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Schrödinger-Type Equations and Unitary Highest Weight Representations of $U(n,n)$

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Abstract. A system of differential equations is defined and the solutions to this system in a certain induced space is shown to be isomorphic to the well-known models of unitary highest weight representations of $U(n,n)$ studied by Kashiwara and Vergne.

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1. Introduction

Many authors have contributed to the highly developed theory of unitarizable highest weight modules. Although the complete classification [5],[11] is algebraic in nature, the analytic view was originally developed by Harish-Chandra [7] who studied spaces of holomorphic sections in vector bundles over G/K. In a seminal article [13], Kashiwara and Vergne realized unitary highest weight spaces as (in the case of $U(n,n)$) $\mathscr{L}^2$-spaces of vector-valued functions. It was later proved in [4] that this construction was exhaustive. This article gives another realization of unitary highest weight spaces as solutions to systems of Schrödinger-type differential equations. The spaces constructed here are shown to be equivalent to the Kashiwara and Vergne realization. However, there are fundamental differences. For example, the $G = U(n,n)$ action in [13] comes from an application of the Stone-von Neumann theorem. As such, this action of $g \in U(n,n)$ is given as an intertwining operator for equivalent irreducible representations of the Heisenberg group. This theme is also present in the Fock model construction of the harmonic oscillator representation [1]. In the present construction, we induce from representations of subgroups of the semi-direct product $U(n,n) \ltimes H_n$, where H_n is the Heisenberg group. Consequently, the action of $U(n,n)$ given here is the natural one arising from the left regular representation.

We obtain the desired subspace of the induced space by taking solutions of systems of Schrödinger-type differential equations. Using differential equations to reduce some initial space is a common theme in the theory of unitary hightest representations. Covariant differential operators ([2], [3], [12]) can be used to find irreducible representations associated to a reduction point. Other differential equations, such as

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the wave equation and Schrödinger equations have been used to find unitary highest weight representations of various groups ([8], [9], [10], [14], [15]). The systems of Schrödinger equations found in [10] were given in terms of real variables and were used to construct representations of the metaplectic group. Due to structural similarities of Lie algebras, there are corresponding similarities between the systems in [10] and the systems found here, the main difference being the equations employed here involve derivatives with respect to complex variables both in position and time. For example (cf. Section 4), we have

$$i\gamma\frac{\partial f}{\partial \bar{t}_{ij}} = \sum_{m=1}^{k} \frac{\partial^2 f}{\partial \bar{z}_{im}\partial z_{jm}}, \text{ for } 1 \le i < j \le n.$$

We briefly outline the contents without specifying all the parameters (see the body of the paper). After setting the notation and defining certain subgroups of $U(n,n)$ and the Heisenberg group, $H_n = \mathbb{C}^{2n\times k} \times \mathbb{R}$, in Section 2, we define an induced representation for $U(n,n) \ltimes H_n$, also carrying an action of $U(k)$, in Section 3. By restricting the functions of the induced representation to $\mathrm{Her}_n \times \mathbb{C}^{n\times k}$, where Her_n denotes the space of Hermitian symmetric matrices, we realize the induced representation, $\mathcal{E}$, as sitting in $\mathscr{C}^\infty\left(\mathrm{Her}_n \times \mathbb{C}^{n\times k}\right)$.

Write $\mathscr{S}ol$ for the solutions to our system of Schrödinger-type equations in the space $\mathscr{C}^\infty\left(\mathrm{Her}_n \times \mathbb{C}^{n\times k}\right)$. The rest of Section 4 establishes the invariance of this system under the action of $U(k)$, H_n, and $U(n,n)$. In Section 5, we define a map T from the Schwartz functions of $\mathscr{L}^2(\mathbb{C}^{n\times k})$ into $\mathscr{S}ol$. The map is constructed via standard Fourier transform techniques and turns out to be an intertwining map for $U(n,n)$.

A point to settle in Section 6 is whether T actually maps into $\mathcal{H} = \mathscr{S}ol \bigcap \mathcal{E}$. This indeed happens for a dense subset of $\mathscr{L}^2$ that is defined in terms of Kashiwara and Vergne's pluriharmonic polynomials.

Generalizing these results, the action of $U(k)$ decomposes $\mathcal{H}$ into a sum of spaces of the form $V_\lambda^* \otimes \mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda)$ where $\lambda \in \hat{U}(k)$, V_λ is the corresponding irreducible representation of $U(k)$ with highest weight λ, and $\mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda)$ are appropriate vector valued solutions to the system of equations that come from the induced representation and have appropriate $U(k)$-invariance. Now $\mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda)$ carries an action of $U(n,n)$. Write $\mathscr{L}^2(\mathbb{C}^{n\times k}, V_\lambda)^{U(k)}$ for the V_λ-valued $\mathscr{L}^2$-functions with an appropriate $U(k)$-invariance. Then there is an analogous intertwining operator, T_λ, defined on the Schwartz functions of $\mathscr{L}^2(\mathbb{C}^{n\times k}, V_\lambda)^{U(k)}$ that maps to $\mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda)$. This operator will complete to an isomorphism of representations for appropriate λ. The inverse map is given by evaluation at $t = 0$ composed with an inverse Fourier transform.

By work of Kashiwara and Vergne, each λ corresponds to highest a weight τ for the maximal subgroup K of G and corresponding irreducible representation W_τ. The space $\mathscr{O}(G/K, W_\tau \otimes \det^k)$ of holomorphic, vector-valued functions on G/K is a classical realization of a unitary highest weight representation for G. Kashiwara and Vergne define an intertwining isomorphism, KV_λ (cf. Proposition 4.4 [13]), from $\mathscr{L}^2(\mathbb{C}^{n\times k}, V_\lambda)^{U(k)}$ to $\mathscr{O}(G/K, W_\tau \otimes \det^k)$.

There is a related noncompact picture of an induced representation for just G, $\mathscr{I}(\mathrm{Her}_n, W_\tau \otimes \det^k)$. Taking boundary values to $z = 0$ gives a map, B, that turns out to be an injective G-map from $\mathscr{O}(G/K, W_\tau \otimes \det^k)$ to $\mathscr{I}(\mathrm{Her}_n, W_\tau \otimes \det^k)$.

Finally, there is a polyomial differential operator that evaluates at $z=0$, D_λ, defined in Section 6 that is a G-map from $\mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda)$ to $\mathscr{I}(\mathrm{Her}_n, W_\tau \otimes \det^k)$.

Putting everything together, we arrive at the following commutative digram of G-maps.

$$\begin{array}{ccc} \overline{\mathcal{H}}(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda) & \xhookrightarrow{D_\lambda} & \mathscr{I}(\mathrm{Her}_n, W_\tau \otimes \det^k) \\ {\scriptstyle T_\lambda}\uparrow & & \uparrow{\scriptstyle B} \\ \mathscr{L}^2(\mathbb{C}^{n\times k}, V_\lambda)^{U(k)} & \xrightarrow[\mathrm{KV}_\lambda]{} & \mathscr{O}(G/K, W_\tau \otimes \det^k) \end{array}$$

The maps T_λ and KV_λ are isomorphisms and the maps $D_\lambda|_{z=0}$ and B are embeddings.

2. Groups

Realizations of $U(n,n)$. Define two groups G and G_1 by

$$G = \left\{ g \in GL(2n,\mathbb{C}) \,\middle|\, g \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix} g^* = \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix} \right\}$$

and
$$G_1 = \left\{ g \in GL(2n,\mathbb{C}) \,\middle|\, g \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix} g^* = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \right\},$$

respectively. The group G_1 is the standard realization of $U(n,n)$. The group G is isomorphic to the group G_1 and the isomorphism is given via the Cayley transform. More precisely, if $\mathbf{c} \in U(2n)$ is the element

$$\mathbf{c} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix},$$

then $G = \mathbf{c}\, G_1 \mathbf{c}^*$. This follows from the identity

$$\mathbf{c} \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix} \mathbf{c}^* = \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix}.$$

Write $g \in \mathbb{C}^{2n\times 2n}$ in block form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},\ A, B, C, D \in \mathbb{C}^{n\times n}. \tag{1}$$

If $g \in G$, one has relations

$$AB^* = BA^*, \quad CD^* = DC^*, \quad AD^* - BC^* = I_n. \tag{2}$$

Since $g \in G$ if and only if $g^* \in G$, one also has

$$A^*C = C^*A, \quad B^*D = D^*B, \quad A^*D - C^*B = I_n. \tag{3}$$

We will consider the maximal compact subgroup

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A + iB \in U(n) \right\},$$

abelian subgroups

$$\overline{N} = \left\{ \overline{n}(c) = \begin{pmatrix} I & 0 \\ c & I \end{pmatrix} \mid c^* = c \right\}, N = \left\{ n(t) = \begin{pmatrix} I_n & t \\ 0 & I_n \end{pmatrix} \mid t^* = t \right\} \quad (4)$$

and
$$MA = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1,*} \end{pmatrix} \mid a \in \mathrm{GL}\,(n, \mathbb{C}) \right\}. \quad (5)$$

Since MA normalizes $\overline{N}$, $MA\overline{N}$ is a subgroup of G.

The Heisenberg group. Let $j = \begin{pmatrix} 0 & -iI_n \\ iI_n & 0 \end{pmatrix}$.

For integers $n, k \geq 1$, define the hermitian form on $\mathbb{C}^{2n\times k}$ by $h(u,v) = \mathrm{Tr}(v^* ju)$ and the skew form $\sigma(u,v) = -\operatorname{Im} h(u,v)$, for $u, v \in \mathbb{C}^{2n\times k}$. The product for the Heisenberg group $H_n = \mathbb{C}^{2n\times k} \times \mathbb{R}$ is given by

$$(u,x).(v,s) = (u+v, x+s+\sigma(u,v)). \quad (6)$$

The group $G \times U(k)$ acts on H_n in a natural way. For $(g,\kappa) \in G \times U(k)$, one has $h(gu\kappa, gv\kappa) = h(u,v)$, and we define an automorphism $\psi_{(g,\kappa)}$ of H_n by

$$\psi_{(g,\kappa)}(u,x) = (gu\kappa^{-1}, x), \text{ for } (g,\kappa) \in G \times U(k), (u,x) \in H_n. \quad (7)$$

For $g \in G$, we will simply write $g.(u,x)$ for (gu,x) and $\kappa.(u,x)$ for $(u\kappa^{-1}, x)$. Write elements $u \in \mathbb{C}^{2n\times k}$ in stacked form: $u = \binom{z}{w}$, for $z, w \in \mathbb{C}^{n\times k}$.
We define abelian subgroups

$$W = \left\{ \omega(w,x) = \left(\begin{pmatrix} 0 \\ w \end{pmatrix}, x \right) \mid w \in \mathbb{C}^{n\times k}, x \in \mathbb{R} \right\},$$
$$Z = \left\{ \zeta(z) = \left(\begin{pmatrix} z \\ 0 \end{pmatrix}, 0 \right) \mid z \in \mathbb{C}^{n\times k} \right\}$$

The two subgroups MA and $\overline{N}$ stabilize W and so $MA\overline{N}$ is a subgroup of $\mathrm{Stab}(W)$. Using (2) and (3) one can show $MA\overline{N} = \mathrm{Stab}(W)$.
Since G consists of automorphisms of H_n, we can form the semidirect product $G \ltimes H_n$, where the group product is given by $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, (g_2^{-1}.h_1)h_2)$.
It follows that $\overline{P} = MA\overline{N} \ltimes W$ is a subgroup of $G \ltimes H_n$.
Observe that N pointwise fixes elements of the abelian subgroup Z so $N \ltimes Z$ is an abelian subgroup of $G \ltimes H_n$. Of course, we have $N \ltimes Z = N \times Z$.
Continuing to write elements of G in block form (1), define

$$G' = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \mid D \text{ is invertible} \right\}.$$

We have the following decompositions, the first being a special case of the second.

Lemma 2.1. *There are decompositions $G' = NMA\overline{N}$ and $G' \ltimes H_n = (N \ltimes Z)\overline{P}$. In particular, $(N \ltimes Z)\overline{P}$ is open and dense in $G \ltimes H_n$.*

Proof. Given $g \in G'$ and $h = (\binom{u_1}{u_2}, s) \in H_n$, one readily checks the equation

$$(g,h) = n(t)\zeta(z)m(a)\overline{n}(c)\omega(w,x), \quad \text{where}$$

$$a = D^{*-1}, \quad c = D^{-1}C, \quad t = BD^{-1}, \quad z = D^{*-1}u_1$$
$$w = u_2 + D^{-1}Cu_1, \quad x = s + \mathrm{Re}(u_1, u_2) + (D^{-1}Cu_1, u_1).$$

Here $(u_1, u_2) = \mathrm{Tr}(u_2^* u_1)$. It follows from group properties (2) and (3) that both c and t are hermitian. The proof of the converse is routine and is omitted. ■

3. Induced spaces

For nonzero $\gamma \in \mathbb{R}$ and integer $k > 0$, define the character $\chi_{\gamma,k} : \overline{P} \to \mathbb{C}$ by

$$\chi_{\gamma,k}(m(a)\overline{n}(c)\omega(w,x)) = \det{}^{-k}(a^*)\, e^{i\gamma x}.$$

We induce from $\overline{P}$ to $G \ltimes H_n$ via $\chi_{\gamma,k}$. The action of $G \ltimes H_n$ is left regular action, $(g.\phi)(g_1) = \phi(g^{-1}g_1)$, on functions ϕ in the induced space $\mathrm{Ind}_{\overline{P}}^{G\ltimes H_n} \chi_{\gamma,k}$ defined as

$$\left\{\Phi \in \mathscr{C}^\infty (G \ltimes H_n, \mathbb{C}) \mid \Phi(g\overline{p}) = \chi_{\gamma,k}(\overline{p})^{-1}\,\Phi(g) \ \ \forall g \in G \ltimes H_n, \overline{p} \in \overline{P}\right\}.$$

The group $U(k)$ acts on this space by $(\kappa.\phi)(g,h) = \phi(g, \kappa^{-1}.h)$. Since $(N \ltimes Z)\overline{P}$ is open and dense (Lemma 2.1), the functions in $\mathrm{Ind}_{\overline{P}}^{G\ltimes H_n} \chi_{\gamma,k}$ are completely determined by their restrictions to $N \ltimes Z = N \times Z$. Let $\mathrm{Her}_n = \{t \in \mathbb{C}^{n\times n} \mid t^* = t\}$ and let $\iota : \mathrm{Her}_n \times \mathbb{C}^{n\times k} \to G \ltimes H_n$ denote the embedding defined by

$$\iota\,(t,z) = n(t)\zeta(z).$$

Define the space of restrictions to $\mathrm{Her}_n \times \mathbb{C}^{n\times k}$ by

$$\mathscr{E}_{\gamma,k} = \{f \in \mathscr{C}^\infty\left(\mathrm{Her}_n \times \mathbb{C}^{n\times k}\right) \mid \exists \phi \in \mathrm{Ind}_{\overline{P}}^{G\ltimes H_n} \chi_{\gamma,k} \text{ with } f = \iota^*\phi\},$$

The following result gives the actions of the groups G, H_n, and $U(k)$ on these restricted functions.

Theorem 3.1. *For $f \in \mathscr{E}_{\gamma,k}$, $g \in G$ acts by*

$$(g \cdot f)(t,z) = \det{}^{-k}((A - tC)^*)\, e^{i\gamma(C(A-tC)^{-1}z,z)} \cdot f((A - tC)^{-1}(tD - B), (A - tC)^{-1} z) \tag{8}$$

when $\det(A - tC) \neq 0$. *The action of* $(\binom{v}{w}, s) \in H_n$ *is*

$$\left(\left(\begin{pmatrix} v \\ w \end{pmatrix}, s\right) \cdot f\right)(t,z) = e^{i\gamma(s+\mathrm{Re}(2z-v,w)+(tw,w))} f(t, z - v + tw). \tag{9}$$

The action of $\kappa \in U(k)$ *is* $(\kappa \cdot f)(t,z) = f(t, z\kappa)$.

Proof. These actions follow from the explicit formulas given in the proof of Lemma 2.1. The group properties (2) and (3) are used, for example, to show that $C(A - tC)^{-1}$ and $(A - tC)^{-1}(tD - B)$ are hermitian. ■

Remark 3.2. For $\gamma \neq 0$, define a representation of H_n on $\mathscr{C}^\infty\left(\mathbb{C}^{n\times k}\right)$ by the formula

$$\left(\left(\begin{pmatrix} v \\ w \end{pmatrix}, s\right) \cdot \psi\right)(z) = e^{i\gamma(s+\mathrm{Re}(2z-v,w))}\psi(z - v). \tag{10}$$

Let E_0 denote evaluation at $t = 0$. Then E_0 intertwines the action in (9) with the action (10). It is well-known that Equation (10) defines an irreducible unitary representation on $\mathscr{L}^2(\mathbb{C}^{n\times k})$ (cf. [6]).

4. An invariant system of Schrödinger-type equations

Schrödinger-type equations. In order to display the relevant system of differential equations, we need to choose coordinates for the real vector space $\mathrm{Her}_n \times \mathbb{C}^{n\times k}$. Let $\mathrm{Sym}_n(\mathbb{R})$ (resp. $\mathrm{Skw}_n(\mathbb{R})$) denote the $n{\times}n$ real symmetric (resp. skew symmetric) matrices. We coordinatize $\mathrm{Sym}_n(\mathbb{R})$ using the basis $\{(E_{ij}+E_{ji}) \mid 1 \le i \le j \le n\}$ and $\mathrm{Skw}_n(\mathbb{R})$ using $\{(E_{ij}-E_{ji}) \mid 1 \le i < j \le n\}$. Each $t \in \mathrm{Her}_n$ can be written $t = s + i\sigma$, where $s = \frac{1}{2}(t+\bar{t}) \in \mathrm{Sym}_n(\mathbb{R})$, and $\sigma = \frac{1}{2i}(t-\bar{t}) \in \mathrm{Skw}_n(\mathbb{R})$.
Define differential operators

$$\frac{\partial}{\partial t_{ij}} = \frac{1}{2}\left(\frac{\partial}{\partial s_{ij}} - i\frac{\partial}{\partial \sigma_{ij}}\right) \quad , \quad \frac{\partial}{\partial \bar{t}_{ij}} = \frac{1}{2}\left(\frac{\partial}{\partial s_{ij}} + i\frac{\partial}{\partial \sigma_{ij}}\right). \tag{11}$$

In like manner, if $z = x + iy \in \mathbb{C}^{n\times k}$, for $x, y \in \mathbb{R}^{n\times k}$, let

$$\frac{\partial}{\partial z_{ij}} = \frac{1}{2}\left(\frac{\partial}{\partial x_{ij}} - i\frac{\partial}{\partial y_{ij}}\right) \quad , \quad \frac{\partial}{\partial \bar{z}_{ij}} = \frac{1}{2}\left(\frac{\partial}{\partial x_{ij}} + i\frac{\partial}{\partial y_{ij}}\right). \tag{12}$$

Observe that for $1 \le i \le j \le n$, we have

$$\frac{\partial}{\partial t_{ij}}(t) = E_{ij} \quad , \quad \frac{\partial}{\partial t_{ij}}(\bar{t}) = E_{ji} \tag{13}$$

and

$$\frac{\partial}{\partial \bar{t}_{ij}}(t) = E_{ji} \quad , \quad \frac{\partial}{\partial \bar{t}_{ij}}(\bar{t}) = E_{ij} \tag{14}$$

For $1 \le i, j \le n$, let

$$D_{ij} = \sum_{m=1}^{k} \frac{\partial}{\partial \bar{z}_{im}} \frac{\partial}{\partial z_{jm}} \tag{15}$$

For γ a nonzero real number, define the system of equations:

$$D_{ij} f = i\gamma \frac{\partial f}{\partial \bar{t}_{ij}} \text{ , for } 1 \le i < j \le n, \tag{16}$$

$$D_{ij} f = i\gamma \frac{\partial f}{\partial t_{ji}} \text{ , for } 1 \le j \le i \le n. \tag{17}$$

Definition 4.1. We set $\mathscr{S}ol_\gamma = \{f \in \mathscr{C}^\infty(\mathrm{Her}_n \times \mathbb{C}^{n\times k}) : f \text{ satisfies (16) and (17)}\}$. We say that functions in $\mathscr{S}ol_\gamma$ are of *Schrödinger-type.*

Invariance of the equations under the groups G, H_n, and $U(k)$. We seek to show that $\mathscr{S}ol_\gamma$ is invariant under the group actions of H_n (9), $U(k)$ and generators MA, N and

$$\tau = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

of G. The formulas for the action of these generators extend to functions in $\mathscr{C}^\infty(\mathrm{Her}_n \times \mathbb{C}^{n\times k})$ and are given by

$$n(b).f(t,z) = f(t-b, z),\ b = b^*, \tag{18}$$

$$(m(a)\cdot f)(t,z) = \det(a)^{-k} f(a^{-1}ta^{*-1}, a^{-1}z),\ a \in \mathrm{GL}\,(n,\mathbb{C})\,, \tag{19}$$

$$(\tau \cdot f)(t,z) = \det(t)^{-k} e^{g(t,z)} f(-t^{-1}, t^{-1}z), \tag{20}$$

where $g(t,z) = -i\gamma(t^{-1}z,z)$, for invertible $t \in \mathrm{Her}_n$. We refer collectively to the elements of MA, N, and τ as the generators of G.

The calculations to appear involve the complex chain rule. If $z = g(w)$, we will write

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial w} + \frac{\partial f}{\partial \overline{z}}\frac{\partial \overline{g}}{\partial w} \quad , \quad \frac{\partial f}{\partial \overline{w}} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial \overline{w}} + \frac{\partial f}{\partial \overline{z}}\frac{\partial \overline{g}}{\partial \overline{w}}.$$

We first list a useful, elementary result.

Lemma 4.2. *If $f, g \in \mathscr{C}^\infty(\mathbb{C}^{n\times k})$, then for $1 \le i,j \le n$,*

(a) $D_{ij}(fg) = (D_{ij}f)\,g + f(D_{ij}g) + \sum_{\ell=1}^k \left(\frac{\partial f}{\partial z_{j\ell}}\frac{\partial g}{\partial \overline{z}_{i\ell}} + \frac{\partial f}{\partial \overline{z}_{i\ell}}\frac{\partial g}{\partial z_{j\ell}}\right)$

(b) $D_{ij}(e^g) = e^g\left[\sum_{\ell=1}^k \frac{\partial g}{\partial z_{j\ell}}\frac{\partial g}{\partial \overline{z}_{i\ell}} + D_{ij}(g)\right]$.

(c) *If $P = (p_{mn}) \in \mathbb{C}^{n\times n}$, then $D_{ij}(f(Pz)) = \sum_{r,m=1}^n (D_{mr}(f))\,(Pz)p_{rj}\overline{p}_{mi}$.*

Proposition 4.3. *If $f \in \mathscr{S}ol_\gamma$, then $g\cdot f \in \mathscr{S}ol_\gamma$ for g a generator of G.*

Proof. The proposition is clear for the subgroup N, since the system is constant coefficient. Now let $m(a) \in MA$, and set $a.(t,z) = (a^{-1}ta^{*-1}, a^{-1}z)$. By Lemma 4.2(c), $(D_{ij}\,(m(a)\cdot f))\,(t,z)$ is

$$\det(a^*)^{-k}\sum_{r,m=1}^n (D_{mr}(f))\,(a.(t,z))(a^{-1})_{rj}\overline{(a^{-1})}_{mi}.$$

On the other hand, if $1 \le i < j \le n$, then by the chain rule one finds that $i\gamma\frac{\partial}{\partial \overline{t}_{ij}}(m(a)\cdot f)(t,z)$ is equal to

$$i\gamma\det(a^*)^{-k}\left(\sum_{r\le m}^n \frac{\partial f(a.(t,z))}{\partial t_{rm}}\frac{\partial (a^{-1}ta^{*-1})_{rm}}{\partial \overline{t}_{ij}} + \sum_{r<m}^n \frac{\partial f(a.(t,z))}{\partial \overline{t}_{rm}}\frac{\partial \overline{(a^{-1}ta^{*-1})}_{rm}}{\partial \overline{t}_{ij}}\right).$$

By (13), $$\frac{\partial (a^{-1}ta^{*-1})_{rm}}{\partial \overline{t}_{ij}} = (a^{-1}E_{ji}a^{*-1})_{rm} = (a^{-1})_{rj}(\overline{a^{-1}})_{mi}.$$

Since $a^{-1}ta^{*-1}$ is hermitian, one has $\overline{(a^{-1}ta^{*-1})} = (a^{-1}ta^{*-1})^T$ and hence, by (13),

$$\frac{\partial \overline{(a^{-1}ta^{*-1})}_{rm}}{\partial \overline{t}_{ij}} = (a^{-1}E_{ji}a^{*-1})_{mr} = (a^{-1})_{mj}(\overline{a^{-1}})_{ri}.$$

Comparing terms, we see that if $f \in \mathscr{C}^\infty(\mathrm{Her}_n \times \mathbb{C}^{n\times k})$ satisfies equation (16), then so does $m(a)\cdot f$. The proof for equation (17) is similar and is omitted.

We turn now to the invariance of the system under τ. From (20) and Lemma 4.2(a), one sees that $D_{ij}\left((\tau\cdot f)\,(t,z)\right)$ is

$$\det(t)^{-k}\left(D_{ij}(e^{g(t,z)})f(-t^{-1},t^{-1}z) + e^{g(t,z)}D_{ij}(f(-t^{-1},t^{-1}z)) + S_{ij}\right),$$

where

$$S_{ij} = \sum_{\ell=1}^k \left(\frac{\partial e^{g(t,z)}}{\partial z_{j\ell}}\frac{\partial\left(f(-t^{-1},t^{-1}z)\right)}{\partial \overline{z}_{i\ell}} + \frac{\partial e^{g(t,z)}}{\partial \overline{z}_{i\ell}}\frac{\partial\left(f(-t^{-1},t^{-1}z)\right)}{\partial z_{j\ell}}\right).$$

We first compute $D_{ij}(e^{g(t,z)})$. Now

$$\frac{\partial g(t,z)}{\partial z_{j\ell}} = -i\gamma(z^*t^{-1})_{\ell j} \quad \text{and} \quad \frac{\partial g(t,z)}{\partial \overline{z}_{i\ell}} = -i\gamma(t^{-1}z)_{i\ell}$$

and so the sum appearing in Lemma 4.2(b) is

$$(i\gamma)^2\sum_{\ell=1}^{k}(t^{-1}z)_{i\ell}(z^*t^{-1})_{\ell j} = (i\gamma)^2(t^{-1}zz^*t^{-1})_{ij}.$$

Next, compute

$$D_{ij}g = \sum_{\ell=1}^{k}\frac{\partial}{\partial \overline{z}_{i\ell}}(-i\gamma(z^*t^{-1})_{\ell j}) = -i\gamma\sum_{\ell=1}^{k}(E_{\ell i}t^{-1})_{\ell j} = -i\gamma\sum_{\ell=1}^{k}(t^{-1})_{ij} = -i\gamma k(t^{-1})_{ij}.$$

From Lemma 4.2, (2), we find

$$D_{ij}(e^{g(t,z)}) = -i\gamma e^{g(t,z)}(-i\gamma(t^{-1}zz^*t^{-1})_{ij} + k(t^{-1})_{ij}).$$

Next, consider $\frac{\partial\left(f(-t^{-1},t^{-1}z)\right)}{\partial \overline{z}_{i\ell}}$. By the chain rule, this is

$$\sum_{r=1}^{n}\sum_{s=1}^{k}\left(\frac{\partial f}{\partial \overline{z}_{rs}}\right)(-t^{-1},t^{-1}z)\frac{\partial(\overline{t^{-1}}\overline{z})_{rs}}{\partial \overline{z}_{i\ell}}.$$

But $\frac{\partial(\overline{t^{-1}}\overline{z})_{rs}}{\partial \overline{z}_{i\ell}} = (\overline{t^{-1}}E_{i\ell})_{rs} = (\overline{t^{-1}})_{ri}\delta_{\ell s}$ so that

$$\frac{\partial\left(f(-t^{-1},t^{-1}z)\right)}{\partial \overline{z}_{i\ell}} = \sum_{r=1}^{n}\left(\frac{\partial f}{\partial \overline{z}_{rs}}\right)(-t^{-1},t^{-1}z)(\overline{t^{-1}})_{ri}.$$

Similarly, one has

$$\frac{\partial\left(f(-t^{-1},t^{-1}z)\right)}{\partial z_{j\ell}} = \sum_{r=1}^{n}\left(\frac{\partial f}{\partial z_{r\ell}}\right)(-t^{-1},t^{-1}z)(t^{-1})_{rj}$$

so that $S_{ij} = -i\gamma e^{g(t,z)}P_{ij}$, where

$$P_{ij} = \sum_{\ell=1}^{k}\sum_{r=1}^{n}\left((z^*t^{-1})_{\ell j}\overline{(t^{-1})}_{ri}\frac{\partial f}{\partial \overline{z}_{r\ell}} + (t^{-1}z)_{i\ell}\left(t^{-1}\right)_{rj}\frac{\partial f}{\partial z_{r\ell}}\right).$$

Here each partial derivative of f is evaluated at $(-t^{-1},t^{-1}z)$. Using Lemma 4.2(c) to write $D_{ij}(f(-t^{-1},t^{-1}z))$, we find

$$D_{ij}\left((\tau\cdot f)(t,z)\right) = \det(t)^{-k}e^{g(t,z)}[-i\gamma R_{ij}f + L_{ij} - i\gamma P_{ij}], \tag{21}$$

where

$$R_{ij} = -i\gamma(t^{-1}zz^*t^{-1})_{ij} + k(t^{-1})_{ij}, \quad L_{ij} = \sum_{r,m=1}^{n}\left(D_{mr}(f)\right)\left(t^{-1}\right)_{rj}\overline{(t^{-1})}_{mi},$$

$$P_{ij} = \sum_{\ell=1}^{k}\sum_{r=1}^{n}\left((z^*t^{-1})_{\ell j}\overline{(t^{-1})}_{ri}\frac{\partial f}{\partial \overline{z}_{r\ell}} + (t^{-1}z)_{i\ell}\left(t^{-1}\right)_{rj}\frac{\partial f}{\partial z_{r\ell}}\right)$$

and f, all partials of f as well as $D_{mr}(f)$ are evaluated at $(-t^{-1},t^{-1}z)$.

We now assume $1 \leq i < j \leq n$ and compute $\frac{\partial((\tau\cdot f)(t,z))}{\partial \bar{t}_{ij}}$. We begin by computing $\frac{\partial}{\partial \bar{t}_{ij}}(\det^{-k}(t))$. From the form of the classical adjoint of a matrix, one finds

$$\frac{\partial}{\partial \bar{t}_{ij}}(\det(t)^{-k}) = -k \det(t)^{-k} \left(t^{-1}\right)_{ij}. \tag{22}$$

Next consider $\displaystyle \frac{\partial(e^{g(t,z)})}{\partial \bar{t}_{ij}} = e^{g(t,z)} \frac{\partial g}{\partial \bar{t}_{ij}} = -i\gamma e^{g(t,z)} (\frac{\partial}{\partial \bar{t}_{ij}} \left(t^{-1}\right) z, z).$

By (14), we have $\displaystyle \frac{\partial}{\partial \bar{t}_{ij}} \left(t^{-1}\right) = -\left(t^{-1}\right) \frac{\partial t}{\partial \bar{t}_{ij}} \left(t^{-1}\right) = -\left(t^{-1}\right) E_{ji} \left(t^{-1}\right)$ and hence

$$\frac{\partial(e^{g(t,z)})}{\partial \bar{t}_{ij}} = i\gamma e^{g(t,z)} (\left(t^{-1}\right) E_{ji} \left(t^{-1}\right) z, z) = i\gamma e^{g(t,z)} (\left(t^{-1}\right) z z^* \left(t^{-1}\right))_{ij}.$$

We now consider $\frac{\partial(f(-t^{-1}, t^{-1}z))}{\partial \bar{t}_{ij}}$. By the chain rule, this is

$$\sum_{p \leq q} \frac{\partial f}{\partial t_{pq}} (-\frac{\partial}{\partial \bar{t}_{ij}} \left(t^{-1}\right)_{pq}) + \sum_{p<q} \frac{\partial f}{\partial \bar{t}_{pq}} (-\frac{\partial}{\partial \bar{t}_{ij}} \overline{(t^{-1})}_{pq}) + \sum_{r,s} \frac{\partial f}{\partial z_{rs}} \frac{\partial}{\partial \bar{t}_{ij}} \left(t^{-1}z\right)_{rs} + \sum_{r,s} \frac{\partial f}{\partial \overline{z}_{rs}} \frac{\partial}{\partial \bar{t}_{ij}} \overline{(t^{-1}z)}_{rs},$$

where each partial derivative of f is evaluated at $(-t^{-1}, t^{-1}z)$. We have

$$\frac{\partial}{\partial \bar{t}_{ij}} \left(t^{-1}\right)_{pq} = (\frac{\partial}{\partial \bar{t}_{ij}} (t^{-1}))_{pq} = -(\left(t^{-1}\right) E_{ji} \left(t^{-1}\right))_{pq} = -(t^{-1})_{pj} (t^{-1})_{iq}$$

and since t is hermitian,

$$\frac{\partial}{\partial \bar{t}_{ij}} \overline{(t^{-1})}_{pq} = (\frac{\partial}{\partial \bar{t}_{ij}} (t^{-1}))_{qp} = -(\left(t^{-1}\right) E_{ji} \left(t^{-1}\right))_{qp} = -(t^{-1})_{qj} (t^{-1})_{ip}.$$

Similarly,

$$\frac{\partial}{\partial \bar{t}_{ij}} \left(t^{-1}z\right)_{rs} = ((\frac{\partial}{\partial \bar{t}_{ij}} \left(t^{-1}\right)) z)_{rs} = -(\left(t^{-1}\right) E_{ji} \left(t^{-1}\right) z)_{rs} = -(t^{-1})_{rj} (t^{-1}z)_{is}$$

and

$$\frac{\partial}{\partial \bar{t}_{ij}} \overline{(t^{-1}z)}_{rs} = ((\frac{\partial}{\partial \bar{t}_{ij}} (\overline{t^{-1}})) \overline{z})_{rs} = -(\left(\overline{t^{-1}}\right) E_{ij} \left(\overline{t^{-1}}\right) \overline{z})_{rs} = -(\overline{t^{-1}})_{ri} (\overline{t^{-1}} \overline{z})_{js}.$$

We then see that $\frac{\partial(f(-t^{-1}, t^{-1}z))}{\partial \bar{t}_{ij}}$ is

$$\sum_{p \leq q} \frac{\partial f}{\partial t_{pq}} (t^{-1})_{pj} (t^{-1})_{iq} + \sum_{p<q} \frac{\partial f}{\partial \bar{t}_{pq}} (t^{-1})_{qj} (t^{-1})_{ip} - \sum_{r,s} \frac{\partial f}{\partial z_{rs}} (t^{-1})_{rj} (t^{-1}z)_{is} - \sum_{r,s} \frac{\partial f}{\partial \overline{z}_{rs}} (\overline{t^{-1}})_{ri} (\overline{t^{-1}} \overline{z})_{js}.$$

Observe that $(\overline{t^{-1}z})_{js} = (z^* t^{-1})_{sj}$. Putting this together, we have for $1 \le i < j \le n$,

$$\frac{\partial\left((\tau\cdot f)(t,z)\right)}{\partial \bar{t}_{ij}} = -k\det(t)^{-k}\left(t^{-1}\right)_{ij} e^{g(t,z)} f(-t^{-1}, t^{-1}z)$$
$$+ \det(t)^{-k}(i\gamma e^{g(t,z)}(\left(t^{-1}\right) zz^* \left(t^{-1}\right))_{ij}) f(-t^{-1}, t^{-1}z) + \det(t)^{-k} e^{g(t,z)} Q_{ij},$$

where

$$Q_{ij} = \sum_{p\le q} \frac{\partial f}{\partial t_{pq}} (t^{-1})_{pj}(t^{-1})_{iq} + \sum_{p<q} \frac{\partial f}{\partial \bar{t}_{pq}} (t^{-1})_{qj}(t^{-1})_{ip}$$
$$- \sum_{r,s} \frac{\partial f}{\partial z_{rs}} (t^{-1})_{rj}(t^{-1}z)_{is} - \sum_{r,s} \frac{\partial f}{\partial \bar{z}_{rs}} (\overline{t^{-1}})_{ri}(z^* t^{-1})_{sj}.$$

From equation (21) and the above calculation, we see that

$$D_{ij}\left((\tau\cdot f)(t,z)\right) - \frac{\partial\left((\tau\cdot f)(t,z)\right)}{\partial \bar{t}_{ij}}$$
$$= \sum_{p\le q}\left(D_{qp}(f) - i\gamma\frac{\partial f}{\partial t_{pq}}\right)(t^{-1})_{pj}(t^{-1})_{iq} + \sum_{p<q}\left(D_{pq}(f) - i\gamma\frac{\partial f}{\partial \bar{t}_{pq}}\right)(t^{-1})_{qj}(t^{-1})_{ip}.$$

Consequently, if $f \in \mathscr{C}^\infty(\mathrm{Her}_n \times \mathbb{C}^{n\times k})$ satisfies equation (16), then so does $\tau \cdot f$. The remaining case (17) is similar and is omitted. ∎

Recall from equation (9) that if $h = (\binom{v}{w}, s) \in H_n$, then

$$(h\cdot f)(t,z) = e^{p(t,z)} f(t, z - v + tw)$$

where $p(t,z) = i\gamma(s + \mathrm{Re}(2z - v, w) + (tw, w))$.

Proposition 4.4. *If $h \in H_n$ and $f \in \mathscr{S}ol_\gamma$, then $h \cdot f \in \mathscr{S}ol_\gamma$.*

Proof. By Lemma 4.2(a),

$$D_{ij}\left((h\cdot f)(t,z)\right) = (D_{ij}e^p) f(t, z - v + tw) + e^p(D_{ij}(f(t, z - v + tw)))$$
$$+ \sum_{\ell=1}^{k}\left(\frac{\partial e^p}{\partial z_{j\ell}}\frac{\partial(f(t,z-v+tw))}{\partial \bar{z}_{i\ell}} + \frac{\partial e^p}{\partial \bar{z}_{i\ell}}\frac{\partial(f(t,z-v+tw))}{\partial z_{j\ell}}\right).$$

Since $D_{ij}(p) = 0$, from Lemma 4.2, (2), we have

$$(D_{ij}e^p) = e^p\left[\sum_{\ell=1}^{k}\frac{\partial p}{\partial z_{j\ell}}\frac{\partial p}{\partial \bar{z}_{i\ell}}\right].$$

Since $\mathrm{Re}(z,w) = \frac{1}{2}((z,w) + (w,z))$, one has $\frac{\partial\,\mathrm{Re}(z,w)}{\partial z_{j\ell}} = \frac{1}{2}\mathrm{Tr}(w^* E_{j\ell}) = \frac{1}{2}(w^*)_{\ell j}$ so that $\dfrac{\partial p}{\partial z_{j\ell}} = i\gamma(w^*)_{\ell j}$. Similarly, $\dfrac{\partial p}{\partial \bar{z}_{i\ell}} = i\gamma(w)_{i\ell}$.

Consequently,
$$\sum_{\ell=1}^{k}\frac{\partial p}{\partial z_{j\ell}}\frac{\partial p}{\partial \bar{z}_{i\ell}} = (i\gamma)^2(ww^*)_{ij}.$$

Since differentiation commutes with translation, we have

$$D_{ij}\left((h\cdot f)(t,z)\right) = e^p\left((i\gamma)^2(ww^*)_{ij}\right)f + e^p(D_{ij}f) + e^p\sum_{\ell=1}^{k}\left((i\gamma(w^*)_{\ell j})\frac{\partial f}{\partial \overline{z}_{i\ell}} + (i\gamma(w^*)_{\ell j})\frac{\partial f}{\partial z_{j\ell}}\right),$$

where f and its derivatives are evaluated at $(t, z-v+tw)$.

We turn now to the computation of $\frac{\partial(h\cdot f)}{\partial \overline{t}_{ij}}$ for $1 \le i < j \le n$. First observe from (14) that $\frac{\partial p}{\partial \overline{t}_{ij}} = i\gamma\operatorname{Tr}(E_{ji}ww^*) = i\gamma(ww^*)_{ij}$ from which we get

$$\frac{\partial e^p}{\partial \overline{t}_{ij}} = i\gamma e^p(ww^*)_{ij}.$$

By the chain rule,

$$\frac{\partial(f(t,z-v+tw))}{\partial \overline{t}_{ij}} = \frac{\partial f}{\partial \overline{t}_{ij}} + \sum_{r,s=1}^{n,k}\left(\frac{\partial f}{\partial z_{rs}}\frac{\partial}{\partial \overline{t}_{ij}}(tw)_{rs} + \frac{\partial f}{\partial \overline{z}_{rs}}\frac{\partial}{\partial \overline{t}_{ij}}\overline{(tw)}_{rs}\right),$$

where the derivatives of f are evaluated at $(t, z-v+tw)$. Since $\frac{\partial}{\partial \overline{t}_{ij}}(tw)_{rs} = \delta_{rj}w_{is}$ and $\frac{\partial}{\partial \overline{t}_{ij}}\overline{(tw)}_{rs} = \delta_{ri}\overline{w}_{js}$, we find $\frac{\partial((h\cdot f)(t,z))}{\partial \overline{t}_{ij}}$ is

$$e^p\left[i\gamma(ww^*)_{ij}f + \frac{\partial f}{\partial \overline{t}_{ij}} + \sum_{s=1}^{k}\left(\frac{\partial f}{\partial z_{js}}w_{is} + \frac{\partial f}{\partial \overline{z}_{is}}\overline{w}_{js}\right)\right].$$

Therefore, we find

$$D_{ij}\left((h\cdot f)(t,z)\right) - i\gamma\frac{\partial((h\cdot f)(t,z))}{\partial \overline{t}_{ij}} = e^p\left(D_{ij}f - i\gamma\frac{\partial f}{\partial \overline{t}_{ij}}\right),$$

from which the proposition follows in the case $1 \le i < j \le n$. The remaining case $i \ge j$ is similar and is omitted. ■

Recall the action of $\kappa \in U(k)$ is $(\kappa\cdot f)(t,z) = f(t, z\kappa)$.

Proposition 4.5. *If $\kappa \in U(k)$ and $f \in \mathscr{S}ol_\gamma$, then $\kappa\cdot f \in \mathscr{S}ol_\gamma$.*

Proof. It follows from the chain rule that $D_{ij}(\kappa\cdot f) = \kappa\cdot(D_{ij}f)$. Clearly, one has $\frac{\partial}{\partial \overline{t}_{ij}}(\kappa\cdot f) = \kappa\cdot(\frac{\partial f}{\partial \overline{t}_{ij}})$ so the proposition follows. ■

5. An invariant map

In this section, we construct an intertwining operator that maps into the space of Schrödinger-type functions.

Definition 5.1. For nonzero $\gamma \in \mathbb{R}$ and $\xi \in \mathbb{C}^{n\times k}$, define the *kernel function* K_ξ^γ on $\operatorname{Her}_n \times \mathbb{C}^{n\times k}$ by $K_\xi^\gamma(t,z) = e^{i\operatorname{Re}(z,\xi) + \frac{i}{4\gamma}(t\xi,\xi)}$.

Remark 5.2. Fix $t \in \operatorname{Her}_n$ and $\xi \in \mathbb{C}^{n\times k}$. Then the function $z \to K_\xi^\gamma(t,z)$ lies in $\mathscr{S}ol_\gamma$. To see this, write $\operatorname{Re}(z,\xi) = \frac{1}{2}[\operatorname{Tr}(\xi^* z) + \operatorname{Tr}(\xi^T\overline{z})]$, so one finds

$$D_{ij}(e^{i\operatorname{Re}(z,\xi)}) = \left(\frac{i}{2}\right)^2(\xi\xi^*)_{ij}e^{i\operatorname{Re}(z,\xi)}, \quad 1 \le i,j \le n.$$

By equation (14) one has

$$\frac{\partial}{\partial \bar{t}_{ij}}(e^{\frac{i}{4\gamma}(t\xi,\xi)}) = \frac{i}{4\gamma}(\xi\xi^*)_{ij}e^{\frac{i}{4\gamma}(t\xi,\xi)} \text{ for } 1 \le i < j \le n,$$

and by equation (13),

$$\frac{\partial}{\partial t_{ij}}(e^{\frac{i}{4\gamma}(t\xi,\xi)}) = \frac{i}{4\gamma}(\xi\xi^*)_{ij}e^{\frac{i}{4\gamma}(t\xi,\xi)} \text{ for } 1 \le j \le i \le n,$$

from which one concludes

$$\frac{\partial}{\partial \bar{t}_{ij}}(K_\xi^\gamma(t,z)) = \frac{i}{4\gamma}(\xi\xi^*)_{ij}K_\xi^\gamma(t,z) \text{ for } 1 \le i < j \le n,$$

and

$$\frac{\partial}{\partial t_{ij}}(K_\xi^\gamma(t,z)) = \frac{i}{4\gamma}(\xi\xi^*)_{ij}K_\xi^\gamma(t,z) \text{ for } 1 \le i, j \le n.$$

So, the function K_ξ^γ lies in $\mathscr{S}ol_\gamma$.

Definition 5.3. Let $\mathscr{S}(\mathbb{C}^{n\times k})$ denote the Schwartz functions on $\mathbb{C}^{n\times k}$. Define the operator $T_\gamma : \mathscr{S}(\mathbb{C}^{n\times k}) \to \mathscr{C}^\infty(\mathrm{Her}_n \times \mathbb{C}^{n\times k})$ by

$$T_\gamma(\psi)(t,z) = (2\pi)^{-nk}\int_{\mathbb{C}^{n\times k}} \psi(\xi)K_\xi^\gamma(t,z)d\xi,$$

where $d\xi$ denotes the Lebesgue measure on $\mathbb{C}^{n\times k}$.

Remark 5.4. (1) By Remark 5.2, the image $T_\gamma(\mathscr{S}(\mathbb{C}^{n\times k}))$ lies in $\mathscr{S}ol_\gamma$.
(2) For fixed $t \in \mathrm{Her}_n$, the map $\psi \to T_\gamma(\psi)(t,\cdot)$ may be viewed as the composition $\mathcal{F}\circ M_\gamma(t)$, of the Fourier transform

$$(\mathcal{F}\psi)(z) = (2\pi)^{-nk}\,\widehat{\psi}(z),$$

where $\widehat{\psi}(z) = \int_{\mathbb{C}^{n\times k}} \psi(\xi)e^{i\,\mathrm{Re}(z,\xi)}d\xi$, and the multiplication operator

$$(M_\gamma(t)\psi)(\xi) = e^{\frac{i}{4\gamma}(t\xi,\xi)}\psi(\xi),$$

both of which are unitary on the space $\mathscr{L}^2\left(\mathbb{C}^{n\times k}\right)$. In particular, we have

$$\|T_\gamma(\psi)(t,\cdot)\|_2 = \|\psi\|_2, \quad \psi \in \mathscr{S}(\mathbb{C}^{n\times k}).$$

(3) Let E_0 denote evaluation at $t = 0$. Since $T_\gamma(\psi)(0,\cdot) = \mathcal{F}(\psi)$, it is clear that $\mathcal{F}^{-1}\circ E_0$ is the left-inverse of T_γ. ■

We define the unitary scaling operator $s_\mu : \mathscr{L}^2\left(\mathbb{C}^{n\times k}\right) \to \mathscr{L}^2\left(\mathbb{C}^{n\times k}\right)$ for a nonzero scalar $\mu \in \mathbb{R}$ by

$$s_\mu(\psi)(\xi) = |\mu|^{nk}\,\psi(\mu\xi).$$

Note that $\tau^2 = -I_{2n}$ and so $\sigma = -\tau$ along with subgroups MA and N will also generate G. Define formulas on $\mathscr{L}^2\left(\mathbb{C}^{n\times k}, \mathbb{C}\right)$ for generators by

$$(L_k^\gamma(m(a))\psi)(\xi) = \det{}^k(a)\,\psi(a^*\xi), \quad (L_k^\gamma(n(x))\psi)(\xi) = e^{-\frac{i}{4\gamma}(x\xi,\xi)}\psi(\xi),$$
$$(L_k^\gamma(\sigma)\psi)(\xi) = (i\,\mathrm{sgn}(\gamma))^{nk}(s_{(2\gamma)^{-1}}\circ\mathcal{F}\psi)(\xi). \tag{23}$$

When $\gamma = \frac{1}{4}$, these formulas agree with those of the represenation of G found in [13]. Note that the Schwartz space $\mathscr{S}(\mathbb{C}^{n\times k})$ is preserved by $L_k^\gamma(g)$ for $g \in G$.

Theorem 5.5. *The operator $T_\gamma\colon \mathscr{S}(\mathbb{C}^{n\times k}) \to \mathscr{C}^\infty(\mathrm{Her}_n \times \mathbb{C}^{n\times k})$ intertwines the generators of G.*

Proof. For $\psi \in \mathscr{S}(\mathbb{C}^{n\times k})$ and $x \in \mathrm{Her}_n$, we have

$$T_\gamma(L_k^\gamma(n(x))\psi)(t,z) = (2\pi)^{-nk}\int_{\mathbb{C}^{n\times k}} e^{-\frac{i}{4\gamma}(x\xi,\xi)}\psi(\xi)e^{i\,\mathrm{Re}(z,\xi)+\frac{i}{4\gamma}(t\xi,\xi)}d\xi$$

$$= (2\pi)^{-nk}\int_{\mathbb{C}^{n\times k}} \psi(\xi)e^{i\,\mathrm{Re}(z,\xi)+\frac{i}{4\gamma}((t-x)\xi,\xi)}d\xi = T_\gamma(\psi)(t-x,z) = (n(x).T_\gamma(\psi))(t,z).$$

If $a \in GL(n,\mathbb{C})$, one has

$$T_\gamma(L_k^\gamma(m(a))\psi)(t,z) = \frac{\det(a)^k}{(2\pi)^{nk}}\int_{\mathbb{C}^{n\times k}} \psi(a^*\xi)e^{i\,\mathrm{Re}(z,\xi)+\frac{i}{4\gamma}(t\xi,\xi)}d\xi.$$

Apply the transformation $\xi \to a^{*,-1}\xi$ to get

$$\frac{\det(a)^k}{(2\pi)^{nk}}\,|\det(a^*)|^{-2k}\int_{\mathbb{C}^{n\times k}} \psi(\xi)e^{i\,\mathrm{Re}(z,a^{*,-1}\xi)+\frac{i}{4\gamma}(ta^{*,-1}\xi,a^{*,-1}\xi)}d\xi$$

$$= \frac{\det(a^*)^{-k}}{(2\pi)^{nk}}\int_{\mathbb{C}^{n\times k}} \psi(\xi)e^{i\,\mathrm{Re}(a^{-1}z,\xi)+\frac{i}{4\gamma}(a^{-1}ta^{*,-1}\xi,\xi)}d\xi = (m(a).T_\gamma(\psi))(t,z).$$

We next show that

$$T_\gamma(L_k^\gamma(\sigma)\psi)(t,z) = (\sigma.T_\gamma(\psi))(t,z) \tag{24}$$

for all $\psi \in \mathscr{S}(\mathbb{C}^{n\times k})$, invertible $t \in \mathrm{Her}_n$, and $z \in \mathbb{C}^{n\times k}$. We shall see that this reduces to the special case (shown below) where $t = \lambda I_n$, for nonzero $\lambda \in \mathbb{R}$. Indeed, assuming this special case for arbitrary n, it certainly holds when $n = 1$. By viewing the variable $z \in \mathbb{C}^{n\times k}$ in terms of its n rows, each of the operators in equation (24) may be viewed as the n-fold tensor product of the corresponding operator on $\mathbb{C}^{1\times k}$ It now follows that (24) will hold for $t = d = \mathrm{diag}(\lambda_1,\dots\lambda_n) \in GL(n,\mathbb{R})$. In particular, equation (24) holds when $\psi \in \mathscr{S}(\mathbb{C}^{n\times k})$ is replaced by $L_k^\gamma(m(u))\psi$ for $u \in U(n)$. Observe that $m(u)$ and σ commute. Upon this replacement and using the fact that T_γ intertwines the MA action, we obtain (after cancelling $\det(u)^{-k}$),

$$T_\gamma(L_k^\gamma(\sigma)\psi)(u^{-1}du,u^{-1}z) = (\sigma.T_\gamma(\psi))(u^{-1}du,u^{-1}z),$$

for all $z \in \mathbb{C}^{n\times k}$, diagonal $d \in GL(n,\mathbb{R})$ and $\psi \in \mathscr{S}(\mathbb{C}^{n\times k})$. Replacing z by uz finishes the proof since any $t \in \mathrm{Her}_n$ can be written as $u^{-1}du$.

It remains to show that equation (24) is true when $t = \lambda I_n \in GL(n,\mathbb{R})$. On one hand, by dominated convergence, we write $T_\gamma(L_k^\gamma(\sigma)\psi)(\lambda I_n,z)$ as

$$C_\gamma \lim_{\epsilon\to 0^+}((s_{(2\gamma)^{-1}}\circ\mathcal{F})\psi, e^{-i\,\mathrm{Re}(z,\cdot)-(\epsilon+\frac{i\lambda}{4\gamma})(\cdot,\cdot)}),$$

where $C_\gamma = (\frac{i\,\mathrm{sgn}(\gamma)}{2\pi})^{nk}$. By the unitary nature of s_μ and $\mathcal{F}$, this is

$$C_\gamma\,|2\gamma|^{nk}\lim_{\epsilon\to 0^+}\Big(\psi, \mathcal{F}^{-1}\Big(e^{-i2\gamma\,\mathrm{Re}(z,\cdot)-(\epsilon+\frac{i\lambda}{4\gamma})4\gamma^2(\cdot,\cdot))}\Big)\Big).$$

Now $(\mathcal{F}^{-1}\left(e^{-i2\gamma\,\mathrm{Re}(z,\cdot)}\varphi\right))(\xi) = \mathcal{F}^{-1}(\varphi)(\xi+2\gamma z)$ and we conclude from the standard result $(e^{-a(\cdot,\cdot)})\hat{}\,(\xi) = (\frac{\pi}{a})^{nk}e^{-(\xi,\xi)/4a}$, $\mathrm{Re}(a) > 0$ and $\xi \in \mathbb{C}^{n\times k}$, that

$$\mathcal{F}^{-1}\left(e^{-i2\gamma\,\mathrm{Re}(z,\cdot)-(\epsilon+\frac{i\lambda}{4\gamma})4\gamma^2(\cdot,\cdot))}\right)(\xi) = \frac{1}{(2\pi)^{nk}}\Big(\frac{\pi}{a(\gamma,\lambda)}\Big)^{nk}e^{-\|\xi+2\gamma z\|^2/4a(\gamma,\lambda)},$$

where $a(\gamma,\lambda) = 4\gamma^2\epsilon + i\gamma\lambda$.

Letting $\epsilon \to 0^+$, we see $T_\gamma(L_k^\gamma(\sigma)\psi)(\lambda I_n, z)$ is

$$C_\gamma \, |2\gamma|^{nk} \, (-2i\gamma\lambda)^{-nk} \int_{\mathbb{C}^{n\times k}} \psi(\xi) e^{\frac{-i}{\gamma\lambda}[\|\xi\|^2/4+\gamma \operatorname{Re}(z,\xi)+\gamma^2\|z\|^2]} d\xi$$
$$= C_\gamma(-i\lambda\gamma sgn(\gamma))^{-nk} e^{\frac{-i\gamma}{\lambda}\|z\|^2} \int_{\mathbb{C}^{n\times k}} \psi(\xi) e^{\frac{-i}{\lambda}\operatorname{Re}(z,\xi)} e^{\frac{-i}{4\gamma\lambda}\|\xi\|^2} d\xi.$$

Recall from equation (8), we have

$$\sigma.f(\lambda I_n, z) = \det(-\lambda I_n)^{-k} e^{i\gamma(-\lambda^{-1}z,z)} f(-\lambda^{-1}I_n, -\lambda^{-1}z)$$

and so

$$(\sigma.T_\gamma(\psi))(\lambda I_n, z) = (\frac{1}{2\pi})^{nk}(-\lambda)^{-nk} e^{i\gamma(-\lambda^{-1}z,z)} \int_{\mathbb{C}^{n\times k}} \psi(\xi) e^{i\operatorname{Re}(-\lambda^{-1}z,\xi)+\frac{i}{4\gamma}(-\lambda^{-1}\xi,\xi)} d\xi.$$

The proof is complete by combining constants. ■

Corollary 5.6. *The space $T_{1/4}(\mathscr{S}(\mathbb{C}^{n\times k}))$ carries a G-action and $T_{1/4}$ is a G-intertwining map on $\mathscr{S}(\mathbb{C}^{n\times k})$.*

Proof. From [13], the formulas for $L_k^{1/4}$ give a genuine representation of G. Since $T_{1/4}$ intertwines the generators of G and $\mathscr{S}(\mathbb{C}^{n\times k})$ is $L_k^{1/4}(G)$-invariant, the result follows. ■

We now turn to the intertwining property of T_γ with respect to the Heisenberg group. We extend the action in equation (9) to $\mathscr{C}^\infty(\mathrm{Her}_n \times \mathbb{C}^{n\times k})$. Next, we conjugate the Heisenberg representation given in equation (10) by the Fourier transform to obtain the representation on $\mathscr{L}^2(\mathbb{C}^{n\times k})$

$$R_\gamma(\begin{pmatrix} v \\ w \end{pmatrix}, s)\psi\,(z) = e^{i\gamma(s+\operatorname{Re}(v,w))-i\operatorname{Re}(z,v)}\psi(z-2\gamma w),\ \psi \in \mathscr{L}^2(\mathbb{C}^{n\times k}).$$

Note that this action preserves the Schwartz space $\mathscr{S}(\mathbb{C}^{n\times k})$.

Proposition 5.7. *The map T_γ intertwines the representation R_γ with the Heisenberg action on $\mathscr{C}^\infty(\mathrm{Her}_n \times \mathbb{C}^{n\times k})$.*

Proof. This is a straightforward calculation involving only the definitions and a change of variable so we omit the details. ■

Although the image of $T_{1/4}$ carries a G-action whose formulas are given by the natural G-action on the representation space $\mathscr{E}_{1/4,k}$, it is yet unclear how $\mathscr{E}_{1/4,k}$ and $T_{1/4}(\mathscr{S}(\mathbb{C}^{n\times k}))$ are related. We show in Section 7 (Proposition 7.5) that there exists a subspace of $\mathscr{S}(\mathbb{C}^{n\times k})$ which is dense in $\mathscr{L}^2(\mathbb{C}^{n\times k})$ for $\gamma > 0$ and maps into $\mathscr{E}_{\gamma,k}$ under T_γ. The construction of this space relies on the space of pluriharmonic functions defined in [13].

6. Polynomials and Differential Operators

Let $X = \mathbb{C}^{n\times k} \times \mathbb{C}^{n\times k}$ and let V be a complex, finite dimensional inner product space with inner product $(\,.\,,\,.\,)_V$. Let $\mathscr{P}(X, V)$ denote the space of V-valued polynomials on X. Then $\mathscr{P}(X, V)$ is an inner product space as well.

Let $du\,dw$ denote the Lebesgue measure on X and $d\mu(u,w)$ the Gaussian measure $ce^{-(\|u\|^2+\|w\|^2)}dudw$, where $c>0$ is chosen so that $\int_X d\mu(u,w)=1$. Define an inner product on $\mathscr{P}(X,V)$ by

$$(p,q)=\int_X (p(u,w),q(u,w))_V d\mu(u,w), \text{ for } p,q\in\mathscr{P}(X,V).$$

Using multi-index notation, each polyomial $p\in\mathscr{P}(X,V)$ has a unique expansion $p(u,w)=\Sigma_{\alpha,\beta}u^\alpha w^\beta v_{\alpha,\beta}$ for $v_{\alpha,\beta}\in V$. Define $p(\partial)$ to be the linear map from $\mathscr{C}^\infty(\mathbb{C}^{n\times k},V)$ to $\mathscr{C}^\infty(\mathbb{C}^{n\times k},\mathbb{C})$ by

$$(p(\partial)F)(z)=\sum_{\alpha,\beta}\frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial\overline{z}^\beta}(F(z),v_{\alpha,\beta}),\ z\in\mathbb{C}^{n\times k}.$$

Note that the map $p\to p(\partial)$ is conjugate linear. Now suppose that $W\subset\mathscr{P}(X,V)$ is a finite dimensional subspace and let $\{p_j\}$ denote an orthonormal basis of W. Define a map $D_W:\mathscr{C}^\infty(\mathbb{C}^{n\times k},V)\to W$ by

$$D_W F=\sum_j (p_j(\partial)F)(0)p_j. \tag{25}$$

Observe that D_W is independent of the choice of the orthonormal basis.

Consider the real subspace $Y=\{(z,\overline{z}):z\in\mathbb{C}^{n\times k}\}$ of X. For $p\in\mathscr{P}(X,V)$, let $p|_Y$ denote the restriction of p to Y. Then $p|_Y$ is a V-valued polynomial on $\mathbb{C}^{n\times k}$. Since each $p\in\mathscr{P}(X,V)$ is holomorphic, the restriction map $p\to p|_Y$ is injective (p. 42, [6]). Moreover, restriction intertwines differentiation in the following sense:

$$\frac{\partial}{\partial z_{ij}}(p(z,\overline{z}))=\left(\frac{\partial p}{\partial u_{ij}}\right)(z,\overline{z}) \text{ and } \frac{\partial}{\partial \overline{z}_{ij}}(p(z,\overline{z}))=\left(\frac{\partial p}{\partial w_{ij}}\right)(z,\overline{z}). \tag{26}$$

Now fix $z\in\mathbb{C}^{n\times k}$ and define point evaluation $E_z\colon W\to V$ by $E_z(p)=p(z,\overline{z})$. Then the adjoint map $E_z^*\colon V\to W$ can be realized in terms of an orthonormal basis $\{p_j\}$ of W:

$$E_z^*(v)=\sum_j (v,p_j(z,\overline{z}))_V p_j. \tag{27}$$

Lemma 6.1. *Suppose $W\subset\mathscr{P}(X,V)$ is a finite dimensional subspace of homogeneous polynomials of degree m. For fixed $v\in V$ and $w\in\mathbb{C}^{n\times k}$, let $F_w(z)=e^{i\,\mathrm{Re}(z,w)}v$. Then $D_W F_w=(i/2)^m E_w^*(v)$.*

Proof. Since $\mathrm{Re}(z,w)=1/2(\mathrm{Tr}(\overline{w}^T z)+\mathrm{Tr}(\overline{z}^T w)$, one has

$$\tfrac{\partial^\alpha}{\partial z^\alpha}(e^{i\,\mathrm{Re}(z,w)})\,|_{z=0}=(i/2)^{|\alpha|}\overline{w}^\alpha \quad\text{and}\quad \tfrac{\partial^\alpha}{\partial \overline{z}^\alpha}(e^{i\,\mathrm{Re}(z,w)})\,|_{z=0}=(i/2)^{|\alpha|}w^\alpha.$$

If $p(u,w)=\Sigma_{\alpha,\beta}u^\alpha w^\beta v_{\alpha,\beta}$ is homogeneous of degree m, we have

$$(p(\partial)F_w)(0)=(i/2)^m\Sigma_{\alpha,\beta}(\overline{w}^\alpha w^\beta v,v_{\alpha,\beta})_V=(i/2)^m(v,p(w,\overline{w}))_V. \tag{28}$$

The lemma now follows by (27) and (25). ■

Definition 6.2. Let $(u,w)\in X$. Define differential operators by

$$\Lambda_{ij}=\sum_{m=1}^k\frac{\partial^2}{\partial u_{im}\partial w_{jm}},\ \text{for } 1\le i,j\le n.$$

We say that a polynomial p is pluriharmonic if $\Lambda_{ij}p=0$, for all $1\le i,j\le n$. Let $\mathscr{H}$ denote the space of pluriharmonic polynomials.

Remark 6.3. By (26), we have $(\Lambda_{ij}(p))|_Y = D_{ij}(p|_Y)$ for any $p \in \mathscr{P}(X)$. Thus, $p \in \mathscr{H}$ iff its restriction $p|_Y$ satisfies $D_{ij}(p|_Y) = 0$, for all $1 \le i, j \le n$. Recall the coordinates $z = x + iy$, $x, y \in \mathbb{R}^{n\times k}$ of $\mathbb{C}^{n\times k}$. Let $\triangle_{ij} = \partial^2/\partial x_{ij}^2 + \partial^2/\partial y_{ij}^2$ and $\triangle = \sum_{i,j} \triangle_{ij}$ denote the Laplace operator. Since $\triangle_{ij} = 4\frac{\partial^2}{\partial \bar{z}_{ij}\partial z_{ij}}$, we have $\triangle = 4\sum_{i=1}^n D_{ii}$. So a polynomial annihilated by all D_{ij} is harmonic. In particular if $p \in \mathscr{H}$, then $p|_Y$ is harmonic. ∎

The groups $K_{\mathbb{C}}$ and $GL(k,\mathbb{C})$ act on $\mathscr{H}$ by the formulas:

$$
\begin{aligned}
(g_1, g_2).p(u, w) &= p(g_1^{-1}u, g_2^T w), \;\; g_1, g_2 \in GL(n,\mathbb{C}), \\
g.p(u, w) &= p(ug, wg^{T,-1}), \;\; g \in GL(k,\mathbb{C}).
\end{aligned} \tag{29}
$$

Let ρ be the representation of $K_{\mathbb{C}}$ given in equation (29).

Let (λ, V_λ) be an irreducible holomorphic representation of $GL(k,\mathbb{C})$, equipped with an inner product $(.,.)_\lambda$ satisfying $\lambda(g)^* = \lambda(g^*)$ for $g \in GL(k,\mathbb{C})$. Let $\mathscr{H}_\lambda$ denote the space of V_λ-valued pluriharmonic polynomials p satisfying the transforming property

$$p(ug, w(g^T)^{-1}) = \lambda(g)^{-1}p(u, w), \text{ for } g \in GL(k,\mathbb{C}).$$

Let Σ denote the set of those (λ, V_λ) for which $\mathscr{H}_\lambda \neq 0$. The set Σ is described explicitly in [13].

By Proposition 5.6 of [13], the space $\mathscr{H}_\lambda$ is irreducible under ρ. Let ρ_λ denote this irreducible action of $K_{\mathbb{C}}$ on $\mathscr{H}_\lambda$. Furthermore, we restrict the above inner product to $\mathscr{H}_\lambda$ and denote this restriction by $(.,.)_{\mathscr{H}_\lambda}$. One then has $\rho_\lambda(g_1, g_2)^* = \rho_\lambda(g_1^*, g_2^*)$ for $(g_1, g_2) \in K_{\mathbb{C}}$. Observe that by Schur's lemma, $\mathscr{H}_\lambda$ is a space of homogeneous polynomials. Let m_λ denote the corresponding homogeneous degree.

In what follows, we will work with the differential operator D_W where $W = \mathscr{H}_\lambda$ and $V = V_\lambda$ for $\lambda \in \Sigma$.

7. The range of T_γ

In this section, we construct a dense subspace of $\mathscr{L}^2(\mathbb{C}^{n\times k})$ that maps to $\mathscr{E}_{\gamma,k}$ under T_γ. The space of pluriharmonic functions $\mathscr{H}$ is central to the construction.

Lemma 7.1. *Suppose that p_m is a homogeneous harmonic polynomial of degree m on $\mathbb{R}^n$. Then for $a > 0$ and $x \in \mathbb{R}^n$, one has*

$$\int_{\mathbb{R}^n} e^{ix\cdot y}p(y)e^{-a\|y\|^2}dy = \left(\frac{\pi}{a}\right)^{n/2}\left(\frac{i}{2a}\right)^m p(x)e^{-\|x\|^2/4a}$$

Proof. Making the change of variable $y \to \sqrt{\frac{\pi}{a}}y$ in the above integral, we obtain

$$
\begin{aligned}
&\left(\frac{\pi}{a}\right)^{(n+m)/2}\int_{\mathbb{R}^n} e^{i\sqrt{\frac{\pi}{a}}x\cdot y}p_m(y)e^{-\pi\|y\|^2}dy \\
&\qquad = \left(\frac{\pi}{a}\right)^{(n+m)/2}\int_{\mathbb{R}^n} e^{-2\pi i\left(\frac{-1}{2\sqrt{\pi a}}x\right)\cdot y}p_m(y)e^{-\pi\|y\|^2}dy.
\end{aligned}
$$

The lemma follows by applying the result [[16], p. 155]

$$\int_{\mathbb{R}^n} e^{-2\pi i x\cdot y}p_m(y)e^{-\pi\|y\|^2}dy = i^{-m}p_m(x)e^{-\pi\|x\|^2}.$$ ∎

Proposition 7.2. *Let $p_m \in \mathscr{H}$ be homogeneous of degree m and assume that $\Omega = \{A \in GL(n,\mathbb{C}) \mid A + A^* > 0\}$. Then for $\gamma > 0$, $z \in \mathbb{C}^{n\times k}$ and $A \in \Omega$,*

$$\int_{\mathbb{C}^{n\times k}} e^{i\,\mathrm{Re}(z,\xi)} p_m(\xi,\overline{\xi}) e^{-\frac{1}{4\gamma}(A\xi,\xi)} d\xi = C \det(A)^{-k} p_m(A^{-1}z, A^{-1,T}\overline{z}) e^{-\gamma(A^{-1}z,z)},$$

where C is a constant depending on m, k, n and γ.

Proof. Both sides of the above equation holomorphically depend on $A \in \Omega$. Assume that $A > 0$ and let $A^{1/2}$ denote the unique hermitian operator with $\left(A^{1/2}\right)^2 = A$. Making the change of variable $\xi \to A^{-1/2}\xi$ in the above integral yields

$$\det(A)^{-k} \int_{\mathbb{C}^{n\times k}} e^{i\,\mathrm{Re}(A^{-1/2}z,\xi)} p_m(A^{-1/2}\xi, (A^{-1/2})^T\overline{\xi}) e^{-\frac{1}{4\gamma}(\xi,\xi)} d\xi$$

Since $\mathscr{H}$ is invariant under $K_{\mathbb{C}}$, the polyomial $(u,w) \to p_m(A^{-1/2}u, (A^{-1/2})^T w)$ lies in $\mathscr{H}$. So its restriction $p_m(A^{-1/2}\xi, (A^{-1/2})^T\overline{\xi})$ is harmonic by Remark 6.3. By Lemma 7.1 the integral is the constant $(4\gamma\pi)^{nk}(2\gamma i)^m$ times

$$\det(A)^{-k} p_m(A^{-1/2}(A^{-1/2}z), (A^{-1/2})^T \overline{A^{-1/2}z}) e^{-\gamma(A^{-1}z,z)},$$

giving the right side of the equation. The proposition now follows by analytic continuation. ∎

Corollary 7.3. *Let $p_m \in \mathscr{H}$ be homogeneous of degree m. Then for $\gamma > 0$ and $(t,z) \in \mathrm{Her}_n \times \mathbb{C}^{n\times k}$,*

$$T_\gamma(p_m(\xi,\overline{\xi}) e^{-\frac{1}{4\gamma}(\xi,\xi)})(t,z) = C \det(I - it)^{-k} p_m((I-it)^{-1}z, (I-it)^{-1,T}\overline{z}) e^{-\gamma((I-it)^{-1}z,z)},$$

where C is a constant depending on m, k, n and γ.

Proof. On one hand,

$$T_\gamma(p_m(\xi,\overline{\xi}) e^{-\frac{1}{4\gamma}(\xi,\xi)})(t,z) = (2\pi)^{-nk} \int_{\mathbb{C}^{n\times k}} e^{i\,\mathrm{Re}(z,\xi)} p_m(\xi,\overline{\xi}) e^{-\frac{1}{4\gamma}((I-it)\xi,\xi)} d\xi,$$

which by the proposition is

$$C \det(I-it)^{-k} p_m((I-it)^{-1}z, (I-it)^{-1,T}\overline{z}) e^{-\gamma((I-it)^{-1}z,z)},$$

since $(I - it) \in \Omega$. ∎

We wish to extend the functions in (t,z) from the above corollary to functions in the induced space $\mathrm{Ind}_{\overline{P}}^{G\ltimes H_n} \chi_{\gamma,k}$. To this end, consider more generally a group H that factors $H = X \cdot Y$, where Y is a subgroup. Let (χ, V) be a representation of Y. If each $h \in H$ has a unique factorization $h = xy$, for $x \in X$, $y \in Y$, then a natural way to lift a function $f : X \to V$ to a function in $\mathrm{Ind}|_Y^H \chi$ is to set $\Phi(h) = \chi(y)^{-1} f(x)$ since the desired transforming property $\Phi(hy) = \chi(y)^{-1}\Phi(h)$ becomes automatic. In our setting, an obstruction to this approach is that the needed factorization $(N \ltimes Z)\overline{P}$ is only open and dense in $G \ltimes H_n$ (cf. Lemma (2.1)), i.e. the matrix D in $g \in G$ (written in block form (1)) must be invertible.

However, for functions of the form in Corollary 7.3, the obstruction may be removed so that these functions admit a smooth extension to $G \ltimes H_n$. To display this extension, recall a natural action of $g \in G$ (written in block form (1)) on the upper (or lower) Segal half-plane is given by linear fractional transformations:

$$g.Z = (AZ + B)(CZ + D)^{-1}.$$

Proposition 7.4. *Let $p \in \mathbb{C}[X]$ be a polynomial. The function*

$$f(t,z) = \det(I - it)^{-k} p((I - it)^{-1} z, (I - it)^{-1,T} \overline{z}) e^{-\gamma((I-it)^{-1}z,z)}$$

extends to a function in $\mathrm{Ind}_{\overline{P}}^{G \ltimes H_n} \chi_{\gamma,k}$*. In particular,* $T_\gamma(\mathscr{H}|_Y e^{-\frac{1}{4\gamma}(\xi,\xi)}) \subset \mathscr{E}_{\gamma,k}$*, for* $\gamma > 0$*.*

Proof. Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $h = \left(\binom{z}{w}, s\right)$. Assume for the moment that $(g,h) \in G' \ltimes H_n$. By Lemma 2.1, $G' \ltimes H_n$ factors as $(N \ltimes Z)\overline{P}$ so we can write $(g,h) = n_{(g,h)} z_{(g,h)} \overline{p}_{(g,h)}$ and, by identifying functions on $N \ltimes Z$ with functions on $\mathrm{Her}_n \times \mathbb{C}^{n\times k}$, define $\Phi(g,h) = \chi_{\gamma,k}(\overline{p}_{(g,h)})^{-1} f(n_{(g,h)} z_{(g,h)})$. By the explicit formulas in the proof of Lemma 2.1, we have

$$\Phi(g,h) = \det(D^*)^{-k} e^{-i\gamma(s + \mathrm{Re}(z,w) + (D^{-1}Cz,z))} f(BD^{-1}, D^{*-1}z).$$

Using the group properties (2) and (3), $\Phi(g,h)$ may be written as

$$\det(D^* - iB^*)^{-k} e^{-i\gamma((g^*.(-iI_n))z,z) + \mathrm{Re}(z,w) + s)} p((D^* - iB^*)^{-1}\, z, (D - iB)^{-1,T}\, \overline{z}).$$

where $g^*.(-iI_n)$ denotes the linear fractional action. Each inverse appearing here exists by the properties of G. This formula then extends to $g \in G$ and so defines an element of $\mathrm{Ind}_{\overline{P}}^{G \ltimes H_n} \chi_{\gamma,k}$. ■

For $\gamma > 0$, the span of $\{p(z)e^{-\gamma(z,z)} \mid p \in \mathscr{P}(\mathbb{C}^{n\times k})\}$ is dense in $\mathscr{L}^2(\mathbb{C}^{n\times k})$. If $z = x + iy \in \mathbb{C}^{n\times k}$, a polynomial $p \in \mathscr{P}(\mathbb{C}^{n\times k})$ defines a polyomial (again denoted by p) in $\mathscr{P}(X)$ by replacing (x,y) by (u,v). Then the restriction of $p_o \in \mathscr{P}(X)$ defined by $p_o(u,v) = p(\frac{u+v}{2}, \frac{u-v}{2i})$ to Y is p. Therefore, $\mathscr{P}(X)|_Y = \mathscr{P}(\mathbb{C}^{n\times k})$. From Lemma 5.3 [13], one knows $\mathscr{P}(X) = \mathbb{C}[uw^T]\mathscr{H}$ so that $\mathscr{P}(\mathbb{C}^{n\times k}) = \mathbb{C}[z\overline{z}^T]\mathscr{H}|_Y$. Hence, $\mathbb{C}[z\overline{z}^T]\mathscr{H}|_Y\, e^{-\gamma(z,z)}$ is dense in $\mathscr{L}^2(\mathbb{C}^{n\times k})$.

Proposition 7.5. *For $\gamma > 0$, the image of $\mathbb{C}[z\overline{z}^T]\,\mathscr{H}|_Y\, e^{-\gamma(z,z)}$ under T_γ lies in $\mathscr{E}_{\gamma,k}$.*

Proof. By equation (23), the action an element of $\mathfrak{n}$, the Lie algebra of N, is to multiply a function by a matrix entry of $z\overline{z}^T$. With $\mathfrak{U}(\mathfrak{n})$ denoting the universal enveloping algebgra of $\mathfrak{n}$, we have

$$dL_k^\gamma(\mathfrak{U}(\mathfrak{n}))(\mathscr{H}|_Y e^{-\gamma(z,z)}) = \mathbb{C}(z\overline{z}^T)\mathscr{H}|_Y e^{-\gamma(z,z)}.$$

By Theorem 5.5, T_γ intertwines $\mathfrak{U}(\mathfrak{n})$ so the proposition follows. ■

8. Representations and intertwining maps

A multiplier representation on G/K. We consider a multiplier representation of G defined on spaces of holomorphic functions on G/K. Following Kashiwara and Vergne [13], we realize G/K in the unbounded picture.

Let $$\mathfrak{D} = \{\zeta = t + is \mid t, s \in \mathrm{Her}_n,\ s \gg 0\}.$$

Set $K_{\mathbb{C}} = GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ and let (τ, W_τ) be a holomorphic representation of $K_{\mathbb{C}}$. Let $\mathscr{O}(\mathfrak{D}, W_\tau)$ denote the complex space of holomorphic functions $F : \mathfrak{D} \to W_\tau$. Recalling the notation in (5) and (4), we display the representation on the generators of G in [13]:

$$(S_\tau(m(a))F)(\zeta) = \tau(a, a^{*,-1})F(a^{-1}\zeta a^{*,-1}), a \in GL(n,\mathbb{C}), \tag{30}$$
$$(S_\tau(n(t))F)(\zeta) = F(\zeta - t), t \in \mathrm{Her}_n\,, \tag{31}$$
$$(S_\tau(\sigma)F)(\zeta) = \tau(-\zeta, -\zeta^{-1})F(-\zeta^{-1}). \tag{32}$$

For $\lambda \in \Sigma$, let $\mathscr{L}^2(\mathbb{C}^{n\times k}, V_\lambda)^{U(k)}$ denote the space of square integrable, V_λ-valued functions satisfying $f(z\kappa) = \lambda(\kappa)^{-1}f(z)$, $\kappa \in U(k)$. Fix the parameter $\gamma = \frac{1}{4}$ in equations (23) to obtain a representation L_k^λ of G on $\mathscr{L}^2(\mathbb{C}^{n\times k}, V_\lambda)^{U(k)}$. Recall from Section 6 that ρ_λ denotes the irreducible action of $K_{\mathbb{C}}$ on $\mathscr{H}_\lambda$. Define a character δ_k on $K_{\mathbb{C}}$ by $\delta_k(g_1, g_2) = \det(g_2)^k$. We take $\tau_\lambda = \rho_\lambda \otimes \delta_k$ and $W_\tau = \mathscr{H}_\lambda$ as our representation of $K_{\mathbb{C}}$ above and write S_λ for the corresponding representation of G.

For $w \in \mathbb{C}^{n\times k}$, recall the point evaluation map $E_w \colon \mathscr{H}_\lambda \to V_\lambda$ defined by the formula $E_w(f) = f(w, \overline{w})$. Define $E_w^* : V_\lambda \to \mathscr{H}_\lambda$ by $(E_w(h), v)_\lambda = (h, E_w^*(v))_{\mathscr{H}_\lambda}$ for $h \in \mathscr{H}_\lambda$, $v \in V_\lambda$.

Finally, define a map $KV_\lambda \colon \mathscr{L}^2(\mathbb{C}^{n\times k}, V_\lambda)^{U(k)} \to \mathscr{O}(\mathfrak{D}, \mathscr{H}_\lambda)$ by

$$KV_\lambda(\psi)(\zeta) = \int_{\mathbb{C}^{n\times k}} e^{i(\zeta w, w)} E_w^*(\psi(w))dw, \psi \in \mathscr{L}^2(\mathbb{C}^{n\times k}, V_\lambda)^{U(k)}, \zeta \in \mathfrak{D}. \tag{33}$$

By Propositions (4.1) and (5.8) [13], we know that KV_λ is injective and intertwines $L_k^\lambda(g)$ and $S_\lambda(g)$, for $g \in G$.

Principal series representations. We now define a principal series representation of G. For $\lambda \in \Sigma$, define a representation μ_λ of MA by setting $\mu_\lambda(a) = \tau_\lambda(m(a))$ and extend the action trivially to $\overline{N}$ to obtain a representation of $MA\overline{N}$. The functions in the induced space $\mathrm{Ind}_{MA\overline{N}}^G \mu_\lambda$ are completely determined by their restrictions to N. Let

$$\mathscr{I}_\lambda = \mathscr{I}_\lambda(\mathrm{Her}_n, \mathscr{H}_\lambda \otimes \det{}^k) = \{f \in \mathscr{C}^\infty(\mathrm{Her}_n, \mathscr{H}_\lambda) \mid \exists \varphi \in \mathrm{Ind}_{MA\overline{N}}^G \mu_\lambda, f(t) = \varphi(n(t))\}.$$

Recall the block form (1) for elements of G.

Proposition 8.1. *For $f \in \mathscr{I}_\lambda$, the element $g \in G$ acts by*

$$(g.f)(t) = \det((A - tC)^*)^{-k}\rho_\lambda(A - tC, (A - tC)^{-1,*})f((A - tC)^{-1}(tD - B)), \tag{34}$$

when $\det(A - tC) \neq 0$.

Proof. This follows immediately from Lemma 2.1 and the hermitian nature of $(A - tC)^{-1}(tD - B)$. ∎

Let $t + is \in \mathfrak{D}$. For a function F defined on $\mathfrak{D}$, define a function $B(F)$ on Her_n by $B(F)(t) = \lim_{s\to 0^+} F(t + is)$. Note that from formulas (34) and (30) - (32), the boundary-value map B intertwines the G actions on $\mathscr{O}(\mathfrak{D}, \mathscr{H}_\lambda)$ and $\mathscr{I}_\lambda$, when such limits exist.

Isotypic Components. Set $\gamma = 1/4$ for the remainder. Define

$$\mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k}) = \mathcal{E}_{\gamma,k} \bigcap \mathscr{S}ol_\gamma.$$

Recall that we have a G-map on the Schwartz functions

$$T : \mathscr{S}(\mathbb{C}^{n\times k}) \to \mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k})$$

that completes to an embedding of $\mathscr{L}^2\left(\mathbb{C}^{n\times k}\right)$. This can be made more precise by using the action of $U(k)$ on $\mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k})$.

For an irreducible representation (λ, V_λ) of $U(k)$, write the dual space as V_λ^*. For $f \in V_\lambda^*$ and $\psi \in \mathcal{C}^\infty(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda)$, define $\langle \psi, f\rangle \in \mathcal{C}^\infty(\mathrm{Her}_n \times \mathbb{C}^{n\times k})$ by $\langle \psi, f\rangle (t,z) := f(\psi(t,z))$. Now define

$$\begin{aligned}&\mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda)\\ &\quad := \left\{ \psi \in \mathcal{C}^\infty(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda) \;\middle|\; \begin{array}{l} \langle \psi, f\rangle \in \mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k})\ \forall f \in V_\lambda' \text{ and} \\ \psi(th, z) = \lambda(h)^{-1}\psi(t,z)\ \forall h \in U(k) \end{array} \right\}.\end{aligned}$$

The $U(k)$-finite vectors in $\mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k})$ decompose into a direct sum of isotypic components. Write $\mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k})_{\lambda'}$ for the V_λ^*-isotypic $U(k)$ component. It is straightforward to see that the map

$$\mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda) \otimes V_\lambda^* \to \mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k})_{\lambda'}$$

induced by $\psi \otimes f \to \langle \psi, f\rangle$ is an isomorphism onto the V_λ'-isotypic $U(k)$-component of $\mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k})$. By the commutivity of G and $U(k)$, $\mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k})$ inherits the structure of a G-module with action given by the same formula as Theorem 3.1.

There is a similar discussion starting with $\mathscr{L}^2\left(\mathbb{C}^{n\times k}\right)$. It ends up giving an isomorphism

$$\mathscr{L}^2\left(\mathbb{C}^{n\times k}, V_\lambda\right)^{U(k)} \otimes V_\lambda^* \to \mathscr{L}^2\left(\mathbb{C}^{n\times k}\right)_{\lambda'}.$$

In [13], it is shown that the isotypic components are nonzero when $\lambda \in \Sigma$.

Finally, by extending to vector valued integration, the same formula as in Definition 5.3 gives a map on the Schwartz functions in $\mathscr{L}^2\left(\mathbb{C}^{n\times k}, V_\lambda\right)^{U(k)}$,

$$T_\lambda : \mathscr{S}(\mathbb{C}^{n\times k}, V_\lambda) \to \mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda).$$

The map T_λ remains a G-map. It follows from [13] that T_λ completes to an isomorphism when $\lambda \in \Sigma$.

A differential intertwining map. Fix $\lambda \in \Sigma$ and let V_λ and $\mathscr{H}_\lambda$ play the role of spaces V and W found in Section 6. Consider the operator $D_{\mathscr{H}_\lambda}$ in equation (25). Since the map $D_{\mathscr{H}_\lambda} : \mathscr{C}^\infty(\mathbb{C}^{n\times k}, V_\lambda) \to \mathscr{H}_\lambda$ differentiates with respect to the z variable, we may trivially extend $D_{\mathscr{H}_\lambda}$ to a map

$$\mathscr{C}^\infty(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda) \to \mathscr{C}^\infty(\mathrm{Her}_n, \mathscr{H}_\lambda),$$

again denoted by $D_{\mathscr{H}_\lambda}$. We rescale this extension and define $D_\lambda = (i/2)^{-m_\lambda} D_{\mathscr{H}_\lambda}$, where m_λ is the homogeneous degree of polynomials in $\mathscr{H}_\lambda$.

Let $\mathscr{S}(\mathbb{C}^{n\times k}, V_\lambda)$ denote space of the V_λ- valued Schwartz functions on $\mathbb{C}^{n\times k}$ inside of $\mathscr{L}^2\left(\mathbb{C}^{n\times k}, V_\lambda\right)^{U(k)}$. Let $\psi \in \mathscr{S}(\mathbb{C}^{n\times k}, V_\lambda)$. Given the expression for E_w^* in equation (27), the function $\zeta \to KV_\lambda(\psi)(\zeta)$ takes the form

$$\sum_j \left(\int_{\mathbb{C}^{n\times k}} e^{i(\zeta w, w)}(\psi(w), p_j(w, \overline{w}))_\lambda dw\right) p_j,\ \zeta \in \mathfrak{D}.$$

In particular, one sees that the boundary-value map B is well-defined on the space $KV_\lambda(\mathscr{S}(\mathbb{C}^{n\times k}, V_\lambda))$.

In the following result, we set $T = T_{1/4}$ in Definition 5.3.

Theorem 8.2. *Let $\lambda \in \Sigma$. If $\psi \in \mathscr{S}(\mathbb{C}^{n\times k}, V_\lambda)$, then*

$$(D_\lambda \circ T)(\psi) = (B \circ KV_\lambda)(\psi).$$

In particular, D_λ is a G-intertwining map on $\mathcal{H}(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda)$.

Proof. From equation (33), we see for $t \in \mathrm{Her}_n$,

$$(B \circ KV_\lambda)(\psi)(t) = \int_{\mathbb{C}^{n\times k}} e^{i(tw,w)} E_w^*(\psi(w))dw.$$

On the other hand,

$$(D_\lambda \circ T)(\psi)(t) = \int_{\mathbb{C}^{n\times k}} e^{i(tw,w)} D_\lambda(e^{i\,\mathrm{Re}(z,w)}\psi(w))dw.$$

But by Lemma 6.1, we have $D_\lambda(e^{i\,\mathrm{Re}(z,w)}\psi(w)) = E_w^*(\psi(w))$. Finally, since B, KV_λ and T (cf. Corollary 5.6) are G-intertwining maps, so is D_λ. ■

This result completes to the following commutative digram of G-maps.

$$\begin{array}{ccc}
\overline{\mathcal{H}}(\mathrm{Her}_n \times \mathbb{C}^{n\times k}, V_\lambda) & \xhookrightarrow{D_\lambda} & \mathscr{I}(\mathrm{Her}_n, W_\tau \otimes \det^k) \\
{\scriptstyle T_\lambda}\uparrow & & \uparrow{\scriptstyle B} \\
\mathscr{L}^2(\mathbb{C}^{n\times k}, V_\lambda)^{U(k)} & \xrightarrow[\mathrm{KV}_\lambda]{} & \mathscr{O}(G/K, W_\tau \otimes \det^k)
\end{array}$$

The maps T_λ and KV_λ are isomorphisms and the maps $D_\lambda|_{z=0}$ and B are embeddings.

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