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Subsemigroups of Nilpotent Lie Groups

Herbert Abels and Ernest B. Vinberg

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Abstract. For a closed subsemigroup S of a simply connected nilpotent Lie group G, we prove that either S is a subgroup, or there is an epimorphism $f\colon G \to \mathbb{R}$ such that $f(s) \geq 0$ for all $s \in S$.

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1. Introduction

A subsemigroup S of a topological group G is called *cocompact* if there exists a compact subset K of G such that $G = SK$. If S is a closed subgroup, this means that the quotient G/S is compact. Note also that a subsemigroup of a vector group is cocompact if and only if it is not contained in a half-space (see Proposition 2.1).

Obviously, the image of a cocompact subsemigroup under an epimorphism of topological groups is also cocompact.

Let now G be a simply connected nilpotent Lie group, and let

$$G^{ab} = G/(G,G)$$

be its abelianization (which is a vector group). The following theorem is known (A.I. Maltsev [4, Theorem 1]).

Theorem 1.1. *If the image of a subgroup H of G is cocompact in G^{ab}, then the subgroup H itself is cocompact in G.*

The main result of the present paper is

Theorem 1.2. *If the image of a subsemigroup S of G is cocompact in G^{ab}, then the subsemigroup S itself is cocompact in G, and its closure is a subgroup.*

The second assertion of the theorem follows from the first one and a general theorem on cocompact subsemigroups of topological groups (Corollary 3.3).

Corollary 1.3. *If the image of a subsemigroup S of G is dense in G^{ab}, then the subsemigroup S itself is dense in G.*

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Corollary 1.4. *If a closed subsemigroup S of G contains a closed subgroup H such that the quotient S/H is compact, then S is a subgroup.*

Proofs of the main theorem and of the corollaries are given in Section 5. In Sections 2–4 we obtain some preliminary results on subsemigroups of topological and Lie groups, which are also of independent interest. In particular, in Section 3 we prove that every closed cocompact subsemigroup of a topological group is a subgroup.

Conventions. In this paper we understand the term “semigroup” as a semigroup with identity element, and the term “neighborhood” as an open neighborhood. All topological spaces are assumed to be Hausdorff.

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2. Subsemigroups of vector groups

Let V be a real vector space considered as a Lie group. By a half-space in V we mean a subset of the form $\{v \in V; l(v) \geq 0\}$ for some non-zero linear function l on V.

Proposition 2.1. *For a subsemigroup S of V, the following conditions are equivalent:*

(1) *S is cocompact;*

(2) *S is not contained in a half-space;*

(3) *S contains vectors $X_0, X_1, \ldots, X_n$, which span the vector space V and are linearly dependent with positive coefficients.*

In particular, any cocompact convex cone coincides with V.

Proof. Let S be contained in the half-space given by the inequality $l \geq 0$, where l is a non-zero linear function on V. If $V = S + K$, where K is a compact subset of V, then $l \geq -c$ for some $c > 0$ on the whole space V, which is impossible. Hence, (1) implies (2).

Consider now the convex cone C generated by S, i.e. the set of (finite) non-negative linear combinations of vectors of S. By the Hahn–Banach theorem, its closure $\overline{C}$ is the intersection of all the half-spaces containing S. In particular, if there are no such half-spaces, then $\overline{C} = V$, which implies that already $C = V$, since every convex set contains the interior of its closure. In particular, if S itself is a convex cone, then $S = V$.

Clearly, the set of non-negative rational linear combinations of vectors of S is dense in C. So, if $C = V$, we can find among such linear combinations vectors $X_0, X_1, \ldots, X_n$, which span the space V and are linearly dependent with positive (not necessarily rational) coefficients. Multiplying these vectors by suitable natural numbers, we can achieve that $X_0, X_1, \ldots, X_n \in S$. Thus, (2) implies (3).

Finally, let S satisfy condition (3). Then any vector $v \in V$ is a linear combination of $X_0, X_1, \ldots, X_n$, whose coefficients can be made non-negative by adding a suitable multiple of the positive linear dependence of these vectors. It follows that the vector v can be represented as a sum of a linear combination of $X_0, X_1, \ldots, X_n$ with non-negative integer coefficients, which belongs to S, and a linear combination with coefficients in the segment $[0,1]$. This means that $V = S + K$, where

$$K = \{c_0X_0 + c_1X_1 + \cdots + c_nX_n; 0 \leq c_0, c_1, \ldots, c_n \leq 1\}.$$

Thus, (3) implies (1). ■

3. Subsemigroups of topological groups

Let a topological semigroup S act on a topological space X. This action is called *cocompact*, if there exists a compact subset K of X such that $X = SK$, and *proper* if for any compact subsets K_1, K_2 of X the subset

$$S(K_1, K_2) := \{s \in S : sK_1 \cap K_2 \neq \emptyset\}$$

is compact. (In fact it suffices to require the latter condition to hold for $K_1 = K_2$.) In particular, if the action is proper, the stabilizer of any point of X is compact.

Define S-orbits as the subsets of the form Sx with $x \in X$. Note that if $X = SK$ for some $K \subset X$, then each S-orbit Sx is contained in an S-orbit of the form Sk with $k \in K$. Namely, if $x = sk\,(s \in S, k \in K)$, then $Sx \subset Sk$.

Proposition 3.1. *If an action of a topological semigroup S on a topological space X is cocompact and proper, then there is a maximal (by inclusion) S-orbit.*

Proof. Let $K \subset X$ be a compact subset such that $X = SK$. We will prove that there is a maximal subset among all the S-orbits of the form Sk with $k \in K$. By the preceding remark, it will automatically be maximal among all S-orbits.

For $k \in K$, define $T(k) := \{k' \in K : Sk \subset Sk'\} = \{k' \in K : k \in Sk'\}$.

Clearly, $T(k) \ni k$ and, if $Sk_1 \subset Sk_2$, then $T(k_1) \supset T(k_2)$. Moreover, the subset $T(k) \subset K$ is compact. Indeed, if $k = sk'\,(s \in S)$ then s belongs to the compact subset $S' = S(\{k'\}, K) \subset S$. The set of pairs $(s, k') \in S' \times K$ with $sk' = k$ is closed in $S' \times K$, and hence compact. Since the set $T(k)$ is the projection of the latter set on the second factor of the direct product, it is also compact.

For any linearly ordered family $\{Sk_i\,;\, i \in I\}$ the family $\{T(k_i)\,;\, i \in I\}$ of non-empty compact subsets of K is also linearly ordered and hence has a non-empty intersection. If k_I is an element of this intersection, then $Sk_i \subset Sk_I$ for any $i \in I$. By Zorn's lemma, this implies that the set of all S-orbits of the form Sk with $k \in K$ has a maximal element ■

Theorem 3.2. *Let a topological group G act on a topological space X, and let S be a subsemigroup of G. Suppose the induced action of S on X is proper and cocompact. Then S is a subgroup of G.*

Proof. By Proposition 3.1, there is a maximal S-orbit, say, Sx_0. For any $s \in S$, we have $Sx_0 \subset Ss^{-1}x_0$ and, hence, $Sx_0 = Ss^{-1}x_0$. This means that $s^{-1}x_0 \in Sx_0$, that is, $s^{-1}x_0 = tx_0$ for some $t \in S$. It follows that st belongs to the stabilizer of x_0 in S, which is compact. It is well known that every compact subsemigroup of a topological group is a subgroup (see, for example, [2, Proposition V.0.17]). Hence, the element st is invertible in S and, all the more, the element s itself is invertible. ∎

Corollary 3.3. *Every closed cocompact subsemigroup S of a topological group G is a subgroup.*

Proof. The proof is obtained by applying the theorem to the action of the group G on itself by left multiplication. The properness of the induced action of S on G follows from the equality $S(K_1, K_2) = K_2K_1^{-1} \cap S$ for $K_1, K_2 \subset G$. ∎

Corollary 3.4. *Let S be a cocompact subsemigroup of a connected topological group G. If S contains interior points, then $S = G$.*

Proof. By the preceding corollary, the closure $\overline{S}$ of S is a subgroup of G and contains interior points, hence is open. Since G is connected, we have $\overline{S} = G$. Now let U be an open subset of S. Since S is dense in G, it contains a point $s \in U^{-1}$. Then $sU \subset S$ is a neighborhood of e in G contained in S. It follows that $S = G$. ∎

The following local check for cocompactness is useful.

Lemma 3.5. *Suppose G is a connected topological group. Let S be a subsemigroup of G and let K be a non-empty compact subset of G. If the set SK contains a neighborhood of K, then $SK = G$.*

Proof. Since $SK \supset s(SK)$ contains a neighborhood of sK for every $s \in S$, the set SK is open in G. We shall show that it is closed, which will imply the lemma. There is a neighborhood W of e in G such that $KW \subset SK$. Let x belong to the closure of SK. We have to show that $x \in SK$. Every neighborhood of x intersects sK for some $s \in S$. In particular, $xW^{-1} \cap sK \neq \emptyset$ for some $s \in S$, whence $x \in sKW \subset sSK \subset SK$, as claimed. ∎

Proposition 3.6. *Every cocompact subsemigroup S of a connected locally compact topological group G contains a finitely generated cocompact subsemigroup. Moreover, there are a compact subset K of G and a finite subset Σ of S such that the set ΣK contains a neighborhood of K (and hence the subsemigroup generated by Σ is cocompact).*

Proof. Let S be a subsemigroup of G such that $SK_0 = G$ for some compact subset K_0 of G. Take any relatively compact neighborhood W of e in G. Then $U = K_0W$ is an open subset of G with compact closure $K = K_0\overline{W}$. Since $SU = G$ covers K, there exists a finite subset Σ of S such that ΣU covers K. The subsemigroup of G generated by Σ fulfills the hypotheses of the preceding lemma and hence is cocompact. ∎

Finally, we have the following stability result.

Theorem 3.7. *Let G be a connected locally compact topological group. Suppose the subsemigroup S of G is cocompact and generated by finitely many elements, say $\sigma_1, \ldots, \sigma_n$. Then the subsemigroup generated by any elements $\sigma'_1, \ldots, \sigma'_n$ sufficiently close to $\sigma_1, \ldots, \sigma_n$ is also cocompact.*

Proof. By Proposition 3.6 there exist a compact subset K_0 of G and a finite subset Σ of S such that

$$\Sigma K_0 \supset K_0 W_0$$

for some neighborhood W_0 of e in G. Since the elements of Σ are products of the generators $\sigma_1, \dots, \sigma_n$ and multiplication in G is continuous, we may assume from the very beginning that

$$\Sigma = \{\sigma_1, \dots, \sigma_n\}.$$

Under this condition, we will prove that there exist a compact set $K \supset K_0$ and a neighborhood $W \subset W_0$ of e such that

$$\Sigma' K \supset KW$$

for any subset $\Sigma' = \{\sigma'_1, \dots, \sigma'_n\}$ with $\sigma'_1, \dots, \sigma'_n$ sufficiently close to $\sigma_1, \dots, \sigma_n$.

Take a relatively compact symmetric neighborhood W of e such that $\overline{W}^2 \subset W_0$ and set $K = K_0\overline{W}$. There exists a symmetric neighborhood U of e contained in W such that $UK_0 \subset K$. Suppose now that

$$\sigma'_i = \sigma_i u_i \quad (i = 1, \dots, n)$$

with $u_i \in U$. Then for any $k \in K$, $w \in W$ and some i we have

$$kw = k_0 w' w = \sigma_i k'_0 = \sigma'_i u_i^{-1} k'_0 \in \sigma'_i K,$$

where $w' \in \overline{W}$ and $k_0, k'_0 \in K_0$. ■

4. Subsemigroups of Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$ and $g \in G$ denote by gX the tangent vector of G at g obtained from X by the left translation by g.

Let now S be a subsemigroup of G. Denote by $\Sigma(S)$ the set of all vectors $X \in \mathfrak{g}$, for which there exists an element $g \in G$ such that gX is the tangent vector of some smooth curve in G passing through g and lying in S.

Theorem 4.1. ([1, Section 3, Corollary 5]) *If the set $\Sigma(S)$ generates the Lie algebra $\mathfrak{g}$, then S contains interior points.*

We only use the following well known corollary, see [2, Chapter V, Section 1].

Corollary 4.2. *If a subset Σ of $\mathfrak{g}$ generates the Lie algebra $\mathfrak{g}$, then the subsemigroup S of G generated by the one-parameter subsemigroups*

$$X^+ = \{\exp tX : t \geq 0\},\ X \in \Sigma,$$

contains interior points.

The following lemma is used in our proof of the next theorem.

Lemma 4.3. *Let H be the 3-dimensional Heisenberg group with center Z, and let $S \subset H$ be a subsemigroup. Suppose that S projects onto H/Z and contains an open subset of Z. Then $S = H$.*

Proof. It suffices to prove that $S \supset Z$. Take any non-commuting elements $h_1, h_2 \in H$. Since S projects onto H/Z, there exist elements $z_1, z_2, w_1, w_2 \in Z$ such that S contains the elements

$$h_1 z_1,\ h_2 z_2,\ h_1^{-1} w_1,\ h_2^{-1} w_2.$$

Then for any $n > 0$ and $z = z_1 z_2 w_1 w_2 \in Z$ the following element is contained in S:

$$(h_1 z_1)^n (h_2 z_2)^n (h_1^{-1} w_1)^n (h_2^{-1} w_2)^n = (h_1^n, h_2^n) z^n = (h_1, h_2)^{n^2} z^n,$$

(Here (x, y) stands for $xyx^{-1}y^{-1}$.) If we consider the group Z as the real line, we can say that for large n the sign of the element $(h_1, h_2)^{n^2} z^n$ coincides with the sign of (h_1, h_2). Permuting h_1 and h_2, we can obtain elements of the opposite sign in $S \cap Z$. Thus, the subsemigroup $S \cap Z$ is cocompact in Z, and even coincides with Z, by Corollary 3.4. ∎

Theorem 4.4. *Let G be a simply connected nilpotent Lie group and let Σ be a subset of $\mathfrak{g}$, whose image in $\mathfrak{g}^{ab}$ is not contained in a half-space. Then the subsemigroup S generated by the one-parameter subsemigroups X^+ with $X \in \Sigma$ coincides with G.*

Proof. Note that the image of Σ in $\mathfrak{g}^{ab}$ spans $\mathfrak{g}^{ab}$ and hence Σ generates the Lie algebra $\mathfrak{g}$. It follows by Corollary 4.2 that the subsemigroup $S \subset G$ contains interior points. If it does not coincide with G, then by a result of J. Lawson [3, Proposition 5.2] (see also [2, Proposition V.5.14]) it is contained in some maximal subsemigroup $M \subset G$. It follows from [3, Corollary 11.2] (see also [2, Theorem V.5.29]) that the image of M in the vector space G^{ab} is a half-space, which contradicts the hypothesis of our theorem. Therefore, $S = G$.

An alternative proof can be given by induction on $\dim G$ with the help of Lemma 4.3. For G abelian S is a convex cone and coincides with G by Proposition 2.1. Suppose now that the nilpotency class of G is $d > 1$, and let

$$\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_{d-1} \supset \mathfrak{g}_d$$

be the descending central series of the Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}_d = [\mathfrak{g}_1, \mathfrak{g}_{d-1}]$, there is a non-zero element of $\mathfrak{g}_d$, which is the commutator of two elements of $\mathfrak{g}$. The corresponding central one-parameter subgroup Z of G can be included into a Heisenberg subgroup H of G. By the induction hypothesis applied to G/Z, the subsemigroup S of G projects onto G/Z. It follows that the subsemigroup $S \cap H$ of H projects onto H/Z.

Finally, we claim that the subsemigroup $S \cap Z$ contains an open subset of Z. Indeed, let s be an interior point of S. There is an element $t \in S$ such that $st \in Z$ Then st is an interior point of S contained in Z, hence $S \cap Z$ contains an open subset of Z. By Lemma 4.3 applied to $S \cap H$, we can conclude that $S \supset H \supset Z$. Together with the induction hypothesis, this means that $S = G$. ∎

5. Proof of the main theorem

Let S be a subsemigroup of a simply connected nilpotent Lie group G such that the image of S in G^{ab} is cocompact. By Corollary 3.3, it suffices to show that S is cocompact.

By Proposition 2.1, there exist elements $X_0, X_1, \dots, X_n \in \log S \subset \mathfrak{g}$ such that the images of $X_1, \dots, X_n$ constitute a basis of $\mathfrak{g}^{ab}$, while

$$X_0 = c_1X_1 + \cdots + c_nX_n + Y \quad \text{with} \quad c_1, \dots c_n < 0,\, Y \in [\mathfrak{g}, \mathfrak{g}].$$

We are going to prove that already the subsemigroup S_0 of S generated by the elements

$$x_0 = \exp X_0,\, x_1 = \exp X_1, \dots,\, x_n = \exp X_n$$

is cocompact.

Let d be the nilpotency class of $\mathfrak{g}$. Consider the free nilpotent Lie algebra $\tilde{\mathfrak{g}}$ of nilpotency class d on the generators

$$\widetilde{X}_1, \dots, \widetilde{X}_n,$$

together with the epimorphism $\phi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ taking $\widetilde{X}_i$ to X_i for $i = 1, \dots, n$, and define $\widetilde{X}_0 \in \tilde{\mathfrak{g}}$ as any preimage of X_0. Let $\widetilde{G}$ be the simply connected Lie group with Lie algebra $\tilde{\mathfrak{g}}$. There is an epimorphism $\Phi : \widetilde{G} \to G$ with differential ϕ, and it suffices to prove that the subsemigroup $\widetilde{S}_0$ of $\widetilde{G}$ generated by the elements

$$\tilde{x}_0 = \exp \widetilde{X}_0,\, \tilde{x}_1 = \exp \widetilde{X}_1, \dots,\, \tilde{x}_n = \exp \widetilde{X}_n,$$

is cocompact.

Changing the notation, assume from the very beginning that the algebra $\mathfrak{g}$ itself is the free nilpotent Lie algebra of nilpotency class d on the generators $X_1, \dots, X_n$. In this case, it has a natural grading by the degrees in the generators.

Set
$$X_0' = c_1X_1 + \dots c_nX_n.$$

By Theorem 4.4, the subsemigroup of G generated by the one-parameter subsemigroups $X_0'^{+}, X_1^+, \dots, X_n^+$ coincides with G. By Proposition 3.6, there is a finite subset

$$\Sigma \subset X_0'^{+} \cup X_1^+ \cup \dots \cup X_n^+$$

such that the subsemigroup of G generated by Σ is cocompact.

For any $c > 0$, there is an automorphism α_c of the Lie algebra $\mathfrak{g}$ multiplying each homogeneous element of degree k by c^k. Denote by A_c the corresponding automorphism of the group G. Note that it leaves invariant every one-parameter subsemigroup X^+ whose tangent vector X is homogeneous of degree 1, multiplying the parameter by c.

Let now $\sigma = \exp tX_i \in \Sigma$ for some $i \in \{1, \dots, n\}$. Taking a sufficiently small $c > 0$ and a suitable natural number m, the element $A_c(x_i)^m = \exp mcX_i$ can be made arbitrarily close to σ. Similarly, the element of the form $A_c(x_0)^m$ can be made arbitrarily close to any given $\sigma = \exp tX_0' \in \Sigma$, taking into account that all the terms of degree > 1 of $m\alpha_c(X_0)$ tend to 0, when c tends to 0 and mc is bounded from above. Note that the number c can be chosen one and the same (sufficiently small) for all $\sigma \in \Sigma$. By Theorem 3.7, we can conclude that the automorphism A_c takes the subsemigroup S' of S generated by some powers of $x_0, x_1, \dots, x_n$ to a cocompact subsemigroup. Thus, the subsemigroup S itself is cocompact.

Proof of Corollary 1.3. Let S be a subsemigroup of G whose image in G^{ab} is dense. Then $H := \overline{S}$ is a closed subgroup of G whose image in G^{ab} is dense. Then $H = G$ as is seen by induction on the nilpotency class d of G using the map $G/G_d \times G_{d-1}/G_d \to G_d$ induced by the commutator map.

Proof of Corollary 1.4. If S is not a subgroup, then by Theorem 1.2 its image in G^{ab} is not cocompact, i.e., is contained in a half-space (see Proposition 2.1). This means that there is an epimorphism $f : G \to \mathbb{R}$ such that $f(S)$ is contained in the ray $I = [0, +\infty)$. The image $f(H)$ of H is a subgroup, hence trivial. The compactness of S/H then implies that $f(S)$ is also trivial. Thus, S is contained in the kernel of f, which is a simply connected nilpotent Lie subgroup of codimension 1. Now the corollary follows by induction on $\dim G$.

Remark 5.1. More generally, let a closed subsemigroup S of a topological group G contain a closed subgroup H such that S/H is compact. Is then S a subgroup of G? The case, when H is a normal subgroup of G, reduces to the case, when H is trivial, and is well known. The case when G is a connected nilpotent Lie group follows from our results. The case, when G is a semidirect product of a connected solvable Lie group and a compact Lie group, and S has interior points, follows from [3, Corollary 11.2]. We do not know about the general case.

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Herbert Abels
Fakultät für Mathematik
Universität Bielefeld
33501 Bielefeld
Germany
abels@math.uni-bielefeld.de

Ernest B. Vinberg
Chair of Algebra
Dept. of Mechanics and Mathematics
Moscow State University
Moscow 119991, Russia
evinberg@gmail.com

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