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Poincaré-Birkhoff-Witt Theorem for Pre-Lie and Post-Lie Algebras

Vsevolod Gubarev

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Abstract. We construct the universal enveloping preassociative and postassociative algebra for a pre-Lie and a post-Lie algebra respectively. We show that the pairs $(\mathrm{preLie}, \mathrm{preAs})$ and $(\mathrm{postLie}, \mathrm{postAs})$ are Poincaré-Birkhoff-Witt-pairs; for the first this is a reproof of the result of V. Dotsenko and P. Tamaroff.

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Key Words: Rota-Baxter operator, Gröbner-Shirshov basis, pre-Lie algebra, post-Lie algebra, preassociative algebra (dendriform algebra), postassociative algebra.

1. Introduction

In the 1960s, pre-Lie algebras appeared independently in affine geometry (E. Vinberg [45], J.-L. Koszul [32]) and ring theory (M. Gerstenhaber [19]). Arising from diverse areas, pre-Lie algebras are known under different names like Vinberg algebras, Koszul algebras, left- or right-symmetric algebras (LSAs or RSAs), Gerstenhaber algebras. Pre-Lie algebras satisfy an identity $(x_1x_2)x_3 - x_1(x_2x_3) = (x_2x_1)x_3 - x_2(x_1x_3)$. See [8, 38] for surveys on pre-Lie algebras.

In 2001, J.-L. Loday [34] defined the *dendriform (di)algebra* (preassociative algebra) as a vector space endowed with two bilinear operations $\succ, \prec$ satisfying

$$(x_1 \succ x_2 + x_1 \prec x_2) \succ x_3 = x_1 \succ (x_2 \succ x_3),\ (x_1 \succ x_2) \prec x_3 = x_1 \succ (x_2 \prec x_3),$$
$$x_1 \prec (x_2 \succ x_3 + x_2 \prec x_3) = (x_1 \prec x_2) \prec x_3.$$

In 1995, J.-L. Loday [33] also defined *Zinbiel algebras* (precommutative algebras), on which the identity $(x_1x_2 + x_2x_1)x_3 = x_1(x_2x_3)$ holds. Every preassociative algebra with the identity $x \succ y = y \prec x$ is a precommutative algebra ($x_1x_2 = x_1 \succ x_2$) and under the product $x \cdot y = x \succ y - y \prec x$ is a pre-Lie algebra.

In 2004, dendriform trialgebra (postassociative algebra) was introduced [37], i.e., an algebra with three bilinear operations $\prec, \succ, \cdot$ satisfying the following seven identities

$$(x \prec y) \prec z = x \prec (y \succ z + y \prec z + y \cdot z), \quad (x \succ y) \prec z = x \succ (y \prec z),$$
$$(x \succ y + y \succ x + x \cdot y) \succ z = x \succ (y \succ z), \quad x \succ (y \cdot z) = (x \succ y) \cdot z,$$
$$(x \prec y) \cdot z = x \cdot (y \succ z), \quad (x \cdot y) \prec z = x \cdot (y \prec z), \quad (x \cdot y) \cdot z = x \cdot (y \cdot z).$$

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A space A with two bilinear operations $[,]$ and $\cdot$ is called a *post-Lie algebra* (B. Vallette, 2007 [44]) if $[,]$ is a Lie bracket and the next identities hold

$$(x\cdot y)\cdot z - x\cdot(y\cdot z) - (y\cdot x)\cdot z + y\cdot(x\cdot z) = [y,x]\cdot z, \quad x\cdot[y,z] = [x\cdot y,z] + [y,x\cdot z].$$

In the last dozen years a large number of articles devoted to post-Lie algebras in different areas has appeared [9, 18, 40]. Every postassociative algebra under the products $x\cdot y = x\succ y - y\prec x$, $[x,y] = x\circ y - y\circ x$ is a *post-Lie algebra* [3].

Let us explain the choice of terminology. Given a binary quadratic operad $\mathcal{P}$, the defining identities for pre- and post-$\mathcal{P}$-algebras can be found in [2]. One can define the operad of pre- and post-$\mathcal{P}$-algebras as $\mathcal{P}\bullet\mathrm{preLie}$ and $\mathcal{P}\bullet\mathrm{postLie}$ respectively. Here preLie and postLie denote the operads (varieties) of pre-Lie algebras and post-Lie algebras respectively, $\bullet$ denotes the black Manin product of operads [20]. By pre- or postalgebra we will mean pre- or post-$\mathcal{P}$-algebra for some operad $\mathcal{P}$.

Before stating the main problem of the work we introduce a very useful tool to deal with pre- and postalgebras, so called Rota-Baxter operators, and the notion of a Poincaré-Birkhoff-Witt pair.

A linear operator R defined on an algebra A over a field $\Bbbk$ is called a *Rota-Baxter operator* (*RB-operator*, for short) of a weight $\lambda\in\Bbbk$ if it satisfies the relation

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy), \quad x,y\in A. \tag{1}$$

In this case, an algebra A is called *Rota-Baxter algebra* (RB-algebra).

G. Baxter defined the notion of what is now called Rota-Baxter operator on a (commutative) algebra in 1960 [5], solving an analytic problem. The relation (1) with $\lambda = 0$ appeared as a generalization of integration by parts formula. G.-C. Rota [42], P. Cartier [10] and others studied different combinatorial properties of RB-operators and RB-algebras. In the 1980s, the deep connection between Lie RB-algebras and Yang-Baxter equation was found [6, 43]. More about Rota-Baxter algebras see in the monograph of L. Guo [29].

In 2000, M. Aguiar [1] stated that an associative algebra with a given Rota-Baxter operator R of weight zero under the operations $a\succ b = R(a)b$, $a\prec b = aR(b)$ is a preassociative algebra. In 2002, K. Ebrahimi-Fard [16] showed that an associative RB-algebra of nonzero weight λ under the same two products $\succ$, $\prec$ and the third operation $a\cdot b = \lambda ab$ is a postassociative algebra. The analogue of the Aguiar construction for the pair of pre-Lie algebras and Lie RB-algebras of weight zero was stated in 2000 by M. Aguiar [1] and by I. Z. Golubchik, V. V. Sokolov [21]. In 2010 [3], this construction for the pair of post-Lie algebras and Lie RB-algebras of nonzero weight was extended.

In 2013 [2], the construction of M. Aguiar and K. Ebrahimi-Fard was generalized for the case of arbitrary variety.

In 2008, the notion of universal enveloping RB-algebras of pre- and postassociative algebras was introduced [17]. In [17], it was also proved that the universal enveloping of a free pre- or postassociative algebra is free.

In 2011, with the help of Gröbner-Shirshov bases [7], Yu. Chen and Q. Mo proved that every preassociative algebra over a field of characteristic zero injectively embeds into its universal enveloping RB-algebra [12].

In 2013 [27], given a variety Var, it was proved that every pre-Var-algebra (post-Var-algebra) injectively embeds into its universal enveloping Var-RB-algebra of weight $\lambda = 0$ ($\lambda \neq 0$). Further, the author constructed universal enveloping RB-algebra for a given pre- or postalgebra in commutative [22], associative [23], and Lie [24] cases. In the associative case it gave an answer to the question of L. Guo [29, p. 148].

The classical Poincaré-Birkhoff-Witt theorem states that given a Lie algebra g with a linear basis $\{x_i \mid i \in I\}$, where I is a well-ordered set, the monomials $x_{i_1} \cdots x_{i_n}$ with $i_1 \leq \ldots \leq i_n$ form a linear basis for the universal enveloping associative algebra $U(g)$. As a consequence, we get that the linear basis of the algebra $U(g)$ does not depend on the product in the Lie algebra g. Such relationships between two varieties $\mathcal{V}, \mathcal{W}$ in the case when there exists a functor $\phi\colon \mathcal{V} \to \mathcal{W}$ associating to every algebra $A \in \mathcal{V}$ an algebra $\phi(A) \in \mathcal{W}$ by changing multiplication in A was generalized by I. Shestakov and A. A. Mikhalev in the term Poincaré-Birkhoff-Witt (PBW-) pair, see details in [39].

Now let us formulate the main problem to which the work is devoted. In what follows, preAs and postAs denote the varieties of pre- and postassociative algebras respectively.

Problem 1.1. (a) Prove that every pre-Lie (post-Lie) algebra injectively embeds into its universal enveloping preassociative (postassociative) algebra.
(b) Clarify if the pairs (preAs, preLie) and (postAs, postLie) are PBW-ones.
(c) Construct the universal enveloping preassociative (postassociative) algebra for a given pre-Lie (post-Lie) algebra.

For pre-Lie algebras, Problem 1.1b and the special version of Problem 1.1c were stated by P. Kolesnikov in [31] in the context of Gröbner-Shirshov bases for preassociative algebras. J.-L. Loday asked V. Dotsenko about the solution of Problem 1.1b around 2009 [15]. The discussion of Problem 1.1 in the case of restricted pre-Lie algebras can be found in [13]. The analogues of Problem 1.1 for Koszul-dual objects, di- and trialgebras, were solved in [36, 28].

Recently, in [26] and [25], the author solved Problem 1.1a in pre- and postalgebra cases with the help of embedding of pre-Lie (post-Lie) algebras into Lie RB-algebras [27] and the Gröbner-Shirshov bases technique developed for associative RB-algebras [30]. Actually, the solution of Problem 1.1a for pre-Lie algebras can be derived from the results concerning Hopf preassociative algebras and so called brace algebras stated in 2002 independently by F. Chapaton [11] and M. Ronco [41]. In 2018, V. Dotsenko and P. Tamaroff solved Problem 1.1b for pre-Lie algebras stating that the pair of varieties (preLie, preAs) is a PBW-pair [15] by means of the general approach arising from category theory. In 2019, independently of the present work, by the same method, V. Dotsenko proved that the pair (postLie, postAs) is a PBW-one [14].

The current work is devoted to the complete solution of Problem 1.1 in both pre- and postalgebra cases. Let us briefly describe the idea of the solution. For this, we need one more embedding problem.

Let A be an associative algebra with an RB-operator R. Then the algebra $A^{(-)}$ is a Lie RB-algebra under the product $[x, y] = xy - yx$ and the same action of R. Thus, we can state the analogue of Problem 1.1 for the varieties of Lie and associative RB-algebras.

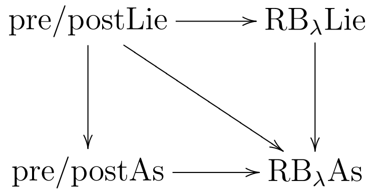
Problem 1.2. (a) Prove that every Lie RB-algebra injectively embeds into its universal enveloping associative RB-algebra.
(b) Construct the universal enveloping associative RB-algebra for a given Lie RB-algebra.

We are not asking whether the pair $(\mathrm{RB}_\lambda\mathrm{As}, \mathrm{RB}_\lambda\mathrm{Lie})$ of the varieties of associative and Lie RB-algebras (of weight λ) forms a PBW-pair, since it can be easily disproved just in the 2-dimensional case. To the moment, we are far from the solution of Problem 1.2, and the current work as well as [25, 26] can be also considered like a step in such direction.

The sketch of the solution of Problem 1.1c is following. Firstly, we embed a pre- or post-Lie algebra C into its universal enveloping Lie RB-algebra L [24]. Secondly, with the help of Gröbner-Shirshov bases we embed the Lie RB-algebra L into its universal enveloping associative RB-algebra A with an RB-operator R. Lastly, we show that a subalgebra $U(C)$ generated by C in the induced pre- or postassociative algebra on the space A is the universal enveloping pre- or postassociative algebra for C.

Actually, the same solution algorithm was used earlier by author in [25, 26] but with the following change: it was considered injective enveloping Lie RB-algebra from [27] instead of L, the universal enveloping one constructed in [24], in the first step. Surprisingly, a posteriori we may say that at least in the pre-Lie algebra case the injective enveloping preassociative algebra of a given pre-Lie algebra obtained earlier in [26] is an universal one.

At the end of Introduction, let us collect all stated in the work connections between universal enveloping algebras of different kind in the following commutative diagram,



where every arrow maps the algebra from corresponding variety into its universal enveloping one. An associative RB-algebra A with an RB-operator P of weight $\lambda = 0$ ($\lambda \neq 0$) is an enveloping for a pre-Lie (post-Lie) algebra C in the sense that the following equalities

$$a \succ b = P(a)b - bP(a), \quad a \prec b = aP(b) - P(b)a \quad (a \cdot b = \lambda ab - \lambda ba) \qquad (2)$$

hold for any $a, b \in C$.

2. Preliminaries

2.1. Some required formulas

Given a Lie algebra L, denote the product $[[\ldots[[y,x],x]\ldots],x] \in L^{p+1}$ by $[y, x^{(p)}]$.

Lemma 2.1 ([26]). *Given a Lie algebra L, the equality*

$$(l+1)yx^l = \sum_{i=2}^{l+1}(-1)^i\binom{l+1}{i}[y, x^{(i-1)}]x^{l+1-i} + \big(yx^l + xyx^{l-1} + \ldots + x^l y\big) \qquad (3)$$

holds in the universal enveloping algebra $U(L)$ for any $x, y \in L$ and $l \geq 0$.

It follows immediately from (1) that

$$R(b_1)\dots R(b_t) - R\bigg(\sum_{i=1}^{t} R(b_1)\dots R(b_{i-1})\hat{R}(b_i)R(b_{i+1})\dots R(b_t)$$
$$+\lambda \sum_{1\leq i_1<i_2\leq t} R(b_1)\dots \hat{R}(b_i)\dots \hat{R}(b_j)\dots R(b_t) + \dots$$
$$+\lambda^{k-1} \sum_{1\leq i_1<i_2<\dots<i_k\leq t} R(b_1)\dots \hat{R}(b_{i_1})\dots \hat{R}(b_{i_k})\dots R(b_t) + \dots + \lambda^{t-1}b_1\dots b_t\bigg) = 0, \quad (4)$$

where the sign ˆ shows the omitting action of R. From (3) and (4), we derive

$$R(R(b_1)R(b_2)\dots R(b_{l-1})b_l R(b_l)^k R(b_{l+k+1}))$$
$$= \frac{1}{k+1}R(b_1)R(b_2)\dots R(b_{l-1})R(b_l)^{k+1}R(b_{l+k+1})$$
$$+ \frac{1}{k+1}R\bigg(\sum_{i=2}^{k+1}(-1)^i\binom{k+1}{i}R(b_1)R(b_2)\dots R(b_{l-1})[b_l, R(b_l)^{(i-1)}]R(b_l)^{k+1-i}R(b_{l+k+1})$$
$$- \sum_{1\leq i\leq l-1,\, i=l+k+1} R(b_1)\dots R(b_{i-1})\hat{R}(b_i)R(b_{i+1})\dots R(b_t) \quad (5)$$
$$- \lambda \sum_{1\leq i_1<i_2\leq l+k+1} R(b_1)\dots \hat{R}(b_i)\dots \hat{R}(b_j)\dots R(b_{l+k+1}) - \dots$$
$$- \lambda^{s-1} \sum_{1\leq i_1<\dots<i_s\leq l+k+1} R(b_1)\dots \hat{R}(b_{i_1})\dots \hat{R}(b_{i_s})\dots R(b_{l+k+1}) - \dots - \lambda^{l+k}b_1\dots b_{l+k+1}\bigg),$$

where $b_l = b_{l+1} = \dots = b_{l+k}$ and $R(b_l)^s$ means $(R(b_l))^s$.
In what follows, we will use the formula (5) with maybe absent $R(b_1)$ and $R(b_{l+k+1})$, it means that we omit all b_1 and $R(b_1)$ in the summands and all indexes i_j start with two when $R(b_1)$ is absent. We do the same if $R(b_{l+k+1})$ is absent.

2.2. Embedding of pre- and postalgebras into RB-algebras

Theorem 2.2 ([1, 2, 4, 16, 21, 35]). *Let A be an RB-algebra of a variety* Var *and weight $\lambda = 0$ ($\lambda \neq 0$). With respect to the operations*

$$x \succ y = R(x)y, \quad x \prec y = xR(y) \quad (x\cdot y = \lambda xy) \quad (6)$$

*A is a pre-*Var*-algebra (post-*Var*-algebra).*

Denote the pre- and post-Var-algebra obtained in Theorem 2.2 as $A_\lambda^{(R)}$.
Given a pre-Var-algebra $\langle C, \succ, \prec\rangle$, universal enveloping RB-Var-algebra U of C is the universal algebra in the class of all RB-Var-algebras of weight zero such that there exists homomorphism from C to $U_0^{(R)}$. The notion of universal enveloping RB-Var-algebra of a post-Var-algebra is defined analogously.

Theorem 2.3 ([27]). *Every pre-*Var*-algebra (post-*Var*-algebra) could be embedded into its universal enveloping RB-algebra of the variety* Var *and weight $\lambda = 0$ ($\lambda \neq 0$).*

Let us briefly describe the idea of the proof of Theorem 2.3. Given a pre-Var-algebra (post-Var-algebra) C, we define the product on the space $\widehat{C} = C \oplus C'$, where C' is a copy of C, in such a way that $\widehat{C}$ is an algebra of the variety Var. Then we define on $\widehat{C}$ the linear operator P which occurs to be a Rota-Baxter operator of weight $\lambda = 0$ ($\lambda \neq 0$). Finally, Theorem 2.3 was stated by embedding C into $\widehat{C}_\lambda^{(P)}$ by the map $c \to c'$.

2.3. Gröbner-Shirshov bases for associative RB-algebras

Let $R\mathrm{As}\langle X\rangle$ denote the free associative algebra generated by a set X with a linear map R in the signature. One can construct a linear basis of $R\mathrm{As}\langle X\rangle$ (see, e.g., [17]) by induction. At first, all elements from $S(X)$, the free semigroup generated by X, lie in the basis. At second, if we have basic elements $a_1, a_2, \ldots, a_k$, $k \geq 1$, then the word $w_1 R(a_1) w_2 \ldots w_k R(a_k) w_{k+1}$ lies in the basis of $R\mathrm{As}\langle X\rangle$. Here $w_2, \ldots, w_k \in S(X)$ and $w_1, w_{k+1} \in S(X) \cup \emptyset$, where $\emptyset$ denotes the empty word. Let us denote the basis obtained as $RS(X)$. Given a word u from $RS(X)$, the number of appearances of the symbol R in u is denoted by $\deg_R(u)$, the R-degree of u. We call an element from $RS(X)$ of the form $R(w)$ as R-letter. By X_∞ we denote the union of X and the set of all R-letters. Given $u \in RS(X)$, define $\deg u$ (degree of u) as the length of u in the alphabet X_∞. In [17], $\deg u$ was called the breadth of u.

Suppose that X is a well-ordered set with respect to $<$. Let us introduce by induction the deg-lex order on $S(X)$. At first, we compare two words u and v by the length: $u < v$ if $|u| < |v|$. At second, when $|u| = |v|$, $u = x_i u'$, $v = x_j v'$, $x_i, x_j \in X$, we have $u < v$ if either $x_i < x_j$ or $x_i = x_j$, $u' < v'$. We compare two words u and v from $RS(X)$ by R-degree: $u < v$ if $\deg_R(u) < \deg_R(v)$. If $\deg_R(u) = \deg_R(v)$, we compare u and v in deg-lex order as words in the alphabet X_∞. Here we define each x from X to be less than all R-letters and $R(a) < R(b)$ if and only if $a < b$.

Let $*$ be a symbol not containing in X. By a $*$-bracketed word on X, we mean a basic word from $R\mathrm{As}\langle X \cup \{*\}\rangle$ with exactly one occurrence of $*$. The set of all $*$-bracketed words on X is denoted by $RS^*(X)$. For $q \in RS^*(X)$ and $u \in R\mathrm{As}\langle X\rangle$, we define $q|_u$ as the bracketed word obtained by replacing the letter $*$ in q by u.

The order defined above is *monomial*, i.e., from $u < v$ it follows that $q|_u < q|_v$ for all $u, v \in RS(X)$ and $q \in RS^*(X)$. Given $f \in R\mathrm{As}\langle X\rangle$, by $\bar{f}$ we mean the leading word in f. We call f monic if the coefficient of $\bar{f}$ in f is 1.

Definition 2.4 ([30]). Let $f, g \in R\mathrm{As}\langle X\rangle$. If there exist $\mu, \nu, w \in RS(X)$ such that $w = \bar{f}\mu = \nu\bar{g}$ with $\deg w < \deg(\bar{f}) + \deg(\bar{g})$, then we define $(f,g)_w$ as $f\mu - \nu g$ and call it the *composition of intersection* of f and g with respect to (μ, ν). If there exist $q \in RS^*(X)$ and $w \in RS(X)$ such that $w = \bar{f} = q|_{\bar{g}}$, then we define $(f,g)_w^q$ as $f - q|_g$ and call it the *composition of inclusion* of f and g with respect to q.

Definition 2.5 ([30]). Let S be a subset of monic elements from $R\mathrm{As}\langle X\rangle$ and $w \in RS(X)$.

(1) For $u, v \in R\mathrm{As}\langle X\rangle$, we call u and v congruent modulo (S, w) and denote this by $u \equiv v \bmod (S, w)$ if $u - v = \sum c_i q_i|_{s_i}$ with $c_i \in \Bbbk$, $q_i \in RS^*(X)$, $s_i \in S$ and $q_i|_{\overline{s_i}} < w$.

(2) For $f, g \in R\mathrm{As}\langle X\rangle$ and suitable w, μ, ν or q that give a composition of intersection $(f,g)_w$ or a composition of inclusion $(f,g)_w^q$, the composition is called trivial modulo (S, w) if $(f,g)_w$ or $(f,g)_w^q \equiv 0 \mod (S, w)$.

(3) The set $S \subset R\mathrm{As}\langle X\rangle$ is called a *Gröbner-Shirshov basis* if, for all $f, g \in S$, all compositions of intersection $(f,g)_w$ and all compositions of inclusion $(f,g)_w^q$ are trivial modulo (S, w).

Theorem 2.6 ([30]). *Let S be a set of monic elements in $R\mathrm{As}\langle X\rangle$, let $<$ be a monomial ordering on $RS(X)$ and let $Id(S)$ be the R-ideal of $R\mathrm{As}\langle X\rangle$ generated by S. If S is a Gröbner-Shirshov basis in $R\mathrm{As}\langle X\rangle$, then $R\mathrm{As}\langle X\rangle = \Bbbk Irr(S)\oplus Id(S)$ where $Irr(S) = RS(X)\setminus\{q|_{\bar{s}} \mid q\in RS^*(X), s\in S\}$ and $Irr(S)$ is a linear basis of $R\mathrm{As}\langle X\rangle/Id(S)$.*

3. PBW-theorem for pre-Lie and post-Lie algebras

Let A be an associative algebra with an RB-operator R. Then the algebra $A^{(-)}$ is a Lie RB-algebra under the product $[x,y] = xy - yx$ and the same action of R.

Let L be a Lie RB-algebra with an RB-operator P of weight λ. Suppose that there exists a subset $X_0 = \{x_\alpha \mid \alpha\in\Omega\}$ in L such that $X = \{x_{\alpha,k} := P^k(x_\alpha) \mid k\in\mathbb{N},\ \alpha\in\Omega\}$ is a linear basis of L. Our goal is to construct the universal enveloping associative RB-algebra of L (via Gröbner-Shirshov bases). This will lead us to the proof of the Poincaré-Birkhoff-Witt (PBW) theorem for the pairs (pre-Lie, preAs) and (postLie, postAs).

We may assume that the set Ω is well-ordered, so we define an order $<$ on the set X: $x_{\alpha,k} < x_{\beta,l}$ if $\alpha<\beta$ or $\alpha=\beta$ and $k<l$.

Consider the set S of the following elements in $R\mathrm{As}\langle X\rangle$:

$$xy - yx - [x,y],\ x>y,\ x,y\in X, \tag{7}$$

$$R(a)R(b) - R(R(a)b + aR(b) + \lambda ab), \tag{8}$$

$$R(R(z_1)\vec{x}_{q_1}R(z_2)\ldots R(z_s)\vec{x}_{q_s}x_{\beta,r}x_{\beta,r+1}^k R(z_{s+1})) - \Delta, \tag{9}$$

where Δ is defined as the RHS of (5) for $l = l_1+\ldots+l_s+s+1$ and

$$b_1 = z_1,\ b_2 = \widetilde{x_{q_1}^1},\ b_3 = \widetilde{x_{q_1}^2},\ \ldots,\ b_{l_1+1} = \widetilde{x_{q_1}^{l_1}},\ b_{l_1+2} = z_2,\ b_{l_1+3} = \widetilde{x_{q_2}^1},\ \ldots,$$
$$b_{l_1+\ldots+l_{s-1}+s} = z_s,\ b_{l_1+\ldots+l_{s-1}+s+1} = \widetilde{x_{q_l}^1},\ \ldots,\ b_{l_1+\ldots+l_s+s} = \widetilde{x_{q_s}^{l_s}},$$
$$b_{l_1+\ldots+l_s+s+1} = \ldots = b_{l_1+\ldots+l_s+s+k+1} = x_{\beta,r},\ b_{l_1+\ldots+l_s+s+k+2} = z_{s+1}.$$

Here $\vec{x}_{q_i} = x_{q_i}^1\ldots x_{q_i}^{l_i}$ for $x_{q_i}^j\in X$ and $\tilde{x}$ denotes $x_{\alpha,t-1}$ for $x = x_{\alpha,t}\in X$, i.e., $R(\tilde{x}) = x$. In (8), (9), we have

$$s\geq 1,\quad k,r\geq 0,\quad a,b\in RS(X),\quad z_2,\ldots,z_s\in RS(X)\setminus S(X),$$
$$\vec{x}_{q_1},\ldots,\vec{x}_{q_{s-1}}\in S(X\setminus X_0),\quad \vec{x}_{q_s}\in S(X\setminus X_0)\cup\emptyset$$

and $x_{\beta,r}$ is greater than any letter from $\vec{x}_{q_s}$.

By $R(z_1)$ we denote either that $z_1\in RS(X)\setminus S(X)$ or that $R(z_1)$ is absent, i.e., $R(z_1)=\emptyset$. The same holds for $R(z_{s+1})$. In particular, the values $s=1$, $k=0$, $R(z_1) = R(z_2) = \vec{x}_{q_1} = \emptyset$, transform (9) to the relation $R(x_{\beta,r}) - x_{\beta,r+1}$.

Remark 3.1. In (9), we use associative words $\vec{x}_\alpha\in S(X)$ instead of ordered polynomials from $\Bbbk[X]$, otherwise we will have to reduce the products of such polynomials from $\Bbbk[X]$ to the ordered ones in all possible compositions from S.

Theorem 3.2. *The set S is a Gröbner-Shirshov basis in $R\mathrm{As}\langle X\rangle$.*

Proof. All compositions between two elements from (7) are trivial, as it is the method to construct the universal enveloping associative algebra for a given Lie algebra. Also, compositions of intersection between (8) and (8) are trivial, it is a way to get the free associative RB-algebra. Thus, all compositions of intersection which are not at the same time compositions of inclusion are trivial.

Let us compute a composition of inclusion between (7) and (9). Let

$$w = R(R(z_1)\vec{x}_{q_1}R(z_2)\dots R(z_s)\vec{x}_{q_s}x_{\beta,r}x_{\beta,r+1}^{k}R(z_{s+1})) \tag{10}$$

satisfy all conditions described above. We apply the relation (7): $xy - yx - [x,y]$ to the subword

$$\vec{x}_{q_j} = \vec{x}'_{q_j}xy\vec{x}''_{q_j}, \quad x > y,\ 1 \le j \le s.$$

Suppose that $j < s$. Since the image of an RB-operator is a subalgebra, $[x,y]$ lies in the linear combination of elements from $S(X \setminus X_0)$. Define $w' = w|_{xy\to yx}$ and $w'' = w|_{xy\to[x,y]}$, here $w|_{\alpha\to\beta}$ means the word w with the subword α replaced by β. On one hand, we have modulo (S,w)

$$w \overset{(7)}{\equiv} w' + w'' \overset{(9)}{\equiv} \frac{1}{k+1}\big(R(z_1)\dots R(z_j)\vec{x}_{q_j}|_{xy\to yx+[x,y]}R(z_{j+1}) \\ \dots R(z_s)\vec{x}_{q_s}x_{\beta,r+1}^{k+1}R(z_{s+1}) + R(\Sigma_{w'}) + R(\Sigma_{w''})\big),$$

where Σ_w is the expression in the RHS in the brackets under the action of R from (5). On the other hand, modulo (S,w)

$$w \overset{(9)}{\equiv} \frac{1}{k+1}\big(R(z_1)\dots R(z_j)\vec{x}_{q_j}R(z_{j+1})\dots R(z_s)\vec{x}_{q_s}x_{\beta,r+1}^{k+1}R(z_{s+1}) + R(\Sigma_w)\big) \\ \overset{(7)}{\equiv} \frac{1}{k+1}\big(R(z_1)\dots R(z_j)\vec{x}_{q_j}|_{xy\to yx+[x,y]}R(z_{j+1})\dots R(z_s)\vec{x}_{q_s}x_{\beta,r+1}^{k+1}R(z_{s+1}) + R(\Sigma_w)\big).$$

So, the composition equals $\frac{1}{k+1}(R(\Sigma_w) - R(\Sigma_{w'}) - R(\Sigma_{w''}))$ and we may rewrite it briefly as

$$\frac{1}{k+1}\sum_{s\in S} R(\alpha_s([\tilde{x},y] + [x,\tilde{y}] + \lambda[\tilde{x},\tilde{y}] - R^{-1}([x,y]))\beta_s)$$

for corresponding index set S and $\alpha_s, \beta_s \in RS(X)$. We get zero, since

$$[x,y] = [R(\tilde{x}), R(\tilde{y})] = R([\tilde{x},y] + [x,\tilde{y}] + \lambda[\tilde{x},\tilde{y}])$$

and R has zero kernel on L.

Consider the case $j = s$. The triviality of the corresponding composition of inclusion one can derive from the following fact. Denote as $A(z_1, \vec{x}_{q_1}, z_2, \dots, z_s, \vec{x}_{q_s}, z_{s+1})$ the expression (4) for $t = l_1 + \dots + l_s + s + 1$,

$$b_1 = z_1,\ b_2 = \widetilde{x_{q_1}^1},\ b_3 = \widetilde{x_{q_1}^2},\ \dots,\ b_{l_1+1} = \widetilde{x_{q_1}^{l_1}},\ b_{l_1+2} = z_2,\ b_{l_1+3} = \widetilde{x_{q_2}^1},\ \dots, \\ b_{l_1+\dots+l_{s-1}+s} = z_s,\ b_{l_1+\dots+l_{s-1}+s+1} = \widetilde{x_{ql}^1},\ \dots,\ b_{l_1+\dots+l_s+s} = \widetilde{x_{q_s}^{l_s}},\ b_{l_1+\dots+l_s+s+1} = z_{s+1},$$

where $z_1, \dots, z_{s+1}$ and the words $\vec{x}_{q_i} = x_{q_i}^1 \dots x_{q_i}^{l_i}$ satisfy the conditions for (9) except the one concerned $x_{\beta,r}$. We also have that the word $\vec{x}_{q_s} = x_{q_1}^1 \dots x_{q_1}^{l_s}$ contains the biggest letter $x_{\beta,r}$, $r \ge 1$, on the positions

$$K = \{k_1, k_2, \dots, k_p \mid k_1 < k_2 < \dots < k_p\} \subset \{1, 2, \dots, l_s\}.$$

So, the composition of inclusion is trivial if

$$A(z_1, \vec{x}_{q_1}, z_2, \dots, z_s, \vec{x}_{q_s}, z_{s+1}) \equiv 0 \mod (S, w) \tag{11}$$

for w greater than all terns involved in $A(\dots)$.

To prove (11), we will proceed on by induction on $l_s = |\vec{x}_{q_s}|$. For $l_s = 1$, we are done by (9). Consider the equality

$$\vec{x}_{q_s} = \vec{x}_{q_s,0} x^p_{\beta,r} + \sum_{i=1}^{p} x^1_{q_s} x^2_{q_s} \dots x^{k_i-1}_{q_s} \dots [x^{k_i}_{q_s}, w_i] x^{p-i}_{\beta,r}, \tag{12}$$

where $\vec{x}_{q_s,0}$ is obtained from $\vec{x}_{q_s}$ by arising all letters $x_{\beta,r}$ with preserving order of all remaining letters, w_i is obtained from the word $x^{k_i+1}_{q_s} \dots x^{l_s}_{q_s}$ by arising all $p-i$ letters $x_{\beta,r}$. For $w_i = w_i^1 w_i^2 \dots w_i^{l_s-k_i}$, $w_i^j \in X$, the bracket $[x^{k_i}_{q_s}, w_i]$ in (12) means

$$[x^{k_i}_{q_s}, w_i] = \sum_{j=1}^{l_s-k_i} w_i^1 \dots w_i^{j-1} [x^{k_i}_{q_s}, w_i^j] w_i^{j+1} \dots w_i^{l_s-k_i}.$$

By (9) and Lemma 2.1, we deduce that

$$A(z_1, \vec{x}_{q_1}, z_2, \dots, z_s, \vec{x}_{q_s}, z_{s+1}) \equiv \sum_{i=1}^{p} \sum_{j=1}^{l_s-k_i} A(z_1, \vec{x}_{q_1}, z_2, \dots, z_s, \vec{x}_{q_s}(i,j), z_{s+1})$$

modulo (S, w), where

$$\vec{x}_{q_s}(i,j) = x^1_{q_s} \dots x^{k_i-1}_{q_s} \dots w_i^1 \dots w_i^{j-1} [x^{k_i}_{q_s}, w_i^j] w_i^{j+1} \dots w_i^{l_s-k_i} x^{p-i}_{\beta,r}.$$

The equality $A(z_1, \dots, z_s, \vec{x}_{q_s}(q,i), z_{s+1}) \equiv 0$ modulo (S, w) follows from the inductive hypothesis.

Consider a composition of inclusion between (8) and (9). Let $w = R(w_0)$ be defined by (10) and $b \in RS(X)$. At first, we have modulo (S, w)

$$\begin{aligned}
&R(w_0)R(b) \overset{(8)}{\equiv} R(R(w_0)b + w_0R(b) + \lambda w_0 b) \\
&\overset{(9)}{\equiv} \frac{1}{k+1} R\big(R(z_1)\vec{x}_{q_1}R(z_2)\dots R(z_s)\vec{x}_{q_s}x^{k+1}_{\beta,r+1}R(z_{s+1})b + R(\Sigma_w)b\big) \\
&\quad + R\big(R(z_1)...R(z_s)\vec{x}_{q_s}x_{\beta,r}x^k_{\beta,r+1}R(z_{s+1})R(b) + \lambda R(z_1)...R(z_s)\vec{x}_{q_s}x_{\beta,r}x^k_{\beta,r+1}R(z_{s+1})b\big) \\
&\overset{(9)}{\equiv} \lambda R\big(R(z_1)\dots R(z_s)\vec{x}_{q_s}x_{\beta,r}x^k_{\beta,r+1}R(z_{s+1})b\big) \\
&\quad + \frac{1}{k+1}\big(R\big(R(z_1)\dots R(z_s)\vec{x}_{q_s}x^{k+1}_{\beta,r+1}R(z_{s+1})b + R(\Sigma_w)b\big) \\
&\quad + R(z_1)\dots R(z_s)\vec{x}_{q_s}x_{\beta,r}x^k_{\beta,r+1}R(z_{s+1})R(b) + R(\Sigma_{w|_{R(z_{s+1})\to R(z_{s+1})R(b)}})\big),
\end{aligned}$$

where Σ_w is the expression in the RHS in the brackets under the action of R from (5). At second, modulo (S, w)

$$\begin{aligned}
R(w_0)R(b) &\overset{(9)}{\equiv} \frac{1}{k+1}\big(R(z_1)\vec{x}_{q_1}R(z_2)\dots R(z_s)\vec{x}_{q_s}x^{k+1}_{\beta,r+1}R(z_{s+1})R(b) + R(\Sigma_w)R(b)\big) \\
&\overset{(8)}{\equiv} \frac{1}{k+1}\big(R(z_1)\dots R(z_s)\vec{x}_{q_s}x^{k+1}_{\beta,r+1}R(z_{s+1})R(b) + R(R(\Sigma_w)b + \Sigma_w R(b) + \lambda\Sigma_w b)\big).
\end{aligned}$$

So, the composition of inclusion multiplied by $(k+1)$ equals

$$u = R\big(\lambda(k+1)R(z_1)\dots R(z_s)\vec{x}_{q_s}x_{\beta,r+1}^{k+1}R(z_{s+1})b + R(z_1)\dots R(z_s)\vec{x}_{q_s}x_{\beta,r+1}^{k+1}R(z_{s+1})b\big)$$
$$+R(\Sigma_{w|_{R(z_{s+1})\to R(z_{s+1})R(b)}}) - R(\Sigma_w R(b) + \lambda\Sigma_w b).$$

Depending on the last factor, Σ_w splits into the sum $\Sigma'_w R(z_{s+1}) + \Sigma''_w z_{s+1}$. Applying the formulas

$$\lambda\Sigma_w b = \lambda\Sigma'_w R(z_{s+1})b + \lambda\Sigma''_w z_{s+1}b,$$
$$\Sigma_w R(b) = \Sigma'_w R(z_{s+1})R(b) + \Sigma''_w z_{s+1}R(b),$$
$$\Sigma_{w|_{R(z_{s+1})\to R(z_{s+1})R(b)}} = \Sigma'_w R(z_{s+1})R(b) + \Sigma''_w(R(z_{s+1})b + z_{s+1}R(b) + \lambda z_{s+1}b),$$

we deduce
$$u = R\big(R(z_1)\dots R(z_s)\vec{x}_{q_s}x_{\beta,r+1}^{k+1}R(z_{s+1})b$$
$$+\lambda\big((k+1)R(z_1)\dots R(z_s)\vec{x}_{q_s}x_{\beta,r}x_{\beta,r+1}^{k}R(z_{s+1})b - \Sigma'_w R(z_{s+1})b\big) + \Sigma''_w R(z_{s+1})b\big).$$

Writing down the sum $R(\Sigma''_w R(z_{s+1})b)$, u equals to zero by the definition of Σ'_w, Σ''_w and Lemma 2.1.

Compute a composition of inclusion between (9) and (9). Suppose that we have w defined by (10) and

$$z_m = R(a_1)\vec{x}_{t_1}R(a_2)\dots R(a_n)\vec{x}_{t_n}x_{\gamma,o}x_{\gamma,o+1}^{m}R(a_{n+1})$$

satisfying all conditions written above for (7).

Consider the case $m \le s$. By Σ_w, as earlier, we mean the expression in the RHS in the brackets under the action of R from (5) for w. By Δ, denote the expression in the RHS in the brackets under the action of R from (5) for z_m. We also define

$$z'_m = R(a_1)\vec{x}_{t_1}R(a_2)\dots R(a_n)\vec{x}_{t_n}x_{\gamma,o+1}^{m+1}R(a_{n+1}),$$
$$\check{\Delta} = \Delta - R(a_1)\dots R(a_n)\vec{x}_{t_n}HR(a_{n+1}),$$
$$H = \sum_{i=2}^{m+1}(-1)^i\binom{m+1}{i}\big[x_{\gamma,o}, x_{\gamma,o+1}^{(i-1)}\big]x_{\gamma,o+1}^{m+1-i}$$
$$+ x_{\gamma,o}x_{\gamma,o+1}^{m} + x_{\gamma,o+1}x_{\gamma,o}x_{\gamma,o+1}^{m-1} + \dots + x_{\gamma,o+1}^{m}x_{\gamma,o},$$

$\Sigma_w = \Sigma'_w + \Sigma''_w$, where we collect all summands from Σ_w with the factor $R(z_m)$ in the sum Σ'_w and all others in Σ''_w. On the one hand, modulo (S,w) we get

$$R\big(R(z_1)\vec{x}_{q_1}R(z_2)\dots R(z_s)\vec{x}_{q_s}x_{\beta,r}x_{\beta,r+1}^{k}R(z_{s+1})\big)$$
$$\overset{(9),\ \text{out}}{\equiv} \frac{1}{k+1}\big(R(z_1)\vec{x}_{q_1}R(z_2)\dots\vec{x}_{q_{m-1}}R(z_m)\vec{x}_{q_m}\dots R(z_s)\vec{x}_{q_s}x_{\beta,r+1}^{k+1}R(z_{s+1}) + R(\Sigma_w)\big)$$
$$\overset{(9),\ \text{in}}{\equiv} \frac{1}{(k+1)(m+1)}\big(R(z_1)\vec{x}_{q_1}R(z_2)\dots\vec{x}_{q_{m-1}}z'_m\vec{x}_{q_m}\dots R(z_s)\vec{x}_{q_s}x_{\beta,r+1}^{k+1}R(z_{s+1})$$
$$+ R(z_1)\vec{x}_{\alpha_1}R(z_2)\dots\vec{x}_{q_{m-1}}R(\Delta)\vec{x}_{q_m}\dots R(z_s)\vec{x}_{q_s}x_{\beta,r+1}^{k+1}R(z_{s+1}) + R(\Sigma'_w|_{R(z_m)\to z'_m})$$
$$+ R(\Sigma'_w|_{R(z_m)\to R(\Delta)}) + (m+1)R(\Sigma''_w). \tag{13}$$

On the other hand, modulo (S,w)

$$
\begin{aligned}
&R\big(R(z_1)\vec{x}_{q_1}R(z_2)\dots R(z_s)\vec{x}_{q_s}x_{\beta,r}x^k_{\beta,r+1}R(z_{s+1})\big)\\
&\overset{(9),\text{ in}}{\equiv} \frac{1}{m+1}R\big(R(z_1)\vec{x}_{q_1}R(z_2)\dots \vec{x}_{q_{m-1}}z'_m\vec{x}_{q_m}\dots R(z_s)\vec{x}_{q_s}x_{\beta,r}x^k_{\beta.r+1}R(z_{s+1})\\
&\quad + R\big(R(z_1)\vec{x}_{\alpha_1}R(z_2)\dots \vec{x}_{q_{m-1}}R(\Delta)\vec{x}_{q_m}\dots R(z_s)\vec{x}_{q_s}x_{\beta,r}x^k_{\beta,r+1}R(z_{s+1})\big)\big)\\
&\overset{(9),\text{ out}}{\equiv} \frac{1}{(k+1)(m+1)}\big(R(z_1)\vec{x}_{\alpha_1}R(z_2)\dots x_{\alpha_{m-1}}z'_m\vec{x}_{\alpha_m}\dots R(z_s)\vec{x}_{\alpha_s}x^{k+1}_{\beta,r+1}R(z_{s+1})\\
&\quad + R(\Sigma'_w|_{R(z_m)\to z'_m}) - R(\Sigma''_w|_{z_m\to\check{\Delta}})\\
&\quad + R(z_1)\dots \vec{x}_{\alpha_{m-1}}R(\Delta)\vec{x}_{\alpha_m}\dots R(z_s)\vec{x}_{\alpha_s}x^{k+1}_{\beta,r+1}R(z_{s+1}) + R(\Sigma_w|_{z_m\to\Delta})\big).
\end{aligned}
\tag{14}
$$

Subtracting (14) from (13), we get $\frac{1}{(k+1)(m+1)}\ R(\Sigma''_w|_{z_m\to C})$, where

$$
C = \Delta - \check{\Delta} - (m+1)z_m = R(a_1)\dots R(a_n)\vec{x}_{t_r}DR(a_{n+1})
$$

for $D = H - (m+1)x_{\gamma,o}x^m_{\gamma,o+1}$. Equality of D to zero follows from Lemma 2.1.

The proof in the case $m = s+1$ is slightly different if only $R(a_1) = \emptyset$ and $n{=}1$. Then we need to apply the equality $A\big(z_1,\vec{x}_{q_1},z_2,\dots,z_s,\vec{x}_{q_s}x_{\beta,r}x^k_{\beta,r+1}\vec{x}_{t_1}x^{m+1}_{\gamma,o+1},a_2\big) \equiv 0$ modulo (S,w). Such application is correct, since all terms involved in it have less R-degree than w.

It is easy to verify that the remaining compositions of inclusion between (7) and (8) are trivial. ∎

Corollary 3.3. *The quotient A of $R\mathrm{As}\langle X\rangle$ by $Id(S)$ is the universal enveloping associative RB-algebra for the Lie algebra L with the RB-operator R. Moreover, L injectively embeds into $A^{(-)}$.*

Proof. By (8), A is an associative RB-algebra. By (7)–(9), we have that A is enveloping of L for both: the Lie bracket $[,]$ and the action of R. Thus, A is an associative enveloping of L.

Let us prove that A is the universal enveloping one. Firstly, A is generated by L. Secondly, all elements from S are identities in the universal enveloping associative RB-algebra $U_{RB}(L)$. Indeed, (7) are enveloping conditions for the product, (8) is the RB-identity, the relations (9) as the application of (5) are direct consequences of (1).

By Theorems 2.6 and 3.2 we get the injectivity of embedding L into $A^{(-)}$. ∎

Let C be a pre- or post-Lie algebra with a linear basis Y. By Theorem 2.3, C can be injectively embedded into the Lie algebra L with the RB-operator P of weight λ and such subset $X_0 = \{x_\alpha \mid \alpha\in\Omega\}\subset L$ that $X = \{x_{\alpha,k} := P^k(x_\alpha) \mid k\in\mathbb{N},\ \alpha\in\Omega\}$ is a linear basis of L. Here Ω is a well-ordered set. Then, by Corollary 3.3, we embed the Lie RB-algebra L into its universal enveloping associative algebra A with the RB-operator R. Thus, the subalgebra (in pre- or postalgebra sense) $U(C)$ in $A^{(R)}_\lambda$ generated by the set Y is an enveloping pre- or postassociative algebra of C.

Now, we state the main result of the work, the analogue of the Poincaré-Birkhoff-Witt theorem for pre-Lie and post-Lie algebras.

Theorem 3.4. (a) *Let C be a pre- or post-Lie algebra, then $U(C)$ is the universal enveloping pre- or postassociative algebra of C.*
(b) [15] *The pairs* $(\mathrm{preLie}, \mathrm{preAs})$ *and* $(\mathrm{postLie}, \mathrm{postAs})$ *are PBW-pairs.*

Proof. (a) Consider the post-Lie algebra case, the proof when C is a pre-Lie algebra is the same. Let Y be a basis of C. It is easy to show that A is the universal enveloping associative RB-algebra for C in the sense of the equalities (2). Indeed, given an enveloping associative RB-algebra D of C, define M as the Lie RB-subalgebra of $D^{(-)}$ generated by the image of C. By the universality of L, we have that M is the homomorphic image of the Lie RB-algebra L. Thus, D as the enveloping of M is the homomorphic image of the universal enveloping associative RB-algebra of M. The last one is the homomorphic image of A.

Consider the universal enveloping postassociative algebra $V(C)$ of C. Due to Theorem 2.3, we may embed $V(C)$ into its universal enveloping associative RB-algebra Z with RB-operator Q. Since Z is the homomorphic image of A, $V(C)$ as the subalgebra in $Z_\lambda^{(Q)}$ generated by Y is the homomorphic image of $U(C)$.

(b) We get it by the construction. ■

Corollary 3.5. *The pairs* $(\mathrm{preLie}, RB_0\mathrm{As})$ *and* $(\mathrm{postLie}, RB_\lambda\mathrm{As})$ *are PBW-pairs.*

Corollary 3.6. *The universal enveloping associative RB-algebra of $U(C)$ is isomorphic to A.*

As another corollary, we obtain the commutative diagram from the introduction.

4. Universal enveloping pre/post-associative algebra

4.1. Post-Lie case

Given a post-Lie algebra C with a linear basis Y, we want to construct a linear basis of the universal enveloping postassociative algebra $U(C)$.

Define $Y^+ = Y \cup Y' \subset X \subset L$ for $Y' = \{P(y) \mid y \in Y\}$.

Given a well-ordered set Z, define

$$\mathrm{Com}(Z) := \{w = w_1 \dots w_k \in S(Z) \mid w_j \in Z,\, w_1 \leq w_2 \leq \dots \leq w_k\}.$$

Let us define a set $E \subset RS(Y)$ by induction. Firstly, $Y \subset E$. Lately, the word

$$a = R(z_1)w_1R(z_2)\dots R(z_s)w_sR(z_{s+1})$$

lies in E for $s \geq 1$, $w_1,\dots,w_s \in \mathrm{Com}(Y^+)$, $z_2,\dots,z_s \in E \cap RS(Y) \setminus Y$, and $z_1 \in E \cap RS(Y) \setminus Y$ or $R(z_1) = \emptyset$ (the same holds for z_{s+1}), if

(1) at least one of w_i contains a letter from Y,

(2) every z_i is not of the following form

$$R(q_1)u_1R(q_2)\dots R(q_t)u_tyR(y)^kR(q_{t+1}) \tag{15}$$

with the same conditions written above for a. Moreover, $u_1,\dots,u_t \in \mathrm{Com}(Y')$ and $y \in Y$ is greater than all letters from u_t.

Theorem 4.1. *The set $\{e + Id(S) \mid e \in E\}$ forms a linear basis of $U(C)$.*

Proof. Firstly, Theorems 2.6 and 3.2 imply that the set $\{e + Id(S) \mid e \in E\}$ is a linearly independent set of elements in $R\mathrm{As}\langle Y\rangle$.

Secondly, we show that $U(C) \subset \Bbbk E + Id(S)$. Let us prove it by induction of the summary R-degree r of factors a, b involved in the process of generating the algebra $U(C)$. By the definition, $Y \subset E$. If $r = 0$, we get only linear combinations of elements of the form $w \in S(Y^+)$ with at least one letter from Y, where we identify $R(y)$ with $P(y)$ for every $y \in Y$. Because of (7), we have $w \in E + Id(S)$.

Let us prove the inductive step for $r > 0$. Suppose that $a, b \in E$, more precisely,

$$a = R(z_1)w_1R(z_2)\dots R(z_s)w_sR(z_{s+1}), \quad b = R(q_1)u_1R(q_2)\dots R(q_t)u_tR(q_{t+1}),$$

where

$$s, t \geq 1, \quad w_1, \dots, w_s, u_1, \dots, u_t \in S(Y^+), \quad z_2, \dots, z_s, q_2, \dots, q_t \in E \cap RS(Y) \setminus Y,$$

by $R(z_1)$, as above, we mean that either $z_1 \in E \cap RS(Y) \setminus Y$ or $R(z_1) = \emptyset$. The same holds for $R(z_{s+1}), R(q_1), R(q_{t+1})$.

To prove the Theorem, it is enough to state that $a \succ b, a \prec b, a \cdot b \in \Bbbk E + Id(S)$. We have modulo $Id(S)$

$$a \cdot b = \begin{cases} R(z_1)w_1\dots R(z_s)w_su_1R(q_2)\dots u_tR(q_{t+1}), & R(z_{s+1}) = R(q_1) = \emptyset, \\ R(z_1)w_1\dots R(z_s)w_sR(z_{s+1}\circ q_1)u_1R(q_2)\dots u_tR(q_{t+1}), & R(z_{s+1}), R(q_1) \neq \emptyset, \\ ab, & \text{otherwise}, \end{cases}$$

where $z_{s+1}\circ q_1 = z_{s+1} \succ q_1 + z_{s+1} \prec q_1 + z_{s+1}\cdot q_1$. In the third case, we have an element from E. In the first one, we need to express $w_su_1 \in U(L)$ via basic elements from $\mathrm{Com}(Y^+)$ and so $a \cdot b \in \Bbbk E$. Finally, in the second case, after rewriting $z_{s+1}\circ q_1$ as a linear combination of elements from $E + Id(S)$, we only need to check the condition 2 of the definition of E. If it's required, we apply the relation (9) and we are done by induction on r.

We have modulo $Id(S)$

$$a \succ b = \begin{cases} R(a)u_1R(q_2)\dots u_tR(q_{t+1}), & R(q_1) = \emptyset, \\ R(a\circ q_1)u_1R(q_2)\dots u_tR(q_{t+1}), & R(q_1) \neq \emptyset. \end{cases}$$

By the inductive hypothesis, $a \circ q_1 \in \Bbbk E + Id(S)$. We need to check the condition 2 of the definition of E for the first R-letter of the product. As above, we apply (9). The case of $a \prec b$ can be considered analogously. ■

Remark 4.2. We may reformulate Theorem 4.1 in terms avoiding $Id(S)$ and any factorization. For this, we need to define by induction the products $\succ, \prec, \cdot$ on the space $\Bbbk E$.

4.2. Pre-Lie case

Given a pre-Lie algebra C with a linear basis Y, we construct a linear basis of the universal enveloping preassociative algebra $U(C)$.

Let us define a set $E \subset RS(Y)$ by induction. Firstly, $Y \subset E$. Secondly, the word

$$a = R(z_1)w_1R(z_2)\dots R(z_s)w_sR(z_{s+1})$$

lies in E for $s \geq 1$, $w_1, \dots, w_s \in \mathrm{Com}(Y^+)$, $z_2, \dots, z_s \in E \cap RS(Y) \setminus Y$, and $z_1 \in E \cap RS(Y) \setminus Y$ or $R(z_1) = \emptyset$ (the same holds for z_{s+1}), if

(1) the only w_i contains a letter from Y,

(2) every z_i is not of the following form

$$R(q_1)u_1R(q_2)\dots R(q_t)u_t y R(y)^k R(q_{t+1}) \tag{16}$$

with the same conditions written above for a; $u_1, \dots, u_t \in \mathrm{Com}(Y')$ and $y \in Y$ is greater than all letters from u_t.

Theorem 4.3. *The set $\{e + Id(S) \mid e \in E\}$ forms a linear basis of $U(C)$.*

Proof. Analogous to the proof of Theorem 4.1. ■

Example 4.4. If C is a one-dimensional pre-Lie algebra with linear basis $\{y\}$, then $\{y(P(y))^k\}$ is a linear basis of $U(C)$. Indeed, because of the condition 2, all basic elements have no R-letters, so we have unique y and several $P(y)$ to form the elements from E. Moreover, in the case $C = \mathrm{Span}\{y\}$ we have the isomorphism $U_{gr}(C) \cong \mathrm{preCom}\langle y\rangle$, where $U_{gr}(C)$ means the associated graded preassociative algebra obtained from $U(C)$ by the filtration by the degree and $\mathrm{preCom}\langle Z\rangle$ means a free precommutative algebra generated by the set Z. If $C = \mathrm{Span}\{y\}$ has the trivial product, then $U(C) \cong \mathrm{preCom}\langle y\rangle$.

Remark 4.5. Note that given a pre-Lie algebra C, the injective enveloping preassociative algebra T constructed in [26] is isomorphic to $U(C)$, so it is the universal enveloping one. In the current paper, we have embedded C into its universal enveloping Lie RB-algebra, but in [26] C was embedded into the enveloping Lie RB-algebra $\widehat{C}$ arisen from the proof of Theorem 2.3. A posteriori, we conclude that it is enough to embed C into the doubling (as a vector space) Lie RB-algebra to preserve all required connections for the construction of the universal enveloping preassociative algebra $U(C)$.

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Vsevolod Gubarev
Sobolev Institute of Mathematics
630090 Novosibirsk, Russia
wsewolod89@gmail.com

and: Faculty of Mathematics
University of Vienna
1090 Vienna
Austria

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