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Block Degeneracy for Graded Lie Superalgebras of Cartan Type

Ke Ou

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Abstract. Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. In this short note, we illustrate a class of Lie superalgebras over $\mathbb{K}$ such that the category of restricted supermodules is of one block. As an application, if $p > 3$ and $\mathfrak{g}$ is a graded restricted Cartan type Lie superalgebra of type W, S and H, then the category of restricted supermodules of $\mathfrak{g}$ is of one block.

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1. Introduction

A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ over $\mathbb{K}$ is called restricted if $(\mathfrak{g}_{\bar{0}}, [p])$ is a restricted Lie algebra with p-mapping $[p] : \mathfrak{g}_{\bar{0}} \to \mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ is a restricted $\mathfrak{g}_{\bar{0}}$ module via the adjoint action (cf. [5]). Let $(\mathfrak{g}, [p])$ be a restricted Lie superalgebra and $U(\mathfrak{g})$ be the enveloping superalgebra of $\mathfrak{g}$. One can define the so-called restricted enveloping superalgebra $u(\mathfrak{g}) = U(\mathfrak{g})/I_p$ where I_p is the $\mathbb{Z}_2$-graded two-sided ideal generated by $\{x^p - x^{[p]} \mid x \in \mathfrak{g}_{\bar{0}}\}$. A $\mathfrak{g}$ supermodule $(V = V_{\bar{0}} \oplus V_{\bar{1}}, \rho)$ is called restricted if ρ satisfies $\rho(x^{[p]}) = \rho(x)^p$ for all $x \in \mathfrak{g}_{\bar{0}}$. All restricted $\mathfrak{g}$-supermodules constitute a full subcategory of the $\mathfrak{g}$-supermodule category which coincides with the $u(\mathfrak{g})$-supermodule category denoted by $u(\mathfrak{g})$-smod. We call that $u(\mathfrak{g})$ is of one block if $u(\mathfrak{g})$-smod is of one block.

Over the past decades, the study of modular representations of restricted Lie (super)algebras in prime characteristic has made significant progress (see [3, 4, 8, 9, 10] for examples). When $\mathfrak{g} = W(0, n)$ over $\mathbb{C}$, Shomron proves in [6] that the category of finite-dimensional representations decomposes into blocks parametrized by $(\mathbb{C}/\mathbb{Z}) \times \mathbb{Z}_2$. In contrast to complex case, if either $\mathfrak{g} = X(m, 1)$ is a Cartan type Lie algebra where $X \in \{W, S, H, K\}$ ([3]) or $\mathfrak{g} = W(0, n, 1)$ is a Cartan type Lie superalgebra ([8]) over $\mathbb{K}$, the category of restricted (super)modules has only one block. In this paper, we generalize this degeneracy phenomenon of restricted supermodules to the so-called graded restricted Cartan type Lie superalgebras $X(m, n, 1)$ where $X \in \{W, S, H\}$.

Our paper is organized as follows. In Section 2, we illustrate a class of Lie superalgebras over $\mathbb{K}$ such that the category of restricted supermodules is of one block. Section 3 is concerned with the structure of the Cartan type Lie superalgebras. Applying the results in Section 2, we obtain the following main theorem in Section 4:

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Theorem 1.1. (see Theorem 4.6) *Let $\mathbb{K}$ be an algebraically closed field with characteristics $p>3$, and $\mathfrak{g}=X(m,n,1)$, $X\in\{W,S,H\}$, be a graded restricted Lie superalgebra of Cartan type over $\mathbb{K}$ except if $X=H$ with $n=4$.*
Then $u(\mathfrak{g})$ is of one block.

As I know, Duan, Shu and Yao obtain similar results in [2] by a different method.
Entire the whole paper, denote $I=\{0,1,\cdots,p-1\}$. For $n\in\mathbb{N}$, set $\mathbb{B}(n)=\cup_{k=0}^{n}\mathbb{B}_k$ where $\mathbb{B}_0=\emptyset$ and $\mathbb{B}_k=\{(i_1,\cdots,i_k)\mid m+1\leq i_1<\cdots<i_k\leq m+n\}$ for $0<k\leq n$.

2. Restricted Lie superalgebras with triangular Decomposition

Let $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ be a finite dimensional restricted Lie superalgebra over $\mathbb{K}$. We say that $\mathfrak{g}$ admits a triangular decomposition relative to a maximal torus $\mathfrak{h}$ of $\mathfrak{g}_{\bar{0}}$ if there is a vector decomposition $\mathfrak{g}=\mathfrak{g}_{\bar{1}}^{-}\oplus\mathfrak{n}^{-}\oplus\mathfrak{h}\oplus\mathfrak{n}^{+}\oplus\mathfrak{g}_{\bar{1}}^{+}$ such that:

(1) $\mathfrak{g}_{\bar{0}}=\mathfrak{n}^{-}\oplus\mathfrak{h}\oplus\mathfrak{n}^{+}$ is a triangular decomposition of $\mathfrak{g}_{\bar{0}}$,

(2) both $\mathfrak{n}^{-}\oplus\mathfrak{g}_{\bar{1}}^{-}$ and $\mathfrak{n}^{+}\oplus\mathfrak{g}_{\bar{1}}^{+}$ are p-nilpotent restricted subalgebras,

(3) $[\mathfrak{h},\mathfrak{n}^{\pm}\oplus\mathfrak{g}_{\bar{1}}^{\pm}]\subseteq\mathfrak{n}^{\pm}\oplus\mathfrak{g}_{\bar{1}}^{\pm}$.

Set $\mathfrak{g}^{\pm}=\mathfrak{g}_{\bar{1}}^{\pm}\oplus\mathfrak{n}^{\pm}$, $\mathfrak{b}_{\mathfrak{g}}^{\pm}=\mathfrak{g}_{\bar{1}}^{\pm}\oplus\mathfrak{n}^{\pm}\oplus\mathfrak{h}$ and $\mathfrak{b}_{\mathfrak{g}_{\bar{0}}}^{\pm}=\mathfrak{n}^{\pm}\oplus\mathfrak{h}$. Analogue to [3], we say that this decomposition for $\mathfrak{g}$ is long if

$$\dim_{\mathbb{K}}(\mathfrak{n}^{-})<\dim_{\mathbb{K}}(\mathfrak{n}^{+})\text{ and }\dim_{\mathbb{K}}(\mathfrak{g}_{\bar{1}}^{-})<\dim_{\mathbb{K}}(\mathfrak{g}_{\bar{1}}^{+}).$$

By [10], the iso-classes of simple restricted $\mathfrak{g}$ modules are parametrized by restricted weights $\Lambda=\{\lambda\in\mathfrak{h}^{*}\mid\lambda(h^{[p]})=\lambda(h)^{p},\ \forall h\in\mathfrak{h}\}$. Precisely, for a given $\lambda\in\Lambda$, there is a one-dimensional restricted $\mathfrak{b}_{\mathfrak{g}}^{+}$ module $\mathbb{K}_\lambda=\mathbb{K}\cdot 1_\lambda$ on which $\mathfrak{h}$ acts as a scalar determined by λ while $\mathfrak{g}_{\bar{1}}^{+}\oplus\mathfrak{n}^{+}$ acts trivially. Then one has the so-called baby Verma module

$$V^{+}(\lambda):=u(\mathfrak{g})\otimes_{u(\mathfrak{b}_{\mathfrak{g}}^{+})}\mathbb{K}_\lambda$$

with simple head $L(\lambda)$. Moreover, For any restricted simple module $\mathfrak{m}$, there is a $\lambda\in\Lambda$, such that $V^{+}(\lambda)\twoheadrightarrow\mathfrak{m}$ (cf. [10]).
Similarly, for each $\mu\in\Lambda$, the one-dimensional restricted $\mathfrak{b}_{\mathfrak{g}}^{-}$ module $\mathbb{K}_\mu$ induces an $u(\mathfrak{g})$ module

$$V^{-}(\mu):=u(\mathfrak{g})\otimes_{u(\mathfrak{b}_{\mathfrak{g}}^{-})}\mathbb{K}_\mu,$$

which is indecomposable with simple head.
If $\dim(\mathfrak{h})=n$, by [7, Theorem 3.6], $\mathfrak{h}$ possesses a basis $\{h_1,\cdots,h_n\}$ such that $h_i^{[p]}=h_i$ for all $i=1,\cdots,n$. Then $\Lambda\simeq I^n=\{(\lambda_1,\cdots,\lambda_n)\mid\lambda_i\in I,i=1,\cdots,n\}$ by sending $\lambda\in\Lambda$ to $(\lambda(h_1),\cdots,\lambda(h_n))\in I^n$.
For $M\in u(\mathfrak{g})$-smod, let $[M]$ denote the formal sum of simple composition factors in the Grothendick ring of $u(\mathfrak{g})$-smod.

Lemma 2.1. *Let $\mathfrak{l}=\mathfrak{l}_{\bar{0}}\oplus\mathfrak{l}_{\bar{1}}$ be a finite dimensional restricted Lie superalgebra which admits a triangular decomposition relative to a maximal torus $\mathfrak{h}_{\mathfrak{l}}$ of $\mathfrak{l}_{\bar{0}}$:*

$$\mathfrak{l}=\mathfrak{l}_{\bar{1}}^{-}\oplus\mathfrak{n}_{\mathfrak{l}}^{-}\oplus\mathfrak{h}_{\mathfrak{l}}\oplus\mathfrak{n}_{\mathfrak{l}}^{+}\oplus\mathfrak{l}_{\bar{1}}^{+}$$

where $\mathfrak{l}_{\bar{0}}=\mathfrak{n}_{\mathfrak{l}}^{-}\oplus\mathfrak{h}_{\mathfrak{l}}\oplus\mathfrak{n}_{\mathfrak{l}}^{+}$. Assume the following:

(1) *$\mathfrak{l}_{\bar{1}}^{-}\oplus\mathfrak{n}_{\mathfrak{l}}^{-}\oplus\mathfrak{n}_{\mathfrak{l}}^{+}\oplus\mathfrak{l}_{\bar{1}}^{+}$ is a p-nilpotent $\mathbb{Z}_2$-graded ideal.*

(2) $\mathfrak{n}_{\mathfrak{l}}^{+}$ *contains* $\dim(\mathfrak{h}_{\mathfrak{l}})$ *weight vectors having linearly independent weights in* Λ.

Then for each $\lambda \in \Lambda$, $[V^{-}(\lambda)]$ *is independent of* λ *and*

$$[V^{-}(\lambda)] = \sum_{\mu\in\Lambda} p^{s}2^{t}[\mathbb{K}_{\mu}],$$

where $s = \dim(\mathfrak{n}_{\mathfrak{l}}^{+}) - \dim(\mathfrak{h}_{\mathfrak{l}})$, $t = \dim(\mathfrak{l}_{\bar{1}}^{+})$ *and* $\mathbb{K}_{\mu}$ *is the one dimensional simple* $u(\mathfrak{l})$ *module of weight* μ.

Proof. Denote $\mathfrak{l}' = \mathfrak{l}_{\bar{1}}^{-} \oplus \mathfrak{n}_{\mathfrak{l}}^{-} \oplus \mathfrak{n}_{\mathfrak{l}}^{+} \oplus \mathfrak{l}_{\bar{1}}^{+}$. By (1), $\mathfrak{l}' < \mathrm{rad}(\mathfrak{l})$. Since $\mathfrak{l}'$ is p-nilpotent and finite dimension, $\mathfrak{l}'$ is strictly triangulizable by [10, Lemma 2.2]. Therefore, each restricted simple representation of $\mathfrak{l}$ is one dimensional and the iso-classes are parametrized by Λ. Let $\{\mathbb{K}_{\mu} \mid \mu \in \Lambda\}$ represent the set of non-isomorphic simple $u(\mathfrak{l})$ modules.
The composition factors of a module can be obtained by computing its weight spaces. By (2), suppose $n = \dim(\mathfrak{h}_{\mathfrak{l}})$, $\mathfrak{l}_{\bar{1}}^{+}$ has basis $\{z_1, \cdots, z_t\}$ and $\mathfrak{n}_{\mathfrak{l}}^{+}$ has basis $\{x_1, \cdots, x_n, y_1, \cdots, y_s\}$, where x_i is of weight $\alpha_i \in \Lambda$ for each $i = 1, \cdots, n$ such that $\alpha_1, \cdots, \alpha_n$ are linear independent. Then $x_1^{i_1} \cdots x_n^{i_n}$ has weight $i_1\alpha_1 + \cdots + i_n\alpha_n$ for each $(i_1, \cdots, i_n) \in I^n$.
For each choice of $\underline{j} \in I^s$ and $\underline{k} \in \mathbb{B}(t)$, as $u(\mathfrak{h}_{\mathfrak{l}})$ module,

$$N = \mathrm{span}_{\mathbb{K}}\{X^{\underline{i}}Y^{\underline{j}}Z^{\underline{k}} \mid \underline{i} \in I^n\}$$

must have all weights occurring with multiplicity 1. Since

$$V^{-}(\lambda) = \mathrm{span}_{\mathbb{K}}\{X^{\underline{i}}Y^{\underline{j}}Z^{\underline{k}} \otimes 1_{\lambda} \mid \underline{i} \in I^n, \underline{j} \in I^s \text{ and } \underline{k} \in \mathbb{B}(t)\},$$

then all possible weights occuring with the same multiplicity p^s2^t in $V^{-}(\lambda)$.
Namely, $[V^{-}(\lambda)] = \sum_{\mu\in\Lambda} p^s2^t[\mathbb{K}_{\mu}]$ which is independent of λ. ∎

Proposition 2.2. *Let* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a finite dimensional restricted Lie superalgebra which admits a long triangular decomposition relative to a maximal torus* $\mathfrak{h}$ *of* $\mathfrak{g}_{\bar{0}}$:

$$\mathfrak{g} = \mathfrak{g}_{\bar{1}}^{-} \oplus \mathfrak{n}^{-} \oplus \mathfrak{h} \oplus \mathfrak{n}^{+} \oplus \mathfrak{g}_{\bar{1}}^{+}, \ \mathfrak{g}_{\bar{0}} = \mathfrak{n}^{-} \oplus \mathfrak{h} \oplus \mathfrak{n}^{+}.$$

Assume the following:

(1) $\mathfrak{g}$ *has a restricted subalgebra* $\mathfrak{l}$ *which satisfies the assumptions of Lemma* 2.1.

(2) $\mathfrak{n}_{\mathfrak{l}}^{-} = \mathfrak{n}^{-}$, $\mathfrak{l}_{\bar{1}}^{-} = \mathfrak{g}_{\bar{1}}^{-}$, $\mathfrak{h}_{\mathfrak{l}} = \mathfrak{h}$, $\mathfrak{n}_{\mathfrak{l}}^{+} = \mathfrak{l} \cap \mathfrak{n}^{+}$, $\mathfrak{l}_{\bar{1}}^{+} = \mathfrak{l} \cap \mathfrak{g}_{\bar{1}}^{+}$. *Hence,* $\mathfrak{b}_{\mathfrak{l}}^{-} = \mathfrak{b}_{\mathfrak{g}}^{-}$.

(3) $\mathfrak{n}_{\mathfrak{l}}^{-} = \mathfrak{n}^{-}$ *has at least* $\dim(\mathfrak{h})$ *weight vectors having linearly independent weights in* Λ.

Then for each $\lambda \in \Lambda$,
$$[V^{-}(\lambda)] = \sum_{\mu\in\Lambda} p^{s}2^{t}[V^{+}(\mu)],$$

where $s = \dim(\mathfrak{n}^{+}) - \dim(\mathfrak{n}^{-}) - \dim(\mathfrak{h})$, $t = \dim(\mathfrak{g}_{\bar{1}}^{+}) - \dim(\mathfrak{g}_{\bar{1}}^{-})$.

Proof. By Lemma 2.1, for each $\lambda \in \Lambda$,

$$[V^{-}(\lambda)] = [u(\mathfrak{g}) \otimes_{u(\mathfrak{l})} [u(\mathfrak{l}) \otimes_{u(\mathfrak{b}_{\mathfrak{l}}^{-})} \mathbb{K}_{\lambda}]] = \sum_{\mu\in\Lambda} p^{\alpha}2^{\beta}[u(\mathfrak{g}) \otimes_{u(\mathfrak{l})} \mathbb{K}_{\mu}],$$

where $\alpha = \dim(\mathfrak{n}_{\mathfrak{l}}^{+}) - \dim(\mathfrak{h}_{\mathfrak{l}})$, $\beta = \dim(\mathfrak{l}_{\bar{1}}^{+})$.
In particular, $[V^{-}(\lambda)]$ is independent of λ.

By assumption (3), $\mathfrak{b}_{\mathfrak{g}}^{\pm}$ satisfy the assumptions of Lemma 2.1. Therefore,

$$[u(\mathfrak{g})\otimes_{u(\mathfrak{h})}\mathbb{K}_\lambda]=[u(\mathfrak{g})\otimes_{u(\mathfrak{b}_{\mathfrak{g}}^{\pm})}[u(\mathfrak{b}_{\mathfrak{g}}^{\pm})\otimes_{u(\mathfrak{h})}\mathbb{K}_\lambda]]$$
$$=\sum_{\mu\in\Lambda}p^{s_\pm}2^{t_\pm}[u(\mathfrak{g})\otimes_{u(\mathfrak{b}_{\mathfrak{g}}^{\pm})}\mathbb{K}_\mu]=\sum_{\mu\in\Lambda}p^{s_\pm}2^{t_\pm}[V^{\pm}(\mu)],$$

where $s_\pm=\dim(\mathfrak{n}^\pm)$, $t_\pm=\dim(\mathfrak{g}_{\bar{1}}^{\pm})$. Since the triangular decomposition of $\mathfrak{g}$ is long, i.e. $s_+>s_-$ and $t_+>t_-$, we have

$$\sum_{\lambda\in\Lambda}[V^-(\lambda)]=\sum_{\mu\in\Lambda}p^{s_+-s_-}2^{t_+-t_-}[V^+(\mu)].$$

Note that $[V^-(\lambda)]$ is independent of λ. Therefore, for all $\lambda\in\Lambda$,

$$p^{\dim(\mathfrak{h})}[V^-(\lambda)]=\sum_{\mu\in\Lambda}p^{s_+-s_-}2^{t_+-t_-}[V^+(\mu)],$$
$$[V^-(\lambda)]=\sum_{\mu\in\Lambda}p^{s_+-s_--\dim(\mathfrak{h})}2^{t_+-t_-}[V^+(\mu)].$$ ■

Proposition 2.3. *Let $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ be a finite dimensional restricted Lie superalgebra which admits a triangular decomposition relative to a maximal torus $\mathfrak{h}$ of $\mathfrak{g}_{\bar{0}}$:*

$$\mathfrak{g}=\mathfrak{g}_{\bar{1}}^-\oplus\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+\oplus\mathfrak{g}_{\bar{1}}^+,\ \mathfrak{g}_{\bar{0}}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+.$$

Assume that there exists a restricted subalgebra $\mathfrak{l}=\mathfrak{l}_{\bar{0}}\oplus\mathfrak{l}_{\bar{1}}$ such that:

(1) *$\mathfrak{l}$ is a classical Lie superalgebra and there is a bijection $\psi:\Lambda\to\Lambda$, such that*

$$[u(\mathfrak{l})\otimes_{u(\mathfrak{b}_{\mathfrak{l}}^-)}\mathbb{K}_\lambda]=[u(\mathfrak{l})\otimes_{u(\mathfrak{b}_{\mathfrak{l}}^+)}\mathbb{K}_{\psi(\lambda)}].$$

(2) *$\mathfrak{h}\subseteq\mathfrak{l}_{\bar{0}}$ and $\mathfrak{l}$ admits a triangular decomposition relative to $\mathfrak{h}$:*

$$\mathfrak{l}=\mathfrak{l}_{\bar{1}}^-\oplus\mathfrak{n}_{\mathfrak{l}}^-\oplus\mathfrak{h}\oplus\mathfrak{n}_{\mathfrak{l}}^+\oplus\mathfrak{l}_{\bar{1}}^+,\ \mathfrak{l}_{\bar{0}}=\mathfrak{n}_{\mathfrak{l}}^-\oplus\mathfrak{h}\oplus\mathfrak{n}_{\mathfrak{l}}^+$$

such that $\mathfrak{n}_{\mathfrak{l}}^-=\mathfrak{n}^-$, $\mathfrak{l}_{\bar{1}}^-=\mathfrak{g}_{\bar{1}}^-$, $\mathfrak{n}_{\mathfrak{l}}^+=\mathfrak{l}\cap\mathfrak{n}^+$, $\mathfrak{l}_{\bar{1}}^+=\mathfrak{l}\cap\mathfrak{g}_{\bar{1}}^+$. Hence, $\mathfrak{b}_{\mathfrak{l}}^-=\mathfrak{b}_{\mathfrak{g}}^-$.

(3) *A vector space complementary to $\mathfrak{l}$, in $\mathfrak{g}$, has at least $\dim(\mathfrak{h})$ weight vectors having linearly independent weights in Λ.*

Then
$$[V^-(\lambda)]=\sum_{\mu\in\Lambda}p^s2^t[V^+(\mu)],$$
where $s=\dim(\mathfrak{n}^+)-\dim(\mathfrak{n}^-)-\dim(\mathfrak{h}),t=\dim(\mathfrak{g}_{\bar{1}}^+)-\dim(\mathfrak{g}_{\bar{1}}^-)$.

Proof. Similar to the proof of Lemma 2.1, for all $\lambda\in\Lambda$, we have

$$[u(\mathfrak{b}_{\mathfrak{g}}^+)\otimes_{u(\mathfrak{b}_{\mathfrak{l}}^+)}\mathbb{K}_\lambda]=\sum_{\mu\in\Lambda}p^s2^t[\mathbb{K}_\mu],$$

where s and t are defined in the proposition. By assumption (1) and (2), we have

$$[u(\mathfrak{g})\otimes_{u(\mathfrak{b}_{\mathfrak{g}}^-)}\mathbb{K}_\lambda]=[u(\mathfrak{g})\otimes_{u(\mathfrak{l})}(u(\mathfrak{l})\otimes_{u(\mathfrak{b}_{\mathfrak{l}}^-)}\mathbb{K}_\lambda)]=[u(\mathfrak{g})\otimes_{u(\mathfrak{l})}(u(\mathfrak{l})\otimes_{u(\mathfrak{b}_{\mathfrak{l}}^+)}\mathbb{K}_{\psi(\lambda)})]$$
$$=[u(\mathfrak{g})\otimes_{u(\mathfrak{b}_{\mathfrak{g}}^+)}(u(\mathfrak{b}_{\mathfrak{g}}^+)\otimes_{u(\mathfrak{b}_{\mathfrak{l}}^+)}\mathbb{K}_{\psi(\lambda)})]=\sum_{\mu\in\Lambda}p^s2^t[u(\mathfrak{g})\otimes_{u(\mathfrak{b}_{\mathfrak{g}}^+)}\mathbb{K}_\mu],$$

as desired. ■

Remark 2.4. The Lie algebra version of Lemma 2.1 and Proposition 2.2 (resp. Proposition 2.3) is investigated in [3] (resp. [4]).

Corollary 2.5. *If $\mathfrak{g}$ is a restricted Lie superalgebra which satisfies all assumptions in Proposition* 2.2 *or* 2.3*, then $u(\mathfrak{g})$ is of one block.*

Proof. Note that $V^-(\lambda)$ is indecomposable with simple head for all $\lambda \in \Lambda$. The proof of Corollary 2.4 in [3] still works. Hence, the corollary holds. ∎

3. Restricted Cartan Type Lie Superalgebras

For given positive integers m and n, put

$$\tau(i) = \begin{cases} 0, & 1 \leq i \leq m; \\ 1, & m+1 \leq i \leq m+n. \end{cases}$$

Let $A(m,1) = \mathbb{K}[T_1, ..., T_m]/(T_1^p, ..., T_m^p)$ be the truncated polynomial algebra over $\mathbb{K}$. Denote by x_i the image of T_i in the quotient. Then $A(m,1)$ has a basis $\{x^\alpha \mid \alpha \in I^m\}$ where $x^\alpha = x_1^{a_1} ... x_m^{a_m}$ if $\alpha = (a_1, ..., a_m)$. Let $\Lambda(n)$ be the Grassmann superalgebra over $\mathbb{K}$ in n variables $x_{m+1}, ..., x_{m+n}$ with basis $\{x^{(\beta)} \mid \beta \in \mathbb{B}(n)\}$ where $x^{(\beta)} = x_{i_1} \cdots x_{i_k}$ if $\beta = (i_1, \cdots, i_k)$.
Denote the tensor product by $A(m,n,1) = A(m,1) \otimes \Lambda(n)$. Then $A(m,n,1)$ is an associative superalgebra with a $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $A(m,1)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$. Finally, denote by $\mathrm{d}(f)$ the parity of $f \in A(m,n,1)$.

Let $D_1, \cdots, D_{m+n}$ be the superderivations of the superalgebra $A(m,n,1)$ such that $D_i(x_j) = \delta_{ij}$ for $1 \leq i,j \leq m+n$. Define

$$W(m,n,1) = \left\{ \sum_{i=1}^{m+n} f_i D_i \mid f_i \in A(m,n,1), 1 \leq i \leq m+n \right\}.$$

Then $W(m,n,1)$ is a $\mathbb{Z}$-graded restricted Lie superalgebra of Witt type. The $\mathbb{Z}$-grading of

$$W(m,n,1) = \oplus_{i \in \mathbb{Z}} W(m,n,1)_i$$

is induced by $|x_i| = 1$ and $|D_i| = -1$ for all $1 \leq i \leq m+n$. Namely,

$$W(m,n,1)_i = \left\{ \sum_{j=1}^{m+n} f_j D_j \mid |f_j| = i+1 \right\}.$$

For each pair $1 \leq i,j \leq m+n$ defines a linear map

$$D_{ij} : A(m,n,1) \to W(m,n,1)$$

by $D_{ij}(f) = f_i D_i + f_j D_j$ where f is homogeneous and

$$f_i = -(-1)^{\mathrm{d}(f)(\tau(i)+\tau(j))} D_j(f), \ f_j = (-1)^{\tau(i)\tau(j)} D_i(f).$$

The special superalgebra $S(m,n,1)$ is defined by

$$S(m,n,1) = \langle D_{ij}(f) \mid f \text{ is homogeneous}, \ 1 \leq i < j \leq m+n \rangle.$$

Then $S(m,n,1)$ is a $\mathbb{Z}$-graded restricted subalgebra of $W(m,n,1)$. The $\mathbb{Z}$-grading structure is given by $S(m,n,1)_i := S(m,n,1) \cap W(m,n,1)_i$.

Next we define the Hamiltonian type Lie superalgebra $H(m,n,1)$, where $m=2l$ is even and $n>3$. Let

$$i' = \begin{cases} i+l, & 1 \le i \le l, \\ i-l, & l+1 \le i \le m, \\ i, & m < i \le m+n; \end{cases} \qquad \sigma(i) = \begin{cases} 1, & 1 \le i \le l, \\ -1, & l+1 \le i \le m, \\ 1, & m < i \le m+n. \end{cases}$$

The Hamiltonian operator D_H is defined as:

$$D_H \colon A(m,n,1) \to W(m,n,1), \quad f \mapsto D_H(f) = \sum_{i=1}^{m+n} f_i D_i,$$

where f is homogeneous and $f_i = \sigma(i')(-1)^{\tau(i')\mathrm{d}(f)} D_{i'}(f)$.

The Hamiltonian superalgebra $H(m,n,1)$ is defined by

$$\bar{H}(m,n,1) = \langle D_H(f) \mid f \text{ is homogeneous} \rangle, \quad H(m,n,1) = [\bar{H}(m,n,1), \bar{H}(m,n,1)].$$

Then $H(m,n,1)$ is a $\mathbb{Z}$-graded restricted subalgebra of $W(m,n,1)$. The $\mathbb{Z}$-grading structure is given by $H(m,n,1)_i := H(m,n,1) \cap W(m,n,1)_i$.

4. Blocks of Cartan type Lie superalgebra

Entire this section, assume $p > 3$.

4.1. Type W. For $W(m,n,1)$, there is no subalgebra $\mathfrak{l}$ satisfying the hypothesis of Proposition 2.2 (in fact, assumption (3) fails). Hence, we need Proposition 2.3.

Let $\mathfrak{l}$ be a restricted Lie superalgebra of classical type with triangular decomposition $\mathfrak{l} = \mathfrak{l}^- \oplus \mathfrak{h}_\mathfrak{l} \oplus \mathfrak{l}^+$ with respect to $\mathfrak{h}$. Suppose σ is an even restricted automorphism of $\mathfrak{l}$ such that $\sigma(\mathfrak{h}_\mathfrak{l}) \subseteq \mathfrak{h}_\mathfrak{l}$. Then it induces $\tilde{\sigma} : \mathfrak{h}_\mathfrak{l}^* \to \mathfrak{h}_\mathfrak{l}^*$ by

$$\tilde{\sigma}(\lambda)(h) = -\lambda(\sigma(h))$$

where $\lambda \in \mathfrak{h}_\mathfrak{l}^*$, $h \in \mathfrak{h}_\mathfrak{l}$. Moreover, $\tilde{\sigma}(\Lambda) \subseteq \Lambda$.

Denote $\mathfrak{b}_\mathfrak{l} = \mathfrak{h}_\mathfrak{l} \oplus \mathfrak{l}^+$ a solvable subalgebra of $\mathfrak{l}$ and $V(\lambda) = u(\mathfrak{l}) \otimes_{u(\mathfrak{b}_\mathfrak{l})} \mathbb{K}_\lambda$ the baby Verma module where $\mathbb{K}_\lambda$ is a one-dimensional $u(\mathfrak{b}_\mathfrak{l})$ module with weight λ. Let $V^\sigma(\lambda)$ be the twisted baby Verma module. Namely, $V^\sigma(\lambda) \simeq V(\lambda)$ as vector spaces while $x \cdot m := \sigma(x)(m)$ for all $x \in u(\mathfrak{l})$, $m \in V^\sigma(\lambda)$.

The following lemma is a straightforward calculation.

Lemma 4.1. *Keep assumptions as above,* $V^\sigma(\lambda) \simeq u(\mathfrak{l}) \otimes_{u(\sigma^{-1}(\mathfrak{b}))} \mathbb{K}_{-\tilde{\sigma}(\lambda)}$ *by sending* $x \otimes 1_\lambda$ *to* $\sigma^{-1}(x) \otimes 1_{-\tilde{\sigma}(\lambda)}$. *In particular,* $[V(\lambda)] = [V^\sigma(\lambda)] = [u(\mathfrak{l}) \otimes_{u(\sigma^{-1}(\mathfrak{b}))} \mathbb{K}_{-\tilde{\sigma}(\lambda)}]$.

Define $\mathfrak{g} = W(m,n,1)$. Let $\mathfrak{h} = \langle h_1, \cdots, h_{m+n} \rangle$ be a maximal torus of $\mathfrak{g}$ where $h_i := x_i D_i$ for $i = 1, \cdots, m+n$. One can check that $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$ is a triangular decomposition related to $\mathfrak{h}$ where

$$\mathfrak{g}^+ = \langle x_i D_j \mid 1 \le i < j \le m+n \rangle \oplus \oplus_{i \ge 1} \mathfrak{g}_i,$$

$$\mathfrak{g}^- = \langle x_j D_i \mid 1 \le i < j \le m+n \rangle \oplus \mathfrak{g}_{-1}.$$

Now, set $$\mathfrak{l} = \langle D_1, \cdots, D_{m+n}\rangle \oplus \mathfrak{g}_0 \oplus \langle p_1, \cdots, p_{m+n}\rangle$$

where $p_i = x_i \sum_{j=1}^{m+n} x_j D_j \in \mathfrak{g}_1$. Thanks to [1, Lemma 3.1], $\mathfrak{l} \simeq \mathfrak{pgl}(m+1|n)$. Now let $e_i := E_{i,i+1}$ and $f_i := E_{i+1,i}$. Then $\mathfrak{pgl}(m+1|n)$ has the generators $\{e_i, f_i \mid i = 1, \cdots, m+n-1\}$. There is an even restricted automorphism α of $\mathfrak{l}$ induced by $\alpha(e_i) = f_i$ and $\alpha(f_i) = e_i$. Note that $\alpha(\mathfrak{b}_{\mathfrak{l}}^{\pm}) = \mathfrak{b}_{\mathfrak{l}}^{\mp}$ and $\tilde{\alpha}$ keeps Λ. By Lemma 4.1, we have

$$[u(\mathfrak{l}) \otimes_{u(\mathfrak{b}_{\mathfrak{l}}^-)} 1_\lambda] = [u(\mathfrak{l}) \otimes_{u(\mathfrak{b}_{\mathfrak{l}}^+)} 1_{-\tilde{\alpha}(\lambda)}].$$

Therefore, $\mathfrak{l}$ is a subalgebra satisfying (1) and (2) of Proposition 2.3.
For each $i = 1, \cdots, m+n$, $x_i^3 D_i$ has weight $2\gamma_i$ where $\gamma_i \in \Lambda$ such that $\gamma_i(h_j) = \delta_{ij}$. Therefore, the complementary of $\mathfrak{l}$ contains weight vectors $\{x_i^3 D_i \mid i = 1, \cdots, m+n\}$ with linear independent weights. Assumption (3) of Proposition 2.3 is satisfied.
To sum above up, all assumptions of Proposition 2.3 hold for $W(m,n,1)$ and hence we have the following proposition by Corollary 3.4.

Proposition 4.2. *Keep assumptions as above, and let $\mathfrak{g} = W(m,n,1)$, then $u(\mathfrak{g})$ is of one block.*

4.2. Type S. Denote $\mathfrak{g} = S(m,n,1)$. Let $\mathfrak{h} = \langle h_i \mid 1 \le i \le m+n-1\rangle$ be a maximal torus of $\mathfrak{g}$ where $h_i := x_i D_i - x_{i+1} D_{i+1}$ for $i = 1, \cdots, m+n-1$. One can check that $\mathfrak{g} = \mathfrak{g}_{\bar{1}}^- \oplus \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{g}_{\bar{1}}^+$ is a triangular decomposition related to $\mathfrak{h}$ where $\mathfrak{n}^\pm = \mathfrak{g}_{\bar{0}} \cap W(m,n,1)^\pm$, $\mathfrak{g}_{\bar{1}}^\pm = \mathfrak{g}_{\bar{1}} \cap W(m,n,1)^\pm$.
It is a direct computation that

$$\mathfrak{g}_{\bar{1}}^- = \langle D_{m+1}, \cdots, D_{m+n}\rangle \oplus \langle x_i D_j \mid 1 \le i \le m < j \le m+n\rangle, \text{ and}$$

$$\mathfrak{n}^- = \langle D_1, \cdots, D_m\rangle \oplus \langle x_i D_j \mid 1 \le i < j \le m \text{ or } m+1 \le i < j \le m+n\rangle.$$

Set $\mathfrak{l} = \mathfrak{g}_{\bar{1}}^- \oplus \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+$ where $\mathfrak{n}_{\mathfrak{l}}^+ = \mathfrak{s} \oplus \mathfrak{t}$,

$$\mathfrak{s} = \langle x^{(a)} x_{m+i} D_{m+n} \mid (a) \in I^m \backslash \{0\}, 1 \le i < n\rangle,$$

$$\mathfrak{t} = \langle x^{(a)} D_m \mid (a) \in I^m, a_m = 0, |a| \ge 2\rangle,$$

$$\mathfrak{l}_{\bar{1}}^+ = \langle x^{(a)} D_{m+n} \mid (a) \in I^m \backslash \{0\}\rangle.$$

One can check the followings:

- $[\mathfrak{h}, \mathfrak{g}_{\bar{1}}^- \oplus \mathfrak{n}^- \oplus \mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+] \subseteq \mathfrak{g}_{\bar{1}}^- \oplus \mathfrak{n}^- \oplus \mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+$;
- $\mathfrak{g}_{\bar{1}}^- \oplus \mathfrak{n}^-$ is p-nilpotent;
- $[\mathfrak{n}^-, \mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+] \subseteq \mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+$; $[\mathfrak{g}_{\bar{1}}^-, \mathfrak{n}_{\mathfrak{l}}^+] \subseteq \mathfrak{l}_{\bar{1}}^+$, $[\mathfrak{g}_{\bar{1}}^-, \mathfrak{l}_{\bar{1}}^+] = 0$;
- $[\mathfrak{s}, \mathfrak{s}] = [\mathfrak{t}, \mathfrak{t}] = [\mathfrak{s}, \mathfrak{l}_{\bar{1}}^+] = [\mathfrak{l}_{\bar{1}}^+, \mathfrak{l}_{\bar{1}}^+] = 0$, $[\mathfrak{s}, \mathfrak{t}] \subseteq \mathfrak{s}$, and $[\mathfrak{t}, \mathfrak{l}_{\bar{1}}^+] \subseteq \mathfrak{l}_{\bar{1}}^+$.

Therefore, $\mathfrak{g}_{\bar{1}}^- \oplus \mathfrak{n}^- \oplus \mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+$ is a p-nilpotent ideal. $\mathfrak{l}$ is a subalgebra satisfying (1) of Lemma 2.1 and (2) of Proposition 2.2
Now define $\gamma_i \in \Lambda$ by $\gamma_i(h_j) = \delta_{ij}$, $1 \le i, j \le m+n-1$.

For $1 \le i \le m$ and $1 \le j \le n-1$, D_i has weight $(1-\delta_{1,i})\gamma_{i-1}-\gamma_i$ while $x_{m+j}D_{m+j+1}$ has weight $-\gamma_{m+j-1}+2\gamma_{m+j}-(1-\delta_{j,n-1})\gamma_{m+j+1}$ with respect to $\mathfrak{h}$. One can check that $\mathfrak{n}^-$ contains $m+n-1$ weight vectors

$$\{D_i,\ x_{m+j}D_{m+j+1} \mid 1 \le i \le m;\ 1 \le j \le n-1\}.$$

with linear independent weights. Therefore, assumption (3) of Proposition 2.2 is satisfied.
For $1 \le i \le m-1,\ 2 \le j \le m,\ x_i^2D_j$ has weight $2\gamma_i - 2(1-\delta_{1,i})\gamma_{i-1}+\gamma_{j-1}-\gamma_j$ while $x_1^3D_j$ has weight $3\gamma_1+\gamma_{j-1}-\gamma_j$.
For $1 \le j \le n-1,\ x_1x_{m+j}D_{m+n}$ has weight $\gamma_1+\gamma_{m+j-1}-\gamma_{m+j}-\gamma_{m+n-1}$.
Hence, $\mathfrak{n}_1^+$ contains $m+n-1$ weight vectors

$$\{x_1^2D_i,\ x_1x_{m+j}D_{m+n} \mid 2 \le i \le m;\ 1 \le j \le n-1\} \cup \{x_1^3D_2\}$$

with linear independent weights. Assumption (2) of Lemma 2.1 is satisfied.
To sum above up, all assumptions of Proposition 2.2 hold for $S(m,n,1)$ and hence we have the following proposition by Corollary 3.4.

Proposition 4.3. *Keep assumptions as above, and let $\mathfrak{g} = S(m,n,1)$, then $u(\mathfrak{g})$ is of one block.*

4.3. Type H Let $\mathfrak{g} := H(m,n,1)$, where $m = 2l,\ n > 3$. Denote $k = [n/2]$. For every $(a) = (a_1,\cdots,a_m) \in I^m$ and $(b) = (b_1,\cdots,b_u) \in \mathbb{B}_u \subseteq \mathbb{B}(n)$, denote

$$X^{(a)}Y^{(b)} = x_1^{a_1}\cdots x_m^{a_m}x_{m+b_1}\cdots x_{m+b_u}.$$

By definition, $\mathfrak{g} = \langle D_H(X^{(a)}Y^{(b)}) \mid X^{(a)}Y^{(b)} \ne x_1^{p-1}\cdots x_m^{p-1}x_{m+1}\cdots x_{m+n}\rangle$.
Fix a maximal torus $\mathfrak{h}$ with basis

$$\{h_i,\ h_{m+j} \mid i = 1,\cdots,l; j = 1,\cdots,k\}$$

where $h_i = D_H(x_ix_{l+i}),\ h_{m+j} = D_H(\sqrt{-1}x_{m+j}x_{m+k+j})$.
For $1 \le i \le k$, set $e_i := x_{m+i}+\sqrt{-1}x_{m+k+i}$ and $f_i := x_{m+i}-\sqrt{-1}x_{m+k+i}$.
Then both $D_H(e_i)$ and $D_H(f_i)$ are homogeneous odd elements of degree -1.
Define the following subspaces of $\mathfrak{g}$:

$$\alpha = \langle D_i \mid m+1 \le i \le m+n\rangle \oplus \langle D_H(x_if_j) \mid 1 \le i \le m; 1 \le j \le k\rangle;$$

$$\beta = \langle D_H(x_ix_{m+n}) \mid l+1 \le i \le m\rangle;$$

$$\mathfrak{n}_1^- = \langle D_H(x_ix_j) \mid l+1 \le i,j \le 2l \text{ or } 1 \le i < l \le j < i+l\rangle \oplus \langle D_1,\cdots,D_m\rangle;$$

$$\mathfrak{n}_2' = \langle D_H(a_{ij}),\ D_H(b_{ij}) \mid 1 \le i < j \le k\rangle \text{ where } a_{ij} = f_ie_j,\ b_{ij} = f_if_j;$$

$$\mathfrak{n}_3' = \langle D_H(f_ix_{m+n}) \mid 1 \le i \le k\rangle.$$

Let $\mathfrak{g} = \mathfrak{g}_{\bar 1}^- \oplus \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{g}_{\bar 1}^+$ be a triangular decomposition related to maximal torus $\mathfrak{h}$ where $\mathfrak{n}^\pm = \mathfrak{g}_{\bar 0} \cap W(m,n,1)^\pm,\ \mathfrak{g}_{\bar 1}^\pm = \mathfrak{g}_{\bar 1} \cap W(m,n,1)^\pm$.
One can check that $\mathfrak{n}^- = \mathfrak{n}_1^- \oplus \mathfrak{n}_2^-$ where

$$\mathfrak{n}_2^- = \begin{cases} \mathfrak{n}_2', & n = 2k, \\ \mathfrak{n}_2' \oplus \mathfrak{n}_3', & n = 2k+1; \end{cases} \quad \text{and} \quad \mathfrak{g}_{\bar 1}^- = \begin{cases} \alpha, & n = 2k, \\ \alpha \oplus \beta, & n = 2k+1. \end{cases}$$

Remark 4.4. Above description for $\mathfrak{n}_2^-$ due to [9, section 2.2].

For each $(c) = (c_1, \cdots, c_u) \in \mathbb{B}(k)$, $0 \leq u \leq k$, denote $f^{(c)} = f_{c_1} \cdots f_{c_u}$. In this case, the parity of $D_H(f^{(c)})$ equals to the parity of $u = c_1 + \cdots + c_u$.
If $n = 2k$ is even, define $\mathfrak{l}^+ = \langle D_H(x^{(a)} f^{(c)}) \mid a_j = 0 \text{ if } 1 \leq j \leq l; |(a)| + |(c)| \geq 3 \rangle$.
If $n = 2k+1$ is odd, define

$$\mathfrak{l}^+ = \langle D_H(x^{(a)} f^{(c)} x_{m+n}^{\delta}) \mid a_j = 0 \text{ if } 1 \leq j \leq l; \delta \in \{0,1\}; |(a)| + |(c)| + \delta \geq 3 \rangle.$$

Let $\mathfrak{n}_{\mathfrak{l}}^+$ (resp. $\mathfrak{l}_{\bar{1}}^+$) be the even (resp. odd) part of $\mathfrak{l}^+$.
One can check that $\mathfrak{l} := \mathfrak{g}_{\bar{1}}^- \oplus \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+$ is a restricted subalgebra of $\mathfrak{g}$. Note that for all $f, g \in A(m, n, 1)$ homogeneous,

$$[D_i, D_H(f)] = D_H(D_i(f)), \quad \text{and}$$

$$[D_H(f), D_H(g)] = D_H \left(\sum_{i=1}^{m+n} \sigma(i)(-1)^{\tau(i)\mathrm{d}(f)} D_i(f) D_{i'}(g) \right).$$

We have the following:

- $[D_H(e_i), D_H(e_j)] = [D_H(f_i), D_H(f_j)] = 0$;
- $[D_H(e_i), D_H(f_j)] = [D_H(f_j), D_H(e_i)] = -2\delta_{ij}$;
- $[\mathfrak{h}, \mathfrak{g}_{\bar{1}}^- \oplus \mathfrak{n}^- \oplus \mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+] \subseteq \mathfrak{g}_{\bar{1}}^- \oplus \mathfrak{n}^- \oplus \mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+$;
- $\mathfrak{g}_{\bar{1}}^- \oplus \mathfrak{n}^-$ is p-nilpotent;
- $[\mathfrak{n}^-, \mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+] \subseteq \mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+$; $[\mathfrak{g}_{\bar{1}}^-, \mathfrak{n}_{\mathfrak{l}}^+] \subseteq \mathfrak{l}_{\bar{1}}^+$, $[\mathfrak{g}_{\bar{1}}^-, \mathfrak{l}_{\bar{1}}^+] = 0$;
- $[\mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+, \mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+] = 0$.

Therefore, $\mathfrak{g}_{\bar{1}}^- \oplus \mathfrak{n}^- \oplus \mathfrak{n}_{\mathfrak{l}}^+ \oplus \mathfrak{l}_{\bar{1}}^+$ is a p-nilpotent ideal. $\mathfrak{l}$ is a subalgebra satisfying (1) of Lemma 2.1 and (2) of Proposition 2.2.
For each $1 \leq i, j \leq l$, $1 \leq u, v \leq k$, defines $\gamma_i, \delta_j \in \Lambda$ by $\gamma_i(h_j) = \delta_{ij}$, and $\delta_u(h_{m+v}) = \delta_{uv}$.
For $1 \leq i \leq l$ and $1 \leq j \leq k$, D_i has weight $-\gamma_i$ while $D_H(e_j)$ (resp. $D_H(f_j)$) has weight δ_j (resp. $-\delta_j$) with respect to $\mathfrak{h}$. Then b_{uv} has weight $-\delta_u - \delta_v$ for each $1 \leq u, v \leq k$ and a_{12} has weight $-\delta_1 + \delta_2$.
Therefore, $\mathfrak{n}^-$ contains $l + k$ linear independent vectors

$$\{D_i, \ D_H(b_{j,j+1}) \mid 1 \leq i \leq l; \ 1 \leq j \leq k-1\} \cup \{D_H(a_{12})\}.$$

with linear independent weights. Assumption (3) of Proposition 2.2 is satisfied.
For $1 \leq i \leq l$, $D_H(x_{l+i}^3)$ has weight $3\gamma_i$. For $1 \leq u < v \leq k-1$, $D_H(x_{l+1}^2 b_{u,v})$ has weight $2\gamma_1 - \delta_u - \delta_v$, and $D_H(x_{l+1}^2 f_k x_{m+n})$ has weight $2\gamma_1 - \delta_k$ if $n = 2k+1$.
Moreover, $\mathfrak{n}^+$ has a subset S containing $l+k$ weight vectors with linear independent weights as following:
If $k \geq 3$ is odd,

$$S = \{D_H(x_{l+i}^3), D_H(x_{l+1}^2 b_{j,j+1}) \mid 1 \leq i \leq l; 1 \leq j \leq k-1\} \cup \{D_H(x_{l+1}^2 b_{1k})\};$$

If $k \geq 3$ is even,

$$S = \{D_H(x_{l+i}^3), D_H(x_{l+1}^2 b_{j,j+1}) \mid 1 \leq i \leq l; 1 \leq j \leq k-1\} \cup \{D_H(x_{l+1}^2 b_{2k})\};$$

If $n = 5$, $S = \{D_H(x_{l+i}^3) \mid 1 \leq i \leq l\} \cup \{D_H(x_{l+1}^2 b_{12}), D_H(x_{l+1}^2 f_{12} x_{m+5})\}$.
Therefore, assumption (2) of Lemma 2.1 is satisfied if $n > 4$.

Proposition 4.5. *Keep assumptions as above, and let $\mathfrak{g} = H(m, n, 1)$ with $n > 4$, then $u(\mathfrak{g})$ is of one block.*

By Propositions 4.2, 4.3 and 4.5, we have our main theorem as follows.

Theorem 4.6. *Let $\mathbb{K}$ be an algebraically closed field with characteristics $p > 3$, and $\mathfrak{g} = X(m, n, 1)$, $X \in \{W, S, H\}$, be a graded restricted Lie superalgebra of Cartan type over $\mathbb{K}$ except if $X = H$ with $n = 4$. Then $u(\mathfrak{g})$ is of one block.*

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Ke Ou
Dept. of Statistics and Mathematics
Yunnan University of
Finance and Economics
Kunming, 650221
China
justinok1311@hotmail.com

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