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Wells Exact Sequence and Automorphisms of Extensions of Lie Superalgebras

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Abstract. Let $0 \to \mathfrak{a} \to \mathfrak{e} \to \mathfrak{g} \to 0$ be an abelian extension of Lie superalgebras. In this article, corresponding to this extension we construct two exact sequences connecting the various automorphism groups and the 0-th homogeneous part of the second cohomology group, $H^2(\mathfrak{g}, \mathfrak{a})_0$. These exact sequences constitute an analogue of the well-known Wells exact sequence for group extensions. From this it follows that the obstruction for a pair of automorphism $(\phi, \psi) \in Aut(\mathfrak{a}) \times Aut(\mathfrak{g})$ to be induced from an automorphism in $Aut_{\mathfrak{a}}(\mathfrak{e})$ lies in $H^2(\mathfrak{g}, \mathfrak{a})_0$. Then we consider the family of Heisenberg Lie superalgebras and show that not all pairs are inducible in this family. We also give some necessary and sufficient conditions for inducibility of pairs arising in this family.

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1. Introduction

The cohomology theory of Lie superalgebras has been considered by several authors and it has found many important applications in physics as well as in mathematics. In this article, we connect the cohomology theory of Lie superalgebras to the well-known problem of inducibility of a pair of automorphisms in an abelian extension of Lie superalgebra.

Let $0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$ be an abelian extension of Lie superalgebras where i is just an inclusion map. Then any $\gamma \in Aut(\mathfrak{e})$ which takes $\mathfrak{a}$ onto $\mathfrak{a}$ induces a pair of automorphisms $(\phi, \psi) \in Aut(\mathfrak{a}) \times Aut(\mathfrak{g})$. We refer to Section 2 to see how (ϕ, ψ) is induced. We call such a pair to be inducible. But it is not clear whether for every pair (ϕ, ψ) in $Aut(\mathfrak{a}) \times Aut(\mathfrak{g})$ there is some $\gamma \in Aut(\mathfrak{e})$ from which the pair is induced. In the present article we consider this problem in detail.

Problem 1.1. Let $0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$ be an abelian extension of Lie superalgebras. Under what conditions a pair $(\phi, \psi) \in Aut(\mathfrak{a}) \times Aut(\mathfrak{g})$ is inducible?

A similar problem in group theory was considered in [18] for not necessarily abelian extensions of groups and in that paper an important exact sequence involving automorphism of group extensions and cohomology of groups was obtained. Since then

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the problem has drawn much attention in group theory and the exact sequence is now popularly known as Wells exact sequence. In recent years, several authors have studied the problem further, see [10], [11], [14], [15] and references therein. In [14], two more exact sequences were obtained along with the Wells exact sequence itself.

But the Wells map, a set map arising in the Wells sequence, was not well understood in full generality until [11] was published. In [11] it was proved that the set map is actually a derivation of the corresponding group action, see Proposition 5.5 for more details. In the present article, we establish all the above mentioned exact sequences for extensions of Lie superalgebras and apply them to study the above problem. It is worth mentioning that though one of the main theorems in [14], which is Theorem 2 was particularly for central extensions, here it is proved for the more general class of abelian extensions. Some of the above results were also obtained in the case of Lie algebras in [2]. In spite of that, we pay our attention to the case of Lie superalgebras separately as neither the structure theory nor the representation theory of Lie superalgebras is very similar to that of Lie algebras. And to the best of our knowledge the above problem for Lie superalgebras has not been yet considered in the literature.

The paper is organized as follows. In Section 2 we present the main definitions including those required for the formulation of the problem along with a couple of elementary lemmas, these should be kept in mind throughout the article. Section 3 contains the first main theorem and its proof proving some necessary and sufficient conditions for a pair to be inducible. The main result of Section 4 is preparation and construction of two important exact sequences which help in constructing the famous Wells exact sequence. In Section 5 we establish the Wells exact sequence which is Theorem 5.2 and discuss some immediate corollaries of this theorem. We also prove that the Wells map is actually a derivation for the corresponding action in the present Lie superalgebra case. Section 6 mainly deals with the case of Heisenberg Lie superalgebras. In this case we show that not all compatible pairs are inducible and we obtain some necessary and sufficient conditions for inducibilty of a pair. At the end in the appendix we give a detailed proof of a famous result connecting extensions and cohomology of Lie superalgebra as it is not readily available in the literature.

2. Preliminaries

In this section, we give some preliminaries on Lie superalgebras and modules over them. We also describe some notation and facts which will be used later in this paper Throughout the article $\mathbb{Z}_2$ will denote the abelian group of two elements, isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Let V be a vector space over a field κ of arbitrary characteristic and not necessarily algebraically closed (except in Section 6). A $\mathbb{Z}_2$-grading on V is a way of expressing V as a direct sum $V = V_0 \oplus V_1$. A vector space V together with a $\mathbb{Z}_2$-grading is called a *vector superspace* (or just a *superspace*). The non-zero elements of $\cup_i V_i$ are called homogeneous elements of V. For a homogeneous element $x \in V_i$, degree(or parity) of x is defined to be i and is denoted by $|x|$.

A vector space homomorphism $\phi\colon V \to W$ between two superspaces $V(= V_0 \oplus V_1)$ and $W(= W_0 \oplus W_1)$ is called *homogeneous of degree* $k \in \mathbb{Z}_2$ if $\phi(V_i) \subseteq W_{k+i}$ $\forall i \in \mathbb{Z}_2$

and the degree is denoted again by $|\phi| = k$. In what follows, if we talk about a homogeneous linear map without mentioning its degree, we mean a homogeneous linear map of degree 0.

A vector space V together with a bilinear map $[-,-]\colon V \times V \to V$ is called an *algebra*. The algebra $(V, [-,-])$ is called $\mathbb{Z}_2$-graded if $[V_i, V_j] \subseteq V_{i+j}$ for $i, j \in \mathbb{Z}_2$.

Definition 2.1. (Lie superalgebras) Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $\mathbb{Z}_2$-graded algebra with respect to the bilinear map $[-,-]$. Then $(\mathfrak{g}, [-,-])$ is called a *Lie superalgebra* if the following two conditions are satisfied:

(1) $[x, y] = -(-1)^{|x||y|}[y, x]$ for all homogeneous $x, y \in \mathfrak{g}$ (Super Skew-Symmetry).

(2) $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$ for all homogeneous $x, y, z \in \mathfrak{g}$ (Super Jacobi Identity).

We denote the Lie superalgebra $(\mathfrak{g}, [-,-])$ simply by $\mathfrak{g}$. ■

A homomorphism between two Lie superalgebras $\mathfrak{g}$ and $\mathfrak{h}$ is a homogeneous linear map $\phi\colon \mathfrak{g} \to \mathfrak{h}$ of degree 0 such that $\phi([x, y]) = [\phi(x), \phi(y)]$ $\forall x, y \in \mathfrak{g}$.

Definition 2.2. (Modules) Let $\mathfrak{g}$ be a Lie superalgebra and V a vector superspace over the same field κ. Then V is called a *module* (or a *supermodule*) over $\mathfrak{g}$ if there is a bilinear map $(-,-)\colon \mathfrak{g} \times V \to V$ satisfying the following conditions:

(1) $(\mathfrak{g}_i, V_j) \subseteq V_{i+j}$ $\forall i, j \in \mathbb{Z}_2$.

(2) $([x, y], v) = (x, (y, v)) - (-1)^{|x||y|}(y, (x, v))$ $\forall\, v \in V$ and homogeneous $x, y \in \mathfrak{g}$.

We call V a $\mathfrak{g}$-*module* and the above action (x, v) of some $x \in \mathfrak{g}$ on some $v \in V$ will be denoted by $x \cdot v$. ■

Now consider the vector space $End(V)$ of all vector space endomorphisms of V. Then $End(V) = End(V)_0 \oplus End(V)_1$ where

$$End(V)_k = \left\{ \phi \in End(V) : \begin{array}{l} \phi(V_i) \subseteq V_{i+k} \text{ for all } i \in \mathbb{Z}_2, \\ \text{i.e. } \phi \text{ is homogeneous of degree } k \end{array} \right\}$$

and $End(V)$ has a natural Lie superalgebra structure when the bracket is defined by $[\phi, \psi] := \phi \circ \psi - (-1)^{|\phi||\psi|}\psi \circ \phi$ for ϕ, ψ homogeneous, and then extended to whole of $End(V)$ linearly. It is then very easy to see that a $\mathfrak{g}$-module V is equivalent to a Lie superalgebra homomorphism $\rho\colon \mathfrak{g} \to End(V)$ given by $\rho(x)(v) := x \cdot v$ for $x \in \mathfrak{g}, v \in V$.

Let $\mathfrak{a}$ and $\mathfrak{g}$ be two Lie superalgebras. Then an extension of $\mathfrak{g}$ by $\mathfrak{a}$ is a short exact sequence of Lie superalgebras $0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$. We also call $\mathfrak{e}$ an *extension* of $\mathfrak{g}$ by $\mathfrak{a}$. We call this extension an *abelian extension* if $\mathfrak{a}$ is an abelian Lie superalgebra i.e., $[\mathfrak{a}, \mathfrak{a}] = 0$. Also, the extension is said to be a *central extension* if $i(\mathfrak{a}) \subseteq \mathcal{Z}(\mathfrak{e})$, the center of $\mathfrak{e}$, defined by $\mathcal{Z}(\mathfrak{e}) := \{x \in \mathfrak{e} \mid [x, e] = 0 \ \forall e \in \mathfrak{e}\}$. A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is said to be a *subalgebra* of the Lie superalgebra $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ and an ideal of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. From the above exactness it follows that $i(\mathfrak{a})$ is an ideal in $\mathfrak{e}$ and hence we will generally identify $\mathfrak{a}$ with $i(\mathfrak{a})$ and consider i to be the inclusion map.

Now let us consider an abelian extension $0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$ where i is just inclusion. Then we get a $\mathfrak{g}$-module structure on $\mathfrak{a}$ in the following way: Let $s\colon \mathfrak{g} \to \mathfrak{e}$

be a section of the map p i.e., s is linear and $ps = 1$. Then we define the action of $\mathfrak{g}$ on $\mathfrak{a}$ by $x \cdot a := [s(x), a]\ \forall x \in \mathfrak{g}, a \in \mathfrak{a}$. Using the fact that $\mathfrak{a}$ is abelian, it can be easily seen that $\mathfrak{a}$ is a $\mathfrak{g}$-module with this action. We will always denote this induced action by $\rho\colon \mathfrak{g} \to End(\mathfrak{a})$ i.e., $\rho(x)(a) = [s(x), a]$ for all $x \in \mathfrak{g}, a \in \mathfrak{a}$. The next lemma shows that this action does not depend on the choice of s.

Lemma 2.3. *The above $\mathfrak{g}$-module structure of $\mathfrak{a}$ does not depend on the choice of the section.*

Proof. Let s and s' be two sections of p. Then for any $x \in \mathfrak{g}$ and $a \in \mathfrak{a}$ we have $ps(x) - ps'(x) = 0$. which implies $s(x) - s'(x) \in ker\, p = \mathfrak{a}$. So $[s(x) - s'(x), a] = 0$ implying $[s(x), a] = [s'(x), a]$. ■

Without loss of generality we shall always take s to be homogeneous of degree 0 in this paper. Such a section $s\ (= s_0 \oplus s_1)$ of p always exists by taking sections s_i of p_i where $p_i = p|_{\mathfrak{e}_i}$ for $i \in \mathbb{Z}_2$.

Also, let $Aut(\mathfrak{g})$ denotes the group of Lie superalgebra automorphisms of a Lie superalgebra $\mathfrak{g}$. Let us set up the following notations parallel to those of [14] for future use, corresponding to the above abelian extension:

$$Aut_{\mathfrak{a}}(\mathfrak{e}) := \{\gamma \in Aut(\mathfrak{e}) \mid \gamma(\mathfrak{a}) = \mathfrak{a}\}, \quad Aut^{\mathfrak{a}}(\mathfrak{e}) := \{\gamma \in Aut(\mathfrak{e}) \mid \gamma(a) = a \text{ for all } a \in \mathfrak{a}\},$$
$$Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e}) := \{\gamma \in Aut(\mathfrak{e}) \mid \gamma(a) = a \text{ for all } a \in \mathfrak{a} \text{ and } \gamma(x) - x \in \mathfrak{a} \text{ for all } x \in \mathfrak{e}\},$$
$$Aut^{\mathfrak{g}}_{\mathfrak{a}}(\mathfrak{e}) := \{\gamma \in Aut(\mathfrak{e}) \mid \gamma(\mathfrak{a}) = \mathfrak{a} \text{ and } \gamma(x) - x \in \mathfrak{a} \text{ for all } x \in \mathfrak{e}\}.$$

Let s be a section of p. Then any $\gamma \in Aut_{\mathfrak{a}}(\mathfrak{e})$ induces a pair $(\gamma|_{\mathfrak{a}}, \bar{\gamma}) \in Aut(\mathfrak{a}) \times Aut(\mathfrak{g})$ where $\bar{\gamma}(x) := p(\gamma(s(x)))$ for $x \in \mathfrak{g}$. It can be seen that $\bar{\gamma}$ is actually an automorphism of $\mathfrak{g}$.
We define a map $\tau\colon Aut_{\mathfrak{a}}(\mathfrak{e}) \to Aut(\mathfrak{a}) \times Aut(\mathfrak{g})$ by $\tau(\gamma) := (\gamma|_{\mathfrak{a}}, \bar{\gamma})$. Then τ is a group homomorphism. But it may seem that τ is depending on the choice of s as construction of $\bar{\gamma}$ involves s. The next lemma shows that this is not the case.

Lemma 2.4. *The map τ defined above doesn't depend on the choice of the section.*

Proof. Let s and s' be two sections of p. Then $s(x) - s'(x) \in \mathfrak{a}$ as before. As $\gamma \in Aut_{\mathfrak{a}}(\mathfrak{e})$, $\gamma(s(x) - s'(x))$ also is in $\mathfrak{a}$. Which implies $p(\gamma(s(x) - s'(x))) = 0$ and the result follows. ■

Now let $\phi \in Aut(\mathfrak{a})$ and $\psi \in Aut(\mathfrak{g})$. The pair $(\phi, \psi) \in Aut(\mathfrak{a}) \times Aut(\mathfrak{g})$ is said to be inducible if there is some $\gamma \in Aut_{\mathfrak{a}}(\mathfrak{e})$ such that $\tau(\gamma) = (\phi, \psi)$.

2.1. Cohomology of Lie superalgebras

Here we give a brief description of some particular low dimensional cohomological spaces of Lie superalgebras, which we need in this paper, without going into the more general constructions. Let $\mathfrak{g}$ be a Lie superalgebra and M a $\mathfrak{g}$-module.

Define $C^1(\mathfrak{g}, M) := \{f\colon \mathfrak{g} \to M \mid f \text{ is linear}\}$, called the space of 1-cochains and $C^2(\mathfrak{g}, M) := \{\sigma\colon \mathfrak{g} \times \mathfrak{g} \to M \mid \sigma \text{ is bilinear and } \sigma(x, y) = -(-1)^{|x||y|}\sigma(y, x)\}$, called the space of 2-cochains.
These spaces are naturally $\mathbb{Z}_2$-graded, given by $C^k(\mathfrak{g}, M) = C^k(\mathfrak{g}, M)_0 \oplus C^k(\mathfrak{g}, M)_1$ for $k = 1, 2$ where the homogeneous parts are described in the following:

$$C^1(\mathfrak{g}, M)_j = \{f \in C^1(\mathfrak{g}, M) : f(\mathfrak{g}_i) \subseteq M_{i+j} \text{ for all } i \in \mathbb{Z}_2 \text{ i.e.,} |f| = j\},$$

$$C^2(\mathfrak{g}, M)_j = \{\sigma \in C^2(\mathfrak{g}, M) : |\sigma(x,y)| = |x| + |y| + j \text{ for all homogeneous } x, y \in \mathfrak{g}\}.$$

In general, elements of $C^n(\mathfrak{g}, M)_j$ are said to have *homogeneous degree* j.
Also, define a map $\delta \colon C^1(\mathfrak{g}, M) \to C^2(\mathfrak{g}, M)$, called the *coboundary map*, by $\delta(f)(x,y) := (-1)^{|x||f|} x \cdot f(y) - (-1)^{|y|(|x|+|f|)} y \cdot f(x) - f([x,y])$ for homogeneous $x, y \in \mathfrak{g}$ and homogeneous $f \in C^1(\mathfrak{g}, M)$, and then extended linearly where $|f|$ denotes the homogeneous degree of f. The image of the map δ is denoted by $B^2(\mathfrak{g}, M)$ and the kernel by $Z^1(\mathfrak{g}, M)$. Here it is important to note that the coboundary map respects the above grading, in particular $\delta(C^1(\mathfrak{g}, M)_0) \subseteq C^2(\mathfrak{g}, M)_0$. This will be used later. We define

$$Z^2(\mathfrak{g}, M) := \left\{ \sigma \in C^2(\mathfrak{g}, M) \;\middle|\; \begin{array}{l} (-1)^{|x||\sigma|} x \cdot \sigma(y,z) - (-1)^{|y|(|x|+|\sigma|)} y \cdot \sigma(x,z) \\ + (-1)^{|z|(|x|+|y|+|\sigma|)} z \cdot \sigma(x,y) - \sigma([x,y],z) \\ + (-1)^{|y||z|} \sigma([x,z],y) + \sigma(x,[y,z]) = 0 \end{array} \right\}$$

where $x, y, z \in \mathfrak{g}$ and σ are all homogeneous. Then $B^2(\mathfrak{g}, M) \subseteq Z^2(\mathfrak{g}, M)$. The quotient space $H^2(\mathfrak{g}, M) = Z^2(\mathfrak{g}, M)/B^2(\mathfrak{g}, M)$ is called the second cohomology group of $\mathfrak{g}$ with coefficients in M. In the above notation, elements of $B^2(\mathfrak{g}, M)$ are called 2-coboundaries and elements of $Z^n(\mathfrak{g}, M)$ are called n-cocycles.

Now for obvious reason, the $\mathbb{Z}_2$-grading of $C^k(\mathfrak{g}, M)$ is inherited by $Z^k(\mathfrak{g}, M)$, and $B^2(\mathfrak{g}, M)$ can be seen as a graded subspace of $Z^2(\mathfrak{g}, M)$. Therefore we have $B^2(\mathfrak{g}, M) = B^2(\mathfrak{g}, M)_0 \oplus B^2(\mathfrak{g}, M)_1$. So $H^2(\mathfrak{g}, M)$ becomes $\mathbb{Z}_2$-graded and

$$H^2(\mathfrak{g}, M) = \frac{Z^2(\mathfrak{g}, M)_0}{B^2(\mathfrak{g}, M)_0} \oplus \frac{Z^2(\mathfrak{g}, M)_1}{B^2(\mathfrak{g}, M)_1} = H^2(\mathfrak{g}, M)_0 \oplus H^2(\mathfrak{g}, M)_1.$$

For details and a more general construction of cohomologies we refer to [3], [17], [16] and to [13], the fundamental paper on the cohomology of Lie superalgebras.

3. Necessary and sufficient conditions for the inducibility of a pair

Let $0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$ be the abelian extension of $\mathfrak{g}$. Let(without loss of generality) s be a homogeneous section of degree 0 of p. Then $|s(x)| = |x|$ and $|s(y)| = |y|$ for homogeneous elements $x, y \in \mathfrak{g}$ where $|.|$ denotes the degree of a homogeneous element introduced in Section 2. Define a map $\theta \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ by $\theta(x,y) := [s(x), s(y)] - s[x,y]$ for all $x, y \in \mathfrak{g}$. Also let $(\phi, \psi) \in Aut(\mathfrak{a}) \times Aut(\mathfrak{g})$. Then it is easy to see that θ, $\phi \circ \theta \circ (\psi^{-1}, \psi^{-1}) \in C^2(\mathfrak{g}, \mathfrak{a})$. Also, both θ and $\phi \circ \theta \circ (\psi^{-1}, \psi^{-1})$ are homogeneous of degree 0. So θ, $\phi \circ \theta \circ (\psi^{-1}, \psi^{-1}) \in C^2(\mathfrak{g}, \mathfrak{a})_0$. Now we are ready to state our first theorem.

Theorem 3.1. *Let $0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$ be an abelian extension. A pair $(\phi, \psi) \in Aut(\mathfrak{a}) \times Aut(\mathfrak{g})$ is inducible if and only if the following conditions hold:*

(1) *The 2-cochains $\phi \circ \theta \circ (\psi^{-1}, \psi^{-1})$ and θ differ by a 2-coboundary (in $B^2(\mathfrak{g}, \mathfrak{a})_0$).*

(2) *The following diagram commutes:*
$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\psi} & \mathfrak{g} \\ \downarrow \rho & & \downarrow \rho \\ End(\mathfrak{a}) & \xrightarrow{f \mapsto \phi \circ f \circ \phi^{-1}} & End(\mathfrak{a}) \end{array}$$

If the second condition holds, we call (ϕ, ψ) a compatible pair. The map ρ is described in Section 2.

Proof. Let (ϕ, ψ) be a pair which is inducible. Then there exists an automorphism $\gamma \in Aut_{\mathfrak{a}}(\mathfrak{e})$ such that $\tau(\gamma) = (\phi, \psi)$. Let $s\colon \mathfrak{g} \to \mathfrak{e}$ be a homogeneous section of degree 0. Now for $x \in \mathfrak{g}$, $p\gamma s(x) = \psi(x) = ps\psi(x)$. So $\gamma s(x) - s\psi(x) \in \mathfrak{a}$. We define $\lambda\colon \mathfrak{g} \to \mathfrak{a}$ by $\lambda(x) := \gamma s(x) - s\psi(x)$. Clearly λ is linear, homogeneous of degree 0. Now for $x, y \in \mathfrak{g}$,

$$\begin{aligned}
\phi \circ \theta \circ (\psi^{-1}, \psi^{-1})(x, y) &= \phi(\theta(\psi^{-1}(x), \psi^{-1}(y))) = \gamma(\theta(\psi^{-1}(x), \psi^{-1}(y))) \\
&= \gamma([s\psi^{-1}(x), s\psi^{-1}(y)] - s[\psi^{-1}(x), \psi^{-1}(y)]) \\
&= [\gamma s\psi^{-1}(x), \gamma s\psi^{-1}(y)] - \gamma s[\psi^{-1}(x), \psi^{-1}(y)] \\
&= [\lambda\psi^{-1}(x) + s\psi(\psi^{-1}(x)), \lambda\psi^{-1}(y) + s\psi(\psi^{-1}(y))] \\
&\qquad - \lambda([\psi^{-1}(x), \psi^{-1}(y)]) - s\psi([\psi^{-1}(x), \psi^{-1}(y)]) \\
&= [s(x), \lambda\psi^{-1}(y)] - (-1)^{|x||y|}[s(y), \lambda\psi^{-1}(x)] + [s(x), s(y)] \\
&\qquad - \lambda([\psi^{-1}(x), \psi^{-1}(y)]) - s[x, y] \\
&= x \cdot (\lambda\psi^{-1}(y)) - (-1)^{|x||y|} y \cdot (\lambda\psi^{-1}(x)) - \lambda([\psi^{-1}(x), \psi^{-1}(y)]) + \theta(x, y) \\
&= \delta(\lambda\psi^{-1})(x, y) + \theta(x, y).
\end{aligned}$$

So $\phi \circ \theta \circ (\psi^{-1}, \psi^{-1})$ and θ differ by a homogeneous 2-coboundary of degree 0. Secondly, for $x \in \mathfrak{g}$ and $a \in \mathfrak{a}$,

$$\begin{aligned}
\phi\rho(x)(\phi^{-1}(a)) &= \phi(x \cdot \phi^{-1}a) = \gamma[s(x), \phi^{-1}a] \\
&= [\gamma s(x), a] = [\lambda(x), a] + [s\psi(x), a] = \rho(\psi(x))(a).
\end{aligned}$$

So $\phi\rho(x)\phi^{-1} = \rho\psi(x)$ for all x and hence the diagram commutes.
For the converse, suppose $\phi \circ \theta \circ (\psi^{-1}, \psi^{-1})$ and θ differ by a homogeneous 2-coboundary of degree 0. Then from the discussion in section 2.1, we can find a map $\lambda\colon \mathfrak{g} \to \mathfrak{a}$ such that $\phi \circ \theta \circ (\psi^{-1}, \psi^{-1}) - \theta = \delta(\lambda)$ and λ is homogeneous of degree 0. So we obtain relations for x,y $\in \mathfrak{g}$:

$$\phi \circ \theta \circ (\psi^{-1}, \psi^{-1})(\psi(x), \psi(y)) - \theta(\psi(x), \psi(y)) = \delta(\lambda)(\psi(x), \psi(y)), \quad \text{or}$$
$$\phi \circ \theta(x, y) - \theta(\psi(x), \psi(y)) = [s\psi(x), \lambda\psi(y)] + [\lambda\psi(x), s\psi(y)] - \lambda([\psi(x), \psi(y)]) \quad (1)$$

Let $s\colon \mathfrak{g} \to \mathfrak{e}$ be a homogeneous section of degree 0, then $\mathfrak{e} = \mathfrak{a} \oplus s(\mathfrak{g})$ as vector spaces. Now we define a map $\gamma\colon \mathfrak{e} \to \mathfrak{e}$ by $\gamma(a + s(x)) = \phi(a) + \lambda\psi(x) + s\psi(x)$ for $a \in \mathfrak{a}, x \in \mathfrak{g}$. It is easy to see that γ is homogeneous of degree 0 and also injective. To see that γ is surjective, let us take some arbitrary $a' + s(x') \in \mathfrak{e}$. Then $\gamma(\phi^{-1}(a' - \lambda(x')) + s\psi^{-1}(x')) = a' + s(x')$. So γ is surjective.

Now we check that this γ induces the pair (ϕ, ψ). It is clear that $\gamma|_{\mathfrak{a}} = \phi$. And for $x \in \mathfrak{g}$, $p\gamma s(x) = p(\lambda\psi(x) + s\psi(x)) = ps\psi(x) = \psi(x)$. So γ induces the pair (ϕ, ψ).

The only thing left is to show that γ is a map of Lie superalgebras i.e., γ respects the bracket. For that, let $e = a + s(x)$ and $e' = a' + s(x') \in \mathfrak{e}$. Then,

$$\begin{aligned}\gamma[e, e'] &= \gamma[a + s(x), a' + s(x')] = \gamma([s(x), a']) + \gamma[a, s(x')] + \gamma([s(x), s(x')]) \\ &= \phi([s(x), a']) - (-1)^{|a||x'|}\phi([s(x'), a]) + \gamma(\theta(x, x') + s[x, x']) \\ &= \phi\rho(x)(a') - (-1)^{|a||x'|}\phi\rho(x')(a) + \phi\theta(x, x') + \gamma s[x, x'].\end{aligned}$$

On the other hand,

$$\begin{aligned}&[\gamma(e), \gamma(e')] = [\gamma(a + s(x)), \gamma(a' + s(x'))] \\ &= [\phi(a) + \lambda\psi(x) + s\psi(x), \phi(a') + \lambda\psi(x') + s\psi(x')] \\ &= [\phi(a), s\psi(x')] + [\lambda\psi(x), s\psi(x')] + [s\psi(x), \phi(a')] + [s\psi(x), \lambda\psi(x')] + [s\psi(x), s\psi(x')] \\ &= \rho\psi(x)(\phi(a')) - (-1)^{|a||x'|}\rho\psi(x')(\phi(a)) + \phi\theta(x, x') - \theta(\psi(x), \psi(x')) + \lambda[\psi(x), \psi(y)]) \\ &\qquad + \theta(\psi(x), \psi(x')) + s[\psi(x), \psi(x')] \text{ [using the above relation (1)]} \\ &= \rho\psi(x)\phi(a') - (-1)^{|a||x'|}\rho\psi(x')\phi(a) + \phi\theta(x, x') + \lambda\psi[x, x'] + s\psi[x, x'] \\ &= \rho\psi(x)\phi(a') - (-1)^{|a||x'|}\rho\psi(x')\phi(a) + \phi\theta(x, x') + \gamma s[x, x'].\end{aligned}$$

Using the compatibility of ϕ and ψ we get $\gamma([e, e']) = [\gamma(e), \gamma(e')]$. ■

4. Construction of the exact sequences

For an abelian extension $0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$, a pair $(\phi, \psi) \in Aut(\mathfrak{a}) \times Aut(\mathfrak{g})$ is said to be *compatible* if the second hypothesis of Theorem 3.1 holds, i.e., $\phi \circ \rho(x) \circ \phi^{-1} = \rho(\psi(x))$ for all $x \in \mathfrak{g}$. We denote the set of all compatible pairs by $\mathcal{C}$. It follows at once that $\mathcal{C}$ is a subgroup of $Aut(\mathfrak{a}) \times Aut(\mathfrak{g})$. Let us define $\mathcal{C}_1$ and $\mathcal{C}_2$ by

$$\mathcal{C}_1 := \{\phi \in Aut(\mathfrak{a}) | (\phi, 1) \in \mathcal{C}\} \quad \text{and} \quad \mathcal{C}_2 := \{\psi \in Aut(\mathfrak{g}) | (1, \psi) \in \mathcal{C}\}.$$

Clearly $\mathcal{C}_1$ and $\mathcal{C}_2$ are subgroups of $Aut(\mathfrak{a})$ and $Aut(\mathfrak{g})$, respectively.
Now for $\phi \in \mathcal{C}_1$ and $\psi \in \mathcal{C}_2$ we define two maps $\theta_\phi, \theta_\psi \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ given by $\theta_\phi(x, y) := \phi\theta(x, y) - \theta(x, y)$ and $\theta_\psi(x, y) := \theta(\psi^{-1}(x), \psi^{-1}(y)) - \theta(x, y)$ where θ is introduced in Section 3. The next lemma will become useful several times. It will also follow from this lemma that $\theta_\phi, \theta_\psi \in Z^2(\mathfrak{g}, \mathfrak{a})_0$:

Lemma 4.1. *The map θ is a homogeneous 2-cocycle of degree 0, i.e., $\theta \in Z^2(\mathfrak{g}, \mathfrak{a})_0$.*

Proof. It is clear from the definition that θ is homogeneous of degree 0 i.e., $|\theta| = 0$. Then from Section 2.1, since $|\theta| = 0$, to show θ is a 2-cocycle, we need to show that

$$\begin{aligned}&x \cdot \theta(y, z) - (-1)^{|y||x|} y \cdot \theta(x, z) + (-1)^{|z|(|x|+|y|)} z \cdot \theta(x, y) \\ &\qquad = \theta([x, y], z) - (-1)^{|y||z|}\theta([x, z], y) - \theta(x, [y, z]).\end{aligned}$$

Now,
$$\begin{aligned}&x \cdot \theta(y, z) - (-1)^{|y||x|} y \cdot \theta(x, z) + (-1)^{|z|(|x|+|y|)} z \cdot \theta(x, y) \\ &= [s(x), [s(y), s(z)] - s[y, z]] - (-1)^{|y||x|}[s(y), [s(x), s(z)] - s[x, z]] \\ &\qquad + (-1)^{|z|(|x|+|y|)}[s(z), [s(x), s(y)] - s[x, y]]\end{aligned}$$

$$
\begin{aligned}
&= [s(x),[s(y),s(z)]] - [s(x),s[y,z]] - (-1)^{|y||x|}[s(y),[s(x),s(z)]] \\
&\qquad + (-1)^{|y||x|}[s(y),s[x,z]] + (-1)^{|z|(|x|+|y|)}[s(z),[s(x),s(y)]] \\
&\qquad - (-1)^{|z|(|x|+|y|)}[s(z),s[x,y]] \\
&= [s(x),[s(y),s(z)]] - (-1)^{|y||x|}[s(y),[s(x),s(z)]] - [[s(x),s(y)],s(z)] \\
&\qquad - [s(x),s[y,z]] + (-1)^{|y||x|}[s(y),s[x,z]] + [s[x,y],s(z)] \\
&= -[s(x),s[y,z]] + (-1)^{|y||x|}[s(y),s[x,z]] + [s[x,y],s(z)] \\
&= -[s(x),s[y,z]] + (-1)^{|y||x|}[s(y),s[x,z]] + [s[x,y],s(z)] \\
&\qquad - s[[x,y],z] + s[x,[y,z]] - (-1)^{|x||y|}s[y,[x,z]] \\
&= [s[x,y],s(z)] - s[[x,y],z] + (-1)^{|y||x|}[s(y),s[x,z]] \\
&\qquad - (-1)^{|x||y|}s[y,[x,z]] - [s(x),s[y,z]] + s[x,[y,z]] \\
&= \theta([x,y],z) + (-1)^{|y||x|}\theta(y,[x,z]) - \theta(x,[y,z]) \\
&= \theta([x,y],z) - (-1)^{|y||z|}\theta([x,z],y) - \theta(x,[y,z]).
\end{aligned}
$$

So θ is a 2-cocycle. ■

Now we use the above lemma to prove the following:

Lemma 4.2. *The maps θ_ϕ and $\theta_\psi \in Z^2(\mathfrak{g},\mathfrak{a})_0$.*

Proof. Clearly θ_ϕ and θ_ψ are homogeneous of degree 0. To show that θ_ϕ is a 2-cocycle, we will first show that for $\phi \in Aut(\mathfrak{a})$, $\phi\theta$ is also a 2-cocycle:

$$
\begin{aligned}
&x \cdot \phi\theta(y,z) - (-1)^{|y||x|}y \cdot \phi\theta(x,z) + (-1)^{|z|(|x|+|y|)}z \cdot \phi\theta(x,y) - \phi\theta([x,y],z) \\
&\qquad + (-1)^{|y||z|}\phi\theta([x,z],y) + \phi\theta(x,[y,z]) \\
&= \rho(x)\phi\theta(y,z) - (-1)^{|y||x|}\rho(y)\phi\theta(x,z) + (-1)^{|z|(|x|+|y|)}\rho(z)\phi\theta(x,y) \\
&\qquad - \phi\theta([x,y],z) + (-1)^{|y||z|}\phi\theta([x,z],y) + \phi\theta(x,[y,z]) \\
&= \phi\rho(x)\theta(y,z) - (-1)^{|y||x|}\phi\rho(y)\theta(x,z) + (-1)^{|z|(|x|+|y|)}\phi\rho(z)\theta(x,y) \\
&\qquad - \phi\theta([x,y],z) + (-1)^{|y||z|}\phi\theta([x,z],y) + \phi\theta(x,[y,z]) \quad [\text{as } \phi \in \mathcal{C}_1\ ] \\
&= \phi(x \cdot \theta(y,z) - (-1)^{|y||x|}y \cdot \theta(x,z) + (-1)^{|z|(|x|+|y|)}z \cdot \theta(x,y) - \theta([x,y],z) \\
&\qquad + (-1)^{|y||z|}\theta([x,z],y) + \theta(x,[y,z])) \\
&= 0 \qquad [\text{as } \theta \text{ is a 2-cocycle}]
\end{aligned}
$$

So $\phi\theta$ is a 2-cocycle. Since θ is also a 2-cocycle, clearly there difference θ_ϕ is a 2-cocycle. Similar calculation shows that θ_ψ is also a 2-cocycle. ■

From the above lemma it follows that θ_ϕ and θ_ψ give two elements in $H^2(\mathfrak{g},\mathfrak{a})_0$ namely $[\theta_\phi]$ and $[\theta_\psi]$, the cohomology classes of θ_ϕ and θ_ψ respectively.
In view of the above fact, we define two functions $\lambda_i \colon \mathcal{C}_i \to H^2(\mathfrak{g},\mathfrak{a})_0$ for $i = 1,2$ by $\lambda_1(\phi) := [\theta_\phi]$ and $\lambda_2(\psi) := [\theta_\psi]$. At first glance it may seem that these functions depend on the choice of θ, but this is not the case:

Lemma 4.3. *The maps λ_1 and λ_2 are well defined.*

Proof. Since θ only depends on the section, we will show that the cohomology classes $[\theta_\phi]$ and $[\theta_\psi]$ do not depend on the choice of the section. For that let $s,t\colon \mathfrak{g} \to \mathfrak{e}$ be two homogeneous sections of degree 0 of p and the corresponding 2-cocycles be θ_s and θ_t respectively. Now since $ps = pt = 1$, $s(x) - t(x) \in \mathfrak{a}$. We define a map $\lambda\colon \mathfrak{g} \to \mathfrak{a}$ by $\lambda(x) := s(x) - t(x)$ for $x \in \mathfrak{g}$. Then $\lambda \in \mathcal{C}^1(\mathfrak{g},\mathfrak{a})_0$. Now,

$$\begin{aligned}\theta_s(x,y) - \theta_t(x,y) &= [s(x),s(y)] - s[x,y] - [t(x),t(y)] + t[x,y]\\ &= [\lambda(x)+t(x),\lambda(y)+t(y)] - [t(x),t(y)] - \lambda[x,y]\\ &= [\lambda(x),t(y)] + [t(x),\lambda(y)] - \lambda[x,y]\\ &= x\cdot\lambda(y) - (-1)^{|x||y|}y\cdot\lambda(x) - \lambda[x,y] = \delta(\lambda)(x,y)\end{aligned}$$

So $\theta_s - \theta_t \in B^2(\mathfrak{g},\mathfrak{a})_0$. Also,

$$\begin{aligned}\phi\theta_s(x,y) - \phi\theta_t(x,y) &= \phi(\theta_s(x,y) - \theta_t(x,y))\\ &= \phi(x\cdot\lambda(y) - (-1)^{|x||y|}y\cdot\lambda(x) - \lambda[x,y])\\ &= \phi\rho(x)(\lambda(y)) - (-1)^{|x||y|}\phi\rho(y)(\lambda(x)) - \phi\lambda[x,y])\\ &= \rho(x)\phi\lambda(y) - (-1)^{|x||y|}\rho(y)\phi\lambda(x) - \phi\lambda[x,y]) \ \ [\text{ as } \phi\in\mathcal{C}_1]\\ &= x\cdot\phi\lambda(y)) - (-1)^{|x||y|}y\cdot\phi\lambda(x) - \phi\lambda[x,y] = \delta(\phi\lambda)(x,y).\end{aligned}$$

Hence $\phi\theta_s - \phi\theta_t \in B^2(\mathfrak{g},\mathfrak{a})_0$. The above two calculations imply $\theta_{s\phi} - \theta_{t\phi} \in B^2(\mathfrak{g},\mathfrak{a})_0$. Therefore λ_1 is independent of the choice of section (hence θ-independent). Similar calculation shows that λ_2 is also independent of the choice of section. Hence the maps λ_1 and λ_2 are well defined. ■

Remark 4.4. The above two maps λ_1 and λ_2 are not in general group homomorphisms but $ker\lambda_i$ will have their usual meaning.

Let $\gamma \in Aut_{\mathfrak{a}}(\mathfrak{e})$ (see Section 2 for the notation) and $\tau(\gamma) = (\phi,\psi)$. Now, in particular if $\gamma \in Aut_{\mathfrak{a}}^{\mathfrak{g}}(\mathfrak{e})$ let us set $\tau_1(\gamma) := \phi$ and if $\gamma \in Aut^{\mathfrak{a}}(\mathfrak{e})$ then $\tau_2(\gamma) := \psi$. The following lemma shows that τ_1 and τ_2 are maps into $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively.

Lemma 4.5. *$\tau_1(Aut_{\mathfrak{a}}^{\mathfrak{g}}(\mathfrak{e})) \subseteq \mathcal{C}_1$ and $\tau_2(Aut^{\mathfrak{a}}(\mathfrak{e})) \subseteq \mathcal{C}_2$.*

Proof. Let $\gamma \in Aut_{\mathfrak{a}}^{\mathfrak{g}}(\mathfrak{e})$, $\tau(\gamma) = (\phi,\psi)$. Then $\psi(x) = p\gamma s(x)$ for some section s. But since $\gamma \in Aut_{\mathfrak{a}}^{\mathfrak{g}}(\mathfrak{e})$, $\gamma(s(x)) = s(x) + a$ for some $a \in \mathfrak{a}$. But then $\psi(x) = p\gamma s(x) = p(s(x)+a) = x$. So $\psi = 1$ and $\tau(\gamma) = (\phi,1)$. Then from Theorem 3.1 it follows that the pair $(\phi,1)$ is compatible and hence $\phi \in \mathcal{C}_1$.

And for $\gamma \in Aut^{\mathfrak{a}}(\mathfrak{e})$ let $\tau(\gamma) = (\phi,\psi)$. Then from the definition of $Aut^{\mathfrak{a}}(\mathfrak{e})$, $\phi = \gamma|_{\mathfrak{a}} = 1$. So $\tau(\gamma) = (1,\psi)$ and again from Theorem 3.1 the pair $(1,\psi)$ is compatible. Therefore $\psi \in \mathcal{C}_2$. Hence the proof of the lemma. ■

With the above set-up in mind we are now ready to state our next theorem.

Theorem 4.6. *Let $0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$ be an abelian extension. Then the following two sequences are exact:*

$$1 \to Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e}) \xrightarrow{\iota} Aut_{\mathfrak{a}}^{\mathfrak{g}}(\mathfrak{e}) \xrightarrow{\tau_1} \mathcal{C}_1 \xrightarrow{\lambda_1} H^2(\mathfrak{g},\mathfrak{a})_0 \tag{2}$$

$$1 \to Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e}) \xrightarrow{\iota} Aut^{\mathfrak{a}}(\mathfrak{e}) \xrightarrow{\tau_2} \mathcal{C}_2 \xrightarrow{\lambda_2} H^2(\mathfrak{g},\mathfrak{a})_0 \tag{3}$$

Proof. Since ι is the inclusion map the sequences are clearly exact at the first term. Now for $\gamma \in Aut^{\mathfrak{g}}_{\mathfrak{a}}(\mathfrak{e})$, $\tau_1(\gamma) = 1$ if and only if $\gamma|_{\mathfrak{a}} = 1$. So $ker\,\tau_1 = Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e})$ and (2) is exact at the second term. Now for the exactness of (2) at the third term, $\tau_1(\gamma) = \phi$ implies $(\phi, 1)$ is an inducible pair. Then from Theorem 3.1 it follows that $\lambda_1(\phi) = \theta_\phi (= \phi\theta - \theta) \in B^2(\mathfrak{g}, \mathfrak{a})_0$. So $Img\,\tau_1 \subseteq ker\lambda_1$. Conversely, for $\phi \in \mathcal{C}_1$, if $\lambda_1(\phi) \in B^2(\mathfrak{g}, \mathfrak{a})_0$ then the conditions of Theorem 3.1 are satisfied, so $(\phi, 1)$ becomes an inducible pair. Now for some γ, if $\tau(\gamma) = (\phi, 1)$ then $p\gamma s(x) = x = ps(x)$ for all $x \in \mathfrak{g}$. Which implies $\gamma(s(x)) - s(x) \in \mathfrak{a}$ for all $x \in \mathfrak{g}$. But since $\mathfrak{e} = \mathfrak{a} \oplus s(\mathfrak{g})$, $\gamma(x) - x \in \mathfrak{a}$ for all $x \in \mathfrak{e}$. So $\gamma \in Aut^{\mathfrak{g}}_{\mathfrak{a}}(\mathfrak{e})$ and $ker\,\lambda_1 \subseteq Img\,\tau_1$. Consequently (2) is exact.

Now we consider (3). If for some $\gamma \in Aut^{\mathfrak{a}}(\mathfrak{e})$, $\tau_2(\gamma) = 1$ then $p\gamma s = 1$. From the above, then it follows that $\gamma(x) - x \in \mathfrak{a}$ for all $x \in \mathfrak{e}$. So in particular $\gamma \in Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e})$ and ker $\tau_2 \subseteq Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e})$. Now if $\gamma \in Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e})$ then $\gamma s(x) = s(x) + a$ for $x \in \mathfrak{g}$ and for some $a \in \mathfrak{a}$. Therefore $\tau_2(\gamma)(x) = p\gamma s(x) = p(s(x) + a) = x$ and $Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e}) \subseteq ker\,\tau_2$. So (3) is exact at the second term. To check exactness at the third term, let $\gamma \in Aut^{\mathfrak{a}}(\mathfrak{e})$ and let $\tau_2(\gamma) = \psi$. Then $(1, \psi)$ becomes an inducible and from Theorem 3.1 it follows that $\lambda_2(\psi) = \theta_\psi \in B^2(\mathfrak{g}, \mathfrak{a})_0$. So $Img\,\tau_2 \subseteq ker\,\lambda_2$. For the converse, let $\psi \in \mathcal{C}_2$ and $\lambda_2(\psi) \in B^2(\mathfrak{g}, \mathfrak{a})_0$. Then both the conditions of Theorem 3.1 are satisfied and $(1, \psi)$ becomes an inducible pair. Let $\tau(\gamma) = (1, \psi)$. Then this implies $\gamma|_{\mathfrak{a}} = 1$ and hence $\gamma \in Aut^{\mathfrak{a}}(\mathfrak{e})$. This proves that $ker\,\lambda_2 \subseteq Img\,\tau_2$ and (3) is exact. ∎

5. Wells exact sequence

In this section, we assemble the results from the previous section with some other results to prove an analog of the well-known Wells exact sequence for group extensions which was established in [18] and was studied further in [10],[11],[14] and [15]. In that exact sequence in [18], the first(non-trivial) term is some group of 1-cocycles but so far in our exact sequences the first term is $Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e})$. The next lemma connects these two to some extent.

Lemma 5.1. *Let $0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$ be an abelian extension of Lie superalgebras. Then $Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e}) \cong Z^1(\mathfrak{g}, \mathfrak{a})_0$ as groups.[Here the obvious additive, abelian group structure of $Z^1(\mathfrak{g}, \mathfrak{a})_0$ is considered.]*

Proof. Let $\gamma \in Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e})$, then $\gamma(x) - x \in \mathfrak{a}$. Now $x = a + s(b)$ for some unique $a \in \mathfrak{a}, b \in \mathfrak{g}$. Define $\lambda_\gamma\colon \mathfrak{g} \to \mathfrak{a}$ by $\lambda_\gamma(b) = \gamma(x) - x$ for $b \in \mathfrak{g}$. Since $\gamma|_{\mathfrak{a}} = 1$, $\lambda_\gamma(b) = \gamma s(b) - s(b)$. Clearly λ_γ is homogeneous of degree 0.

Now we want to show that the above map λ_γ belongs to $Z^1(\mathfrak{g}, \mathfrak{a})_0$. For that purpose we only have to show that λ_γ is a 1-cocycle. Let $x, y \in \mathfrak{g}$. Then,

$$\begin{aligned}\lambda_\gamma[x, y] &= \gamma s[x, y] - s[x, y] = \gamma([s(x), s(y)] - \theta(x, y)) + \theta(x, y) - [s(x), s(y)]\\ &= [\gamma s(x), \gamma s(y)] - [s(x), s(y)] = [\lambda_\gamma(x) + s(x), \lambda_\gamma(y) + s(y)] - [s(x), s(y)]\\ &= [\lambda_\gamma(x), s(y)] + [s(x), \lambda_\gamma(y)].\end{aligned}$$

So λ_γ is a 1-cocycle and therefore $\lambda_\gamma \in Z^1(\mathfrak{g}, \mathfrak{a})_0$. Let us define a function $\chi\colon Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e}) \to Z^1(\mathfrak{g}, \mathfrak{a})_0$ by $\chi(\gamma) := \lambda_\gamma$. We shall show that χ is actually a group isomorphism. To see that it is a group homomorphism, let $\gamma_1, \gamma_2 \in Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e})$. Then for $x \in \mathfrak{g}$,

$$\begin{aligned}\lambda_{\gamma_1\circ\gamma_2}(x) &= \gamma_1 \circ \gamma_2 s(x) - s(x) = \gamma_1 \circ \gamma_2 s(x) - \gamma_1 s(x) + \gamma_1 s(x) - s(x) \\ &= \gamma_1(\gamma_2 s(x) - s(x)) + \gamma_1 s(x) - s(x) = \gamma_1(\lambda_{\gamma_2}(x)) + \lambda_{\gamma_1}(x) \\ &= \lambda_{\gamma_2}(x) + \lambda_{\gamma_1}(x).\end{aligned}$$

So $\lambda_{\gamma_1\circ\gamma_2} = \lambda_{\gamma_1} + \lambda_{\gamma_2}$ and χ is a group homomorphism.

Now if $\chi(\gamma)$ is the zero map for some γ then $\gamma(s(x)) = s(x)$ for all $x \in \mathfrak{g}$. Also $\gamma|_{\mathfrak{a}} = 1$. Which implies γ is the identity map on $\mathfrak{e}(= \mathfrak{a} \oplus s(\mathfrak{g}))$ and χ is injective.

To show that χ is surjective, let $\lambda \in Z^1(\mathfrak{g}, \mathfrak{a})_0$. We define the function $\gamma\colon \mathfrak{e} \to \mathfrak{e}$ by $\gamma(a + s(x)) := a + \lambda(x) + s(x)$ for all $a \in \mathfrak{a}$, $x \in \mathfrak{g}$ then $\chi(\gamma)$ is obviously λ. Clearly γ is homogeneous of degree 0. It is also clear from the definition that $\gamma|_{\mathfrak{a}} = 1$ and $\gamma(e) - e \in \mathfrak{a}$ for all $e \in \mathfrak{e}$. Now $\gamma(a + s(x)) = 0$ implies $s(x) = 0$. Since s is injective, $x = 0$, which in turn implies $a = 0$. So γ is injective. Also $\gamma(a - \lambda(x) + s(x)) = a + s(x)$ implies γ is surjective.

The only thing left to show is that γ respects the bracket. To show this let $e_1, e_2 \in \mathfrak{e}$ and $e_1 = a_1 + s(x_1), e_2 = a_2 + s(x_2)$. Then,

$$\begin{aligned}\gamma[e_1, e_2] &= \gamma[a_1 + s(x_1), a_2 + s(x_2)] = \gamma([a_1, s(x_2)] + [s(x_1), a_2] + [s(x_1), s(x_2)]) \\ &= [a_1, s(x_2)] + [s(x_1), a_2] + \gamma[s(x_1), s(x_2)] \quad (\text{as } \gamma|_{\mathfrak{a}} = 1) \\ &= [a_1, s(x_2)] + [s(x_1), a_2] + \theta(x_1, x_2) + \gamma s[x_1, x_2] \\ &= [a_1, s(x_2)] + [s(x_1), a_2] + \theta(x_1, x_2) + \lambda[x_1, x_2] + s[x_1, x_2] \\ &= [a_1, s(x_2)] + [s(x_1), a_2] + \theta(x_1, x_2) + [\lambda(x_1), s(x_2)] + [s(x_1), \lambda(x_2)] + s[x_1, x_2] \\ &= [a_1, s(x_2)] + [s(x_1), a_2] + [s(x_1), s(x_2)] + [\lambda(x_1), s(x_2)] + [s(x_1), \lambda(x_2)] \\ &= [\gamma(e_1), \gamma(e_2)].\end{aligned}$$

So $\gamma \in Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e})$, which completes the proof of the lemma. ■

With all this work done, we are now finally ready to state the Wells exact sequence for Lie superalgebras, which is our next theorem.

Theorem 5.2. *Let $0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$ be an abelian extension of Lie superalgebras. Then there is a set map Λ so that*

$$0 \to Z^1(\mathfrak{g}, \mathfrak{a})_0 \xrightarrow{i} Aut_{\mathfrak{a}}(\mathfrak{e}) \xrightarrow{\tau} \mathcal{C} \xrightarrow{\Lambda} H^2(\mathfrak{g}, \mathfrak{a})_0.$$

is an exact sequence of groups.

The map τ was introduced in Section 2 and by Theorem 3.1, the image of τ lies in $\mathcal{C}$ which is the set of all compatible pairs defined in Section 4. The map Λ here is called the Wells map.

Proof. The map i is the inclusion map via the isomorphism of Lemma 5.1, so clearly the sequence is exact at the first term. Now, if for some $\gamma \in Aut_{\mathfrak{a}}(\mathfrak{e})$, $\tau(\gamma) = (1, 1)$ then clearly $\gamma \in Aut^{\mathfrak{a}}(\mathfrak{e})$ and $\tau_2(\gamma) = 1$. From the second exact sequence of Theorem 4.6 then it follows that $\gamma \in Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e})$. So $ker\,\tau \subseteq Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e})$. Conversely, if $\gamma \in Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e})$ then $\tau_i(\gamma) = 1$ for $i = 1, 2$. Which implies $\tau(\gamma) = (1, 1)$ and therefore $Aut^{\mathfrak{a},\mathfrak{g}}(\mathfrak{e}) \subseteq ker\,\tau$. So exactness follows at the second term.

To show exactness at the third term we have to define the map Λ. For that we take help of the two maps θ_ϕ and θ_ψ (defined in Section 4) for $(\phi,\psi)\in\mathcal{C}$, and define $\theta_{\phi,\psi}\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{a}$ by

$$\theta_{\phi,\psi}(x,y) := \phi\theta(\psi^{-1}(x),\psi^{-1}(y)) - \theta(x,y).$$

Some calculation similar to Lemma 4.2 shows that $\theta_{\phi,\psi}$ is actually a homogeneous 2-cocycle of degree 0.
The map Λ is given by $\Lambda(\phi,\psi) := [\theta_{\phi,\psi}]$, the cohomology class of $\theta_{\phi,\psi}$. Again one can check that this map is also well-defined, i.e. it doesn't depend on the choice of section. Now if $\Lambda(\phi,\psi)$ is a coboundary then from Theorem 3.1 it follows that the pair is inducible. So there is some $\gamma\in Aut_{\mathfrak{a}}(\mathfrak{e})$ such that $\tau(\gamma)=(\phi,\psi)$ implying $ker\,\Lambda\subseteq Img\,\tau$. Conversely if $(\phi,\psi)\in Img\,\tau$, i.e. (ϕ,ψ) is inducible then again from Theorem 3.1, $\theta_{\phi,\psi}\in Z^2(\mathfrak{g},\mathfrak{a})_0$ which implies $\Lambda(\phi,\psi)=0$. Therefore $Img\,\tau\subseteq ker\,\Lambda$ and consequently the sequence is exact at the third term, proving the theorem. ■

Here are some immediate corollaries of the above theorem.

Corollary 5.3. *If* $0\to\mathfrak{a}\xrightarrow{i}\mathfrak{e}\xrightarrow{p}\mathfrak{g}\to 0$ *is a split extension then every compatible pair gives rise to an automorphism of* $\mathfrak{e}$.

Proof. The above extension is called a split when there is a section s of p which is also a Lie superalgebra map. In that case clearly $\theta_{\phi,\psi}=0$. So the map Λ is trivial and then by the theorem, $\mathcal{C}=Img\,\tau$ implying the result. ■

Corollary 5.4. *In the extension* $0\to\mathbb{C}\xrightarrow{i}\mathfrak{e}\xrightarrow{p}\mathfrak{g}\to 0$, *if* $\mathfrak{g}$ *is a complex basic classical Lie superalgebra except* $\mathfrak{sl}(n,n)$ *then every compatible pair can be extended to an automorphism of* $\mathfrak{e}$. *Here* $\mathbb{C}$ *is considered to be the trivial module over* $\mathfrak{g}$.

For the description of the basic classical Lie superalgebras or for a complete classification of finite dimensional simple Lie superalgebras, we refer to [12].
From [5](also see [6],[7]), we get to know some remarkable results on cohomology of the basic classical Lie superalgebras which will be used here to prove the corollary. In particular, the cohomologies of these Lie superalgebras were given in terms of those of simple Lie algebras.

Proof. If $\mathfrak{g}=\mathfrak{sl}(m,n); m\neq n$, then from Theorem (2.6.1) of [6](also see [5]), $H^2(\mathfrak{g},\mathbb{C}) = H^2(\mathfrak{sl}_{sup\{m,n\}},\mathbb{C})$ but the latter is 0 by the Whitehead's second lemma. Therefore from the theorem it follows that Λ is the zero map and hence the result. For $\mathfrak{g}$ from one of the families $B(m,n), C(n+1)$ or $D(m,n)$ it follows at once, e.g. from Theorem (5.3) in [7] and using the Whiteshead's second lemma, that $H^2(\mathfrak{g},\mathbb{C})=0$ which proves the assertion of the corollary for these algebras. Similarly for other basic classical Lie superalgebras, it follows from Theorem (5.4) in [7] that $H^2(\mathfrak{g},\mathbb{C})=0$, which in turn implies $\Lambda=0$. This completes the proof. ■

We remark that the map Λ in Theorem 5.2 is not in general a group homomorphism but there is an action of the group $\mathcal{C}$ on $H^2(\mathfrak{g},\mathfrak{a})_0$ such that Λ becomes a principal derivation. We discuss this action below:
For (ϕ,ψ) in $\mathcal{C}$, define an action of (ϕ,ψ) on some $\Theta\in Z^2(\mathfrak{g},\mathfrak{a})_0$ by

$$(\phi,\psi)\cdot\Theta(x,y) := \phi(\Theta(\psi^{-1}(x),\psi^{-1}(y))) \text{ for } x,y\in\mathfrak{g}.$$

Here we recall that (ϕ,ψ) is a compatible pair if and only if $\phi\circ\rho(x)\circ\phi^{-1}=\rho(\psi(x))$ for all $x\in\mathfrak{g}$. Some routine calculation(along with the fact that ϕ and ψ are compatible) shows that $(\phi,\psi)\cdot\Theta$ is also in $Z^2(\mathfrak{g},\mathfrak{a})_0$. Moreover, if $\Theta\in B^2(\mathfrak{g},\mathfrak{a})_0$ then $(\phi,\psi)\cdot\Theta\in B^2(\mathfrak{g},\mathfrak{a})_0$, follows for the following reason: if $\Theta\in B^2(\mathfrak{g},\mathfrak{a})_0$ then $\Theta=\delta(\lambda)$ for some $\lambda\in C^1(\mathfrak{g},\mathfrak{a})_0$. Then,

$$\begin{aligned}(\phi,\psi)\cdot\Theta(x,y)&=\phi(\Theta(\psi^{-1}(x),\psi^{-1}(y)))=\phi(\delta(\lambda)(\psi^{-1}(x),\psi^{-1}(y)))\\&=\phi(\psi^{-1}(x)\cdot\lambda(\psi^{-1}(y))-(-1)^{|\psi^{-1}(x)||\psi^{-1}(y)|}\psi^{-1}(y)\cdot\lambda(\psi^{-1}(x))-\lambda[\psi^{-1}(x),\psi^{-1}(y)])\\&=\phi\rho(\psi^{-1}(x))(\lambda(\psi^{-1}(y)))-(-1)^{|x||y|}\phi\rho(\psi^{-1}(y))(\lambda(\psi^{-1}(x)))-\phi\lambda[\psi^{-1}(x),\psi^{-1}(y)]\\&=\rho(\psi\psi^{-1}(x))\phi\lambda\psi^{-1}(y)-(-1)^{|x||y|}\rho(\psi\psi^{-1}(y))\phi\lambda(\psi^{-1}(x))-\phi\lambda\psi^{-1}[x,y]\\&=x\cdot\phi\lambda\psi^{-1}(y)-(-1)^{|x||y|}y\cdot\phi\lambda\psi^{-1}(x)-\phi\lambda\psi^{-1}[x,y]\\&=\delta(\phi\lambda\psi^{-1})(x,y)\in B^2(\mathfrak{g},\mathfrak{a})_0.\end{aligned}$$

So the action keeps $B^2(\mathfrak{g},\mathfrak{a})_0$ invariant and therefore induces an action on $H^2(\mathfrak{g},\mathfrak{a})_0$ defined by $(\phi,\psi)\cdot[\Theta]:=[(\phi,\psi)\cdot\Theta]$ for $\Theta\in Z^2(\mathfrak{g},\mathfrak{a})_0$ where $[-]$ denotes cohomology class.

The next proposition shows that Λ is a principal derivation with respect to this action. For the definition of a principal derivation we refer the reader to Chapter IV, p. 89, in [4].

Proposition 5.5. *The map Λ in the Wells exact sequence is a principal derivation with respect to the above action.*

Proof. Let $(\phi,\psi)\in\mathcal{C}$. Then,

$$\theta_{\phi,\psi}(x,y)=\phi\theta(\psi^{-1}(x),\psi^{-1}(y))-\theta(x,y)=(\phi,\psi)\cdot\theta(x,y)-\theta(x,y).$$

Which implies $\Lambda(\phi,\psi)=[\theta_{\phi,\psi}]=[(\phi,\psi)\cdot\theta-\theta]=(\phi,\psi)\cdot[\theta]-[\theta]$. Consequently Λ is a principal derivation. ∎

So far we have considered only a particular extension $0\to\mathfrak{a}\xrightarrow{i}\mathfrak{e}\xrightarrow{p}\mathfrak{g}\to 0$ which induces the action of $\mathfrak{g}$ on $\mathfrak{a}$ given by ρ. But there may be many such extensions inducing the same ρ. Henceforth, we denote by $Ext_\rho(\mathfrak{g},\mathfrak{a})$, the set of all equivalence classes of extensions of $\mathfrak{g}$ by $\mathfrak{a}$ which induce ρ.

Here two extensions $\mathcal{E}_1\colon 0\to\mathfrak{a}\xrightarrow{i_1}\mathfrak{e}_1\xrightarrow{p_1}\mathfrak{g}\to 0$ and $\mathcal{E}_2\colon 0\to\mathfrak{a}\xrightarrow{i_2}\mathfrak{e}_2\xrightarrow{p_2}\mathfrak{g}\to 0$ are said to be equivalent if there exists a Lie superalgebra map $\gamma\colon\mathfrak{e}_1\to\mathfrak{e}_2$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}\mathcal{E}_1:0&\longrightarrow&\mathfrak{a}&\xrightarrow{i_1}&\mathfrak{e}_1&\xrightarrow{p_1}&\mathfrak{g}&\longrightarrow&0\\&&\|&&\downarrow{\scriptstyle\gamma}&&\|&&\\\mathcal{E}_2:0&\longrightarrow&\mathfrak{a}&\xrightarrow{i_2}&\mathfrak{e}_2&\xrightarrow{p_2}&\mathfrak{g}&\longrightarrow&0\end{array}$$

Once there is such a map γ, it is easy to see that γ automatically becomes an automorphism. It is an important fact that

$$Ext_\rho(\mathfrak{g},\mathfrak{a})\simeq H^2(\mathfrak{g},\mathfrak{a})_0 \tag{4}$$

as sets. This fact will be proved latter in the appendix. Now we conclude this section with an important corollary of the above proposition whose proof uses this fact.

Corollary 5.6. *Let $\rho\colon \mathfrak{g} \to End(\mathfrak{a})$ be a given $\mathfrak{g}$-module structure on $\mathfrak{a}$ and $(\phi,\psi) \in \mathcal{C}_\rho$. Then (ϕ,ψ) is inducible in each extension inducing ρ if and only if (ϕ,ψ) acts on $H^2(\mathfrak{g},\mathfrak{a})_0$ trivially.*

Here $\mathcal{C}_\rho$ is the set of all pairs $(\phi,\psi) \in Aut(\mathfrak{a}) \times Aut(\mathfrak{g})$ such that for the given ρ, $\phi \circ \rho(x) \circ \phi^{-1} = \rho(\psi(x))$ for all $x \in \mathfrak{g}$.

Proof. Let $\mathcal{E}$ be an extension in $Ext_\rho(\mathfrak{g},\mathfrak{a})$. Then from the proof of (4) in A, there is a 2-cocycle θ in $Z^2(\mathfrak{g},\mathfrak{a})_0$ such that $\mathcal{E}$ corresponds to $[\theta]$ in $H^2(\mathfrak{g},\mathfrak{a})_0$ and also Λ for that extension is given by $\Lambda(\phi,\psi) = [\theta_{\phi,\psi}]$. Let us suppose (ϕ,ψ) is inducible in this extension then $\Lambda(\phi,\psi) = 0$. From the above proposition then it follows that (ϕ,ψ) acts trivially on $[\theta]$. Now if (ϕ,ψ) is inducible throughout all the extensions in $Ext_\rho(\mathfrak{g},\mathfrak{a})$ then it clearly follows from (4) that (ϕ,ψ) acts trivially on $H^2(\mathfrak{g},\mathfrak{a})_0$.

Conversely, if (ϕ,ψ) acts trivially on $H^2(\mathfrak{g},\mathfrak{a})_0$ then from the proposition we get $\Lambda(\phi,\psi) = [\theta_{\phi,\psi}] = 0$ for all $\theta \in Z^2(\mathfrak{g},\mathfrak{a})_0$. Which implies by (4) that (ϕ,ψ) is inducible in all the extensions of $Ext_\rho(\mathfrak{g},\mathfrak{a})$. This completes the proof of the corollary. ■

6. Automorphisms of Heisenberg Lie superalgebras

From Theorem 3.1 we know that for an abelian extension $0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$ of Lie superalgebras, the compatibility condition is necessary for any pair $(\phi,\psi) \in Aut(\mathfrak{a}) \times Aut(\mathfrak{g})$ to be inducible. In this section we will encounter a particular situation where not all compatible pairs can be induced from automorphisms of the extension Lie superalgebra $\mathfrak{e}$. For this purpose we consider the Heisenberg Lie superalgebras and see them as 1-dimensional extensions of some abelian Lie superalgebras. It turns out that not all compatible pairs are inducible in this case. Also, as an application of Theorem 3.1, we obtain some necessary and sufficient conditions for those pairs to become inducible.

In the present section we assume our underlying field (say $\mathbb{F}$) to be algebraically closed and of characteristic different from 2 and 3. In that case, the class of finite dimensional Heisenberg Lie superalgebras (which are not Lie algebras) splits precisely into the following two families(see [1]):

- $\mathfrak{h}_{2m,n}$ having basis $\{z; x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_{2m} \,|\, y_1, y_2, \ldots, y_n\}$ where $0 \leq m \in \mathbb{Z}$, $1 \leq n \in \mathbb{Z}$ and the only non-zero brackets are given by $[x_i, x_{m+i}] = z$, $[y_j, y_j] = z$ for $1 \leq i \leq m$, $1 \leq j \leq n$.
- $\mathfrak{ba}_n$ having basis $\{x_1, x_2, \ldots, x_n \,|\, z; y_1, y_2, \ldots, y_n\}$ where $1 \leq n \in \mathbb{Z}$ and the only non-zero brackets are given by $[x_j, y_j] = z$ for $1 \leq j \leq n$.

Clearly each Lie superalgebra from any of the above two families has one dimensional center spanned by z. Also the bar in the description of the basis elements above separates the even elements from the odds, the elements left to the bar being even and right to the bar being odd. Now each of these Lie superalgebras can be seen as abelian extension of some abelian Lie superalgebra in the following way:

$$0 \to \mathcal{Z}(\mathfrak{g}) \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \frac{\mathfrak{g}}{\mathcal{Z}(\mathfrak{g})} \to 0 \tag{5}$$

where $\mathfrak{g}$ is one of the form $\mathfrak{h}_{2m,n}$ or $\mathfrak{ba}_n$. Here i and p are the obvious inclusion and projection map respectively. Also it is clear that $\mathcal{Z}(\mathfrak{g})$ and $\mathfrak{g}/\mathcal{Z}(\mathfrak{g})$ are abelian Lie superalgebras. So in particular, the above extension is an abelian extension, it is also central. Let $\widetilde{\mathfrak{g}} = \mathfrak{g}/\mathcal{Z}(\mathfrak{g})$. Now,

- if $\mathfrak{g} = \mathfrak{h}_{2m,n}$ then $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_0 \oplus \widetilde{\mathfrak{g}}_1$ where $\widetilde{\mathfrak{g}}_0 = span\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m, \bar{x}_{m+1}, \ldots, \bar{x}_{2m}\}$ and $\widetilde{\mathfrak{g}}_1 = span\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n\}$. Also in this case, $\mathcal{Z}(\mathfrak{g})_0 = span\{z\}$ and $\mathcal{Z}(\mathfrak{g})_1 = \{0\}$.
- if $\mathfrak{g} = \mathfrak{ba}_n$ then $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_0 \oplus \widetilde{\mathfrak{g}}_1$ where $\widetilde{\mathfrak{g}}_0 = span\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ and $\widetilde{\mathfrak{g}}_1 = span\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n\}$. But in this case $\mathcal{Z}(\mathfrak{g})_0 = \{0\}$ and $\mathcal{Z}(\mathfrak{g})_1 = span\{z\}$.

In the above, $\bar{x}_i = p(x_i)$ and $\bar{y}_i = p(y_i)$. Here we note that any automorphism $\phi \in Aut(\mathcal{Z}(\mathfrak{g}))$ is determined by some element $\kappa \in \mathbb{F}$, $\kappa \neq 0$, given by $\phi(z) = \kappa z$. Also, if $\psi \in Aut(\widetilde{\mathfrak{g}})$ i.e., $\psi \colon \widetilde{\mathfrak{g}}_0 \oplus \widetilde{\mathfrak{g}}_1 \to \widetilde{\mathfrak{g}}_0 \oplus \widetilde{\mathfrak{g}}_1$ and ψ is an automorphism then $\psi = \psi_0 \oplus \psi_1$ such that $\psi_0 \in Aut(\widetilde{\mathfrak{g}}_0)$ and $\psi_1 \in Aut(\widetilde{\mathfrak{g}}_1)$ for unique ψ_0 and ψ_1. We will denote the matrix of ψ_0 by $[\psi_0]$ with respect to the basis $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m, \bar{x}_{m+1}, \ldots, \bar{x}_{2m}\}$ if $\mathfrak{g} = \mathfrak{h}_{2m,n}$ and with respct to the basis $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ if $\mathfrak{g} = \mathfrak{ba}_n$. The matrix of ψ_1 will be denoted by $[\psi_1]$ with respect to the basis $\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n\}$ in both the cases. Keeping the above discussion in mind, we now state our next theorem which is the main result of this section:

Theorem 6.1. *Consider the abelian extension* $0 \to \mathcal{Z}(\mathfrak{g}) \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \widetilde{\mathfrak{g}} \to 0$ *where* $\mathfrak{g}$ *is of the form* $\mathfrak{h}_{2m,n}$ *or* $\mathfrak{ba}_n$. *Let* $(\phi, \psi) \in Aut(\mathcal{Z}(\mathfrak{g})) \times Aut(\widetilde{\mathfrak{g}})$ *and let* ϕ *be determined by* $\kappa(\neq 0)$. *Also let* $[\psi_0]$ *and* $[\psi_1]$ *be the matrices of* ψ_0 *and* ψ_1 *respectively, described in the preceding discussion. Then* (ϕ, ψ) *is an inducible pair if and only if*

(*Case* $\mathfrak{g} = \mathfrak{h}_{2m,n}$) *the following two conditions hold:*

(1) $A^tD - C^tB = \kappa I_{m\times m}$, *and* A^tC *and* B^tD *are symmetric, where*

$$[\psi_0]_{2m\times 2m} = \begin{bmatrix} A_{m\times m} & B_{m\times m} \\ C_{m\times m} & D_{m\times m} \end{bmatrix}.$$

(2) $[\psi_1]^t[\psi_1] = \kappa I_{n\times n}$.

(*Case* $\mathfrak{g} = \mathfrak{ba}_n$) $[\psi_0]^t[\psi_1] = \kappa I_{n\times n}$.

Proof. Throughout the proof we fix a homogeneous section $s \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ of degree 0 given by $s(\bar{x}_i) = x_i$ and $s(\bar{y}_i) = y_i$ for all possible i. Now as the induced action of $\widetilde{\mathfrak{g}}$ on $\mathcal{Z}(\mathfrak{g})$ is independent of the choice of the section, it must be given by $x \cdot a := [s(x), a]$ for all $x \in \widetilde{\mathfrak{g}}$, $a \in \mathcal{Z}(\mathfrak{g})$. But as $a \in \mathcal{Z}(\mathfrak{g})$, $x \cdot a = 0$ for all x, a. Therefore the representation $\rho \colon \widetilde{\mathfrak{g}} \to \mathtt{End}(\mathcal{Z}(\mathfrak{g}))$ coming out of this action is given by $\rho(x) = 0$ for all x. From this it is clear that for any pair $(\phi, \psi) \in Aut(\mathcal{Z}(\mathfrak{g})) \times Aut(\widetilde{\mathfrak{g}})$, the corresponding diagram in the second condition of Theorem 3.1 commutes. So any pair in our consideration is a campatible pair. Also, since the Lie superalgebra $\widetilde{\mathfrak{g}}$ is abelian and the action of $\widetilde{\mathfrak{g}}$ on $\mathcal{Z}(\mathfrak{g})$ is trivial, any 2-coboundary is trivial in this case. Hence by Theorem 3.1, a pair (ϕ, ψ) is inducible if and only if $\phi \circ \theta \circ (\psi^{-1}, \psi^{-1})(x, y) = \theta(x, y)$, or equivalently

$$\phi \circ \theta(x, y) = \theta \circ (\psi, \psi)(x, y) \ \forall \text{ homogeneous basis elements } x, y \in \widetilde{\mathfrak{g}}. \tag{6}$$

Below we show that this condition is equivalent to the above conditions mentioned in the theorem.

• (Case $\mathfrak{g} = \mathfrak{h}_{2m,n}$). Suppose the condition (6) holds. Then $\phi \circ \theta(\bar{x}_i, \bar{x}_j) = \theta(\psi(\bar{x}_i), \psi(\bar{x}_j))$. Now $\theta(\bar{x}_i, \bar{x}_j) = [s(\bar{x}_i), s(\bar{x}_j)] - s[\bar{x}_i, \bar{x}_j] = [x_i, x_j]$ implies that

$$\theta(\psi(\bar{x}_i), \psi(\bar{x}_j)) = \phi([x_i, x_j]) \text{ for all } 1 \leq i, j \leq 2m. \tag{7}$$

Let the matrices A, B, C, D be given by $(a_{k,l}), (b_{k,l}), (c_{k,l}), (d_{k,l}); 1 \leq k, l \leq m$, respectively, where

$$[\psi_0] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

as mentioned above. Now if $1 \leq i \leq m\,;\, m+1 \leq j \leq 2m$ then

$$\psi(\bar{x}_i) = \psi_0(\bar{x}_i) = \sum_{k=1}^{m} a_{k,i}\bar{x}_k + \sum_{k=1}^{m} c_{k,i}\bar{x}_{m+k}$$

and
$$\psi(\bar{x}_j) = \psi_0(\bar{x}_j) = \sum_{k=1}^{m} b_{k,j-m}\bar{x}_k + \sum_{k=1}^{m} d_{k,j-m}\bar{x}_{m+k}.$$

From this we get, $\theta(\psi(\bar{x}_i), \psi(\bar{x}_j)) =$

$$\begin{aligned} &= \left[s\left(\sum_{k=1}^{m} a_{k,i}\bar{x}_k + \sum_{k=1}^{m} c_{k,i}\bar{x}_{m+k} \right),\ s\left(\sum_{k=1}^{m} b_{k,j-m}\bar{x}_k + \sum_{k=1}^{m} d_{k,j-m}\bar{x}_{m+k} \right) \right] \\ &= \left[\sum_{k=1}^{m} a_{k,i}x_k + \sum_{k=1}^{m} c_{k,i}x_{m+k}\,,\ \sum_{k=1}^{m} b_{k,j-m}x_k + \sum_{k=1}^{m} d_{k,j-m}x_{m+k} \right] \\ &= \sum_{k=1}^{m} (a_{k,i}d_{k,j-m} - c_{k,i}b_{k,j-m})[x_k, x_{m+k}] = \sum_{k=1}^{m} (a_{k,i}d_{k,j-m} - c_{k,i}b_{k,j-m})z. \end{aligned}$$

Then from (7) it follows that

$$\sum_{k=1}^{m} (a_{k,i}d_{k,j-m} - c_{k,i}b_{k,j-m}) = \begin{cases} \kappa & \text{if } j = m+i; \\ 0 & \text{otherwise} \end{cases}$$

or equivalently

$$\sum_{k=1}^{m} (a_{k,i}d_{k,l} - c_{k,i}b_{k,l}) = \begin{cases} \kappa & \text{if } l = i; \\ 0 & \text{otherwise} \end{cases}.$$

The last equation is equivalent to the fact that

$$A^t D - C^t B = \kappa I_{m \times m}. \tag{8}$$

If $1 \leq i, j \leq m$ then $\psi(\bar{x}_i) = \psi_0(\bar{x}_i) = \sum_{k=1}^{m} a_{k,i}\bar{x}_k + \sum_{k=1}^{m} c_{k,i}\bar{x}_{m+k}$ as before but $\psi(\bar{x}_j) = \psi_0(\bar{x}_j) = \sum_{k=1}^{m} a_{k,j}\bar{x}_k + \sum_{k=1}^{m} c_{k,j}\bar{x}_{m+k}$. Now in a similar way we get,

$$\theta(\psi(\bar{x}_i), \psi(\bar{x}_j)) = \sum_{k=1}^{m} (a_{k,i}c_{k,j} - c_{k,i}a_{k,j})z.$$

Then again from (7) it follows that $\sum_{k=1}^{m}(a_{k,i}c_{k,j} - c_{k,i}a_{k,j}) = 0$ for all $1 \leq i, j \leq m$. Clearly this last condition is equivalent to saying

$$A^tC - C^tA = 0, \text{ i.e., } A^tC \text{ is symmetric.} \tag{9}$$

Now the only case left is $m+1 \leq i, j \leq 2m$. In this case

$$\psi(\bar{x}_i) = \psi_0(\bar{x}_i) = \sum_{k=1}^{m} b_{k,i-m}\bar{x}_k + \sum_{k=1}^{m} d_{k,i-m}\bar{x}_{m+k}$$

and $$\psi(\bar{x}_j) = \psi_0(\bar{x}_j) = \sum_{k=1}^{m} b_{k,j-m}\bar{x}_k + \sum_{k=1}^{m} d_{k,j-m}\bar{x}_{m+k}.$$

Then proceeding exactly as above we get the condition

$$B^tD - D^tB = 0 \text{ or } B^tD \text{ is symmetric.} \tag{10}$$

The only remaining condition in this case is $[\psi_1]^t[\psi_1] = \kappa I_{n\times n}$, which can be obtained by considering the basis elements $\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n\}$ as follows:
Let the matrix of ψ_1 be given by $[\psi_1] = (u_{i,j})$; $1 \leq i, j \leq n$. Then

$$\psi(\bar{y}_i) = \psi_1(\bar{y}_i) = \sum_{k=1}^{n} u_{k,i}\bar{y}_k \text{ for all } 1 \leq i \leq n.$$

Therefore, following a similar method as above, we write,

$$\theta(\psi(\bar{y}_i), \psi(\bar{y}_j)) = \left[\sum_{k=1}^{n} u_{k,i}y_k\, ,\, \sum_{l=1}^{n} u_{l,j}y_l\right] = \sum_{k=1}^{n} u_{k,i}u_{k,j}[y_k, y_k] = \sum_{k=1}^{n} u_{k,i}u_{k,j}z.$$

From this relation using (6) we obtain

$$\sum_{k=1}^{n} u_{k,i}u_{k,j} = \begin{cases} \kappa & \text{if } i = j; \\ 0 & \text{for } i \neq j \end{cases}$$

which is clearly equivalent to the condition

$$[\psi_1]^t[\psi_1] = \kappa I_{n\times n}. \tag{11}$$

For the converse, it is very easy to see that the conditions (8),(9) and (10) are actually equivalent to (7) or in turn, to (6) for all $x, y \in \widetilde{\mathfrak{g}}_0$. Also the condition (11) is equivalent to (6) for all $x, y \in \widetilde{\mathfrak{g}}_1$. Now if $x, y \in \widetilde{\mathfrak{g}}$ are homogeneous and of different degree i.e., $|x| \neq |y|$ then obviously $|\psi(x)| \neq |\psi(y)|$ and therefore $|s(x)| \neq |s(y)|$ and $|s(\psi(x))| \neq |s(\psi(y))|$. From the structure of $\mathfrak{g}$ then it clearly follows that $[s(x), s(y)] = [s(\psi(x)), s(\psi(y))] = 0$ which in turn implies $\theta(x, y) = \theta(\psi(x), \psi(y)) = 0$. So (6) automatically holds for all homogeneous $x, y \in \widetilde{\mathfrak{g}}$ with $|x| \neq |y|$.
This proves the theorem in the case of $\mathfrak{g} = \mathfrak{h}_{2m,n}$.

• (Case $\mathfrak{g} = \mathfrak{ba}_n$). Assume again that condition (6) holds in this case. Also, let the matrices of ψ_0 and ψ_1 be given by $[\psi_0] = (a_{i,j})$ and $[\psi_1] = (b_{i,j})$ for $1 \leq i, j \leq n$. Then clearly $\psi(\bar{x}_i) = \psi_0(\bar{x}_i) = \sum_{k=1}^{n} a_{k,i}\bar{x}_k$ and $\psi(\bar{y}_j) = \psi_1(\bar{y}_j) = \sum_{l=1}^{n} b_{l,j}\bar{y}_l$. Now,

$$\theta(\psi(\bar{x}_i), \psi(\bar{y}_j)) = \left[\sum_{k=1}^{n} a_{k,i}x_k\, ,\, \sum_{l=1}^{n} b_{l,j}y_j\right] = \sum_{k=1}^{n} a_{k,i}b_{k,j}[x_k, y_k] = \sum_{k=1}^{n} a_{k,i}b_{k,j}z.$$

On the other hand, as before, $\phi \circ \theta(\bar{x}_i, \bar{y}_j) = \phi([s(\bar{x}_i), s(\bar{y}_j)] - s[\bar{x}_i, \bar{y}_j]) = \phi([x_i, x_j])$. Then from (6) we get $\sum_{k=1}^{n} a_{k,i} b_{k,j} = \begin{cases} \kappa & \text{if } i = j; \\ 0 & \text{otherwise} \end{cases}$ which is equivalent to

$$[\psi_0]^t [\psi_1] = \kappa I_{n \times n}. \tag{12}$$

Conversely, this condition clearly implies (6) for $x, y \in \widetilde{\mathfrak{g}}$ with $|x| \neq |y|$. Now if $x, y \in \widetilde{\mathfrak{g}}_0$ then so is $\psi(x), \psi(y)$ and therefore $s(x), s(y), s(\psi(x)),\ s(\psi(y))$ all belong to $\mathfrak{g}_0$. From the structure of $\mathfrak{g}$ then $\theta(x, y) = \theta(\psi(x), \psi(y)) = 0$ which in turn implies that (6) holds for $x, y \in \widetilde{\mathfrak{g}}_0$. Similar arguments show that the same is true for $x, y \in \widetilde{\mathfrak{g}}_1$. This completes the proof of the theorem for $\mathfrak{g} = \mathfrak{ba}_n$. ■

A. Abelian extensions and the second cohomology

Several proofs of the fact (4) have appeared in the literature already, for example see [8],[17]. But in these papers the fact was proved particularly for central extensions, in which case the action of $\mathfrak{g}$ on $\mathfrak{a}$ is trivial. Also the proof in [8] contained some error which was later fixed in [9]. Therefore, we present here a detailed proof of the above fact (4) for a more general class of extensions, namely the abelian extensions.

Theorem A.1. *Let $\mathfrak{a}$ be a $\mathfrak{g}$-module where the action is given by $\rho\colon \mathfrak{g} \to \mathit{End}(\mathfrak{a})$. Let $Ext_\rho(\mathfrak{g}, \mathfrak{a})$ be the set of all equivalence classes of extensions of $\mathfrak{g}$ by $\mathfrak{a}$ inducing the action ρ considering $\mathfrak{a}$ as an abelian Lie superalgebra. Then $Ext_\rho(\mathfrak{g}, \mathfrak{a}) \simeq H^2(\mathfrak{g}, \mathfrak{a})_0$ as sets.*

Proof. We denote the action of $\mathfrak{g}$ on $\mathfrak{a}$ by ρ, so $\rho(x)(a) = x \cdot a$ for $x \in \mathfrak{g}, a \in \mathfrak{a}$. Now let $\mathcal{E}\colon 0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \to 0$ be an extension in $Ext_\rho(\mathfrak{g}, \mathfrak{a})$ and let s be a homogeneous section of p of degree 0. Then $x \cdot a = [s(x), a]$ for all $x \in \mathfrak{g}, a \in \mathfrak{a}$. Let us define $\theta(x, y) = [s(x), s(y)] - s[x, y]$. Then from Lemma 4.1 we know that $\theta \in Z^2(\mathfrak{g}, \mathfrak{a})_0$. So, to each extension $\mathcal{E}$ in $Ext_\rho(\mathfrak{g}, \mathfrak{a})$, we can assign an element $[\theta]$ of $H^2(\mathfrak{g}, \mathfrak{a})_0$ where $[\theta]$ represnts the cohomology class of θ. Now we show that this assignment does not depend on the choice of the section s. For that let s_1, s_2 be two such sections of p and θ_1, θ_2 be the corresponding 2-cocycles in $Z^2(\mathfrak{g}, \mathfrak{a})_0$. Now,

$$\begin{aligned} \delta(s_1 - s_2)(x, y) &= x \cdot (s_1 - s_2)(y) - (-1)^{|x||y|} y \cdot (s_1 - s_2)(x) - (s_1 - s_2)([x, y]) \\ &= [s_1(x), (s_1 - s_2)(y)] - (-1)^{|x||y|} [s_2(y), (s_1 - s_2)(x)] - (s_1 - s_2)([x, y]) \\ &= [s_1(x), s_1(y)] - [s_1(x), s_2(y)] - [s_2(x), s_2(y)] + [s_1(x), s_2(y)] - (s_1 - s_2)([x, y]) \\ &= \theta_1(x, y) - \theta_2(x, y). \end{aligned}$$

Here we have used the fact that the action does not depend on the choice of section which was proved in Lemma 2.3. This shows that θ_1 and θ_2 differ by a 2-coboundary and therefore $[\theta_1] = [\theta_2]$. Hence, we have a well-defined set map $\mathcal{E} \mapsto [\theta] \in H^2(\mathfrak{g}, \mathfrak{a})_0$. Conversely, corresponding to each 2-cocycle $\theta \in Z^2(\mathfrak{g}, \mathfrak{a})_0$, we construct an extension in $Ext_\rho(\mathfrak{g}, \mathfrak{a})$ in the following way: first we construct a Lie superalgebra whose underlying vector space is $\mathfrak{g} \oplus \mathfrak{a}$ and the bracket is given by

$$[(x, a), (y, b)] = ([x, y], x \cdot b - (-1)^{|y||a|} y \cdot a + \theta(x, y)) \text{ for } x \in \mathfrak{g}, a \in \mathfrak{a}.$$

One can check that this bracket does make $\mathfrak{g} \oplus \mathfrak{a}$ into a Lie superalgebra, we denote this Lie superalgebra by $\mathfrak{g} \oplus_\theta \mathfrak{a}$.

Consider the sequence $\mathcal{E}_\theta\colon 0 \to \mathfrak{a} \xrightarrow{i} \mathfrak{g} \oplus_\theta \mathfrak{a} \xrightarrow{p} \mathfrak{g} \to 0$ where $i(a) = (0,a)$ and $p(x,a) = x$. Clearly i and p are Lie superalgebra maps which make this sequence exact. Also $s(x) := (x,0)$ is a section of p and the induced action of $\mathfrak{g}$ on $\mathfrak{a}$ is given by $[s(x),a] = [(x,0),(0,a)] = (0, x\cdot a) = \rho(x)(a)$ (here we identify a with $i(a)$). Which implies $\mathcal{E}_\theta \in Ext_\rho(\mathfrak{g},\mathfrak{a})$ and this gives a set map $\theta \mapsto \mathcal{E}_\theta \in Ext_\rho(\mathfrak{g},\mathfrak{a})$. Now if we start with some $\mathcal{E} \in Ext_\rho(\mathfrak{g},\mathfrak{a})$ and $\mathcal{E} \mapsto [\theta]$ then it can be easily checked that $\theta \mapsto \mathcal{E}$ itself under this map. Consequently, $\theta \mapsto \mathcal{E}_\theta$ gives a map onto $Ext_\rho(\mathfrak{g},\mathfrak{a})$.

Now we shall prove that if for two 2-cocycles $\theta_1,\theta_2 \in Z^2(\mathfrak{g},\mathfrak{a})_0$, $\theta_1 - \theta_2$ is a 2-coboundary in $B^2(\mathfrak{g},\mathfrak{a})_0$ then $\mathcal{E}_{\theta_1}$ and $\mathcal{E}_{\theta_2}$ are equivalent extensions. For that, let $\theta_1 - \theta_2 = \delta(\lambda)$ for some $\lambda \in C^1(\mathfrak{g},\mathfrak{a})_0$. From the discussion in Section 2.1, such a λ always exists.

Let us define a map $\gamma\colon \mathfrak{g}\oplus_{\theta_1}\mathfrak{a} \to \mathfrak{g}\oplus_{\theta_2}\mathfrak{a}$ by $\gamma(x,a) := (x, a+\lambda(x))$. Clearly, with this γ the following diagram is commutative:

$$\begin{array}{ccccccccc}
\mathcal{E}_{\theta_1} : 0 & \longrightarrow & \mathfrak{a} & \xrightarrow{i} & \mathfrak{g}\oplus_{\theta_1}\mathfrak{a} & \xrightarrow{p} & \mathfrak{g} & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle\gamma} & & \| & & \\
\mathcal{E}_{\theta_2} : 0 & \longrightarrow & \mathfrak{a} & \xrightarrow{i} & \mathfrak{g}\oplus_{\theta_2}\mathfrak{a} & \xrightarrow{p} & \mathfrak{g} & \longrightarrow & 0
\end{array}$$

where i and p are described as above. To show these extensions are equivalent, what else we need to show is that the map γ is a map of Lie superalgebras. Let us take two homogeneous elements $(x,a),(y,b) \in \mathfrak{g}\oplus_{\theta_1}\mathfrak{a}$. We note that, (x,a) is homogeneous implies $|x| = |a| = |a+\lambda(x)|$. Now,

$$\begin{aligned}
\gamma[(x,a),(y,b)] &= \gamma([x,y], x\cdot b - (-1)^{|y||a|} y\cdot a + \theta_1(x,y))\\
&= ([x,y], x\cdot b - (-1)^{|y||a|} y\cdot a + \theta_1(x,y) + \lambda[x,y]).
\end{aligned} \tag{13}$$

On the other hand,

$$\begin{aligned}
[\gamma(x,a),\gamma(y,b)] &= [(x, a+\lambda(x)),(y, b+\lambda(y))]\\
&= ([x,y], x\cdot(b+\lambda(y)) - (-1)^{|y||a+\lambda(x)|} y\cdot(a+\lambda(x)) + \theta_2(x,y))\\
&= ([x,y], x\cdot b - (-1)^{|y||a|} y\cdot a + x\cdot\lambda(y) - (-1)^{|y||x|} y\cdot\lambda(x) + \theta_2(x,y))\\
&= ([x,y], x\cdot b - (-1)^{|y||a|} y\cdot a + \delta(\lambda)(x,y) + \lambda[x,y] + \theta_2(x,y))\\
&= ([x,y], x\cdot b - (-1)^{|y||a|} y\cdot a + (\theta_1-\theta_2)(x,y) + \lambda[x,y] + \theta_2(x,y))\\
&= ([x,y], x\cdot b - (-1)^{|y||a|} y\cdot a + \theta_1(x,y) + \lambda[x,y]).
\end{aligned} \tag{14}$$

From (13) and (14) it follows that γ is a Lie superalgebra homomorphism and hence become an automorphism by the commutativity of the above diagram. Also, it is very easy to directly see that γ is bijective. All these imply $\mathcal{E}_{\theta_1}$ and $\mathcal{E}_{\theta_2}$ are equivalent. As a consequence we get a well-defined set map $[\theta] \mapsto \mathcal{E}_\theta$ from $H^2(\mathfrak{g},\mathfrak{a})_0$ onto $Ext_\rho(\mathfrak{g},\mathfrak{a})$.

To complete the proof of the theorem we are just left to show that $[\theta] \mapsto \mathcal{E}_\theta$ is one-one. To show this let $\mathcal{E}_{\theta_1}$ and $\mathcal{E}_{\theta_2}$ are equivalent for some $[\theta_1],[\theta_2] \in H^2(\mathfrak{g},\mathfrak{a})_0$. Then there is an automorphism $\gamma\colon \mathfrak{g}\oplus_{\theta_1}\mathfrak{a} \to \mathfrak{g}\oplus_{\theta_2}\mathfrak{a}$ such that the corresponding diagram is commutative. Clearly then $\gamma(0,a) = (0,a)$ and $\gamma(x,0) = (x,\varphi(x))$ for some $\varphi\colon \mathfrak{g}\to\mathfrak{a}$. It is easy to see that $\varphi \in C^1(\mathfrak{g},\mathfrak{a})_0$. Now,

$$([x,y], x\cdot\varphi(y) - (-1)^{|x||y|} y\cdot\varphi(x) + \theta_2(x,y)) = [(x,\varphi(x)),(y,\varphi(y))]$$
$$= [\gamma(x,0),\gamma(y,0)] = \gamma[(x,0),(y,0)] = \gamma([x,y],\theta_1(x,y)) = ([x,y],\theta_1(x,y)+\varphi[x,y]).$$

This implies

$$x\cdot\varphi(y) - (-1)^{|x||y|} y\cdot\varphi(x) + \theta_2(x,y) = \theta_1(x,y) + \varphi[x,y]$$

i.e., $\theta_1(x,y) - \theta_2(x,y) = x\cdot\varphi(y) - (-1)^{|x||y|} y\cdot\varphi(x) - \varphi[x,y] = \delta(\varphi)(x,y).$

So $\theta_1 - \theta_2 \in B^2(\mathfrak{g},\mathfrak{a})_0$, hence $[\theta_1] = [\theta_2]$. This completes the proof of the theorem. ∎

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