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On the Symmetries of Five-Dimensional Solvable Lie Groups

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Abstract. We consider the five-dimensional solvable Lie group, equipped with left-invariant Riemannian metric. We obtain a full classification of Killing and affine vector fields as well as Ricci, curvature and matter collineations.

Mathematics Subject Classification: 53C50, 53B30.

Key Words: Solvable Lie group, Killing vector fields, affine vector fields, Ricci collineations, curvature collineations.

1. Introduction and preliminaries

In [2], Božek constructed special examples of unimodular solvable Lie groups equipped with left-invariant metrics in arbitrary odd dimension. Such spaces are described in the following way (see [2], [13]): For any integer $n \geq 1$, let G_n be the matrix group of all matrices of the form

$$\begin{pmatrix} e^{u_0} & 0 & \cdots & 0 & x_0 \\ 0 & e^{u_1} & \cdots & 0 & x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{u_n} & x_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

where $(x_0, x_1, ..., x_n, u_1, ..., u_n) \in \mathbb{R}^{2n+1}$ and $u_0 = -(u_1 + ... + u_n)$. The Lie group G_n is unimodular and solvable. The underlying manifold for G_n is $\mathbb{R}^{2n+1}$. The geometry of these Lie groups have been investigated by many authors. In [4], Calvaruso, Kowalski and Marinosci, studied the set of homogeneous geodesics of each solvable Lie group G_n with left-invariant Riemannian metric. In [1], Aghasi and Nasehi considered the five-dimensional solvable Lie group G_2 and investigated other geometrical properties. In particular, they proved the non-existence of left-invariant Ricci solitons on both Lorentzian and Riemannian solvable Lie group G_2. In addition, they showed that the space-like energy on the Lorentzian Lie group G_2 does not have a critical point and there is no left invariant almost complex structure on $G_2 \times \mathbb{R}$. Recently, in [14], it was proved the existence of non-trivial (i.e., not Einstein) Ricci solitons on both Lorentzian and Riemannian five-dimensional solvable Lie group. Moreover, there are no gradient Ricci solitons.

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Let (M, g) be a pseudo-Riemannian manifold, a Killing vector field is a vector field on (M, g) that preserves the metric. Killing fields are the infinitesimal generators of isometries; that is, flows generated by Killing fields are continuous isometries of the manifold. More simply, the flow generates a symmetry, in the sense that moving each point on an object the same distance in the direction of the Killing vector will not distort distances on the object. Specifically, a vector field X is a Killing field if the Lie derivative with respect to X of the metric g vanishes: $L_X g = 0$. In terms of the Levi-Civita connection, this is equivalent to $g(\nabla_Y X, Z) = -g(\nabla_Z X, Y)$ for all vector fields Y, Z. Therefore, it is sufficient to establish it in a preferred coordinate system in order to have it hold in all coordinate systems. The Killing fields on a manifold M form a Lie subalgebra of vector fields on M. This is the Lie algebra of the isometry group of the manifold if M is complete.

A typical use of the Killing field is to express a symmetry in General relativity (in which the geometry of space-time as distorted by gravitational fields is viewed as a 4-dimensional pseudo-Riemannian manifold). In a static configuration, in which nothing changes with time, the time vector will be a Killing vector, and thus the Killing field will point in the direction of forward motion in time.

On the other hand, a vector field X tangent to (M, g) is said to be *affine* if it satisfies $L_X \nabla = 0$, where ∇ is the Levi Civita connection of (M, g) (or equivalently, if $[X, \nabla_Y Z] = \nabla_{[X,Y]} Z + \nabla_Y [X, Z]$ for all vector fields Y, Z) which means that the local fluxes of X given by affine maps. Obviously Killing vector field is also affine. However, the converse does not hold in general. In particular, if (M, g) is a simply connected space-time, the existence of a non Killing affine vector field implies the existence of a second-order covariantly constant symmetric tensor, nowhere vanishing, not proportional to g. As a consequence, the holonomy group of the manifold is reducible (see for example [15]).

A curvature (resp. Ricci) collineation is vector field which preserves the Riemann tensor R (resp. the Ricci tensor ϱ) in the sense that, $L_X R = 0$ (resp. $L_X \varrho = 0$), where L denotes the Lie derivative. The set of all smooth curvature collineations forms a Lie algebra under the Lie bracket operation, which may be infinite-dimensional. Every affine vector field is a curvature collineation.

A matter collineation is a vector field X that satisfies the condition $L_X \mathcal{T} = 0$, where $\mathcal{T}$ is the energy-momentum tensor given by $\mathcal{T} = \varrho - \frac{1}{2}\tau g$ with τ denotes the scalar curvature. The relation between geometry and physics may be highlighted here, as the vector field X is regarded as preserving certain physical quantities along the flow lines of X, this being true for any two observers. In connection with this, it may be shown that every Killing vector field is a matter collineation (by the Einstein field equations, with or without cosmological constant). Thus, a vector field that preserves the metric necessarily preserves the corresponding energy-momentum tensor. When the energy-momentum tensor represents a perfect fluid, every Killing vector field preserves the energy density, pressure and the fluid flow vector field. When the energy-momentum tensor represents an electromagnetic field, a Killing vector field does not necessarily preserve the electric and magnetic fields.

More general, a collineation or a symmetry of a tensor field S on a pseudo-Riemannian manifold (M, g) is a one-parameter group of diffeomorphisms of (M, g), which leaves S invariant. Therefore, each symmetry corresponds to a vector field X which satisfies $L_X S = 0$. Symmetries of the metric tensor g which correspond to the Killing vector fields. Symmetries of the Levi-Civita connection ∇ which correspond to the affine vector fields. Since symmetries are more significant from physical aspects, and they have been studied on several kinds of space-times (see [5, 6, 7, 8, 9, 10, 11]).

The aim of this paper is to study symmetries of the five-dimensional solvable Lie group G_2, equipped with a left-invariant Riemannian metric. We shall treat a single case of left-invariant metric on the solvable Lie group G_2, this metric has some relevance as it appears as one of the possibilities of five-dimensional generalized symmetric spaces. We shall essentially show that besides Ricci collineations all other conditions reduce to Killing vectors and there are no other than right-invariant vector fields.

The paper is organized in the following way. In Section 2, we shall report some basic information about five-dimensional solvable Lie group and its left-invariant metrics in global coordinates, we shall describe their Levi-Civita connection, the curvature and the Ricci tensor. In Section 3, affine and Killing vector fields of solvable Lie group G_2 are characterized via a system of partial differential equations. Then, in Section 4 we shall respectively classify Ricci, curvature and matter collineations on the five-dimensional solvable Lie group G_2 equipped with Riemannian left-invariant metric.

2. Connection and curvature of solvable Lie groups

Consider the five-dimensional solvable Lie group G_2 which is diffeomorphic to the Cartesian space $\mathbb{R}^5(x_0, x_1, x_2, u_1, u_2)$. Throughout the paper, we will denote the coordinate basis $\left\{\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right\}$ by $\{\partial_{x_0}, \partial_{x_1}, \partial_{x_2}, \partial_{u_1}, \partial_{u_2}\}$ and we shall endow G_2 with the left-invariant Riemannian metric g, given by

$$g = \sum_{i=0}^{2} e^{-2u_i}(\mathrm{d}x_i)^2 + a \sum_{\alpha,\beta=1}^{2} \mathrm{d}u_\alpha \mathrm{d}u_\beta, \quad \text{with} \quad a > 0, \tag{1}$$

Let ∇ denote the *Levi-Civita connection* of (G_2, g), and R its *curvature tensor*, taken with the sign convention:

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

The *Ricci tensor* ϱ of (G_2, g), is defined by

$$\varrho(X,Y) = \mathrm{tr}(Z \to R(Z,X)Y)$$

Starting from (1), we can describe the Levi-Civita connection ∇, and then the curvature of (G_2, g), with respect to the basis of coordinate vector fields. Explicitly, we get that the Levi-Civita connection is completely determined by the following possibly non-vanishing components:

$$\begin{aligned}
\nabla_{\partial_{x_0}}\partial_{x_0} &= -\frac{2}{3a}e^{2(u_1+u_2)}\left(\partial_{u_1} + \partial_{u_2}\right), & \nabla_{\partial_{x_0}}\partial_{u_1} &= \nabla_{\partial_{x_0}}\partial_{u_2} = \partial_{x_0},\\
\nabla_{\partial_{x_1}}\partial_{x_1} &= \frac{2}{3a}e^{-2u_1}\left(2\partial_{u_1} - \partial_{u_2}\right), & \nabla_{\partial_{x_1}}\partial_{u_1} &= -\partial_{x_1},\\
\nabla_{\partial_{x_2}}\partial_{x_2} &= \frac{2}{3a}e^{-2u_2}\left(-\partial_{u_1} + 2\partial_{u_2}\right), & \nabla_{\partial_{x_2}}\partial_{u_2} &= -\partial_{x_2}.
\end{aligned} \tag{2}$$

The non-vanishing curvature components are:

$$\begin{aligned}
R(\partial_{x_0},\partial_{x_1})\partial_{x_0} &= -\frac{2}{3a}e^{2(u_1+u_2)}\partial_{x_1}, & R(\partial_{x_0},\partial_{x_1})\partial_{x_1} &= \frac{2}{3a}e^{-2u_1}\partial_{x_0},\\
R(\partial_{x_0},\partial_{x_2})\partial_{x_0} &= -\frac{2}{3a}e^{2(u_1+u_2)}\partial_{x_2}, & R(\partial_{x_0},\partial_{x_2})\partial_{x_2} &= \frac{2}{3a}e^{-2u_2}\partial_{x_0},\\
R(\partial_{x_0},\partial_{u_1})\partial_{x_0} &= \frac{2}{3a}e^{2(u_1+u_2)}\left(\partial_{u_1}+\partial_{u_2}\right), & R(\partial_{x_0},\partial_{u_1})\partial_{u_1} &= -\partial_{x_0},\\
R(\partial_{x_0},\partial_{u_2})\partial_{x_0} &= \frac{2}{3a}e^{2(u_1+u_2)}\left(\partial_{u_1}+\partial_{u_2}\right), & R(\partial_{x_0},\partial_{u_1})\partial_{u_2} &= -\partial_{x_0},\\
R(\partial_{x_0},\partial_{u_2})\partial_{u_1} &= -\partial_{x_0}, & R(\partial_{x_0},\partial_{u_2})\partial_{u_2} &= -\partial_{x_0},\\
R(\partial_{x_1},\partial_{x_2})\partial_{x_1} &= -\frac{2}{3a}e^{-2u_1}\partial_{x_2}, & R(\partial_{x_1},\partial_{x_2})\partial_{x_2} &= \frac{2}{3a}e^{-2u_2}\partial_{x_1},\\
R(\partial_{x_1},\partial_{u_1})\partial_{x_1} &= \frac{2}{3a}e^{-2u_1}\left(2\partial_{u_1}-\partial_{u_2}\right), & R(\partial_{x_1},\partial_{u_1})\partial_{u_1} &= -\partial_{x_1},\\
R(\partial_{x_2},\partial_{u_2})\partial_{x_2} &= \frac{2}{3a}e^{-2u_2}\left(-\partial_{u_1}+2\partial_{u_2}\right), & R(\partial_{x_2},\partial_{u_2})\partial_{u_2} &= -\partial_{x_2},
\end{aligned} \tag{3}$$

and the ones obtained from them using the symmetries of the curvature tensor. The non-zero components of the Ricci tensor are:

$$\varrho_{u_1u_1} = \varrho_{u_2u_2} = -2 \quad \text{and} \quad \varrho_{u_1u_2} = -1. \tag{4}$$

The scalar curvature is then given by $$\tau = -\frac{4}{a}. \tag{5}$$

3. Affine and Killing vector fields of 5-dimensional solvable Lie groups

In this section we completely classify affine and Killing vector fields of five-dimensional Riemannian solvable Lie group (G_2, g). Let $V = X\partial_{x_0} + Y\partial_{x_1} + Z\partial_{x_2} + S\partial_{u_1} + T\partial_{u_2}$ be a vector field on G_2, where X, Y, Z, S, T are smooth functions of the variables x_0, x_1, x_2, u_1, u_2.

Long but direct calculation yields the following description of the Lie derivative of the Levi-Civita connection (2) with respect to V, as follows

$$\begin{aligned}
(L_V\nabla)(\partial_{x_0},\partial_{x_0}) &= \left(\partial^2_{x_0}X + 2\partial_{x_0}S + 2\partial_{x_0}T + \frac{2}{3a}e^{2u_1+2u_2}(\partial_{u_1}X + \partial_{u_2}X)\right)\partial_{x_0}\\
&+ \left(\partial^2_{x_0}Y + \frac{2}{3a}e^{2u_1+2u_2}(\partial_{u_1}Y + \partial_{u_2}Y)\right)\partial_{x_1}\\
&+ \left(\partial^2_{x_0}Z + \frac{2}{3a}e^{2u_1+2u_2}(\partial_{u_1}Z + \partial_{u_2}Z)\right)\partial_{x_2}\\
&+ \left(\partial^2_{x_0}S + \frac{2}{3a}e^{2u_1+2u_2}(\partial_{u_1}S + \partial_{u_2}S - 2S - 2T - 2\partial_{x_0}X)\right)\partial_{u_1}\\
&+ \left(\partial^2_{x_0}T + \frac{2}{3a}e^{2u_1+2u_2}(\partial_{u_1}T + \partial_{u_2}T - 2S - 2T - 2\partial_{x_0}X)\right)\partial_{u_2},\\
(L_V\nabla)(\partial_{x_0},\partial_{x_1}) &= (\partial_{x_0}\partial_{x_1}X + \partial_{x_1}S + \partial_{x_1}T)\,\partial_{x_0} + (\partial_{x_0}\partial_{x_1}Y - \partial_{x_0}S)\,\partial_{x_1}\\
&+ (\partial_{x_0}\partial_{x_1}Z)\,\partial_{x_2} + \left(\partial_{x_0}\partial_{x_1}S + \frac{2}{3a}(2e^{-2u_1}\partial_{x_0}Y - e^{2u_1+2u_2}\partial_{x_1}X)\right)\partial_{u_1}\\
&+ \left(\partial_{x_0}\partial_{x_1}T - \frac{2}{3a}(e^{-2u_1}\partial_{x_0}Y + e^{2u_1+2u_2}\partial_{x_1}X)\right)\partial_{u_2},
\end{aligned}$$

$$
\begin{aligned}
(L_V\nabla)(\partial_{x_0},\partial_{x_2}) &= (\partial_{x_0}\partial_{x_2}X+\partial_{x_2}S+\partial_{x_2}T)\,\partial_{x_0}+(\partial_{x_0}\partial_{x_2}Y)\,\partial_{x_1}\\
&\quad+(\partial_{x_0}\partial_{x_2}Z-\partial_{x_0}T)\,\partial_{x_2}+\left(\partial_{x_0}\partial_{x_2}S-\frac{2}{3a}(e^{-2u_2}\partial_{x_0}Z+e^{2u_1+2u_2}\partial_{x_2}X)\right)\partial_{u_1}\\
&\quad+\left(\partial_{x_0}\partial_{x_2}T+\frac{2}{3a}(2e^{-2u_2}\partial_{x_0}Z-e^{2u_1+2u_2}\partial_{x_2}X)\right)\partial_{u_2},\\
(L_V\nabla)(\partial_{x_0},\partial_{u_1}) &= (\partial_{x_0}\partial_{u_1}X+\partial_{u_1}S+\partial_{u_1}T)\,\partial_{x_0}+(\partial_{x_0}\partial_{u_1}Y-2\partial_{x_0}Y)\,\partial_{x_1}\\
&\quad+(\partial_{x_0}\partial_{u_1}Z-\partial_{x_0}Z)\,\partial_{x_2}+\left(\partial_{x_0}\partial_{u_1}S-\partial_{x_0}S-\frac{2}{3a}e^{2u_1+2u_2}\partial_{u_1}X\right)\partial_{u_1}\\
&\quad+\left(\partial_{x_0}\partial_{u_1}T-\partial_{x_0}T-\frac{2}{3a}e^{2u_1+2u_2}\partial_{u_1}X\right)\partial_{u_2},\\
(L_V\nabla)(\partial_{x_0},\partial_{u_2}) &= (\partial_{x_0}\partial_{u_2}X+\partial_{u_2}S+\partial_{u_2}T)\,\partial_{x_0}+(\partial_{x_0}\partial_{u_2}Y-\partial_{x_0}Y)\,\partial_{x_1}\\
&\quad+(\partial_{x_0}\partial_{u_2}Z-2\partial_{x_0}Z)\,\partial_{x_2}+\left(\partial_{x_0}\partial_{u_2}S-\partial_{x_0}S-\frac{2}{3a}e^{2u_1+2u_2}\partial_{u_2}X\right)\partial_{u_1}\\
&\quad+\left(\partial_{x_0}\partial_{u_2}T-\partial_{x_0}T-\frac{2}{3a}e^{2u_1+2u_2}\partial_{u_2}X)\right)\partial_{u_2},\\
(L_V\nabla)(\partial_{x_1},\partial_{x_1}) &= \left(\partial^2_{x_1}X+\frac{2}{3a}e^{-2u_1}(\partial_{u_2}X-2\partial_{u_1}X)\right)\partial_{x_0}\\
&\quad+\left(\partial^2_{x_1}Y-2\partial_{x_1}S+\frac{2}{3a}e^{-2u_1}(\partial_{u_2}Y-2\partial_{u_1}Y)\right)\partial_{x_1}\\
&\quad+\left(\partial^2_{x_1}Z+\frac{2}{3a}e^{-2u_1}(\partial_{u_2}Z-2\partial_{u_1}Z)\right)\partial_{x_2}\\
&\quad+\left(\partial^2_{x_1}S+\frac{2}{3a}e^{-2u_1}(4\partial_{x_1}Y+\partial_{u_2}S-4S-2\partial_{u_1}S)\right)\partial_{u_1}\\
&\quad+\left(\partial^2_{x_1}T+\frac{2}{3a}e^{-2u_1}(\partial_{u_2}T-2\partial_{x_1}Y+2S-2\partial_{u_1}T)\right)\partial_{u_2},\\
(L_V\nabla)(\partial_{x_1},\partial_{x_2}) &= (\partial_{x_1}\partial_{x_2}X)\,\partial_{x_0}+(\partial_{x_1}\partial_{x_2}Y-\partial_{x_2}S)\,\partial_{x_1}+(\partial_{x_1}\partial_{x_2}Z-\partial_{x_1}T)\,\partial_{x_2}\\
&\quad+\left(\partial_{x_1}\partial_{x_2}S+\frac{2}{3a}(2e^{-2u_1}\partial_{x_2}Y-e^{-2u_2}\partial_{x_1}Z)\right)\partial_{u_1}\\
&\quad+\left(\partial_{x_1}\partial_{x_2}T+\frac{2}{3a}(2e^{-2u_2}\partial_{x_1}Z-e^{-2u_1}\partial_{x_2}Y)\right)\partial_{u_2},\\
(L_V\nabla)(\partial_{x_1},\partial_{u_1}) &= (\partial_{x_1}\partial_{u_1}X+2\partial_{x_1}X)\,\partial_{x_0}+(\partial_{x_1}\partial_{u_1}Y-\partial_{u_1}S)\,\partial_{x_1}\\
&\quad+(\partial_{x_1}\partial_{u_1}Z+\partial_{x_1}Z)\,\partial_{x_2}+\left(\partial_{x_1}\partial_{u_1}S+\partial_{x_1}S+\frac{4}{3a}e^{-2u_1}\partial_{u_1}Y\right)\partial_{u_1}\\
&\quad+\left(\partial_{x_1}\partial_{u_1}T+\partial_{x_1}T-\frac{2}{3a}e^{-2u_1}\partial_{u_1}Y\right)\partial_{u_2},\\
(L_V\nabla)(\partial_{x_1},\partial_{u_2}) &= (\partial_{x_1}\partial_{u_2}X+\partial_{x_1}X)\,\partial_{x_0}+(\partial_{x_1}\partial_{u_2}Y-\partial_{u_2}S)\,\partial_{x_1}\\
&\quad+(\partial_{x_1}\partial_{u_2}Z-\partial_{x_1}Z)\,\partial_{x_2}+\left(\partial_{x_1}\partial_{u_2}S+\frac{4}{3a}e^{-2u_1}\partial_{u_2}Y\right)\partial_{u_1}\\
&\quad+\left(\partial_{x_1}\partial_{u_2}T-\frac{2}{3a}e^{-2u_1}\partial_{u_2}Y\right)\partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{u_1}\right) &= \left(\partial_{x_2}\partial_{u_1}X+\partial_{x_2}X\right)\partial_{x_0}+\left(\partial_{x_2}\partial_{u_1}Y-\partial_{x_2}Y\right)\partial_{x_1}\\
&\quad+\left(\partial_{x_2}\partial_{u_1}Z-\partial_{u_1}T\right)\partial_{x_2}+\left(\partial_{x_2}\partial_{u_1}S-\frac{2}{3a}e^{-2u_2}\partial_{u_1}Z\right)\partial_{u_1}\\
&\quad+\left(\partial_{x_2}\partial_{u_1}T+\frac{4}{3a}e^{-2u_2}\partial_{u_1}Z\right)\partial_{u_2},\\
\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{x_2}\right) &= \left(\partial^2_{x_2}X+\frac{2}{3a}e^{-2u_2}\left(\partial_{u_1}X-2\partial_{u_2}X\right)\right)\partial_{x_0}\\
&\quad+\left(\partial^2_{x_2}Y+\frac{2}{3a}e^{-2u_2}\left(\partial_{u_1}Y-2\partial_{u_2}Y\right)\right)\partial_{x_1}\\
&\quad+\left(\partial^2_{x_2}Z-2\partial_{x_2}T+\frac{2}{3a}e^{-2u_2}\left(\partial_{u_1}Z-2\partial_{u_2}Z\right)\right)\partial_{x_2}\\
&\quad+\left(\partial^2_{x_2}S+\frac{2}{3a}e^{-2u_2}\left(\partial_{u_1}S-2\partial_{u_2}S+2T-2\partial_{x_2}Z\right)\right)\partial_{u_1}\\
&\quad+\left(\partial^2_{x_2}T+\frac{2}{3a}e^{-2u_2}\left(\partial_{u_1}T-2\partial_{u_2}T-4T+4\partial_{x_2}Z\right)\right)\partial_{u_2},\\
\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{u_2}\right) &= \left(\partial_{x_2}\partial_{u_2}X+2\partial_{x_2}X\right)\partial_{x_0}+\left(\partial_{x_2}\partial_{u_2}Y+\partial_{x_2}Y\right)\partial_{x_1}\\
&\quad+\left(\partial_{x_2}\partial_{u_2}Z-\partial_{u_2}T\right)\partial_{x_2}+\left(\partial_{x_2}\partial_{u_2}S+\partial_{x_2}S-\frac{2}{3a}e^{-2u_2}\partial_{u_2}Z\right)\partial_{u_1}\\
&\quad+\left(\partial_{x_2}\partial_{u_2}T+\partial_{x_2}T+\frac{4}{3a}e^{-2u_2}\partial_{u_2}Z\right)\partial_{u_2},\\
\left(L_V\nabla\right)\left(\partial_{u_1},\partial_{u_1}\right) &= \left(\partial^2_{u_1}X+2\partial_{u_1}X\right)\partial_{x_0}+\left(\partial^2_{u_1}Y-2\partial_{u_1}Y\right)\partial_{x_1}\\
&\quad+\left(\partial^2_{u_1}Z\right)\partial_{x_2}+\left(\partial^2_{u_1}S\right)\partial_{u_1}+\left(\partial^2_{u_1}T\right)\partial_{u_2},\\
\left(L_V\nabla\right)\left(\partial_{u_1},\partial_{u_2}\right) &= \left(\partial_{u_1}\partial_{u_2}X+\partial_{u_1}X+\partial_{u_2}X\right)\partial_{x_0}+\left(\partial_{u_1}\partial_{u_2}Y-\partial_{u_2}Y\right)\partial_{x_1}\\
&\quad+\left(\partial_{u_1}\partial_{u_2}Z-\partial_{u_1}Z\right)\partial_{x_2}+\left(\partial_{u_1}\partial_{u_2}S\right)\partial_{u_1}+\left(\partial_{u_1}\partial_{u_2}T\right)\partial_{u_2},\\
\left(L_V\nabla\right)\left(\partial_{u_2},\partial_{u_2}\right) &= \left(\partial^2_{u_2}X+2\partial_{u_2}X\right)\partial_{x_0}+\left(\partial^2_{u_2}Y\right)\partial_{x_1}\\
&\quad+\left(\partial^2_{u_2}Z-2\partial_{u_2}Z\right)\partial_{x_2}+\left(\partial^2_{u_2}S\right)\partial_{u_1}+\left(\partial^2_{u_2}T\right)\partial_{u_2}.
\end{aligned}
$$

In order to determine the affine vector fields, we must solve the system of PDEs obtained by requiring that all the coefficients in the above Lie derivative are equal to zero. From $\left(L_V\nabla\right)\left(\partial_{u_1},\partial_{u_1}\right)=\left(L_V\nabla\right)\left(\partial_{u_1},\partial_{u_2}\right)=\left(L_V\nabla\right)\left(\partial_{u_2},\partial_{u_2}\right)=0$ we obtain

$$
\begin{aligned}
S&=\overline{S}\left(x_0,x_1,x_2\right)u_1+\overline{\overline{S}}\left(x_0,x_1,x_2\right)u_2+\widehat{S}\left(x_0,x_1,x_2\right),\\
T&=\overline{T}\left(x_0,x_1,x_2\right)u_1+\overline{\overline{T}}\left(x_0,x_1,x_2\right)u_2+\widehat{T}\left(x_0,x_1,x_2\right),
\end{aligned}
$$

where $\overline{S},\overline{T},\overline{\overline{S}},\overline{\overline{T}},\widehat{T},\widehat{S}$ are real-valued smooth functions depending on x_0,x_1,x_2. We now we derive, with respect to u_1, the equations given by

$$
\mathrm{d}u_1\left[\left(L_V\nabla\right)\left(\partial_{x_1},\partial_{u_1}\right)\right]=0,\quad\text{and}\quad \mathrm{d}u_2\left[\left(L_V\nabla\right)\left(\partial_{x_1},\partial_{u_1}\right)\right]=0.
$$

Using the fact that $\mathrm{d}x_1\left[\left(L_V\nabla\right)\left(\partial_{u_1},\partial_{u_1}\right)\right]=0$, we deduce $\partial_{x_1}\overline{S}=\partial_{x_1}\overline{T}=0$. Then, taking the derivative, with respect to u_2, of the equations given by

$$
\mathrm{d}u_1\left[\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{u_2}\right)\right]=0,\quad\text{and}\quad \mathrm{d}u_2\left[\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{u_2}\right)\right]=0,
$$

and taking into account that $dx_2\left[\left(L_V\nabla\right)\left(\partial_{u_2},\partial_{u_2}\right)\right]=0$, we get $\partial_{x_2}\overline{\overline{S}}=\partial_{x_2}\overline{\overline{T}}=0$. By deriving, with respect to u_2, the equation given by $du_1\left[\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{u_1}\right)\right]=0$ and using the equations obtained from

$$du_1\left[\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{u_1}\right)\right]=0,\quad du_2\left[\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{u_1}\right)\right]=0,\quad dx_2\left[\left(L_V\nabla\right)\left(\partial_{u_1},\partial_{u_2}\right)\right]=0,$$

we prove that $\partial_{u_1}Z=\partial_{x_2}\overline{S}=\partial_{x_2}\overline{T}=0$.

Again, deriving, with respect to u_1, the equation given by $du_1\left[\left(L_V\nabla\right)\left(\partial_{x_1},\partial_{u_2}\right)\right]=0$ and using the equations obtained from

$$du_1\left[\left(L_V\nabla\right)\left(\partial_{x_1},\partial_{u_2}\right)\right]=0,\quad du_2\left[\left(L_V\nabla\right)\left(\partial_{x_1},\partial_{u_2}\right)\right]=0,\quad dx_1\left[\left(L_V\nabla\right)\left(\partial_{u_1},\partial_{u_2}\right)\right]=0,$$

we have $\partial_{u_2}Y=\partial_{x_1}\overline{\overline{S}}=\partial_{x_1}\overline{\overline{T}}=0$. Therefore, S and T reduce to

$$\begin{aligned}S&=\overline{S}\left(x_0\right)u_1+\overline{\overline{S}}\left(x_0\right)u_2+\widehat{S}\left(x_0,x_1,x_2\right),\\T&=\overline{T}\left(x_0\right)u_1+\overline{\overline{T}}\left(x_0\right)u_2+\widehat{T}\left(x_0,x_1,x_2\right).\end{aligned}$$

Next, the equation $dx_1\left[\left(L_V\nabla\right)\left(\partial_{x_0},\partial_{u_2}\right)\right]=0$ together with the equation given by $dx_2\left[\left(L_V\nabla\right)\left(\partial_{x_0},\partial_{u_1}\right)\right]=0$ lead us to prove that $\partial_{x_0}Y=\partial_{x_0}Z=0$.

Thus, from the equations $dx_1\left[\left(L_V\nabla\right)\left(\partial_{x_0},\partial_{x_1}\right)\right]=0$ and $dx_2\left[\left(L_V\nabla\right)\left(\partial_{x_0},\partial_{x_2}\right)\right]=0$, we get $\partial_{x_0}S=\partial_{x_0}T=0$.

Now, using the equations $dx_1\left[\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{u_2}\right)\right]=dx_1\left[\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{x_2}\right)\right]=0$, we obtain $\partial_{x_2}Y=\partial_{u_1}Y=0$.

We then use the equations given by

$$\begin{aligned}&dx_1\left[\left(L_V\nabla\right)\left(\partial_{x_1},\partial_{u_1}\right)\right]=0, &&dx_1\left[\left(L_V\nabla\right)\left(\partial_{x_1},\partial_{u_2}\right)\right]=0,\\&dx_1\left[\left(L_V\nabla\right)\left(\partial_{x_1},\partial_{x_2}\right)\right]=0, &&du_1\left[\left(L_V\nabla\right)\left(\partial_{x_1},\partial_{u_1}\right)\right]=0,\end{aligned}$$

to obtain $\partial_{x_1}S=\partial_{x_2}S=\partial_{u_1}S=\partial_{u_2}S=0$.

Again, the equation $dx_2\left[\left(L_V\nabla\right)\left(\partial_{x_1},\partial_{u_1}\right)\right]=du_1\left[\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{u_2}\right)\right]=0$ leads us to prove $\partial_{x_1}Z=\partial_{u_2}Z=0$. Hence, $S=c_1$ for $c_1\in\mathbb{R}$, $Y=Y(x_1)$, and $Z=Z(x_2)$.

Now, using the equations obtained from

$$\begin{aligned}&dx_2\left[\left(L_V\nabla\right)\left(\partial_{x_1},\partial_{x_2}\right)\right]=0, &&du_2\left[\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{u_2}\right)\right]=0,\\&dx_2\left[\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{u_1}\right)\right]=0, &&dx_2\left[\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{u_2}\right)\right]=0,\end{aligned}$$

we get $\partial_{x_1}T=\partial_{x_2}T=\partial_{u_1}T=\partial_{u_2}T=0$, and so, $T=c_2\in\mathbb{R}$. Thus, by using the equations

$$du_1\left[\left(L_V\nabla\right)\left(\partial_{x_1},\partial_{x_1}\right)\right]=0,\quad\text{and}\quad du_1\left[\left(L_V\nabla\right)\left(\partial_{x_2},\partial_{x_2}\right)\right]=0,$$

we deduce that $Y=c_1x_1+c_3$ and $Z=c_2x_2+c_4$ for some real constants c_3,c_4.

Next, from the equations

$$\begin{aligned}&du_1\left[\left(L_V\nabla\right)\left(\partial_{x_0},\partial_{x_1}\right)\right]=0, &&du_1\left[\left(L_V\nabla\right)\left(\partial_{x_0},\partial_{x_2}\right)\right]=0,\\&du_2\left[\left(L_V\nabla\right)\left(\partial_{x_0},\partial_{u_1}\right)\right]=0, &&du_2\left[\left(L_V\nabla\right)\left(\partial_{x_0},\partial_{u_2}\right)\right]=0,\end{aligned}$$

we conclude that $\partial_{x_1} X = \partial_{x_2} X = \partial_{u_1} X = \partial_{u_2} X = 0$, which together with the equation obtained from $\mathrm{d}u_2 \left[(L_V \nabla) (\partial_{x_0}, \partial_{x_0}) \right] = 0$ gives

$$X = -(c_1 + c_2)x_0 + c_5, \quad c_5 \in \mathbb{R}.$$

We now determine the Lie derivative of the metric (1) with respect to V. We have the following

$$\begin{aligned}
(L_V g)(\partial_{x_0}, \partial_{x_0}) &= 2e^{2(u_1+u_2)} (S + T + \partial_{x_0} X), \\
(L_V g)(\partial_{x_0}, \partial_{x_1}) &= e^{-2u_1} \partial_{x_0} Y + e^{2(u_1+u_2)} \partial_{x_1} X, \\
(L_V g)(\partial_{x_0}, \partial_{x_2}) &= e^{-2u_2} \partial_{x_0} Z + e^{2(u_1+u_2)} \partial_{x_2} X, \\
(L_V g)(\partial_{x_0}, \partial_{u_1}) &= a\partial_{x_0} S + \frac{a}{2} \partial_{x_0} T + e^{2(u_1+u_2)} \partial_{u_1} X, \\
(L_V g)(\partial_{x_0}, \partial_{u_2}) &= a\partial_{x_0} T + \frac{a}{2} \partial_{x_0} S + e^{2(u_1+u_2)} \partial_{u_2} X, \\
(L_V g)(\partial_{x_1}, \partial_{x_1}) &= 2e^{-2u_1} (-S + \partial_{x_1} Y), \\
(L_V g)(\partial_{x_1}, \partial_{x_2}) &= e^{-2u_2} \partial_{x_1} Z + e^{-2u_1} \partial_{x_2} Y, \\
(L_V g)(\partial_{x_1}, \partial_{u_1}) &= a\partial_{x_1} S + \frac{a}{2} \partial_{x_1} T + e^{-2u_1} \partial_{u_1} Y, \\
(L_V g)(\partial_{x_1}, \partial_{u_2}) &= a\partial_{x_1} T + \frac{a}{2} \partial_{x_1} S + e^{-2u_1} \partial_{u_2} Y, \\
(L_V g)(\partial_{x_2}, \partial_{x_2}) &= 2e^{-2u_2} (-T + \partial_{x_2} Z), \\
(L_V g)(\partial_{x_2}, \partial_{u_1}) &= a\partial_{x_2} S + \frac{a}{2} \partial_{x_2} T + e^{-2u_2} \partial_{u_1} Z, \\
(L_V g)(\partial_{x_2}, \partial_{u_2}) &= a\partial_{x_2} T + \frac{a}{2} \partial_{x_2} S + e^{-2u_2} \partial_{u_2} Z, \\
(L_V g)(\partial_{u_1}, \partial_{u_1}) &= 2a\partial_{u_1} S + a\partial_{u_1} T, \\
(L_V g)(\partial_{u_1}, \partial_{u_2}) &= a\partial_{u_1} T + a\partial_{u_2} S + \frac{a}{2} \partial_{u_1} S + \frac{a}{2} \partial_{u_2} T, \\
(L_V g)(\partial_{u_2}, \partial_{u_2}) &= 2a\partial_{u_2} T + a\partial_{u_2} S.
\end{aligned} \tag{6}$$

In order to determine the Killing vector fields, we must solve the system of PDEs obtained by requiring that all the coefficients in the above Lie derivative are equal to zero. A straightforward computations lead to prove the following:

Theorem 3.1. *Let V be an arbitrary vector field on the Riemannian Lie group (G_2, g). Then, the following conditions are equivalent:*

- *V is an affine vector field*
- *V is a Killing vector field*
- *V is given by*

$$V = \left[-(c_1 + c_2)x_0 + c_5 \right] \partial_{x_0} + (c_1 x_1 + c_3)\, \partial_{x_1} + (c_2 x_2 + c_4)\, \partial_{x_2} + c_1 \partial_{u_1} + c_2 \partial_{u_2},$$

for some real constants $c_1, \ldots, c_5$.

Remark 3.2. The above Theorem, giving the explicit form of any Killing vector field, shows that the space of Killing vector fields is five-dimensional. Since right-invariant vector fields on Lie groups are Killing for any left-invariant metric, one has that no additional Killing vector fields may exist in (G_2, g). A same remark holds true for affine vector fields. Since any Killing vector field is affine, one has that there are no additional ones.

4. Ricci, curvature and matter collineations

In this section, we shall investigate symmetries of the five-dimensional Riemannian solvable Lie group (G_2, g) where g is given by (1).. Let $V = X\partial_{x_0} + Y\partial_{x_1} + Z\partial_{x_2} + S\partial_{u_1} + T\partial_{u_2}$ denote an arbitrary vector field on G_2 where X, Y, Z, S, T are smooth functions of the variables x_0, x_1, x_2, u_1, u_2. We first determine the Lie derivative of the Ricci tensor ϱ. We have the following

$$
\begin{aligned}
(L_V\varrho)(\partial_{x_0}, \partial_{u_1}) &= -(\partial_{x_0}T + 2\partial_{x_0}S),\\
(L_V\varrho)(\partial_{x_0}, \partial_{u_2}) &= -(2\partial_{x_0}T + \partial_{x_0}S),\\
(L_V\varrho)(\partial_{x_1}, \partial_{u_1}) &= -(\partial_{x_1}T + 2\partial_{x_1}S),\\
(L_V\varrho)(\partial_{x_1}, \partial_{u_2}) &= -(2\partial_{x_1}T + \partial_{x_1}S),\\
(L_V\varrho)(\partial_{x_2}, \partial_{u_1}) &= -(\partial_{x_2}T + 2\partial_{x_2}S),\\
(L_V\varrho)(\partial_{x_2}, \partial_{u_2}) &= -(2\partial_{x_2}T + \partial_{x_2}S),\\
(L_V\varrho)(\partial_{u_1}, \partial_{u_1}) &= -2(\partial_{u_1}T + 2\partial_{u_1}S),\\
(L_V\varrho)(\partial_{u_1}, \partial_{u_2}) &= -(\partial_{u_2}T + 2\partial_{u_2}S + 2\partial_{u_1}T + \partial_{u_1}S),\\
(L_V\varrho)(\partial_{u_2}, \partial_{u_2}) &= -2(2\partial_{u_2}T + \partial_{u_2}S),\\
(L_V\varrho)\left(\partial_{x_i}, \partial_{x_j}\right) &= 0 \quad \text{for all } i, j = 0, 1, 2.
\end{aligned}
$$

Next, using (3), we calculate the Lie derivative of the curvature tensor R, with respect to the coordinate basis. We have the following

$$
\begin{aligned}
(L_VR)(\partial_{x_0}, \partial_{x_1}, \partial_{x_0}) &= \frac{2}{3a}\left(e^{-2u_1}\partial_{x_0}Y + e^{2u_1+2u_2}\partial_{x_1}X\right)\partial_{x_0}\\
&- \frac{4}{3a}e^{2u_1+2u_2}(\partial_{x_0}X + S + T)\partial_{x_1} + \frac{2}{3a}e^{2u_1+2u_2}(2\partial_{x_1}S + \partial_{x_1}T)\partial_{u_1}\\
&+ \frac{2}{3a}e^{2u_1+2u_2}(\partial_{x_1}S + 2\partial_{x_1}T)\partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
(L_VR)(\partial_{x_0}, \partial_{x_1}, \partial_{x_1}) &= \frac{4}{3a}e^{-2u_1}(\partial_{x_1}Y - S)\partial_{x_0} - \frac{2}{3a}\left(e^{-2u_1}\partial_{x_0}Y + e^{2u_1+2u_2}\partial_{x_1}X\right)\partial_{x_1}\\
&- \frac{2}{a}e^{-2u_1}(\partial_{x_0}S)\partial_{u_1} - \frac{2}{3a}e^{-2u_1}(\partial_{x_0}T - \partial_{x_0}S)\partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
(L_VR)(\partial_{x_0}, \partial_{x_1}, \partial_{x_2}) &= \frac{2}{3a}\left(e^{-2u_1}\partial_{x_2}Y + e^{-2u_2}\partial_{x_1}Z\right)\partial_{x_0}\\
&- \frac{2}{3a}\left(e^{2u_1+2u_2}\partial_{x_2}X + e^{-2u_2}\partial_{x_0}Z\right)\partial_{x_1},
\end{aligned}
$$

$$
\begin{aligned}
(L_VR)(\partial_{x_0}, \partial_{x_1}, \partial_{u_1}) &= \left(\frac{2}{3a}e^{-2u_1}\partial_{u_1}Y - \partial_{x_1}S - \partial_{x_1}T\right)\partial_{x_0}\\
&+ \left(-\frac{2}{3a}e^{2u_1+2u_2}\partial_{u_1}X + \partial_{x_0}S\right)\partial_{x_1},
\end{aligned}
$$

$$(L_V R)(\partial_{x_0}, \partial_{x_1}, \partial_{u_2}) = \left(\frac{2}{3a}e^{-2u_1}\partial_{u_2}Y - \partial_{x_1}S - \partial_{x_1}T\right)\partial_{x_0} - \frac{2}{3a}e^{2u_1+2u_2}(\partial_{u_2}X)\partial_{x_1},$$

$$(L_V R)(\partial_{x_0}, \partial_{x_2}, \partial_{x_0}) = \frac{2}{3a}\left(e^{-2u_2}\partial_{x_0}Z + e^{2u_1+2u_2}\partial_{x_2}X\right)\partial_{x_0}$$
$$- \frac{4}{3a}e^{2u_1+2u_2}(\partial_{x_0}X + S + T)\partial_{x_2} + \frac{2}{3a}e^{2u_1+2u_2}(2\partial_{x_2}S + \partial_{x_2}T)\partial_{u_1}$$
$$+ \frac{2}{3a}e^{2u_1+2u_2}(\partial_{x_2}S + 2\partial_{x_2}T)\partial_{u_2},$$

$$(L_V R)(\partial_{x_0}, \partial_{x_2}, \partial_{x_1}) = \frac{2}{3a}\left(e^{-2u_1}\partial_{x_2}Y + e^{-2u_2}\partial_{x_1}Z\right)\partial_{x_0}$$
$$- \frac{2}{3a}\left(e^{2u_1+2u_2}\partial_{x_1}X + e^{-2u_1}\partial_{x_0}Y\right)\partial_{x_2},$$

$$(L_V R)(\partial_{x_0}, \partial_{x_2}, \partial_{x_2}) = \frac{4}{3a}e^{-2u_2}(\partial_{x_2}Z - T)\partial_{x_0} - \frac{2}{3a}\left(e^{-2u_2}\partial_{x_0}Z + e^{2u_1+2u_2}\partial_{x_2}X\right)\partial_{x_2}$$
$$- \frac{2}{3a}e^{-2u_2}(\partial_{x_0}S - \partial_{x_0}T)\partial_{u_1} - \frac{2}{a}e^{-2u_2}(\partial_{x_0}T)\partial_{u_2},$$

$$(L_V R)(\partial_{x_0}, \partial_{x_2}, \partial_{u_1}) = \left(\frac{2}{3a}e^{-2u_2}\partial_{u_1}Z - \partial_{x_2}S - \partial_{x_2}T\right)\partial_{x_0} - \frac{2}{3a}e^{2u_1+2u_2}(\partial_{u_1}X)\partial_{x_2},$$

$$(L_V R)(\partial_{x_0}, \partial_{x_2}, \partial_{u_2}) = \left(\frac{2}{3a}e^{-2u_2}\partial_{u_2}Z - \partial_{x_2}S - \partial_{x_2}T\right)\partial_{x_0}$$
$$+ \left(-\frac{2}{3a}e^{2u_1+2u_2}\partial_{u_2}X + \partial_{x_0}T\right)\partial_{x_2},$$

$$(L_V R)(\partial_{x_0}, \partial_{u_1}, \partial_{x_0}) = -\left(\frac{2}{3a}e^{2u_1+2u_2}(\partial_{u_1}X + \partial_{u_2}X) + \partial_{x_0}S + \partial_{x_0}T\right)\partial_{x_0}$$
$$- \frac{2}{3a}e^{2u_1+2u_2}(2\partial_{u_1}Y + \partial_{u_2}Y)\partial_{x_1} - \frac{2}{3a}e^{2u_1+2u_2}(2\partial_{u_1}Z + \partial_{u_2}Z)\partial_{x_2}$$
$$+ \frac{2}{3a}e^{2u_1+2u_2}(2\partial_{x_0}X + 2S + 2T + \partial_{u_1}T - \partial_{u_2}S)\partial_{u_1}$$
$$+ \frac{2}{3a}e^{2u_1+2u_2}(2\partial_{x_0}X + 2S + 2T + \partial_{u_1}S - \partial_{u_2}T)\partial_{u_2},$$

$$(L_V R)(\partial_{x_0}, \partial_{u_1}, \partial_{x_1}) = \left(\frac{2}{3a}e^{-2u_1}\partial_{u_1}Y - \partial_{x_1}S - \partial_{x_1}T\right)\partial_{x_0}$$
$$+ \frac{2}{3a}\left(2e^{-2u_1}\partial_{x_0}Y + e^{2u_1+2u_2}\partial_{x_1}X\right)\partial_{u_1} + \frac{2}{3a}\left(-e^{-2u_1}\partial_{x_0}Y + e^{2u_1+2u_2}\partial_{x_1}X\right)\partial_{u_2},$$

$$(L_V R)(\partial_{x_0}, \partial_{u_1}, \partial_{x_2}) = \frac{2}{3a}\left(e^{-2u_2}\partial_{u_1}Z - \partial_{x_2}S - \partial_{x_2}T\right)\partial_{x_0}$$
$$+ \frac{2}{3a}e^{2u_1+2u_2}(\partial_{x_2}X)\partial_{u_1} + \frac{2}{3}e^{2u_1+2u_2}(\partial_{x_2}X)\partial_{u_2}$$

$$(L_V R)(\partial_{x_0}, \partial_{u_1}, \partial_{u_1}) = -2(\partial_{u_1}S + \partial_{u_1}T)\partial_{x_0} + (\partial_{x_0}Z)\partial_{x_2}$$
$$+ \left(\frac{2}{3a}e^{2u_1+2u_2}\partial_{u_1}X + \partial_{x_0}S\right)\partial_{u_1} + \left(\frac{2}{3a}e^{2u_1+2u_2}\partial_{u_1}X + \partial_{x_0}T\right)\partial_{u_2},$$

$$(L_V R)(\partial_{x_0}, \partial_{u_1}, \partial_{u_2}) = -(\partial_{u_1}S + \partial_{u_1}T + \partial_{u_2}S + \partial_{u_2}T)\partial_{x_0} + (\partial_{x_0}Y)\partial_{x_1} + (\partial_{x_0}Z)\partial_{x_2}$$
$$+ \left(\frac{2}{3a}e^{2u_1+2u_2}\partial_{u_2}X + \partial_{x_0}S\right)\partial_{u_1} + \left(\frac{2}{3a}e^{2u_1+2u_2}\partial_{u_2}X + \partial_{x_0}T\right)\partial_{u_2},$$

$$
\begin{aligned}
\left(L_V R\right)\left(\partial_{x_0}, \partial_{u_2}, \partial_{x_0}\right) = & -\left(\frac{2}{3a} e^{2u_1+2u_2}\left(\partial_{u_1} X+\partial_{u_2} X\right)+\partial_{x_0} S+\partial_{x_0} T\right) \partial_{x_0} \\
& -\frac{2}{3a} e^{2u_1+2u_2}\left(\partial_{u_1} Y+2 \partial_{u_2} Y\right) \partial_{x_1}-\frac{2}{3a} e^{2u_1+2u_2}\left(\partial_{u_1} Z+2 \partial_{u_2} Z\right) \partial_{x_2} \\
& +\frac{2}{3a} e^{2u_1+2u_2}\left(2 \partial_{x_0} X+2S+2T+\partial_{u_2} T-\partial_{u_1} S\right) \partial_{u_1} \\
& +\frac{2}{3a} e^{2u_1+2u_2}\left(2 \partial_{x_0} X+2S+2T+\partial_{u_2} S-\partial_{u_1} T\right) \partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
\left(L_V R\right)\left(\partial_{x_0}, \partial_{u_2}, \partial_{x_1}\right) = & \left(\frac{2}{3a} e^{-2u_1} \partial_{u_2} Y-\partial_{x_1} S-\partial_{x_1} T\right) \partial_{x_0} \\
& +\frac{2}{3a} e^{2u_1+2u_2}\left(\partial_{u_1} X\right) \partial_{u_1}+\frac{2}{3a} e^{2u_1+2u_2}\left(\partial_{x_1} X\right) \partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
\left(L_V R\right)\left(\partial_{x_0}, \partial_{u_2}, \partial_{x_2}\right) = & \left(\frac{2}{3a} e^{-2u_2} \partial_{u_2} Z-\partial_{x_2} S-\partial_{x_2} T\right) \partial_{x_0} \\
& +\frac{2}{3a}\left(e^{2u_1+2u_2} \partial_{x_2} X-e^{-2u_2} \partial_{x_0} Z\right) \partial_{u_1}+\frac{2}{3a}\left(e^{2u_1+2u_2} \partial_{x_2} X+2 e^{-2u_2} \partial_{x_0} Z\right) \partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
\left(L_V R\right)\left(\partial_{x_0}, \partial_{u_2}, \partial_{u_1}\right) = & -\left(\partial_{u_1} S+\partial_{u_1} T+\partial_{u_2} S+\partial_{u_2} T\right) \partial_{x_0}+\left(\partial_{x_0} Y\right) \partial_{x_1}+\left(\partial_{x_0} Z\right) \partial_{x_2} \\
& +\left(\frac{2}{3a} e^{2u_1+2u_2} \partial_{u_1} X+\partial_{x_0} S\right) \partial_{u_1}+\left(\frac{2}{3a} e^{2u_1+2u_2} \partial_{u_1} X+\partial_{x_0} T\right) \partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
\left(L_V R\right)\left(\partial_{x_0}, \partial_{u_2}, \partial_{u_2}\right) = & -2\left(\partial_{u_2} S+\partial_{u_2} T\right) \partial_{x_0}+\left(\partial_{x_0} Y\right) \partial_{x_1} \\
& +\left(\frac{2}{3a} e^{2u_1+2u_2} \partial_{u_2} X+\partial_{x_0} S\right) \partial_{u_1}+\left(\frac{2}{3a} e^{2u_1+2u_2} \partial_{u_2} X+\partial_{x_0} T\right) \partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
\left(L_V R\right)\left(\partial_{x_1}, \partial_{x_2}, \partial_{x_0}\right) = & \frac{2}{3a}\left(e^{2u_1+2u_2} \partial_{x_2} X+e^{-2u_2} \partial_{x_0} Z\right) \partial_{x_1} \\
& -\frac{2}{3a}\left(e^{2u_1+2u_2} \partial_{x_1} X+e^{-2u_2} \partial_{x_0} Y\right) \partial_{x_2},
\end{aligned}
$$

$$
\begin{aligned}
\left(L_V R\right)\left(\partial_{x_1}, \partial_{x_2}, \partial_{x_1}\right) = & \frac{2}{3a}\left(e^{-2u_1} \partial_{x_2} Y+e^{-2u_2} \partial_{x_1} Z\right) \partial_{x_1}+\frac{4}{3a} e^{-2u_1}\left(S-\partial_{x_1} Y\right) \partial_{x_2} \\
& +\frac{2}{a} e^{-2u_1}\left(\partial_{x_2} S\right) \partial_{u_1}+\frac{2}{3a} e^{-2u_1}\left(\partial_{x_2} T-\partial_{x_2} S\right) \partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
\left(L_V R\right)\left(\partial_{x_1}, \partial_{x_2}, \partial_{x_2}\right) = & \frac{4}{3a} e^{-2u_2}\left(\partial_{x_2} Z-T\right) \partial_{x_1}-\frac{2}{3a}\left(e^{-2u_2} \partial_{x_1} Z+e^{-2u_1} \partial_{x_2} Y\right) \partial_{x_2} \\
& -\frac{2}{3a} e^{-2u_2}\left(\partial_{x_1} S-\partial_{x_1} T\right) \partial_{u_1}-\frac{2}{a} e^{-2u_2}\left(\partial_{x_1} T\right) \partial_{u_2},
\end{aligned}
$$

$$
\left(L_V R\right)\left(\partial_{x_1}, \partial_{x_2}, \partial_{u_1}\right) = \left(\frac{2}{3a} e^{-2u_2} \partial_{u_1} Z-\partial_{x_2} S\right) \partial_{x_1}-\frac{2}{3a} e^{-2u_1}\left(\partial_{u_1} Y\right) \partial_{x_2},
$$

$$
\left(L_V R\right)\left(\partial_{x_1}, \partial_{x_2}, \partial_{u_2}\right) = \frac{2}{3a} e^{-2u_2}\left(\partial_{u_2} Z\right) \partial_{x_1}+\left(-\frac{2}{3a} e^{-2u_1} \partial_{u_2} Y+\partial_{x_1} T\right) \partial_{x_2},
$$

$$
\begin{aligned}
\left(L_V R\right)\left(\partial_{x_1}, \partial_{u_1}, \partial_{x_0}\right) = & \left(\frac{2}{3a} e^{2u_1+2u_2} \partial_{u_1} X-\partial_{x_0} S\right) \partial_{x_1} \\
& +\frac{2}{3a}\left(e^{2u_1+2u_2} \partial_{x_1} X+2 e^{-2u_1} \partial_{x_0} Y\right) \partial_{u_1}+\frac{2}{3a}\left(e^{2u_1+2u_2} \partial_{x_1} X-e^{-2u_1} \partial_{x_0} Y\right) \partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
(L_V R)(\partial_{x_1},\partial_{u_1},\partial_{x_1}) &= \frac{2}{3a}e^{-2u_1}(\partial_{u_2}X - 3\partial_{u_1}X)\partial_{x_0} \\
&+ \left(\frac{2}{3a}e^{-2u_1}(\partial_{u_2}Y - 2\partial_{u_1}Y\ ) - \partial_{x_1}S\right)\partial_{x_1} \\
&+ \frac{2}{3a}e^{-2u_1}(\partial_{u_2}Z - 3\partial_{u_1}Z)\partial_{x_2} + \frac{2}{3a}e^{-2u_1}(4\partial_{x_1}Y - 4S - \partial_{u_2}S)\partial_{u_1} \\
&+ \frac{2}{3a}e^{-2u_1}(-2\partial_{x_1}Y + 2S + \partial_{u_2}T - 2\partial_{u_1}T - \partial_{u_1}S)\partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
(L_V R)(\partial_{x_1},\partial_{u_1},\partial_{x_2}) &= \left(\frac{2}{3a}e^{-2u_2}\partial_{u_1}Z - \partial_{x_2}S\right)\partial_{x_1} \\
&+ \frac{4}{3a}e^{-2u_1}(\partial_{x_2}Y)\partial_{u_1} - \frac{2}{3a}e^{-2u_1}(\partial_{x_2}Y)\partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
(L_V R)(\partial_{x_1},\partial_{u_1},\partial_{u_1}) &= -2(\partial_{u_1}S)\partial_{x_1} + (\partial_{x_1}Z)\partial_{x_2} \\
&+ \left(\partial_{x_1}S + \frac{4}{3a}e^{-2u_1}\partial_{u_1}Y\right)\partial_{u_1} + \left(\partial_{x_1}T - \frac{2}{3a}e^{-2u_1}\partial_{u_1}Y\right)\partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
(L_V R)(\partial_{x_1},\partial_{u_1},\partial_{u_2}) &= -(\partial_{x_1}X)\partial_{x_0} - (\partial_{u_2}S)\partial_{x_1} \\
&+ \frac{4}{3a}e^{-2u_1}(\partial_{u_2}Y)\partial_{u_1} - \frac{2}{3a}e^{-2u_1}(\partial_{u_2}Y)\partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
(L_V R)(\partial_{x_1},\partial_{u_2},\partial_{x_0}) &= \frac{2}{3a}e^{2u_1+2u_2}(\partial_{u_2}X)\partial_{x_1} \\
&+ \frac{2}{3a}e^{2u_1+2u_2}(\partial_{x_1}X)\partial_{u_1} + \frac{2}{3a}e^{2u_1+2u_2}(\partial_{x_1}X)\partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
(L_V R)(\partial_{x_1},\partial_{u_2},\partial_{x_1}) &= -\frac{2}{3a}e^{-2u_1}(\partial_{u_2}X)\partial_{x_0} - \frac{2}{3a}e^{-2u_1}(\partial_{u_2}Z)\partial_{x_2} \\
&+ \frac{4}{3a}e^{-2u_1}(\partial_{u_2}S)\partial_{u_1} - \frac{2}{3a}e^{-2u_1}(\partial_{u_2}S)\partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
(L_V R)(\partial_{x_1},\partial_{u_2},\partial_{x_2}) &= \frac{2}{3a}e^{-2u_2}(\partial_{u_2}Z)\partial_{x_1} \\
&- \frac{2}{3a}e^{-2u_2}(\partial_{x_1}Z)\partial_{u_1} + \frac{4}{3a}e^{-2u_2}(\partial_{x_1}Z)\partial_{u_2},
\end{aligned}
$$

$$
(L_V R)(\partial_{x_1},\partial_{u_2},\partial_{u_1}) = -(\partial_{x_1}X)\partial_{x_0} - (\partial_{u_2}S)\partial_{x_1},
$$

$$
(L_V R)(\partial_{x_1},\partial_{u_2},\partial_{u_2}) = -(\partial_{x_1}X)\partial_{x_0} - (\partial_{u_2}Z)\partial_{x_2},
$$

$$
\begin{aligned}
(L_V R)(\partial_{x_2},\partial_{u_1},\partial_{x_0}) &= \frac{2}{3a}e^{2u_1+2u_2}(\partial_{u_1}X)\partial_{x_2} \\
&+ \frac{2}{3a}e^{2u_1+2u_2}(\partial_{x_2}X)\partial_{u_1} + \frac{2}{3a}e^{2u_1+2u_2}(\partial_{x_2}X)\partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
(L_V R)(\partial_{x_2},\partial_{u_1},\partial_{x_1}) &= \frac{2}{3a}e^{-2u_1}(\partial_{u_1}Y)\partial_{x_2} \\
&+ \frac{4}{3a}e^{-2u_1}(\partial_{x_2}Y)\partial_{u_1} - \frac{2}{3a}e^{-2u_1}(\partial_{x_2}Y)\partial_{u_2},
\end{aligned}
$$

$$
\begin{aligned}
(L_V R)(\partial_{x_2},\partial_{u_1},\partial_{x_2}) &= -\frac{2}{3a}e^{-2u_2}(\partial_{u_1}X)\partial_{x_0} - \frac{2}{3a}e^{-2u_2}(\partial_{u_1}Y)\partial_{x_1} \\
&- \frac{2}{3a}e^{-2u_2}(\partial_{u_1}T)\partial_{u_1} + \frac{4}{3a}e^{-2u_2}(\partial_{u_1}T)\partial_{u_2},
\end{aligned}
$$

$$(L_V R)(\partial_{x_2}, \partial_{u_1}, \partial_{u_1}) = -(\partial_{x_2} X)\partial_{x_0} - (\partial_{x_2} Y)\partial_{x_1},$$

$$(L_V R)(\partial_{x_2}, \partial_{u_1}, \partial_{u_2}) = -(\partial_{x_2} X)\partial_{x_0} - (\partial_{u_1} Z)\partial_{x_2},$$

$$\begin{aligned}(L_V R)(\partial_{x_2}, \partial_{u_2}, \partial_{x_0}) &= \left(\frac{2}{3a}e^{2u_1+2u_2}\partial_{u_2} X - \partial_{x_0} T\right)\partial_{x_2} \\ &+ \frac{2}{3a}\left(e^{2u_1+2u_2}\partial_{x_2} X - e^{-2u_2}\partial_{x_0} Z\right)\partial_{u_1} + \frac{2}{3a}\left(e^{2u_1+2u_2}\partial_{x_2} X + 2e^{-2u_2}\partial_{x_0} Z\right)\partial_{u_2},\end{aligned}$$

$$\begin{aligned}(L_V R)(\partial_{x_2}, \partial_{u_2}, \partial_{x_1}) &= \left(\frac{2}{3a}e^{-2u_1}\partial_{u_2} Y - \partial_{x_1} T\right)\partial_{x_2} \\ &- \frac{2}{3a}e^{-2u_2}(\partial_{x_1} Z)\partial_{u_1} + \frac{4}{3a}e^{-2u_2}(\partial_{x_1} Z)\partial_{u_2},\end{aligned}$$

$$\begin{aligned}(L_V R)(\partial_{x_2}, \partial_{u_2}, \partial_{x_2}) &= \frac{2}{3a}e^{-2u_2}(\partial_{u_1} X - 3\partial_{u_2} X)\partial_{x_0} \\ &+ \frac{2}{3a}e^{-2u_2}(\partial_{u_1} Y - 3\partial_{u_2} Y\ )\partial_{x_1} + \left(\frac{2}{3a}e^{-2u_2}(\partial_{u_1} Z - 2\partial_{u_2} Z) - \partial_{x_2} T\right)\partial_{x_2} \\ &+ \frac{2}{3a}e^{-2u_2}(2T - 2\partial_{x_2} Z - \partial_{u_2} T + \partial_{u_1} S - 2\partial_{u_2} S)\partial_{u_1} \\ &+ \frac{2}{3a}e^{-2u_2}(4\partial_{x_2} Z - 4T + \partial_{u_1} T)\partial_{u_2},\end{aligned}$$

$$\begin{aligned}(L_V R)(\partial_{x_2}, \partial_{u_2}, \partial_{u_1}) &= -(\partial_{x_2} X)\partial_{x_0} - (\partial_{u_1} T)\partial_{x_2} \\ &- \frac{2}{3a}e^{-2u_2}(\partial_{u_1} Z)\partial_{u_1} + \frac{4}{3a}e^{-2u_2}(\partial_{u_1} Z)\partial_{u_2},\end{aligned}$$

$$\begin{aligned}(L_V R)(\partial_{x_2}, \partial_{u_2}, \partial_{u_2}) &= (\partial_{x_2} Y)\partial_{x_1} - 2(\partial_{u_2} T)\partial_{x_2} \\ &+ \left(\partial_{x_2} S - \frac{2}{3a}e^{-2u_2}\partial_{u_2} Z\right)\partial_{u_1} + \left(\partial_{x_2} T + \frac{4}{3a}e^{-2u_2}\partial_{u_2} Z\right)\partial_{u_2},\end{aligned}$$

$$\begin{aligned}(L_V R)(\partial_{u_1}, \partial_{u_2}, \partial_{x_0}) &= \frac{2}{3a}e^{2u_1+2u_2}(\partial_{u_1} X - \partial_{u_2} X)\partial_{u_1} \\ &+ \frac{2}{3a}e^{2u_1+2u_2}(\partial_{u_1} X - \partial_{u_2} X)\partial_{u_2},\end{aligned}$$

$$(L_V R)(\partial_{u_1}, \partial_{u_2}, \partial_{x_1}) = -\frac{4}{3a}e^{-2u_1}(\partial_{u_2} Y)\partial_{u_1} + \frac{2}{3a}e^{-2u_1}(\partial_{u_2} Y)\partial_{u_2},$$

$$(L_V R)(\partial_{u_1}, \partial_{u_2}, \partial_{x_2}) = -\frac{2}{3a}e^{-2u_2}(\partial_{u_1} Z)\partial_{u_1} + \frac{4}{3a}e^{-2u_2}(\partial_{u_1} Z)\partial_{u_2},$$

$$(L_V R)(\partial_{u_1}, \partial_{u_2}, \partial_{u_1}) = (\partial_{u_2} X - \partial_{u_1} X)\partial_{x_0} + (\partial_{u_2} Y)\partial_{x_1},$$

$$(L_V R)(\partial_{u_1}, \partial_{u_2}, \partial_{u_2}) = (\partial_{u_2} X - \partial_{u_1} X)\partial_{x_0} - (\partial_{u_1} Z)\partial_{x_2}.$$

Finally, we study matter collineations, using equations (1), (4) and (5).
We determine the tensor field $\mathcal{T} = \varrho - \frac{\tau}{2}g$ as follows

$$\mathcal{T} = \begin{pmatrix} \frac{2}{a}e^{2(u_1+u_2)} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{a}e^{-2u_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{a}e^{-2u_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Lie derivative of the above tensor field $\mathcal{T}$ is then given by

$$\begin{aligned}
(L_V\mathcal{T})(\partial_{x_0},\partial_{x_0}) &= \frac{4}{a}e^{2(u_1+u_2)}\left(S+T+\partial_{x_0}X\right),\\
(L_V\mathcal{T})(\partial_{x_0},\partial_{x_1}) &= \frac{2}{a}\left(e^{-2u_1}\partial_{x_0}Y+e^{2(u_1+u_2)}\partial_{x_1}X\right),\\
(L_V\mathcal{T})(\partial_{x_0},\partial_{x_2}) &= \frac{2}{a}\left(e^{-2u_2}\partial_{x_0}Z+e^{2(u_1+u_2)}\partial_{x_2}X\right),\\
(L_V\mathcal{T})(\partial_{x_0},\partial_{u_1}) &= \frac{2}{a}e^{2(u_1+u_2)}\partial_{u_1}X,\\
(L_V\mathcal{T})(\partial_{x_0},\partial_{u_2}) &= \frac{2}{a}e^{2(u_1+u_2)}\partial_{u_2}X,\\
(L_V\mathcal{T})(\partial_{x_1},\partial_{x_1}) &= \frac{4}{a}e^{-2u_1}(-S+\partial_{x_1}Y),\\
(L_V\mathcal{T})(\partial_{x_1},\partial_{x_2}) &= \frac{2}{a}\left(e^{-2u_2}\partial_{x_1}Z+e^{-2u_1}\partial_{x_2}Y\right),\\
(L_V\mathcal{T})(\partial_{x_1},\partial_{u_1}) &= \frac{2}{a}e^{-2u_1}\partial_{u_1}Y,\\
(L_V\mathcal{T})(\partial_{x_1},\partial_{u_2}) &= \frac{2}{a}e^{-2u_1}\partial_{u_2}Y,\\
(L_V\mathcal{T})(\partial_{x_2},\partial_{x_2}) &= \frac{4}{a}e^{-2u_1}(-T+\partial_{x_2}Z),\\
(L_V\mathcal{T})(\partial_{x_2},\partial_{u_1}) &= \frac{2}{a}e^{-2u_2}\partial_{u_1}Z,\\
(L_V\mathcal{T})(\partial_{x_2},\partial_{u_2}) &= \frac{2}{a}e^{-2u_2}\partial_{u_2}Z,\\
(L_V\mathcal{T})\left(\partial_{u_\alpha},\partial_{u_\beta}\right) &= 0 \quad \text{for all } \alpha,\beta=1,2.
\end{aligned}$$

Ricci, curvature and Matter collineations are then calculated by solving the system of PDE obtained by requiring that all the above components of $L_V\varrho, L_VR$ and $L_V\mathcal{T}$ vanish, respectively. A very long computations lead to prove the following:

Theorem 4.1. *Let V be an arbitrary vector field on the Riemannian Lie group (G_2,g) where g is given by* (1)*. Then,*

- *V is a Ricci collineation if and only if*

  $$V = X\partial_{x_0}+Y\partial_{x_1}+Z\partial_{x_2}+\left(c_1(u_1+2u_2)+c_2\right)\partial_{u_1}+\left(-c_1(2u_1+u_2)+c_3\right)\partial_{u_2},$$

  where $c_1,c_2,c_3\in\mathbb{R}$ and X,Y and Z are smooth functions.
- *V is a curvature collineation if and only if V is matter collineation if and only if V is Killing.*

Remark 4.2. The situation is different as concern distinct collineations. In fact, note that the space of curvature collineations is not necessarily finite dimensional. This is indeed the case when considering Ricci collineations which are shown to be of infinite dimension. In contrast, curvature collineations are shown to be five-dimensional, they reduce to Killing vector fields and the same is true for matter-collineations. Therefore, besides Ricci-collineations, all remaining symmetries reduce to Killing vector fields.

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