains the smooth $F \times F$-module $k(F)$ as a direct summand). Consequently, for any left $\mathcal{D}$-comodule $\mathcal{N}$, applying the functor of coextension of scalars $\mathcal{N} \longmapsto {}_{\mathcal{C}}\mathcal{N} = \mathcal{C} \,\square_{\mathcal{D}}\, \mathcal{N} = \mathrm{ind}_H^G \mathcal{N}$ produces a $\mathcal{C}$-comodule whose underlying $\mathcal{D}$-comodule contains $\mathcal{N}$ as a direct summand. Similarly, for any left $\mathcal{D}$-contramodule $\mathfrak{Q}$, applying the functor of contraextension of scalars $\mathfrak{Q} \longmapsto {}^{\mathcal{C}}\mathfrak{Q} = \mathrm{Cohom}_{\mathcal{D}}(\mathcal{C}, \mathfrak{Q})$ produces a $\mathcal{C}$-contramodule whose underlying $\mathcal{D}$-contramodule contains $\mathfrak{Q}$ as a direct summand.

Part (a): assume that the $\mathcal{C}$-comodule $_{\mathcal{C}}\mathcal{N}$ is quasi-finitely cogenerated. As explained in the proof of Proposition 5.10, it follows that the $\mathcal{D}$-comodule $_{\mathcal{C}}\mathcal{N}$ is quasi-finitely cogenerated. Since the class of quasi-finitely cogenerated comodules over any coalgebra is closed under direct summands, the assertion follows. The argument for part (b) is similar, using Proposition 5.10(a); and parts (c-d) are also similar. ■

Due to the isomorphism of coalgebras $\mathcal{C} = k(H) \simeq \mathcal{C}^{\mathrm{op}}$, there is no difference between the "left" and "right" versions of the properties appearing in the next corollary.

Corollary 5.13. *Let H be a profinite group, $F \subset H$ be an open subgroup, and k be a field. Then*

(a) *the ring $k[[H]] = k(H)^{\vee}$ is Noetherian if and only if the ring $k[[F]] = k(F)^{\vee}$ is Noetherian;*

(b) *the coalgebra $k(H)$ is quasi-contra-Noetherian if and only if the coalgebra $k(F)$ is quasi-contra-Noetherian;*

(c) *the coalgebra $k(H)$ is quasi-cocoherent if and only if the coalgebra $k(F)$ is quasi-cocoherent.*

Proof. Part (a) is essentially well-known; cf. [1, Exercise 7.6]. We will prove part (b), parts (a) and (c) being similar. The argument is a straightforward consequence of the results of Section 5.5 and this section. As above, we set $\mathcal{C} = k(H)$ and $\mathcal{D} = k(F)$.

"If": let $\mathfrak{P}$ be a quasi-finitely generated $\mathcal{C}$-contramodule and $\mathfrak{Q} \subset \mathfrak{P}$ be a subcontramodule. In other words, this means that $\mathfrak{P}$ is a contraadmissible H-contramodule over k (see Lemma 5.9(b)). Hence $\mathfrak{P}$ is also a contraadmissible F-contramodule over k, i. e., a quasi-finitely generated $\mathcal{D}$-contramodule.

By assumption, $\mathcal{D}$ is quasi-contra-Noetherian, so $\mathfrak{Q}$ is a quasi-finitely generated $\mathcal{D}$-contramodule, too. In other words, $\mathfrak{Q}$ is a contraadmissible F-contramodule over k, hence a contraadmissible H-contramodule. Thus $\mathfrak{Q}$ is a quasi-finitely generated $\mathcal{C}$-contramodule.

"Only if": let $\mathfrak{P}$ be a quasi-finitely generated $\mathcal{D}$-contramodule and $\mathfrak{Q} \subset \mathfrak{P}$ be a subcontramodule. Since $\mathcal{C}$ is an injective left $\mathcal{D}$-comodule, the functor of contraextension of scalars $\mathfrak{P} \longmapsto {}^{\mathcal{C}}\mathfrak{P} = \mathrm{Cohom}_{\mathcal{D}}(\mathcal{C}, \mathfrak{P})$ is exact. So ${}^{\mathcal{C}}\mathfrak{P}$ is a $\mathcal{C}$-contramodule and ${}^{\mathcal{C}}\mathfrak{Q} \subset {}^{\mathcal{C}}\mathfrak{P}$ is a $\mathcal{C}$-subcontramodule. By Lemma 5.12(c), the $\mathcal{C}$-contramodule ${}^{\mathcal{C}}\mathfrak{P}$ is quasi-finitely generated. By assumption, $\mathcal{C}$ is quasi-contra-Noetherian, so the $\mathcal{C}$-contramodule ${}^{\mathcal{C}}\mathfrak{Q}$ is quasi-finitely generated, too. Again by Lemma 5.12(c), the $\mathcal{D}$-contramodule $\mathfrak{Q}$ is quasi-finitely generated.

In part (c), one has to use Proposition 5.10(a) and Lemma 5.12(b).

In part (a), the implication "if" is a particular case of [5, Corollary 1.5]. To prove the "only if", use the fact that any left $k[[F]]$-module M is a direct summand of the left $k[[F]]$-module $k[[H]] \otimes_{k[[F]]} M$, since the $k[[F]]$-$k[[F]]$-bimodule $k[[F]]$ is a direct summand of the $k[[F]]$-$k[[F]]$-bimodule $k[[H]]$. ∎

Let G be a locally profinite group, $H \subset G$ be a compact open subgroup, and k be a field. We will say that G is *locally Noetherian over k* if the ring $k[[H]] = k(H)^{\vee}$ is Noetherian. Similarly, we will say that G is *locally quasi-contra-Noetherian over k* if the coalgebra $k(H)$ is quasi-contra-Noetherian, and we will say that G is *locally quasi-cocoherent over k* if the coalgebra $k(H)$ is quasi-cocoherent. In view of Corollary 5.13, these properties of a locally profinite group G do not depend on the choice of a compact open subgroup $H \subset G$. When speaking of such properties in application to profinite groups H, we will drop the adverb "locally".

According to Example 5.8(3), any locally Noetherian (locally profinite) group is locally quasi-contra-Noetherian. Following the discussion in Section 5.4, any locally quasi-contra-Noetherian group is locally quasi-cocoherent (over the same field k). Moreover, for a locally profinite group G that is locally quasi-contra-Noetherian over a field k, any admissible smooth G-module over k is admissibly copresented and any contraadmissible G-contramodule over k is contraadmissibly presented.

Examples 5.14. (1) Following Example 5.8(1), any locally profinite group G which has a compact open subgroup H of the proorder not divisible by the characteristic of k is locally quasi-contra-Noetherian over k. In particular, over a field k of characteristic 0 any locally profinite group is locally quasi-contra-Noetherian.

(2) According to [1, Corollary 7.25 and Theorem 8.32], over a field k of characteristic p, any p-adic Lie group G is locally Noetherian.

6. Involutive triangulated duality

6.1. Quasi-cocoherent coalgebras. Let $\mathcal{C}$ be a coalgebra over a field k. Assume that the coalgebra $\mathcal{C}$ is left quasi-cocoherent and right quasi-cocoherent (see Sections 5.1–5.2 and 5.4 for the relevant definitions).

Then the category of quasi-finitely copresented left $\mathcal{C}$-comodules $\mathcal{C}\text{--}\mathsf{comod}_{\mathsf{qfc}}$ is an abelian subcategory of the abelian category $\mathcal{C}\text{--}\mathsf{comod}$ with an exact identity embedding functor $\mathcal{C}\text{--}\mathsf{comod}_{\mathsf{qfc}} \longrightarrow \mathcal{C}\text{--}\mathsf{comod}$. The abelian category $\mathcal{C}\text{--}\mathsf{comod}_{\mathsf{qfc}}$ has

enough injective objects, which form the full subcategory

$$\mathcal{C}\text{-comod}_{\mathsf{qfc}} \cap \mathcal{C}\text{-comod}^{\mathsf{inj}} = \mathcal{C}\text{-comod}_{\mathsf{qfc}}^{\mathsf{inj}} \subset \mathcal{C}\text{-comod};$$

so the injective objects of $\mathcal{C}\text{-comod}_{\mathsf{qfc}}$ are also injective in $\mathcal{C}\text{-comod}$. The category of quasi-finitely copresented right $\mathcal{C}$-comodules $\mathsf{comod}_{\mathsf{qfc}}\text{-}\mathcal{C} \subset \mathsf{comod}\text{-}\mathcal{C}$ has similar properties.

Furthermore, the category of quasi-finitely presented left $\mathcal{C}$-contramodules $\mathcal{C}\text{-contra}_{\mathsf{qfp}}$ is an abelian subcategory of the abelian category $\mathcal{C}\text{-contra}$ with an exact identity embedding functor $\mathcal{C}\text{-contra}_{\mathsf{qfp}} \longrightarrow \mathcal{C}\text{-contra}$. The abelian category $\mathcal{C}\text{-contra}_{\mathsf{qfp}}$ has enough projective objects, which form the full subcategory

$$\mathcal{C}\text{-contra}_{\mathsf{qfp}} \cap \mathcal{C}\text{-contra}^{\mathsf{proj}} = \mathcal{C}\text{-contra}_{\mathsf{qfp}}^{\mathsf{proj}} \subset \mathcal{C}\text{-contra};$$

so the projective objects of $\mathcal{C}\text{-contra}_{\mathsf{qfp}}$ are also projective in $\mathcal{C}\text{-contra}$.

Recall from the discussion in Section 4.1 that, for any coalgebra $\mathcal{C}$, the three abelian categories $\mathcal{C}\text{-comod}$, $\mathsf{comod}\text{-}\mathcal{C}$, and $\mathcal{C}\text{-contra}$ have the same homological dimension (called the homological dimension of $\mathcal{C}$). For any coalgebra $\mathcal{C}$, the equivalence of additive categories $\mathcal{C}\text{-comod}^{\mathsf{inj}} \simeq \mathcal{C}\text{-contra}^{\mathsf{proj}}$ (28–29) induces an equivalence of the homotopy categories $\mathsf{Hot}(\mathcal{C}\text{-comod}^{\mathsf{inj}}) \simeq \mathsf{Hot}(\mathcal{C}\text{-contra}^{\mathsf{proj}})$ of the additive categories of injective left $\mathcal{C}$-comodules and projective left $\mathcal{C}$-contramodules. For a coalgebra $\mathcal{C}$ of finite homological dimension, this leads to an equivalence of the derived categories

$$\mathbb{R}\Psi_{\mathcal{C}}\colon \mathsf{D}(\mathcal{C}\text{-comod}) = \mathsf{Hot}(\mathcal{C}\text{-comod}^{\mathsf{inj}}) \simeq \mathsf{Hot}(\mathcal{C}\text{-contra}^{\mathsf{proj}}) = \mathsf{D}(\mathcal{C}\text{-contra}) :\mathbb{L}\Phi_{\mathcal{C}} \quad (48)$$

(cf. the discussion in Section 1.4).

For any left and right quasi-cocoherent coalgebra $\mathcal{C}$, the bounded derived category of quasi-finitely copresented left $\mathcal{C}$-comodules is a full subcategory of the derived category of left $\mathcal{C}$-comodules, and the bounded derived category of quasi-finitely presented left $\mathcal{C}$-contramodules is a full subcategory of the derived category of left $\mathcal{C}$-contramodules,

$$\mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{-comod}_{\mathsf{qfc}}) \subset \mathsf{D}(\mathcal{C}\text{-comod}) \quad \text{and} \quad \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{-contra}_{\mathsf{qfp}}) \subset \mathsf{D}(\mathcal{C}\text{-contra}).$$

Now let $\mathcal{C}$ be a left and right quasi-cocoherent coalgebra of finite homological dimension. Then the equivalence of additive categories $\mathcal{C}\text{-comod}_{\mathsf{qfc}}^{\mathsf{inj}} \simeq \mathcal{C}\text{-contra}_{\mathsf{qfp}}^{\mathsf{proj}}$ from Proposition 5.3 shows that the triangulated equivalence (48) restricts to an equivalence between the bounded derived categories of quasi-finitely (co)presented comodules and contramodules,

$$\mathbb{R}\Psi_{\mathcal{C}}\colon \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{-comod}_{\mathsf{qfc}}) = \mathsf{Hot}^{\mathsf{b}}(\mathcal{C}\text{-comod}_{\mathsf{qfc}}^{\mathsf{inj}})$$
$$\simeq \mathsf{Hot}^{\mathsf{b}}(\mathcal{C}\text{-contra}_{\mathsf{qfp}}^{\mathsf{proj}}) = \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{-contra}_{\mathsf{qfp}}) :\mathbb{L}\Phi_{\mathcal{C}}. \quad (49)$$

Furthermore, following Proposition 5.4, the dualization functor $\mathcal{N} \longmapsto \mathcal{N}^{\vee}$ provides an anti-equivalence of abelian categories $(\mathsf{comod}_{\mathsf{qfc}}\text{-}\mathcal{C})^{\mathsf{op}} \simeq \mathcal{C}\text{-contra}_{\mathsf{qfp}}$, inducing an anti-equivalence of the bounded derived categories

$$\mathsf{D}^{\mathsf{b}}(\mathsf{comod}_{\mathsf{qfc}}\text{-}\mathcal{C})^{\mathsf{op}} \simeq \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{-contra}_{\mathsf{qfp}}). \quad (50)$$

Composing the triangulated anti-equivalence $\mathcal{N} \longmapsto \mathcal{N}^{\vee}$ (50) with the triangulated equivalence $\mathbb{L}\Phi_{\mathcal{C}}$ (49), we obtain an anti-equivalence between the bounded derived categories of quasi-finitely copresented left and right $\mathcal{C}$-comodules,

$$\Delta_{\mathcal{C}}\colon \mathsf{D}^{\mathsf{b}}(\mathsf{comod}_{\mathsf{qfc}}\text{-}\mathcal{C})^{\mathsf{op}} \simeq \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{-comod}_{\mathsf{qfc}}). \quad (51)$$

Proposition 6.1. *For any left and right quasi-cocoherent coalgebra $\mathcal{C}$ of finite homological dimension, the triangulated anti-equivalences $\Delta_{\mathcal{C}}\colon \mathsf{D}^{\mathsf{b}}(\mathsf{comod}_{\mathsf{qfc}}\text{--}\mathcal{C})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{--}\mathsf{comod}_{\mathsf{qfc}})$ and $\Delta_{\mathcal{C}^{\mathsf{op}}}\colon \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{--}\mathsf{comod}_{\mathsf{qfc}})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{comod}_{\mathsf{qfc}}\text{--}\mathcal{C})$ are inverse to each other.*

Sketch of proof. More generally, for any coalgebra $\mathcal{C}$ of finite homological dimension and any k-vector space one can compose the contravariant triangulated functor

$$\mathrm{Hom}_k(-,V)\colon \mathsf{D}(\mathsf{comod}\text{--}\mathcal{C})^{\mathsf{op}} \longrightarrow \mathsf{D}(\mathcal{C}\text{--}\mathsf{contra})$$

with the triangulated equivalence $\mathbb{L}\Phi_{\mathcal{C}}$ (48), obtaining a contravariant triangulated functor

$$\Delta_{\mathcal{C}}^{V}\colon \mathsf{D}(\mathsf{comod}\text{--}\mathcal{C})^{\mathsf{op}} \longrightarrow \mathsf{D}(\mathcal{C}\text{--}\mathsf{comod}).$$

Arguing similarly to Section 4.2 (cf. Section 1.8), one shows that the contravariant functors $\Delta_{\mathcal{C}}^{V}$ and

$$\Delta_{\mathcal{C}^{\mathsf{op}}}^{V}\colon \mathsf{D}(\mathcal{C}\text{--}\mathsf{comod})^{\mathsf{op}} \longrightarrow \mathsf{D}(\mathsf{comod}\text{--}\mathcal{C})$$

are *right adjoint to each other*. This means that for every complex of left $\mathcal{C}$-comodules $\mathcal{M}^{\bullet}$ and every complex of right $\mathcal{C}$-comodules $\mathcal{N}^{\bullet}$ there is a natural isomorphism of Hom spaces in the derived categories

$$\mathrm{Hom}_{\mathsf{D}(\mathcal{C}\text{--}\mathsf{comod})}(\mathcal{M}^{\bullet}, \Delta_{\mathcal{C}}^{V}\mathcal{N}^{\bullet}) \simeq \mathrm{Hom}_{\mathsf{D}(\mathsf{comod}\text{--}\mathcal{C})}(\mathcal{N}^{\bullet}, \Delta_{\mathcal{C}^{\mathsf{op}}}^{V}\mathcal{M}^{\bullet}). \tag{52}$$

Similarly to the proof of Proposition 4.2, one can compute both the left- and the right-hand side of (52) as the degree-zero cohomology vector space of what is denoted in [11] by $\mathrm{Hom}_k(\mathrm{Cotor}^{\mathcal{C}}(\mathcal{M}^{\bullet}, \mathcal{N}^{\bullet}), V)$ (see [11, Section 0.2.2]). One can also construct the adjunction morphisms $\mathcal{M}^{\bullet} \longrightarrow \Delta_{\mathcal{C}}^{V}\Delta_{\mathcal{C}^{\mathsf{op}}}^{V}(\mathcal{M}^{\bullet})$ in $\mathsf{D}(\mathcal{C}\text{--}\mathsf{comod})$ and $\mathcal{N}^{\bullet} \longrightarrow \Delta_{\mathcal{C}^{\mathsf{op}}}^{V}\Delta_{\mathcal{C}}^{V}(\mathcal{N}^{\bullet})$ in $\mathsf{D}(\mathsf{comod}\text{--}\mathcal{C})$ in a way similar to the construction in Section 4.2.

Now specializing to the vector space $V = k$ and restricting to the full subcategories $\mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{--}\mathsf{comod}_{\mathsf{qfc}}) \subset \mathsf{D}(\mathcal{C}\text{--}\mathsf{comod})$ and $\mathsf{D}^{\mathsf{b}}(\mathsf{comod}_{\mathsf{qfc}}\text{--}\mathcal{C}) \subset \mathsf{D}(\mathsf{comod}\text{--}\mathcal{C})$, we have a pair of right adjoint contravariant functors $\Delta_{\mathcal{C}}\colon \mathsf{D}^{\mathsf{b}}(\mathsf{comod}_{\mathsf{qfc}}\text{--}\mathcal{C})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{--}\mathsf{comod}_{\mathsf{qfc}})$ and $\Delta_{\mathcal{C}^{\mathsf{op}}}\colon \mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{--}\mathsf{comod}_{\mathsf{qfc}})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{comod}_{\mathsf{qfc}}\text{--}\mathcal{C})$. We already know that the latter two functors are triangulated anti-equivalences (51). Any two adjoint equivalences are mutually inverse. ∎

6.2. Profinite groups. Let k be a field and H be a profinite group of finite k-cohomological dimension. Assume further that H is quasi-cocoherent over k, i. e., the coalgebra $\mathcal{C} = k(H)$ is quasi-cocoherent (see Section 5.6).

Recall that the inverse element map $H \longrightarrow H$ induces an isomorphism $\mathcal{C} \simeq \mathcal{C}^{\mathsf{op}}$, so the categories of left and right $\mathcal{C}$-comodules are equivalent to each other. They are also equivalent to the category $H\text{--}\mathsf{discr}_k$ of discrete H-modules over k, while the category of (left or right) $\mathcal{C}$-contramodules is equivalent to the category $H\text{--}\mathsf{contra}_k$ of H-contramodules over k.

We also recall the notation $H\text{--}\mathsf{discr}_{k,\mathsf{acp}} \subset H\text{--}\mathsf{discr}_k$ and $H\text{--}\mathsf{contra}_{k,\mathsf{caap}} \subset H\text{--}\mathsf{contra}_k$ for the full subcategories of admissibly copresented discrete H-modules and contraadmissibly presented H-contramodules over k (see Section 5.5 for a more general discussion for a locally profinite group G). Under our quasi-cocoherence assumption on the group H, these two categories are abelian. When the profinite group H is

quasi-contra-Noetherian over k, these full subcategories coincide with the full subcategories of admissible discrete H-modules and contraadmissible H-contramodules over k, respectively (see Section 5.4).

Specializing the discussion in Section 6.1 to the case of the coalgebra $\mathcal{C} = k(H)$, we see that the derived equivalence

$$\mathbb{R}\Psi_H \colon \mathsf{D}(H\text{–}\mathsf{discr}_k) \simeq \mathsf{D}(H\text{–}\mathsf{contra}_k) \colon \mathbb{L}\Phi_H \tag{53}$$

restricts to an equivalence between the bounded derived categories of admissibly copresented discrete modules and contraadmissibly presented contramodules $\mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{discr}_{k,\mathsf{acp}}) \subset \mathsf{D}(H\text{–}\mathsf{discr}_k)$ and $\mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{contra}_{k,\mathsf{caap}}) \subset \mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{contra}_k)$,

$$\mathbb{R}\Psi_H \colon \mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{discr}_{k,\mathsf{acp}}) \simeq \mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{contra}_{k,\mathsf{caap}}) \colon \mathbb{L}\Phi_H. \tag{54}$$

Furthermore, the dualization functor $\mathcal{M} \longmapsto \mathcal{M}^{\vee}$ provides an anti-equivalence of abelian categories $(H\text{–}\mathsf{discr}_{k,\mathsf{acp}})^{\mathsf{op}} \simeq H\text{–}\mathsf{contra}_{k,\mathsf{caap}}$ (cf. Proposition 5.11), inducing an anti-equivalence of the bounded derived categories

$$\mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{discr}_{k,\mathsf{acp}})^{\mathsf{op}} \simeq \mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{contra}_{k,\mathsf{caap}}). \tag{55}$$

As in Section 6.1, we compose the triangulated anti-equivalence $\mathcal{M} \longmapsto \mathcal{M}^{\vee}$ (55) with the triangulated equivalence $\mathbb{L}\Phi_H$ (54), producing an auto-anti-equivalence of the bounded derived category of admissibly copresented discrete H-modules

$$\Delta_H \colon \mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{discr}_{k,\mathsf{acp}})^{\mathsf{op}} \simeq \mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{discr}_{k,\mathsf{acp}}). \tag{56}$$

Theorem 6.2.   *For any field k and a profinite group H of finite k-cohomological dimension that is quasi-cocoherent over k, the functor Δ_H is an involutive triangulated auto-anti-equivalence of the bounded derived category of admissibly copresented discrete H-modules over k.*

Proof.   The equivalences of categories

$$k(H)\text{–}\mathsf{comod}_{\mathsf{qfc}} \simeq H\text{–}\mathsf{discr}_{k,\mathsf{acp}} \simeq \mathsf{comod}_{\mathsf{qfc}}\text{–}k(H)$$

identify the functor Δ_H with both the functors $\Delta_{\mathcal{C}}$ and $\Delta_{\mathcal{C}^{\mathsf{op}}}$ for $\mathcal{C} = k(H)$, so the assertion follows from Proposition 6.1. Alternatively, one can put $V = k$ and observe that the functor Δ_H is a restriction of the functor $\Delta_H^k \colon \mathsf{D}(H\text{–}\mathsf{discr}_k)^{\mathsf{op}} \longrightarrow \mathsf{D}(H\text{–}\mathsf{discr}_k)$ (40), which is right self-adjoint by Proposition 4.2. So the functor Δ_H is a right self-adjoint auto-anti-equivalence. Hence the adjunction morphism $\mathcal{M}^{\bullet} \longrightarrow \Delta_H\Delta_H(\mathcal{M}^{\bullet})$ is an isomorphism in the derived category for any complex $\mathcal{M}^{\bullet} \in \mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{discr}_{k,\mathsf{acp}})$, and the functor Δ_H is involutive. $\blacksquare$

6.3. Locally profinite groups.   Let k be a field and G be a locally profinite group locally of finite k-cohomological dimension (see Section 4.1). Assume further that G is locally quasi-cocoherent over k (see Section 5.6).

We consider the full subcategory $\mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{–}\mathsf{smooth}_k) \subset \mathsf{D}(G\text{–}\mathsf{smooth}_k)$ in the derived category of smooth G-modules over k consisting of all the bounded complexes of smooth G-modules *with admissibly copresented cohomology modules*. Similarly, we also consider the full subcategory $\mathsf{D}^{\mathsf{b}}_{\mathsf{caap}}(G\text{–}\mathsf{contra}_k) \subset \mathsf{D}(G\text{–}\mathsf{contra}_k)$ in the derived

category of G-contramodules over k consisting of all the bounded complexes of G-contramodules *with contraadmissibly presented cohomology contramodules.*

Let $H \subset G$ be a compact open subgroup of finite k-cohomological dimension. It follows from the construction of the triangulated equivalence in Theorem 4.1 and the commutative diagrams (37) that the derived equivalences (38) for the groups G and H form a commutative diagram with the forgetful functors; see diagram (12) in Section 1.12. According to (53–54), the triangulated equivalence in the lower line of (12) restricts to a triangulated equivalence (54). Passing to the full preimages of (54) with respect to the vertical forgetful functors in (12), we obtain a commutative diagram of triangulated equivalences and forgetful functors

$$
\begin{array}{ccc}
\mathbb{R}\Psi_G\colon \mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{–}\mathsf{smooth}_k) & \!\!=\!\!\!=\!\!\!=\!\! & \mathsf{D}^{\mathsf{b}}_{\mathsf{caap}}(G\text{–}\mathsf{contra}_k) \colon \mathbb{L}\Phi_G \\
\downarrow & & \downarrow \\
\mathbb{R}\Psi_H\colon \mathsf{D}(H\text{–}\mathsf{discr}_{k,\mathsf{acp}}) & \!\!=\!\!\!=\!\!\!=\!\! & \mathsf{D}(H\text{–}\mathsf{contra}_{k,\mathsf{caap}}) \colon \mathbb{L}\Phi_H
\end{array}
\tag{57}
$$

Furthermore, the dualization functor $\mathcal{M} \longmapsto \mathcal{M}^{\vee}\colon (G\text{–}\mathsf{smooth}_k)^{\mathsf{op}} \longrightarrow G\text{–}\mathsf{contra}_k$ takes $G\text{–}\mathsf{smooth}_{k,\mathsf{acp}}$ into $G\text{–}\mathsf{contra}_{k,\mathsf{caap}}$, and therefore, induces a contravariant triangulated functor

$$
\mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{–}\mathsf{smooth}_k)^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{caap}}(G\text{–}\mathsf{contra}_k).
\tag{58}
$$

The functor (58) forms a commutative diagram with the triangulated anti-equivalence (55) and the forgetful functors,

$$
\begin{array}{ccc}
\mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{–}\mathsf{smooth}_k)^{\mathsf{op}} & \longrightarrow & \mathsf{D}^{\mathsf{b}}_{\mathsf{caap}}(G\text{–}\mathsf{contra}_k) \\
\downarrow & & \downarrow \\
\mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{discr}_{k,\mathsf{acp}})^{\mathsf{op}} & \!\!=\!\!\!=\!\!\!=\!\! & \mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{contra}_{k,\mathsf{caap}})
\end{array}
\tag{59}
$$

Composing the horizontal functors in (59) with the triangulated equivalences $\mathbb{L}\Phi_G$ and $\mathbb{L}\Phi_H$ (57), we obtain a commutative diagram of triangulated functors

$$
\begin{array}{ccc}
\Delta_G\colon \mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{–}\mathsf{smooth}_k)^{\mathsf{op}} & \longrightarrow & \mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{–}\mathsf{smooth}_k) \\
\downarrow & & \downarrow \\
\Delta_H\colon \mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{discr}_{k,\mathsf{acp}})^{\mathsf{op}} & \!\!=\!\!\!=\!\!\!=\!\! & \mathsf{D}^{\mathsf{b}}(H\text{–}\mathsf{discr}_{k,\mathsf{acp}})
\end{array}
\tag{60}
$$

with an equivalence in the lower horizontal line (cf. diagram (15) in Section 1.15 of the introduction).

Theorem 6.3. *For any field k and a locally profinite group G locally of finite k-cohomological dimension that is locally quasi-cocoherent over k, the functor (58) is a triangulated anti-equivalence. The functor Δ_G is an involutive triangulated auto-anti-equivalence of the bounded derived category of smooth G-modules over k with admissibly copresented cohomology modules $\mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{–}\mathsf{smooth}_k)$.*

Proof. As in the proof of Theorem 6.2, we put $V = k$ and observe that the functor Δ_G is a restriction of the functor $\Delta_G^k \colon \mathsf{D}(G\text{–smooth}_k)^{\mathsf{op}} \longrightarrow \mathsf{D}(G\text{–smooth}_k)$ (40), which is right self-adjoint by Proposition 4.2. Hence the functor Δ_G is right self-adjoint, too. In order to show that Δ_G is an involutive auto-anti-equivalence, it remains to check that the adjunction morphism $\boldsymbol{\mathcal{M}}^\bullet \longrightarrow \Delta_G\Delta_G(\boldsymbol{\mathcal{M}}^\bullet)$ is an isomorphism in the category $\mathsf{D}_{\mathsf{acp}}^{\mathsf{b}}(G\text{–smooth}_k)$ for any complex $\boldsymbol{\mathcal{M}}^\bullet \in \mathsf{D}_{\mathsf{acp}}^{\mathsf{b}}(G\text{–smooth}_k)$. Indeed, one can see from the construction of the adjunction morphism in Section 4.2 that for any complex of smooth G-modules $\boldsymbol{\mathcal{N}}^\bullet$ over k the forgetful functor $\mathsf{D}(G\text{–smooth}_k) \longrightarrow \mathsf{D}(H\text{–discr}_k)$ takes the adjunction morphism $\boldsymbol{\mathcal{N}}^\bullet \longrightarrow \Delta_G^k\Delta_G^k(\boldsymbol{\mathcal{N}}^\bullet)$ to the adjunction morphism $\boldsymbol{\mathcal{N}}^\bullet \longrightarrow \Delta_H^k\Delta_H^k(\boldsymbol{\mathcal{N}}^\bullet)$. Now for a complex $\boldsymbol{\mathcal{M}}^\bullet \in \mathsf{D}_{\mathsf{acp}}^{\mathsf{b}}(G\text{–smooth}_k)$, the image of $\boldsymbol{\mathcal{M}}^\bullet$ under the forgetful functor is (isomorphic to) a complex from $\mathsf{D}^{\mathsf{b}}(H\text{–discr}_{k,\mathsf{acp}})$. By Theorem 6.2, the adjunction morphism $\boldsymbol{\mathcal{M}}^\bullet \longrightarrow \Delta_H\Delta_H(\boldsymbol{\mathcal{M}}^\bullet)$ is an isomorphism in $\mathsf{D}^{\mathsf{b}}(H\text{–discr}_{k,\mathsf{acp}})$. Since the forgetful functor $\mathsf{D}(G\text{–smooth}_k) \longrightarrow \mathsf{D}(H\text{–discr}_k)$ is conservative, it follows that the adjunction morphism $\boldsymbol{\mathcal{M}}^\bullet \longrightarrow \Delta_G\Delta_G(\boldsymbol{\mathcal{M}}^\bullet)$ is an isomorphism in $\mathsf{D}_{\mathsf{acp}}^{\mathsf{b}}(G\text{–smooth}_k)$.

We have shown that the functor Δ_G is an (involutive) auto-anti-equivalence. Since the functor $\mathbb{L}\Phi_G$ (57) is an equivalence, it follows that the functor (58) is an anti-equivalence. ∎

In conclusion, we recall that when the group G is locally quasi-contra-Noetherian over the field k, all admissible smooth G-modules over k are admissibly copresented, so $\mathsf{D}_{\mathsf{acp}}^{\mathsf{b}}(G\text{–smooth}_k)$ is the bounded derived category of smooth G-modules over k with admissible cohomology modules. In particular, following Examples 5.14, a p-adic Lie group G is locally quasi-contra-Noetherian over any field k. A p-adic Lie group is also of locally of finite k-cohomological dimension for any field k, so Thereom 6.3 applies.

Over a field k of characteristic 0, any locally profinite group is locally of cohomological dimension 0 and locally quasi-contra-Noetherian, so Theorem 6.3 is applicable in this case, too.

Remark 6.4. Notice the difference between Theorems 6.2 and 6.3: while the former provides an involutive duality on the bounded derived category of admissibly copresented discrete H-modules, the latter establishes an involutive duality on the bounded derived category of complexes of *arbitrary* smooth G-modules *with admissibly copresented cohomology modules.*

One can also consider the bounded derived category of the abelian category of admissibly copresented smooth G-modules, $\mathsf{D}^{\mathsf{b}}(G\text{–smooth}_{k,\mathsf{acp}})$. There is a natural triangulated functor $\mathsf{D}^{\mathsf{b}}(G\text{–smooth}_{k,\mathsf{acp}}) \longrightarrow \mathsf{D}_{\mathsf{acp}}^{\mathsf{b}}(G\text{–smooth}_k)$, but generally speaking this functor is far from being an equivalence. It is *neither* fully faithful *nor* surjective on objects. (It suffices to consider the case of a *discrete* group G, which is, of course, locally Noetherian and locally of cohomological dimension 0 over any field k. Now the admissible G-modules are simply the *finite-dimensional* representations of the group G over k; and, generally speaking, the bounded derived category of finite-dimensional representations of an infinite discrete group is very different from the bounded derived category of complexes of infinite-dimensional representations with finite-dimensional cohomology modules.)

Our constructions do *not* seem to allow to obtain an involutive duality on the derived category of admissible or admissibly copresented G-modules $\mathsf{D}^{\mathsf{b}}(G\text{–smooth}_{k,\mathsf{acp}})$.

The problem is that the construction of the derived functor $\mathbb{L}\Phi_G$ requres replacing a given complex of G-contramodules over k with a complex of G-contramodules that are projective as contramodules over a certain compact open subgroup $H \subset G$ (see the proof of Theorem 4.1). The known construction of such resolutions [11, Section 3.3.3] leads outside of the class of contraadmissible G-contramodules.

Remark 6.5. This paper is inspired by Kohlhaase's paper [8], and in the case of a p-adic Lie group G in the natural characteristic char $k = p$ the theory developed in the present paper is closely related to the theory developed in [8]. Without going into a detailed comparison, let us mention one obvious difference between our approaches: no derived categories are mentioned in [8]. The author of [8] avoids derived categories by considering the sequence of functors Ext in each cohomological degree separately. So he has to consider the corresponding sequence of subquotient categories of the category of admissible smooth G-modules, and have an involutive duality on each of these subquotient categories separately. In this paper we obtain an involutive duality on a triangulated category containing the whole abelian category of admissible smooth G-modules, but this achievement comes with a price: we have to consider the bounded derived category $\mathsf{D}^{\mathsf{b}}_{\mathsf{acp}}(G\text{–}\mathsf{smooth}_k)$ of complexes of nonadmissible smooth G-modules with admissible cohomology G-modules.

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