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Manin Triples of 3-Lie Algebras Induced by Involutive Derivations

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Abstract. Any involutive derivation D on a 3-Lie algebra A induces a local cocycle 3-Lie bialgebra $(A \ltimes_{\mathrm{ad}^*} A^*, \Delta)$. We give precise formulas of the 3-Lie algebra $((A \oplus A^*)^*, \Delta^*)$ and show that the local cocycle 3-Lie bialgebra $(A \ltimes_{\mathrm{ad}^*} A^*, \Delta)$ induced by the involutive derivation D gives rise to a Manin triple of 3-Lie algebras. We give examples of 12-dimensional and 16-dimensional Manin triples using involutive derivations on certain 3-dimensional and 4-dimensional 3-Lie algebras.

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Key Words: 3-Lie algebra, involutive derivation, semi-direct product 3-Lie algebra, Manin triple, 3-Lie bialgebra.

1. Introduction

The notion of an n-Lie algebra, which is closely related to the fields of mathematics and physics, was introduced by Filippov in [10]. n-Lie algebras are the algebraic structure underlying Nambu mechanics [1, 7, 8, 13, 22]. In particular, as a special case of n-Lie algebras, 3-Lie algebras are extensively studied, because they play a significant role in string theory and M-theory [2, 12, 15, 20]. For example, the basic model of Bagger-Lambert-Gustavsson theory is based on the structure of metric 3-Lie algebras, and the Jacobi equation of 3-Lie algebras is the foundation for defining the $N = 8$ supersymmetry action.

Usually the bialgebra theory for an algebraic structure is very important. Associative bialgebras and Lie bialgebras are two of the most famous examples of bialgebras, which have important applications in both mathematics and mathematical physics. A Lie bialgebra is defined to be a vector space endowed with a Lie algebra structure $(A, [\cdot, \cdot])$ and a Lie coalgebra structure (A, Δ) (where $\Delta\colon A \to \wedge^2 A$ is the comultiplication) satisfying the compatibility condition which is proposed based on the Hamiltonian dynamics and Poisson Lie groups [8, 16, 18, 19]. Lie bialgebras have a rich structure theory since it contains the coboundary case, which gives the connection to the classical Yang-Baxter equation [14]. Note that there are three equivalent compatibility conditions in the definition of a Lie bialgebra: requiring the comultiplication Δ to be a derivation; requiring the comultiplication Δ to be a 1-cocycle and requiring $(A \oplus A^*, A, A^*)$ to be a Manin triple.

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Due to the importance of the bialgebra theory for an algebraic structure, some studies of the bialgebra structure for 3-Lie algebras are carried out recently. In particular, the authors introduced the notion of a 3-Lie bialgebra in [4] by requiring the comultiplication to be certain derivation; while in [3], the authors introduced the notion of a local cocycle 3-Lie bialgebra and the notion of a double construction 3-Lie bialgebra, which enjoy the coboundary theory and the Manin triple theory respectively. However, a solution of the classical 3-Lie Yang-Baxter equation can not give rise to a Manin triple of 3-Lie algebras in general. This fact is totally different from the case of Lie bialgebras, and it is still unknown why this happens. Nevertheless, this shows that 3-Lie algebras enjoy their own properties that different from Lie algebras.

The purpose of this paper is to find some special solutions of the classical 3-Lie Yang-Baxter equation so that they also give rise to Manin triples of 3-Lie algebras. We show that using an involutive derivation D on a 3-Lie algebra A, we can obtain a skew-symmetric solution of the classical 3-Lie Yang-Baxter equation in the semidirect 3-Lie algebra $A \ltimes_{\mathrm{ad}^*} A^*$. Moreover, the corresponding local cocycle 3-Lie bialgebra $(A \ltimes_{\mathrm{ad}^*} A^*, \Delta)$ induced by the involutive derivation D gives rise to a Manin triple of 3-Lie algebras. Some examples of 12-dimensional and 16-dimensional Manin triples are given using involutive derivations on 3-dimensional and 4-dimensional 3-Lie algebras.

The paper is organized as follows. In Section 2, we recall some elementary facts on 3-Lie algebras, and give precise formulas of the semi-direct product 3-Lie algebra associated to the coadjoint representation. In Section 3, by means of involutive derivations, we construct a class of local cocycle 3-Lie bialgebras $(A \ltimes_{\mathrm{ad}^*} A^*, \Delta)$. We further give the precise formula of the 3-Lie algebra $((A \oplus A^*)^*, \Delta^*)$. In Section 4, we show that the local cocycle 3-Lie bialgebras $(A \ltimes_{\mathrm{ad}^*} A^*, \Delta)$ induced by an involutive derivation give rise to a Manin triple as well as a matched pair of 3-Lie algebras. In Section 5, we construct 12-dimensional and 16-dimensional Manin triples using involutive derivations on certain 3-dimensional and 4-dimensional 3-Lie algebras.

In the paper, we suppose that all algebras and vector spaces are over a field $\mathbb{F}$ of characteristic zero, and for a subset S of a vector space V, we use $\langle S \rangle$ to denote the subspace of V spanned by S.

2. The semi-direct product 3-Lie algebra $A \ltimes_{\mathrm{ad}^*} A^*$

A 3-Lie algebra is a vector space A with a multiplication (or 3-Lie bracket) of the form $[\cdot,\cdot,\cdot]\colon \wedge^3 A \to A$ satisfying for all $x_i \in A, 1 \le i \le 5$:

$$[x_1, x_2, [x_3, x_4, x_5]] = [[x_1, x_2, x_3], x_4, x_5] + [x_3, [x_1, x_2, x_4], x_5] + [x_3, x_4, [x_1, x_2, x_5]]. \quad (1)$$

A derivation on a 3-Lie algebra A is a linear map $D\colon A \to A$ satisfying

$$D[x_1, x_2, x_3] = [Dx_1, x_2, x_3] + [x_1, Dx_2, x_3] + [x_1, x_2, Dx_3], \quad \forall\, x_1, x_2, x_3 \in A. \quad (2)$$

Thanks to (1), for all $x_1, x_2 \in A$, the map $\mathrm{ad}\colon \wedge^2 A \to \mathfrak{gl}(A)$ defined by

$$\mathrm{ad}_{x_1,x_2} x = [x_1, x_2, x], \quad \forall\, x \in A, \quad (3)$$

satisfies

$$\mathrm{ad}_{x_1,x_2}[x_3, x_4, x_5] = [\mathrm{ad}_{x_1,x_2} x_3, x_4, x_5] + [x_3, \mathrm{ad}_{x_1,x_2} x_4, x_5] + [x_3, x_4, \mathrm{ad}_{x_1,x_2} x_5]. \quad (4)$$

Therefore, ad_{x_1,x_2} is a derivation on $(A,[\cdot,\cdot,\cdot])$, which is called an inner derivation. The notion of a representation of an n-algebra was introduced in ([17]). See also ([9, 11]) for more information. A representation (or an A-module) of a 3-Lie algebra A is a pair $(V;\rho)$, where V is a vector space, and $\rho\colon \wedge^2 A \to \mathfrak{gl}(V)$ is a linear map satisfying, for all $x_1,x_2,x_3,x_4 \in A$,

$$[\rho(x_1,x_2),\rho(x_3,x_4)] = \rho([x_1,x_2,x_3],x_4) + \rho(x_3,[x_1,x_2,x_4]), \tag{5}$$

$$\rho([x_1,x_2,x_3],x_4) = \rho(x_1,x_2)\rho(x_3,x_4) + \rho(x_2,x_3)\rho(x_1,x_4) + \rho(x_3,x_1)\rho(x_2,x_4). \tag{6}$$

lt is straightforward to obtain

Lemma 2.1. *Let A be a 3-Lie algebra. Then $(V;\rho)$ is a representation of A if and only if there is a 3-Lie algebra structure (called the semi-direct product) on the direct sum $A \oplus V$ of vector spaces, defined for $x_i \in A$, $v_i \in V$, $1 \le i \le 3$, by*

$$[x_1+v_1,x_2+v_2,x_3+v_3]_{A\oplus V} = [x_1,x_2,x_3]+\rho(x_1,x_2)v_3+\rho(x_3,x_1)v_2+\rho(x_2,x_3)v_1, \tag{7}$$

We denote this semi-direct product 3-Lie algebra *by $A \ltimes_\rho V$.*

Let $(V;\rho)$ be a representation of a 3-Lie algebra A. Define $\rho^*\colon \wedge^2 A \to \mathfrak{gl}(V^*)$ by

$$\langle \rho^*(x_1,x_2)\alpha, v\rangle = -\langle \alpha, \rho(x_1,x_2)v\rangle, \qquad \forall\, x_1,x_2 \in A,\ \alpha \in V^*,\ v \in V. \tag{8}$$

Lemma 2.2. *Let $(V;\rho)$ be a representation of a 3-Lie algebra $(A,[\cdot,\cdot,\cdot])$. Then $(V^*;\rho^*)$ is a representation of the 3-Lie algebra $(A,[\cdot,\cdot,\cdot])$, which is called the* dual representation.

Example 2.3. Let $(A,[\cdot,\cdot,\cdot])$ be a 3-Lie algebra. Then $(A;\mathrm{ad})$ is a representation of A, which is called the *adjoint representation* of A. The dual representation $(A^*;\mathrm{ad}^*)$ of the adjoint representation $(A;\mathrm{ad})$ of a 3-Lie algebra A is called the *coadjoint representation.* ■

Let $(V;\rho)$ be a representation of a 3-Lie algebra $(A,[\cdot,\cdot,\cdot])$. Denote by $C^n_{3\text{-}Lie}(A;V)$ the space of n-cochains:

$$C^n_{3\text{-}Lie}(A;V) := \mathrm{Hom}(\underbrace{\wedge^2 A\otimes\cdots\otimes\wedge^2 A}_{(n-1)}\wedge A, V) \qquad (n \ge 1).$$

The coboundary operator $\mathrm{d}\colon C^n_{3\text{-}Lie}(A;V) \to C^{n+1}_{3\text{-}Lie}(A;V)$ is defined by

$$\begin{aligned}(\mathrm{d}f)(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}) = &\sum_{1\le j<k\le n}(-1)^j f(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_{k-1},\\ &[x_j,y_j,x_k]\wedge y_k + x_k\wedge[x_j,y_j,y_k],\mathfrak{X}_{k+1},\cdots,\mathfrak{X}_n,x_{n+1})\\ &+\sum_{j=1}^n(-1)^j f(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,[\mathfrak{X}_j,x_{n+1}])\\ &+\sum_{j=1}^n(-1)^{j+1}\rho(\mathfrak{X}_j)f(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,x_{n+1})\\ &+(-1)^{n+1}(\rho(y_n,x_{n+1})f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},x_n)+\rho(x_{n+1},x_n)f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},y_n)),\end{aligned}$$

for all $\mathfrak{X}_i = x_i\wedge y_i \in \wedge^2 A$, $i=1,2,\ldots,n$, and $x_{n+1}\in A$. It was proved in [6, 23] that $\mathrm{d}\circ\mathrm{d} = 0$.

Definition 2.4. Let A be a 3-Lie algebra and $(V;\rho)$ be a representation of A. A linear map $f\colon A\to V$ is called a 1-cocycle on A associated to $(V;\rho)$ if it satisfies

$$f([x_1,x_2,x_3])=\rho(x_1,x_2)f(x_3)+\rho(x_2,x_3)f(x_1)+\rho(x_3,x_1)f(x_2) \tag{9}$$

for all $x_1,x_2,x_3\in A$. ■

For convenience the semi-direct product 3-Lie algebra $A\ltimes_{\mathrm{ad}^*}A^*$ associated with $(A^*;\mathrm{ad}^*)$ is denoted by $B_1=(A\ltimes_{\mathrm{ad}^*}A^*,[\cdot,\cdot,\cdot]_{B_1})$. More precisely, for all $x_i\in A$, $x_i^*\in A^*$, $1\le i\le 3$,

$$[x_1+x_1^*,x_2+x_2^*,x_3+x_3^*]_{B_1}=[x_1,x_2,x_3]+\mathrm{ad}^*_{x_1,x_2}x_3^*+\mathrm{ad}^*_{x_3,x_1}x_2^*+\mathrm{ad}^*_{x_2,x_3}x_1^*. \tag{10}$$

Let A be a 3-Lie algebra with a basis $\{e_1,\cdots,e_n\}$, $\{e_1^*,\cdots,e_n^*\}$ be the dual basis of A, that is, $\{e_1^*,\cdots,e_n^*\}$ is a basis of the dual space A^* satisfying

$$\langle e_i,e_j^*\rangle=\delta_{ij}=\begin{cases}1, & i=j,\quad 1\le i,j\le n,\\ 0, & i\ne j,\quad 1\le i,j\le n.\end{cases} \tag{11}$$

It is clear that the entire multiplication table of A can be recovered from the structure constants Γ^k_{abc} which occur in the expressions

$$[e_a,e_b,e_c]=\sum_{k=1}^n\Gamma^k_{abc}e_k,\quad \Gamma^k_{abc}\in\mathbb{F},\quad 1\le a,b,c,k\le n. \tag{12}$$

We have the following result.

Theorem 2.5. *Let $(A,[\cdot,\cdot,\cdot])$ be a 3-Lie algebra with the basis $\{e_1,\cdots,e_n\}$, the multiplication of A be given by* (12). *Then the multiplication table of the semi-direct product 3-Lie algebra $B_1=(A\ltimes_{\mathrm{ad}^*}A^*,[\cdot,\cdot,\cdot]_{B_1})$ is given by*

$$[e_a,e_b,e_c]_{B_1} = \sum_{k=1}^n\Gamma^k_{abc}e_k,\quad \forall\ 1\le a,b,c\le n, \tag{13}$$

$$[e_a,e_b,e_c^*]_{B_1} = -\sum_{k=1}^n\Gamma^c_{abk}e_k^*,\quad \forall\ 1\le a,b,c\le n, \tag{14}$$

$$[e_a,e_b^*,e_c^*]_{B_1} = [e_a^*,e_b^*,e_c^*]_{B_1}=0,\quad \forall\ 1\le a,b,c\le n. \tag{15}$$

Proof. By (10) and (12), the (13) and (15) is obvious.
For any $e_a,e_b,e_t\in A, e_c^*\in A^*$, assume

$$[e_a,e_b,e_c^*]_{B_1}=\mathrm{ad}^*_{e_a,e_b}e_c^*=\sum_{k=1}^n\lambda^k_{abc}e_k^*,\quad \lambda^k_{abc}\in F,\quad 1\le a,b,c,k\le n.$$

By (8), we have

$$\langle[e_a,e_b,e_c^*]_{B_1},e_t\rangle=\langle\sum_{k=1}^n\lambda^k_{abc}e_k^*,e_t\rangle=\lambda^t_{abc},\quad \lambda^k_{abc}\in F,$$

$$\langle[e_a,e_b,e_c^*]_{B_1},e_t\rangle=-\langle e_c^*,\mathrm{ad}_{e_a,e_b}e_t\rangle=-\langle e_c^*,\sum_{k=1}^n\Gamma^k_{abt}e_k\rangle=-\Gamma^c_{abt},\quad \Gamma^c_{abt}\in F.$$

Therefore, for $1 \leq a, b, c, t \leq n,\ \lambda^t_{abc} = -\Gamma^c_{abt}$. Thus, we have

$$[e_a, e_b, e^*_c]_{B_1} = \sum_{k=1}^{n} \lambda^k_{abc} e^*_k = -\sum_{k=1}^{n} \Gamma^c_{abk} e^*_k, \quad \forall\ 1 \leq a, b, c, t \leq n.$$

The proof is finished. ■

3. The local cocycle 3-Lie bialgebras induced by involutive derivations

In this section, we construct a local cocycle 3-Lie bialgebra structure on the semi-direct product 3-Lie algebra $A \ltimes_{\mathrm{ad}^*} A^*$ by an involutive derivation on a 3-Lie algebra A, and obtain a class of 3-Lie algebra structure on the dual space $(A \oplus A^*)^*$. First let us recall some definitions.

Let A be a 3-Lie algebra, A^* be the dual space of A, and $\Delta\colon A \to A \wedge A \wedge A$ be a linear map. The dual map Δ^* of Δ is a linear map $\Delta^*\colon A^* \wedge A^* \wedge A^* \to A^*$ defined by

$$\langle \Delta^*(\alpha, \beta, \gamma), x \rangle = \langle \alpha \wedge \beta \wedge \gamma, \Delta(x) \rangle, \quad \forall\ \alpha, \beta, \gamma \in A^*,\ x \in A. \tag{16}$$

Definition 3.1 ([3])**.** A local cocycle 3-Lie bialgebra is a pair (A, Δ), where A is a 3-Lie algebra and

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3\colon A \to A \wedge A \wedge A$$

is a linear map, and the following conditions are satisfied:

- (A^*, Δ^*) is a 3-Lie algebra;
- Δ_1 is a 1-cocycle associated to the representation $(A \otimes A \otimes A, \mathrm{ad} \otimes 1 \otimes 1)$;
- Δ_2 is a 1-cocycle associated to the representation $(A \otimes A \otimes A, 1 \otimes \mathrm{ad} \otimes 1)$;
- Δ_3 is a 1-cocycle associated to the representation $(A \otimes A \otimes A, 1 \otimes 1 \otimes \mathrm{ad})$.

Let $(A, [\cdot, \cdot, \cdot])$ be a 3-Lie algebra. For any $r = \sum_i x_i \otimes y_i \in A \otimes A$, define

$$\begin{aligned}[][[r, r, r]] := \sum_{i,j,k} \big(&[x_i, x_j, x_k] \otimes y_i \otimes y_j \otimes y_k + x_i \otimes [y_i, x_j, x_k] \otimes y_j \otimes y_k \\ &+ x_i \otimes x_j \otimes [y_i, y_j, x_k] \otimes y_k + x_i \otimes x_j \otimes x_k \otimes [y_i, y_j, y_k] \big).\end{aligned} \tag{17}$$

The equation

$$[[r, r, r]] = 0 \tag{18}$$

is called the 3-Lie classical Yang-Baxter equation[3].

For any $r = \sum_i x_i \otimes y_i \in A \otimes A$ and $x \in A$, set

$$\begin{cases} \Delta_1(x) := \sum_{i,j} [x, x_i, x_j] \otimes y_j \otimes y_i; \\ \Delta_2(x) := \sum_{i,j} y_i \otimes [x, x_i, x_j] \otimes y_j; \\ \Delta_3(x) := \sum_{i,j} y_j \otimes y_i \otimes [x, x_i, x_j]. \end{cases} \tag{19}$$

In the Lie bialgebra theory, a skew-symmetric solution of the classical Yang-Baxter equation gives a Lie bialgebra. Similarly, we have

Lemma 3.2. [3] *Let A be a 3-Lie algebra and $r \in A \otimes A$ skew-symmetric. If*

$$[[r, r, r]] = 0, \tag{20}$$

then Δ^* defines a 3-Lie algebra structure on A^*, where $\Delta = \Delta_1 + \Delta_2 + \Delta_3 \colon A \to A \otimes A \otimes A$, in which $\Delta_1, \Delta_2, \Delta_3$ are induced by r as in (19). *Furthermore, (A, Δ) is a local cocycle 3-Lie bialgebra.*

Definition 3.3. [3] Let $(A, [\cdot,\cdot,\cdot])$ be a 3-Lie algebra, and $(V; \rho)$ be a representation of A. A linear operator $T \colon V \to A$ is called an $\mathcal{O}$-operator associated to $(V; \rho)$ if T satisfies

$$[Tu, Tv, Tw] = T(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v), \quad \forall\, u, v, w \in V. \tag{21}$$

Lemma 3.4. [3] *Let $T \colon V \to A$ be a linear map and $\overline{T} \in V^* \otimes A$ the corresponding tensor. Then T is an $\mathcal{O}$-operator if and only if*

$$r = \overline{T} - \sigma_{12}\overline{T} \tag{22}$$

is a skew-symmetric solution of the 3-Lie classical Yang-Baxter equation in the semi-direct product 3-Lie algebra $A \ltimes_{\rho^*} V^*$, where σ_{12} is the exchanging operator.

For a 3-Lie algebra A, if a derivation D on A satisfies $D^2 = \mathrm{Id}$, then D is called an *involutive derivation* on A ([5]).

By (2), for all $x, y, z \in A$, the involutive derivation satisfies

$$[Dx, Dy, Dz] = D([Dx, Dy, z] + [Dy, Dz, x] + [Dz, Dx, y]), \tag{23}$$

which implies that the involutive derivation D on A is an $\mathcal{O}$-operator associated to the adjoint representation $(A; \mathrm{ad})$.

Lemma 3.5. [5] *Let A be a 3-Lie algebra. Then there is an involutive derivation D on A if and only if A has a decomposition as vector spaces*

$$A \;=\; A_1 \oplus A_{-1}, \tag{24}$$

where A_1 and A_{-1} are abelian subalgebras of A, such that

$$[A_1, A_1, A_{-1}] \subseteq A_1, \quad [A_1, A_{-1}, A_{-1}] \subseteq A_{-1}.$$

Remark 3.6. In fact, such an involutive derivation D is a special product structure on this 3-Lie algebra. See [21] for more details about product structures on 3-Lie algebras. ■

Let $(A, [\cdot,\cdot,\cdot])$ be a 3-Lie algebra, D be an involutive derivation on A. Let $\{e_1, \cdots, e_n\}$ be the basis of A such that $e_1, \cdots, e_s \in A_1$, and $e_{s+1}, \cdots, e_n \in A_{-1}$. Define the tensor $\overline{D} \in A^* \otimes A$ by

$$\overline{D}(x, \xi) = \langle \xi, Dx \rangle, \quad \forall\, x \in A, \xi \in A^*. \tag{25}$$

More precisely

$$\overline{D} = \sum_{i=1}^{n} e_i^* \otimes De_i \in A^* \otimes A. \tag{26}$$

By Lemma 3.4, we have

$$r=\overline{D}-\sigma_{12}\overline{D}=\sum_{i=1}^{n}e_i^*\otimes De_i-\sum_{i=1}^{n}De_i\otimes e_i^*\in A^*\otimes A \tag{27}$$

is a skew-symmetric solution of 3-Lie classical Yang-Baxter equation in the semi-direct product 3-Lie algebra $B_1=(A\ltimes_{\mathrm{ad}^*}A^*,[\cdot,\cdot,\cdot]_{B_1})$.

Proposition 3.7. *With the above notations, (B_1,Δ) is a local cocycle 3-Lie bialgebra, where B_1 is the semidirect product 3-Lie algebra $A\ltimes_{\mathrm{ad}^*}A^*$ given in Theorem 2.5, Δ is given by (19), for r given by (27). More precisely, for all $X\in B_1$, we have*

$$\begin{aligned}
\Delta\ (X)&=\Delta_1(X)+\Delta_2(X)+\Delta_3(X),\\
\Delta_1(X)&=\sum_{i=1}^{s}\sum_{j=1}^{s}[X,e_i^*,-e_j]\otimes e_j^*\otimes e_i+\sum_{i=1}^{s}\sum_{j=s+1}^{n}[X,e_i^*,e_j]\otimes e_j^*\otimes e_i\\
&+\sum_{i=s+1}^{n}\sum_{j=1}^{s}[X,e_i^*,-e_j]\otimes e_j^*\otimes(-e_i)+\sum_{i=s+1}^{n}\sum_{j=s+1}^{n}[X,e_i^*,e_j]\otimes e_j^*\otimes(-e_i)\\
&+\sum_{i=1}^{s}\sum_{j=1}^{s}[X,-e_i,e_j^*]\otimes e_j\otimes e_i^*+\sum_{i=1}^{s}\sum_{j=s+1}^{n}[X,-e_i,e_j^*]\otimes(-e_j)\otimes e_i^*\\
&+\sum_{i=s+1}^{n}\sum_{j=1}^{s}[X,e_i,e_j^*]\otimes e_j\otimes e_i^*+\sum_{i=s+1}^{n}\sum_{j=s+1}^{n}[X,e_i,e_j^*]\otimes(-e_j)\otimes e_i^*\\
&+\sum_{i=1}^{s}\sum_{j=1}^{s}[X,e_i,e_j]\otimes e_j^*\otimes e_i^*+\sum_{i=1}^{s}\sum_{j=s+1}^{n}[X,e_i,-e_j]\otimes e_j^*\otimes e_i^*\\
&+\sum_{i=s+1}^{n}\sum_{j=1}^{s}[X,-e_i,e_j]\otimes e_j^*\otimes e_i^*+\sum_{i=s+1}^{n}\sum_{j=s+1}^{n}[X,e_i,e_j]\otimes e_j^*\otimes e_i^*,\\
\Delta_2(X)&=\sigma_{13}\sigma_{12}\Delta_1(X),\\
\Delta_3(X)&=\sigma_{12}\sigma_{13}\Delta_1(X).
\end{aligned}\tag{28}$$

Proof. By (19) and (27), we can obtain

$$\begin{aligned}
\Delta_1(X)=&\sum_{i,j=1}^{n}[X,e_i^*,-De_j]\otimes e_j^*\otimes De_i+\sum_{i,j=1}^{n}[X,-De_i,e_j^*]\otimes De_j\otimes e_i^*\\
&+\sum_{i,j=1}^{n}[X,De_i,De_j]\otimes e_j^*\otimes e_i^*.
\end{aligned}\tag{29}$$

By Lemma 3.5, (29) is equivalent to (28). Hence the conclusion holds. ∎

Corollary 3.8. *According to Lemma 3.4, $\big((A\ltimes_{\mathrm{ad}^*}A^*)^*,\Delta^*\big)=((A\oplus A^*)^*,\Delta^*)$ is a 3-Lie algebra which is induced by an involutive derivation D on a 3-Lie algebra A.*

For convenience, we denote it by $B_2=((A\oplus A^*)^*,\Delta^*)$, where

$$\Delta^*\colon (A\oplus A^*)^*\wedge(A\oplus A^*)^*\wedge(A\oplus A^*)^*\to(A\oplus A^*)^*$$

is the dual map of $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ defined by (16), and Δ_i is given by Proposition 3.7.

Remark 3.9. Since $A \oplus A^* = (A \oplus A^*)^*$, so the underlying vector spaces of the 3-Lie algebra B_1 and B_2 are the same. This fact will be frequently used in the sequel.

Theorem 3.10. *Let A be a 3-Lie algebra with an involutive derivation D. Then the involutive derivation D induces a 3-Lie algebra B_2, where the 3-Lie bracket $[\cdot,\cdot,\cdot]_{B_2}$ is given by*

$$[X,Y,Z]_{B_2} = \Delta^*(X,Y,Z) = -[X,Y,Z]_{B_1}, \quad \forall\ X,Y,Z \in B_2 = B_1. \tag{30}$$

Proof. Let A be a 3-Lie algebra with an involutive derivation D. Suppose that

$$\Pi_1 = \{e_1, \cdots, e_s, e_{s+1}, \cdots, e_n, e_1^*, \cdots, e_s^*, e_{s+1}^*, \cdots, e_n^*\} \tag{31}$$

is a basis of $B_1 = (A \ltimes_{\mathrm{ad}^*} A^*, [\cdot,\cdot,\cdot]_{B_1})$, and

$$\Pi_2 = \{\mathfrak{e}_1, \cdots, \mathfrak{e}_s, \mathfrak{e}_{s+1}, \cdots, \mathfrak{e}_n, \mathfrak{e}_1^*, \cdots, \mathfrak{e}_s^*, \mathfrak{e}_{s+1}^*, \cdots, \mathfrak{e}_n^*\} \tag{32}$$

is a basis of $B_2 = ((A \oplus A^*)^*, \Delta^*)$, where $e_i \in A, e_i^* \in A^*$ for $1 \le i \le n$ and $e_k \in A_1$, $e_l \in A_{-1}$, $1 \le k \le s$, $s+1 \le l \le n$, satisfying

$$\langle e_i, \mathfrak{e}_j^* \rangle = \langle e_i^*, \mathfrak{e}_j \rangle = \langle e_i, e_j^* \rangle = \delta_{ij}, \quad \langle e_i, \mathfrak{e}_j \rangle = \langle e_i^*, \mathfrak{e}_j^* \rangle = 0, \quad 1 \le i,j \le n. \tag{33}$$

Thanks to Theorem 2.5, (29) and (28), for all $1 \le t \le n$, we have

$$\begin{aligned}
\Delta(e_t) = \Big(&-\sum_{i,j=1}^{s}\sum_{k=s+1}^{n} - \sum_{i,j,k=1}^{s} + \sum_{i,k=1}^{s}\sum_{j=s+1}^{n} + \sum_{i,k=s+1}^{n}\sum_{j=1}^{s} + \sum_{i,j=s+1}^{n}\sum_{k=1}^{s} + \sum_{i,j,k=s+1}^{n}\Big) \\
&\Gamma_{tjk}^{i}(e_k^* \otimes e_j^* \otimes e_i + e_i \otimes e_k^* \otimes e_j^* + e_j^* \otimes e_i \otimes e_k^*) \\
&+ \Big(\sum_{i,j=1}^{s}\sum_{k=s+1}^{n} + \sum_{i,j,k=1}^{s} - \sum_{i}^{s}\sum_{j,k=s+1}^{n} - \sum_{i=s+1}^{n}\sum_{j,k=1}^{s} + \sum_{i,j=s+1}^{n}\sum_{k=1}^{s} + \sum_{i,j,k=s+1}^{n}\Big) \\
&\Gamma_{tik}^{j}(e_i^* \otimes e_k^* \otimes e_j + e_j \otimes e_i^* \otimes e_k^* + e_k^* \otimes e_j \otimes e_i^*) \\
&+ \Big(\sum_{i,j,k=1}^{s} - \sum_{i,k=1}^{s}\sum_{j=s+1}^{n} - \sum_{i}^{s}\sum_{j,k=s+1}^{n} - \sum_{i=s+1}^{n}\sum_{j,k=1}^{s} - \sum_{i,k=s+1}^{n}\sum_{j=1}^{s} + \sum_{i,j,k=s+1}^{n}\Big) \\
&\Gamma_{tij}^{k}(e_i^* \otimes e_j^* \otimes e_k + e_k \otimes e_j^* \otimes e_i^* + e_i^* \otimes e_k \otimes e_j^*),
\end{aligned}$$

$$\begin{aligned}
\Delta(e_t^*) = \Big(&\sum_{i=1}^{s}\sum_{j=s+1}^{n}\sum_{k=1}^{s} + \sum_{i=1}^{s}\sum_{j=s+1}^{n}\sum_{k=s+1}^{n} + \sum_{i=s+1}^{n}\sum_{j=1}^{s}\sum_{k=1}^{s} \\
&+ \sum_{i=1}^{s}\sum_{j=s+1}^{n}\sum_{k=1}^{s} - \sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{k=s+1}^{n} - \sum_{i=s+1}^{n}\sum_{j=s+1}^{n}\sum_{k=1}^{s}\Big) \\
&\Gamma_{ijk}^{t}(e_k^* \otimes e_j^* \otimes e_i^* + e_i^* \otimes e_k^* \otimes e_j^* + e_j^* \otimes e_i^* \otimes e_k^*).
\end{aligned}$$

First, we suppose

$$\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c)=\sum_{k=1}^{n}\lambda_{abc}^k\mathfrak{e}_k+\sum_{k=1}^{n}\mu_{abc}^k\mathfrak{e}_k^*,\quad \lambda_{abc}^k,\quad \mu_{abc}^k\in F,\quad 1\le a,b,c\le n. \tag{34}$$

Thanks to (33), for $1\le a,b,c,t\le n$,

$$\langle\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c),e_t\rangle=\Big\langle\sum_{k=1}^{n}\lambda_{abc}^k\mathfrak{e}_k+\sum_{k=1}^{n}\mu_{abc}^k\mathfrak{e}_k^*,e_t\Big\rangle=\mu_{abc}^t,$$

$$\langle\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c),e_t^*\rangle=\Big\langle\sum_{k=1}^{n}\lambda_{abc}^k\mathfrak{e}_k+\sum_{k=1}^{n}\mu_{abc}^k\mathfrak{e}_k^*,e_t^*\Big\rangle=\lambda_{abc}^t.$$

We can now give the proof for the 3-Lie structure $[\cdot,\cdot,\cdot]_{B_2}$.

Case 1: if $1\le a,b,c\le s, 1\le t\le n$, by the expression of $\Delta(e_t)$ and $\Delta(e_t^*)$, then

$$\langle\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c),e_t\rangle=\langle\mathfrak{e}_a\otimes\mathfrak{e}_b\otimes\mathfrak{e}_c,\Delta(e_t)\rangle=0,$$

$$\langle\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c),e_t^*\rangle=\langle\mathfrak{e}_a\otimes\mathfrak{e}_b\otimes\mathfrak{e}_c,\Delta(e_t^*)\rangle=0.$$

Therefore, $\lambda_{abc}^t=\mu_{abc}^t=0$, by (34), $\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c)=0$.

Case 2: if $1\le a,b\le s<c\le n, 1\le t\le n$, then

$$\begin{aligned}\langle\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c),e_t\rangle&=\langle\mathfrak{e}_a\otimes\mathfrak{e}_b\otimes\mathfrak{e}_c,\Delta(e_t)\rangle=0,\\ \langle\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c),e_t^*\rangle&=\langle\mathfrak{e}_a\otimes\mathfrak{e}_b\otimes\mathfrak{e}_c,\Delta(e_t^*)\rangle=\begin{cases}-\Gamma_{abc}^t, & 1\le t\le s,\\ 0, & s+1\le t\le n.\end{cases}\end{aligned}$$

Therefore, $\mu_{abc}^t=0,\quad \lambda_{abc}^t=-\Gamma_{abc}^t$. Thus we have $\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c)=-\sum\limits_{k=1}^{s}\Gamma_{abc}^k\mathfrak{e}_k$.

Case 3: if $1\le a\le s<b,c\le n, 1\le t\le n$, then

$$\begin{aligned}\langle\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c),e_t\rangle&=\langle\mathfrak{e}_a\otimes\mathfrak{e}_b\otimes\mathfrak{e}_c,\Delta(e_t)\rangle=0,\\ \langle\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c),e_t^*\rangle&=\langle\mathfrak{e}_a\otimes\mathfrak{e}_b\otimes\mathfrak{e}_c,\Delta(e_t^*)\rangle=\begin{cases}0, & 1\le t\le s,\\ -\Gamma_{abc}^t, & s+1\le t\le n.\end{cases}\end{aligned}$$

We get $\mu_{abc}^t=0,\quad \lambda_{abc}^t=-\Gamma_{abc}^t$, and $\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c)=-\sum\limits_{k=s+1}^{n}\Gamma_{abc}^k\mathfrak{e}_k$.

Thus the multiplication table of $\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c)$ is given by

$$\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c)\ =\ \begin{cases}-\sum_{k=1}^{s}\Gamma_{abc}^k\mathfrak{e}_k, & 1\le a,b\le s<c\le n,\\ -\sum_{k=s+1}^{n}\Gamma_{abc}^k\mathfrak{e}_k, & 1\le a\le s<b,c\le n,\\ 0, & 1\le a,b,c\le s\quad or\\ & s+1\le a,b,c\le n.\end{cases}$$

Similarly, we suppose

$$\Delta^*(\mathfrak{e}_a,\mathfrak{e}_b,\mathfrak{e}_c^*)=\sum_{k=1}^{n}\lambda_{abc}^k\mathfrak{e}_k+\sum_{k=1}^{n}\mu_{abc}^k\mathfrak{e}_k^*,\quad \lambda_{abc}^k,\mu_{abc}^k\in F,\quad 1\le a,b,c\le n.$$

Thanks to (33), for all $1 \leq a, b, c, t \leq n$,

$$\langle \Delta^*(\mathfrak{e}_a, \mathfrak{e}_b, \mathfrak{e}_c^*), e_t \rangle = \langle \mathfrak{e}_a \otimes \mathfrak{e}_b \otimes \mathfrak{e}_c^*), \Delta(e_t) \rangle = \mu_{abc}^t,$$

$$\langle \Delta^*(\mathfrak{e}_a, \mathfrak{e}_b, \mathfrak{e}_c^*), e_t^* \rangle = \langle \mathfrak{e}_a \otimes \mathfrak{e}_b \otimes \mathfrak{e}_c^*), \Delta(e_t^*) \rangle = \lambda_{abc}^t.$$

Case 1': if $1 \leq a, b, c \leq s, 1 \leq t \leq n$, by the expression of $\Delta(e_t)$ and $\Delta(e_t^*)$, then we have

$$\begin{aligned} \langle \Delta^*(\mathfrak{e}_a, \mathfrak{e}_b, \mathfrak{e}_c^*), e_t \rangle &= \langle \mathfrak{e}_a \otimes \mathfrak{e}_b \otimes \mathfrak{e}_c^*, \Delta(e_t) \rangle = \begin{cases} 0, & 1 \leq t \leq s, \\ \Gamma_{abt}^c, & s+1 \leq t \leq n; \end{cases} \\ \langle \Delta^*(\mathfrak{e}_a, \mathfrak{e}_b, \mathfrak{e}_c^*), e_t^* \rangle &= \langle \mathfrak{e}_a \otimes \mathfrak{e}_b \otimes \mathfrak{e}_c^*, \Delta(e_t^*) \rangle = 0. \end{aligned}$$

Hence, $\lambda_{abc}^t = 0, \quad \mu_{abc}^t = \Gamma_{abt}^c, \quad \Delta^*(\mathfrak{e}_a, \mathfrak{e}_b, \mathfrak{e}_c^*) = \sum\limits_{k=s+1}^{n} \Gamma_{abk}^c \mathfrak{e}_k^*.$

Case 2': if $1 \leq a, c \leq s < b \leq n, 1 \leq t \leq n,$ then

$$\begin{aligned} \langle \Delta^*(\mathfrak{e}_a, \mathfrak{e}_b, \mathfrak{e}_c^*), e_t \rangle &= \langle \mathfrak{e}_a \otimes \mathfrak{e}_b \otimes \mathfrak{e}_c^*, \Delta(e_t) \rangle = \begin{cases} \Gamma_{abt}^c, & 1 \leq t \leq s, \\ 0, & s+1 \leq t \leq n; \end{cases} \\ \langle \Delta^*(\mathfrak{e}_a, \mathfrak{e}_b, \mathfrak{e}_c^*), e_t^* \rangle &= \langle \mathfrak{e}_a \otimes \mathfrak{e}_b \otimes \mathfrak{e}_c^*, \Delta(e_t^*) \rangle = 0. \end{aligned}$$

Therefore, $\lambda_{abc}^t = 0, \quad \mu_{abc}^t = \Gamma_{abt}^c, \quad \Delta^*(\mathfrak{e}_a, \mathfrak{e}_b, \mathfrak{e}_c^*) = \sum\limits_{k=1}^{s} \Gamma_{abk}^c \mathfrak{e}_k^*.$

Case 3': if $1 \leq a \leq s < b, c \leq n, 1 \leq t \leq n,$ then

$$\begin{aligned} \langle \Delta^*(\mathfrak{e}_a, \mathfrak{e}_b, \mathfrak{e}_c^*), e_t \rangle &= \langle \mathfrak{e}_a \otimes \mathfrak{e}_b \otimes \mathfrak{e}_c^*, \Delta(e_t) \rangle = \begin{cases} 0, & 1 \leq t \leq s, \\ \Gamma_{abt}^c, & s+1 \leq t \leq n; \end{cases} \\ \langle \Delta^*(\mathfrak{e}_a, \mathfrak{e}_b, \mathfrak{e}_c^*), e_t^* \rangle &= \langle \mathfrak{e}_a \otimes \mathfrak{e}_b \otimes \mathfrak{e}_c^*, \Delta(e_t^*) \rangle = 0. \end{aligned}$$

Therefore, $\lambda_{abc}^t = 0, \quad \mu_{abc}^t = \Gamma_{abt}^c, \quad \Delta^*(\mathfrak{e}_a, \mathfrak{e}_b, \mathfrak{e}_c^*) = \sum\limits_{k=s+1}^{n} \Gamma_{abk}^c \mathfrak{e}_k^*.$ Thus the multiplication table of $\Delta^*(\mathfrak{e}_a, \mathfrak{e}_b, \mathfrak{e}_c^*)$ is given by

$$\Delta^*(\mathfrak{e}_a, \mathfrak{e}_b, \mathfrak{e}_c^*) = \begin{cases} \sum_{k=s+1}^{n} \Gamma_{abk}^c \mathfrak{e}_k^*, & 1 \leq a, b, c \leq s \ \text{ or } \ 1 \leq a \leq s < b, c \leq n \\ \sum_{k=1}^{s} \Gamma_{abk}^c \mathfrak{e}_k^*, & s+1 \leq a, b, c \leq n \text{ or } \ 1 \leq a, c \leq s < b \leq n, \\ 0, & 1 \leq a, b \leq s < c \leq n \text{ or } 1 \leq c \leq s < a, b \leq n. \end{cases}$$

By a similar argumentation we can get

$$\Delta^*(\mathfrak{e}_a, \mathfrak{e}_b^*, \mathfrak{e}_c^*) = \Delta^*(\mathfrak{e}_a^*, \mathfrak{e}_b^*, \mathfrak{e}_c^*) = 0, \qquad 1 \leq a, b, c \leq n.$$

Hence, from the multiplication table of the 3-Lie algebra $(B_2, [\cdot, \cdot, \cdot]_{B_2})$ in the basis $\{\mathfrak{e}_1, \cdots, \mathfrak{e}_s, \mathfrak{e}_{s+1}, \cdots, \mathfrak{e}_n, \mathfrak{e}_1^*, \cdots, \mathfrak{e}_s^*, \mathfrak{e}_{s+1}^*, \cdots, \mathfrak{e}_n^*\}$ we conclude that it is given by (30). ■

4. Manin triples of 3-Lie algebras induced by involutive derivations

In this section, first we recall the notion of Manin triples of 3-Lie algebras and its related matched pairs. Next, we use the 3-Lie algebras discussed in the previous two sections to construct Manin triples and matched pairs of 3-Lie algebras.

Proposition 4.1. *Let $(A, [\cdot,\cdot,\cdot])$ and $(A', [\cdot,\cdot,\cdot]')$ be 3-Lie algebras. Suppose that there are skew-symmetric linear maps $\rho\colon A\wedge A\to \mathrm{Der}(A')$ and $\chi\colon A'\wedge A'\to \mathrm{Der}(A)$ such that $(A';\rho)$ and $(A;\chi)$ are representations of A and A' respectively, and the following conditions are satisfied*

$$\begin{aligned}&\mathrm{ad}_{x_1,x_2}\chi(a_1,a_2)x_3\\ &\qquad= \chi(a_1,a_2)\mathrm{ad}_{x_1,x_2}x_3+\chi(\rho(x_1,x_2)a_1,a_2)x_3+\chi(a_1,\rho(x_1,x_2)a_2)x_3; \qquad (35)\\ &\mathrm{ad}'_{a_1,a_2}\rho(x_1,x_2)a_3\\ &\qquad= \rho(x_1,x_2)\mathrm{ad}'_{a_1,a_2}a_3+\rho(\chi(a_1,a_2)x_1,x_2)a_3+\rho(x_1,\chi(a_1,a_2)x_2)a_3; \qquad (36)\\ &\mathrm{ad}_{x_1,x_2}\chi(a_1,a_2)x_3\\ &\qquad= \chi(\rho(x_1,x_2)a_1,a_2)x_3-\chi(a_1,\rho(x_3,x_1)a_2)x_2-\chi(a_1,\rho(x_2,x_3)a_2)x_1; \qquad (37)\\ &\mathrm{ad}'_{a_1,a_2}\rho(x_1,x_2)a_3\\ &\qquad= \rho(\chi(a_1,a_2)x_1,x_2)a_3-\rho(x_1,\chi(a_3,a_1)x_2)a_2-\rho(x_1,\chi(a_2,a_3)x_2)a_1, \qquad (38)\end{aligned}$$

then $(A\oplus A', [\cdot,\cdot,\cdot]_{A\oplus A'})$ is a 3-Lie algebra, where $[\cdot,\cdot,\cdot]_{A\oplus A'}$ is given by

$$\begin{aligned}&[x_1+a_1,x_2+a_2,x_3+a_3]_{A\oplus A'}\\ &\quad= [x_1,x_2,x_3]+\rho(x_1,x_2)a_3+\rho(x_3,x_1)a_2+\rho(x_2,x_3)a_1\\ &\qquad+[a_1,a_2,a_3]'+\chi(a_1,a_2)x_3+\chi(a_3,a_1)x_2+\chi(a_2,a_3)x_1.\end{aligned}$$

Definition 4.2. [3] Let $(A, [\cdot,\cdot,\cdot])$ and $(A', [\cdot,\cdot,\cdot]')$ be 3-Lie algebras. Suppose that there are skew-symmetric linear maps $\rho\colon A\wedge A\to \mathrm{Der}(A')$ and $\chi\colon A'\wedge A'\to\mathrm{Der}(A)$, and ρ,χ satisfy (35)–(38). Then the 4-tuple (A,A',ρ,χ) is called a *matched pair of 3-Lie algebras.* ■

Definition 4.3. Let A be a 3-Lie algebra. A non-degenerate symmetric bilinear form $S(\cdot,\cdot)\colon A\otimes A\to\mathbb{F}$ on A is called *invariant* if it satisfies

$$S([x_1,x_2,x_3],x_4)+S([x_1,x_2,x_4],x_3)=0,\qquad \forall x_1,x_2,x_3,x_4\in A. \qquad (39)$$

The pair $(A,S(\cdot,\cdot))$ is called *a metric 3-Lie algebra.* ■

If there are two subalgebras A_1 and A_2 of $(A,S(\cdot,\cdot))$ such that $A=A_1\oplus A_2$ as the direct sum of vector spaces, $S(A_1,A_1)=0$, $S(A_2,A_2)=0$, $[A_1,A_1,A_2]\subseteq A_2$ and $[A_2,A_2,A_1]\subseteq A_1$, then the 5-tuple

$$(A,[\cdot,\cdot,\cdot],S(\cdot,\cdot),A_1,A_2)\quad (\text{ or 4-tuple } (A,S(\cdot,\cdot),A_1,A_2))$$

is called *a Manin triple of 3-Lie algebras* ([3]).

Let $(A,S(\cdot,\cdot),A_1,A_2)$ be a Manin triple, and $(A',S'(\cdot,\cdot))$ be a metric 3-Lie algebra. If there is a 3-Lie algebra isomorphism $f\colon A\to A'$ satisfying

$$S(x,y)=S'(f(x),f(y)),\quad \forall x,y\in A,$$

then $(A',S'(\cdot,\cdot),f(A_1),f(A_2))$ is also a Manin triple. And in this case we say that $(A,S(\cdot,\cdot),A_1,A_2)$ is isomorphic to $(A',S'(\cdot,\cdot),A'_1,A'_2)$, where $f\big(A_1\big)=A'_1$, $f\big(A_2\big)=A'_2$.

Let $(A,[\cdot,\cdot,\cdot])$ and $(A^*,[\cdot,\cdot,\cdot]_*)$ be 3-Lie algebras, where A^* is the dual space of A. There is a natural non-degenerate symmetric bilinear form $\mathcal{S}(\cdot,\cdot)$ on $A\oplus A^*$ given by

$$\mathcal{S}(x+\xi,y+\eta)=\langle x,\eta\rangle+\langle \xi,y\rangle,\quad \forall x,y\in A,\ \xi,\eta\in A^*. \tag{40}$$

Define a multiplication $[\cdot,\cdot,\cdot]_{A\oplus A^*}\colon\ \wedge^3(A\oplus A^*)\to A\oplus A^*$ for $x_i\in A,\ y_i\in A^*$, $1\le i\le 3,$ by

$$\begin{aligned}&[x_1+y_1,x_2+y_2,x_3+y_3]_{A\oplus A^*}\\ &\quad=[x_1,x_2,x_3]+\mathrm{ad}^*_{x_1,x_2}y_3+\mathrm{ad}^*_{x_3,x_1}y_2+\mathrm{ad}^*_{x_2,x_3}y_1\\ &\qquad+\mathfrak{ad}^*_{y_1,y_2}x_3+\mathfrak{ad}^*_{y_3,y_1}x_2+\mathfrak{ad}^*_{y_2,y_3}x_1+[y_1,y_2,y_3]_*,\end{aligned} \tag{41}$$

where $(A^*;\mathrm{ad}^*)$ and $(A;\mathfrak{ad}^*)$ are the coadjoint representations of the 3-Lie algebras $(A,[\cdot,\cdot,\cdot])$, and $(A^*,[\cdot,\cdot,\cdot]_*)$ respectively. Then by (40) and (41), for all $x_i\in A$, $y_i\in A^*$, $1\le i\le 4$, we have

$$\begin{aligned}&\mathcal{S}([x_1+y_1,x_2+y_2,x_3+y_3]_{A\oplus A^*},x_4+y_4)+\mathcal{S}(x_3+y_3,[x_1+y_1,x_2+y_2,x_4+y_4]_{A\oplus A^*})\\ &=\langle [x_1,x_2,x_3]+\mathfrak{ad}^*_{y_1,y_2}x_3+\mathfrak{ad}^*_{y_3,y_1}x_2+\mathfrak{ad}^*_{y_2,y_3}x_1,y_4\rangle\\ &\quad+\langle \mathrm{ad}^*_{x_1,x_2}y_3+\mathrm{ad}^*_{x_3,x_1}y_2+\mathrm{ad}^*_{x_2,x_3}y_1+[y_1,y_2,y_3]_*,x_4\rangle\\ &\quad+\langle y_3,[x_1,x_2,x_4]+\mathfrak{ad}^*_{y_1,y_2}x_4+\mathfrak{ad}^*_{y_4,y_1}x_2+\mathfrak{ad}^*_{y_2,y_4}x_1\rangle\\ &\quad+\langle x_3,\mathrm{ad}^*_{x_1,x_2}y_4+\mathrm{ad}^*_{x_4,x_1}y_2+\mathrm{ad}^*_{x_2,x_4}y_1+[y_1,y_2,y_4]_*\rangle\\ &=\langle [x_1,x_2,x_3],y_4\rangle-\langle x_3,[y_1,y_2,y_4]_*\rangle-\langle x_2,[y_3,y_1,y_4]_*\rangle-\langle x_1,[y_2,y_3,y_4]_*\rangle\\ &\quad+\langle x_4,[y_1,y_2,y_3]_*\rangle-\langle [x_1,x_2,x_4],y_3\rangle-\langle [x_3,x_1,x_4],y_2\rangle-\langle [x_2,x_3,x_4],y_1\rangle\\ &\quad+\langle [x_1,x_2,x_4],y_3\rangle-\langle x_4,[y_1,y_2,y_3]_*\rangle-\langle x_2,[y_4,y_1,y_3]_*\rangle-\langle x_1,[y_2,y_4,y_3]_*\rangle\\ &\quad+\langle x_3,[y_1,y_2,y_4]_*\rangle-\langle [x_1,x_2,x_3],y_4\rangle-\langle [x_4,x_1,x_3],y_2\rangle-\langle [x_2,x_4,x_3],y_1\rangle=0,\end{aligned}$$

which implies that $\mathcal{S}(\cdot,\cdot)$ is invariant and $A,\ A^*$ are isotropic.

Therefore, if $(A\oplus A^*,[\cdot,\cdot,\cdot]_{A\oplus A^*})$ is a 3-Lie algebra, then $(A\oplus A^*,[\cdot,\cdot,\cdot]_{A\oplus A^*},\mathcal{S}(\cdot,\cdot))$ is a metric 3-Lie algebra with isotropic subalgebras A and A^*, and

$$(A\oplus A^*,[\cdot,\cdot,\cdot]_{A\oplus A^*},\mathcal{S}(\cdot,\cdot),A,A^*)$$

is a Manin triple, called *the standard Manin triple of 3-Lie algebras.*

By Definition 4.1 and (41), there exists a close relationship between Manin triples and matched pairs of 3-Lie algebras.

Proposition 4.4. *Let $(A,[\cdot,\cdot,\cdot])$ and $(A^*,[\cdot,\ \cdot,\cdot]_*)$ be 3-Lie algebras. Then the 5-tuple $(A\oplus A^*,\ [\cdot,\cdot,\cdot]_{A\oplus A^*},\mathcal{S}(\cdot,\cdot),A,A^*)$ is a standard Manin triple if and only if $(A,A^*,\mathrm{ad}^*,\mathfrak{ad}^*)$ is a matched pair.*

Consider the semi-direct product 3-Lie algebra $B_1=(A\ltimes_{\mathrm{ad}^*}A^*,[\cdot,\cdot,\cdot]_{B_1})$ given in Theorem 2.5, and the 3-Lie algebra $B_2=((A\oplus A^*)^*,[\cdot,\cdot,\cdot]_{B_2})$ given in Theorem 3.10. We have the following conclusions.

Theorem 4.5. *Let A be a 3-Lie algebra with an involutive derivation D. Then $(B_1\oplus B_2,[\cdot,\cdot,\cdot]_{B_1\oplus B_2},\ \overline{\mathcal{S}}(\cdot,\cdot),B_1,B_2)$ is a Manin triple of 3-Lie algebras, where for all $X_a,\ X_b,\ X_c\in B_1,\ \Theta_a,\ \Theta_b,\ \Theta_c\in B_2$, the multiplication $[\cdot,\cdot,\cdot]_{B_1\oplus B_2}$ is defined by*

$$\begin{aligned}[X_a+\Theta_a, X_b+\Theta_b, X_c+\Theta_c]_{B_1\oplus B_2} \\ &= [X_a, X_b, X_c]_{B_1} + \mathrm{ad}^*_{B_1}(X_a, X_b)\Theta_c + \mathrm{ad}^*_{B_1}(X_b, X_c)\Theta_a + \mathrm{ad}^*_{B_1}(X_c, X_a)\Theta_b \\ &\quad + \mathrm{ad}^*_{B_2}(\Theta_a, \Theta_b)X_c + \mathrm{ad}^*_{B_2}(\Theta_b, \Theta_c)X_a + \mathrm{ad}^*_{B_2}(\Theta_c, \Theta_a)X_b + [\Theta_a, \Theta_b, \Theta_c]_{B_2},\end{aligned}$$

and $\overline{\mathcal{S}}(X_a+\Theta_a, X_b+\Theta_b) = \mathcal{S}(X_a, \Theta_b) + \mathcal{S}(X_b, \Theta_a)$.

To prove Theorem 4.5, we give the following lemmas.

Lemma 4.6. *Let* $\mathrm{ad}^*_{B_1}: B_1 \wedge B_1 \to \mathfrak{gl}(B_2)$ *be the coadjoint representation of the 3-Lie algebra* $(B_1, [\cdot,\cdot,\cdot]_{B_1})$ *on* B_2*, i.e.*

$$\langle \mathrm{ad}^*_{B_1}(X_a, X_b)\Theta_c, X_c\rangle = -\langle \Theta_c, [X_a, X_b, X_c]_{B_1}\rangle, \quad \forall X_a, X_b, X_c, \in B_1, \Theta_c \in B_2. \tag{42}$$

Then we have
$$\mathrm{ad}^*_{B_1}(X_a, X_b)\Theta_c = [X_a, X_b, \Theta_c]_{B_1}. \tag{43}$$

Proof. Suppose that $\Pi_1 = \{e_1, \cdots, e_n, e_1^*, \cdots, e_n^*\}$ and $\Pi_2 = \{\mathfrak{e}_1, \cdots, \mathfrak{e}_n, \mathfrak{e}_1^*, \cdots, \mathfrak{e}_n^*\}$ are basis of B_1 and B_2 respectively. First, we set

$$\mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t = \sum_{k=1}^n \lambda^k_{abt}\mathfrak{e}_k + \sum_{k=1}^n \nu^k_{abt}\mathfrak{e}^*_k, \quad \lambda^k_{abt}, \nu^k_{abt} \in F.$$

By (33), for all $1 \le a, b, c, t \le n$, we have

$$\begin{aligned}\langle \mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t, e_c\rangle &= \left\langle \sum_{k=1}^n \lambda^k_{abt}\mathfrak{e}_k + \sum_{k=1}^n \nu^k_{abt}\mathfrak{e}^*_k, e_c\right\rangle = \nu^c_{abt}, \\ \langle \mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t, e^*_c\rangle &= \left\langle \sum_{k=1}^n \lambda^k_{abt}\mathfrak{e}_k + \sum_{k=1}^n \nu^k_{abt}\mathfrak{e}^*_k, e^*_c\right\rangle = \lambda^c_{abt}.\end{aligned}$$

Case 1: if $1 \le a, b \le s, 1 \le t \le n$, for all $1 \le c \le n$, by (13)–(14), (33) and (42), then we obtain

$$\begin{aligned}&\langle \mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t, e_c\rangle = -\langle \mathfrak{e}_t, [e_a, e_b, e_c]_{B_1}\rangle = 0, \quad 1 \le a, b \le s, 1 \le t \le n, \\ &\langle \mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t, e^*_c\rangle = -\langle \mathfrak{e}_t, [e_a, e_b, e^*_c]_{B_1}\rangle = \begin{cases} 0, & 1 \le a, b \le s, \quad 1 \le t \le s, \\ \Gamma^c_{abt}, & 1 \le a, b \le s, \quad s+1 \le t \le n,\end{cases}\end{aligned}$$

Hence, $\nu^c_{abt} = 0, \lambda^c_{abt} = \Gamma^c_{abt}$, for all $1 \le a, b \le s, 1 \le t \le n$. Thus we have

$$\mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t = \begin{cases} 0, & 1 \le a, b, t \le s, \\ \sum_{k=1}^s \Gamma^k_{abt}\mathfrak{e}_k, & 1 \le a, b \le s < t \le n.\end{cases}$$

Case 2: if $s+1 \le a, b \le n$, $1 \le t \le n$, for all $1 \le c \le n$, then we have

$$\begin{aligned}&\langle \mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t, e_c\rangle = -\langle \mathfrak{e}_t, [e_a, e_b, e_c]_{B_1}\rangle = 0, \quad s+1 \le a, b \le n, 1 \le t \le n \\ &\langle \mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t, e^*_c\rangle = -\langle \mathfrak{e}_t, [e_a, e_b, e^*_c]_{B_1}\rangle = \begin{cases} \Gamma^c_{abt}, & 1 \le t \le s+1 \le a, b \le n, \\ 0, & s+1 \le a, b, t \le n.\end{cases}\end{aligned}$$

Hence $\nu^c_{abt} = 0, \lambda^c_{abt} = \Gamma^c_{abt}$, for all $s+1 \le a, b \le n,\ 1 \le t \le n$. Therefore,

$$\mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t = \begin{cases} 0, & s+1 \le a, b, t \le n, \\ \sum_{k=s+1}^n \Gamma^k_{abt}\mathfrak{e}_k, & 1 \le t \le s < a, b \le n. \end{cases}$$

Case (3): if $1 \le a \le s < b \le n,\ 1 \le t \le n$, for all $1 \le c \le n$, then we have

$$\langle \mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t, e_c\rangle = -\langle \mathfrak{e}_t, [e_a, e_b, e_c]_{B_1}\rangle = 0, \quad 1 \le t \le n,$$
$$\langle \mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t, e^*_c\rangle = -\langle \mathfrak{e}_t, [e_a, e_b, e^*_c]_{B_1}\rangle = \begin{cases} \Gamma^c_{abt}, & 1 \le t \le s, \\ \Gamma^c_{abt}, & s+1 \le t \le n. \end{cases}$$

Hence $\nu^c_{abt} = 0, \lambda^c_{abt} = \Gamma^c_{abt}$, for all $1 \le a \le s < b \le n,\ 1 \le t \le n$. Therefore,

$$\mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t = \begin{cases} \sum_{k=1}^s \Gamma^k_{abt}\mathfrak{e}_k, & s+1 \le a, t \le s < b \le n, \\ \sum_{k=s+1}^n \Gamma^k_{abt}\mathfrak{e}_k, & 1 \le a \le s < b, t \le n. \end{cases}$$

Thus we obtain the multiplication table of $\mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t$ in the basis $\Pi_1 \cup \Pi_2$:

$$\mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}_t = \begin{cases} \sum_{k=1}^s \Gamma^k_{abt}\mathfrak{e}_k, & 1 \le a, b \le s < t \le n \quad \text{or} \\ & 1 \le a, t \le s < b \le n; \\ \sum_{k=s+1}^n \Gamma^k_{abt}\mathfrak{e}_k, & 1 \le t \le s < a, b \le n \quad \text{or} \\ & 1 \le a \le s < b, t \le n; \\ 0, & 1 \le a, b, t \le s \quad \text{or} \\ & s+1 \le a, b, t \le n. \end{cases}$$

By the similar discussion, the multiplication of $\mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}^*_t$ and $\mathrm{ad}^*_{B_1}(e^*_a, e_b)\mathfrak{e}_t$ are given by

$$\mathrm{ad}^*_{B_1}(e_a, e_b)\mathfrak{e}^*_t = \begin{cases} -\sum_{k=s+1}^n \Gamma^t_{abk}\mathfrak{e}^*_k, & 1 \le a, b, t \le s \quad \text{or} \\ & 1 \le a \le s < b, t \le n; \\ -\sum_{k=1}^s \Gamma^t_{abk}\mathfrak{e}^*_k, & s+1 \le a, b, t \le n \quad \text{or} \\ & 1 \le a, t \le s < b \le n; \\ 0, & 1 \le a, b \le s < t \le n \quad \text{or} \\ & 1 \le t \le s < a, b \le n; \end{cases}$$

$$\mathrm{ad}^*_{B_1}(e^*_a, e_b)\mathfrak{e}_t = \begin{cases} \sum_{k=s+1}^n \Gamma^a_{bkt}\mathfrak{e}^*_k, & 1 \le a, b, t \le s, \quad 1 \le t \le s < a, b \le \ n \text{ or} \\ & 1 \le b \le s < a, t \le n, \\ \sum_{k=1}^s \Gamma^a_{bkt}\mathfrak{e}^*_k, & s+1 \le a, b, t \le n, \quad 1 \le a, b \le s < t \le n \text{ or} \\ & 1 \le a, t \le s < b \le n, \\ 0, & 1 \le a \le s < b, t \le n \quad \text{or} \\ & 1 \le b, t \le s < a \le n. \end{cases}$$

From the multiplication table of $\mathrm{ad}^*_{B_1}$, we can get that it is given by (43). ∎

Lemma 4.7. *Let* $\mathrm{ad}^*_{B_2}\colon B_2\wedge B_2\to \mathfrak{gl}(B_1)$ *be the coadjoint representation of the* 3*-Lie algebra* $(B_2,[\cdot,\cdot,\cdot]_{B_2})$ *on* B_1*, i.e.*

$$\langle \mathrm{ad}^*_{B_2}(\Theta_a,\Theta_b)X_c,\Theta_c\rangle=-\langle X_c,[\Theta_a,\Theta_b,\Theta_c]_{B_2}\rangle,\quad \forall\ X_c\in B_1,\Theta_a,\Theta_b,\Theta_c\in B_2. \tag{44}$$

Then we have

$$\mathrm{ad}^*_{B_2}(\Theta_a,\Theta_b)X_c=[\Theta_a,\Theta_b,X_c]_{B_2}. \tag{45}$$

Proof. It follows from a proof similar for Lemma 4.6. The multiplication table of $\mathrm{ad}^*_{B_2}$ in the basis $\Pi_1\cup\Pi_2$ is given by

$$\mathrm{ad}^*_{B_2}(\mathfrak{e}_a,\mathfrak{e}_b)e_t=\begin{cases}-\sum_{k=1}^{s}\Gamma^k_{abt}e_k, & 1\le a,b\le s<t\le n \quad\text{or}\\ & 1\le a,t\le s<b\le n,\\ -\sum_{k=s+1}^{n}\Gamma^k_{abt}e_k, & 1\le t\le s<a,b\le n\quad\text{or}\\ & 1\le a\le s<b,t\le n\\ 0, & 1\le a,b,t\le s\quad\text{or}\quad s+1\le a,b,t\le n;\end{cases}$$

$$\mathrm{ad}^*_{B_2}(\mathfrak{e}_a,\mathfrak{e}_b)e^*_t=\begin{cases}\sum_{k=s+1}^{n}\Gamma^t_{abk}e^*_k, & 1\le a,b,t\le s\quad\text{or}\\ & 1\le a\le s<b,t\le n\\ \sum_{k=1}^{s}\Gamma^t_{abk}e^*_k, & s+1\le a,b,t\le n\quad\text{or}\\ & 1\le a,t\le s<b\le n,\\ 0, & 1\le a,b\le s<t\le n\quad\text{or}\quad 1\le t\le s<a,b\le n;\end{cases}$$

$$\mathrm{ad}^*_{B_2}(\mathfrak{e}^*_a,\mathfrak{e}_b)e_t=\begin{cases}-\sum_{k=s+1}^{n}\Gamma^a_{bkt}e^*_k, & 1\le a,b,t\le s,\quad 1\le t\le s<a,b\le n\quad\text{or}\\ & 1\le b\le s<a,t\le n,\\ -\sum_{k=1}^{s}\Gamma^a_{bkt}e^*_k, & s+1\le a,b,t\le n\quad\text{or}\\ & 1\le a,b\le s<t\le n,\\ 0, & 1\le a\le s<b,t\le n\quad\text{or}\quad 1\le b,t\le s<a\le n.\end{cases}$$

From the multiplication table of $\mathrm{ad}^*_{B_2}$, we can get that it is given by (45). Hence the conclusion holds. ■

Next we can give the proof of Theorem 4.5.

Proof of Theorem 4.5. Combining Theorem 3.10, Lemma 4.6 and Lemma 4.7, for all $X_a,\ X_b,\ X_c\in B_1,\ \Theta_a,\Theta_b,\Theta_c\in B_2$, we have

$$\begin{aligned}[X_a+\Theta_a,X_b+\Theta_b,X_c+\Theta_c]_{B_1\oplus B_2}&=[X_a,X_b,X_c]_{B_1}+[\Theta_a,\Theta_b,\Theta_c]_{B_2}\\ &\quad+[X_a,X_b,\Theta_c]_{B_1}+[\Theta_a,\Theta_b,X_c]_{B_2}+c.p.(a,b,c).\end{aligned}$$

Then it is straightforward to deduce that $(B_1\oplus B_2,[\cdot,\cdot,\cdot]_{B_1\oplus B_2})$ is a 3-Lie algebra, and $(B_1\oplus B_2,[\cdot,\cdot,\cdot]_{B_1\oplus B_2},\overline{\mathcal{S}}(\cdot,\cdot),\ B_1,B_2)$ is a standard Manin triple of 3-Lie algebras. ■

Corollary 4.8. *Let* $(B_1,[\cdot,\cdot,\cdot]_{B_1})$ *and* $(B_2,[\cdot,\cdot,\cdot]_{B_2})$ *be* 3*-Lie algebras which are given by Theorem* 2.5 *and Theorem* 3.10*. Then* $(B_1,B_2,\mathrm{ad}^*_{B_1},\mathrm{ad}^*_{B_2})$ *is a matched pair of* 3*-Lie algebras.*

Proof. By Proposition 4.4 and Theorem 4.5, the conclusion holds. ■

5. Examples

In this section, we end the paper by using involutive derivations on 3-dimensional and 4-dimensional 3-Lie algebras to construct some examples of Manin triples of 3-Lie algebras. For convenience, we denote the 3-Lie algebra defined in Theorem 4.5 by B, that is $B=(B_1\oplus B_2,[\cdot,\cdot,\cdot]_{B_1\oplus B_2})$. As a first example, we have

Example 5.1. Let A be a 3-dimensional 3-Lie algebra given with respect to a basis $\{e_1,e_2,e_3\}$ by $[e_1,e_2,e_3]=e_1$.

Then the linear map $D\colon A\to A$ defined by $De_1=e_1$, $De_2=e_2$ and $De_3=-e_3$ is an involutive derivation on A. lt is obvious that $A_1=\langle e_1,e_2\rangle$ and $A_{-1}=\langle e_3\rangle$.

By Theorem 4.5, $(B=B_1\oplus B_2,[\cdot,\cdot,\cdot]_B,\overline{\mathcal{S}}(\cdot,\cdot),B_1,B_2)$ is a 12-dimensional Manin triple of 3-Lie algebras in the basis $\{e_1,e_2,e_3,e_1^*,e_2^*,e_3^*,\mathfrak{e}_1,\mathfrak{e}_2,\mathfrak{e}_3,\mathfrak{e}_1^*,\mathfrak{e}_2^*,\mathfrak{e}_3^*\}$, where $B_1=\langle e_1,e_2,e_3,e_1^*,e_2^*,\ e_3^*\rangle$, $B_2=\langle \mathfrak{e}_1,\mathfrak{e}_2,\mathfrak{e}_3,\mathfrak{e}_1^*,\mathfrak{e}_2^*,\mathfrak{e}_3^*\rangle$. By Theorem 4.5, Lemma 4.6 and Lemma 4.7, the 3-Lie bracket $[\cdot,\cdot,\cdot]_B$ of the Manin triple of 3-Lie algebras is given by:

$$
\begin{array}{llll}
[e_1,e_2,e_3]_B=e_1, & [e_2,e_2,e_1^*]_B=e_3^*, & [e_1,e_3,e_1^*]_B=e_2^*, & [e_3,e_2,e_1^*]_B=e_1^*,\\
[\mathfrak{e}_1,\mathfrak{e}_3,\mathfrak{e}_2]_B=\mathfrak{e}_1, & [\mathfrak{e}_1,\mathfrak{e}_2,\mathfrak{e}_1^*]_B=\mathfrak{e}_3^*, & [\mathfrak{e}_3,\mathfrak{e}_1,\mathfrak{e}_1^*]_B=\mathfrak{e}_2^*, & [\mathfrak{e}_2,\mathfrak{e}_3,\mathfrak{e}_1^*]_B=\mathfrak{e}_1^*,\\
[e_1,e_2,\mathfrak{e}_3]_B=\mathfrak{e}_1, & [e_2,e_3,\mathfrak{e}_1]_B=\mathfrak{e}_1, & [e_3,e_1,\mathfrak{e}_2]_B=\mathfrak{e}_1, & [e_2,e_1,\mathfrak{e}_1^*]_B=\mathfrak{e}_3^*,\\
[\mathfrak{e}_2,\mathfrak{e}_1,e_3]_B=e_1, & [\mathfrak{e}_3,\mathfrak{e}_2,e_1]_B=e_1, & [\mathfrak{e}_1,\mathfrak{e}_3,e_2]_B=e_1, & [\mathfrak{e}_1,\mathfrak{e}_2,e_1^*]_B=\mathfrak{e}_3^*,\\
[e_3,e_2,\mathfrak{e}_1^*]_B=\mathfrak{e}_1^*, & [e_1,e_3,\mathfrak{e}_1^*]_B=\mathfrak{e}_2^*, & [e_1^*,e_2,\mathfrak{e}_1]_B=\mathfrak{e}_3^*, & [e_1^*,\mathfrak{e}_1,e_3]_B=\mathfrak{e}_2^*,\\
[\mathfrak{e}_2,\mathfrak{e}_3,e_1^*]_B=e_1^*, & [\mathfrak{e}_3,\mathfrak{e}_1,e_1^*]_B=e_2^*, & [\mathfrak{e}_1^*,e_1,\mathfrak{e}_2]_B=e_3^*, & [\mathfrak{e}_1^*,\mathfrak{e}_3,e_1]_B=e_2^*,\\
[e_1^*,\mathfrak{e}_2,e_1]_B=\mathfrak{e}_3^*, & [e_1^*,e_1,\mathfrak{e}_3]_B=\mathfrak{e}_2^*, & [e_1^*,\mathfrak{e}_3,e_2]_B=\mathfrak{e}_1^*, & [e_1^*,e_3,\mathfrak{e}_2]_B=\mathfrak{e}_1^*,\\
[\mathfrak{e}_1^*,\mathfrak{e}_1,e_2]_B=e_3^*, & [\mathfrak{e}_1^*,e_3,\mathfrak{e}_1]_B=e_2^*, & [\mathfrak{e}_1^*,e_2,e_3]_B=e_1^*, & [\mathfrak{e}_1^*,e_2,\mathfrak{e}_3]_B=e_1^*.
\end{array}
$$

Example 5.2. Let A be the 4-dimensional 3-Lie algebra given with respect to a basis $\{e_1,e_2,e_3,e_4\}$ by $[e_2,e_3,e_4]=e_1$.

Then the linear map $D\colon A\to A$ defined by $De_i=e_i$ for $i=1,2,3$ and $De_4=-e_4$ is an involutive derivation on A, and satisfies $e_1,e_2,e_3\in A_1$ and $e_4\in A_{-1}$.

By Theorem 4.5, $(B=B_1\oplus B_2,[\cdot,\cdot,\cdot]_B,\overline{\mathcal{S}}(\cdot,\cdot),B_1,B_2)$ is a 16-dimensional Manin triple of 3-Lie algebras in the basis

$$\{e_1,e_2,e_3,e_4,e_1^*,e_2^*,e_3^*,e_4^*,\mathfrak{e}_1,\mathfrak{e}_2,\mathfrak{e}_3,\mathfrak{e}_4,\mathfrak{e}_1^*,\mathfrak{e}_2^*,\mathfrak{e}_3^*,\mathfrak{e}_4^*\},$$

where $B_1=\langle e_1,e_2,e_3,e_4,e_1^*,e_2^*,e_3^*,e_4^*\rangle$, $B_2=\langle \mathfrak{e}_1,\mathfrak{e}_2,\mathfrak{e}_3,\mathfrak{e}_4,\mathfrak{e}_1^*,\mathfrak{e}_2^*,\mathfrak{e}_3^*,\mathfrak{e}_4^*\rangle$. Then the 3-Lie bracket $[\cdot,\cdot,\cdot]_B$ of the Manin triple is given by:

$$
\begin{array}{llll}
[e_2,e_3,e_4]_B=e_1, & [e_3,e_2,e_1^*]_B=e_4^*, & [e_2,e_4,e_1^*]_B=e_3^*, & [e_4,e_3,e_1^*]_B=e_2^*,\\
[e_2,e_3,\mathfrak{e}_4]_B=\mathfrak{e}_1, & [e_3,e_4,\mathfrak{e}_2]_B=\mathfrak{e}_1, & [e_4,e_2,\mathfrak{e}_3]_B=\mathfrak{e}_1, & [e_3,e_2,\mathfrak{e}_1^*]_B=\mathfrak{e}_4^*,\\
[e_4,e_3,\mathfrak{e}_1^*]_B=\mathfrak{e}_2^*, & [e_2,e_4,\mathfrak{e}_1^*]_B=\mathfrak{e}_3^*, & [e_1^*,\mathfrak{e}_4,e_3]_B=\mathfrak{e}_2^*, & [e_1^*,e_3,\mathfrak{e}_2]_B=\mathfrak{e}_4^*,\\
[e_1^*,\mathfrak{e}_2,e_4]_B=\mathfrak{e}_3^*, & [e_1^*,e_4,\mathfrak{e}_3]_B=\mathfrak{e}_2^*, & [e_1^*,e_2,\mathfrak{e}_4]_B=\mathfrak{e}_3^*, & [e_1^*,\mathfrak{e}_3,e_2]_B=\mathfrak{e}_4^*,\\
[\mathfrak{e}_2,\mathfrak{e}_4,\mathfrak{e}_3]_B=\mathfrak{e}_1, & [\mathfrak{e}_2,\mathfrak{e}_3,\mathfrak{e}_1^*]_B=\mathfrak{e}_4^*, & [\mathfrak{e}_4,\mathfrak{e}_2,\mathfrak{e}_1^*]_B=\mathfrak{e}_3^*, & [\mathfrak{e}_3,\mathfrak{e}_4,\mathfrak{e}_1^*]_B=\mathfrak{e}_2^*,\\
[\mathfrak{e}_3,\mathfrak{e}_2,e_4]_B=e_1, & [\mathfrak{e}_4,\mathfrak{e}_3,e_2]_B=e_1, & [\mathfrak{e}_2,\mathfrak{e}_4,e_3]_B=e_1, & [\mathfrak{e}_2,\mathfrak{e}_3,e_1^*]_B=e_4^*,\\
[\mathfrak{e}_3,\mathfrak{e}_4,e_1^*]_B=e_2^*, & [\mathfrak{e}_4,\mathfrak{e}_2,e_1^*]_B=e_3^*, & [\mathfrak{e}_1^*,\mathfrak{e}_3,e_4]_B=e_2^*, & [\mathfrak{e}_1^*,\mathfrak{e}_3,e_2]_B=e_4^*,\\
[\mathfrak{e}_1^*,\mathfrak{e}_4,e_2]_B=e_3^*, & [\mathfrak{e}_1^*,e_3,\mathfrak{e}_4]_B=e_2^*, & [\mathfrak{e}_1^*,e_4,\mathfrak{e}_2]_B=e_3^*, & [\mathfrak{e}_1^*,\mathfrak{e}_2,e_3]_B=e_4^*,
\end{array}
$$

Example 5.3. Let A be the 4-dimensional 3-Lie algebra given with respect to a basis $\{e_1, e_2, e_3, e_4\}$ by $[e_1, e_3, e_4] = e_2,\quad [e_2, e_3, e_4] = e_1$.

Then the linear map $D\colon A \to A$ defined by $De_i = e_i$ for $1 \le i \le 3$ and $De_4 = -e_4$ is an involutive derivation of A. lt is obvious that $A_1 = \langle e_1, e_2, e_3\rangle$ and $A_{-1} = \langle e_4\rangle$. By Theorem 4.5, $(B = B_1 \oplus B_2, [\cdot,\cdot,\cdot]_B, \overline{\mathcal{S}}(\cdot,\cdot), B_1, B_2)$ is a 16-dimensional Manin triple of 3-Lie algebras in the basis

$$\{e_1, e_2, e_3, e_4, e_1^*, e_2^*, e_3^*, e_4^*, \mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4, \mathfrak{e}_1^*, \mathfrak{e}_2^*, \mathfrak{e}_3^*, \mathfrak{e}_4^*\},$$

where $B_1 = \langle e_1, e_2, e_3, e_4, e_1^*, e_2^*, e_3^*, e_4^*\rangle$, $B_2 = \langle \mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4, \mathfrak{e}_1^*, \mathfrak{e}_2^*, \mathfrak{e}_3^*, \mathfrak{e}_4^*\rangle$. The multiplication $[\cdot,\cdot,\cdot]_B$ of the Manin triple of 3-Lie algebras is given by:

$$\begin{array}{llll}
[e_2, e_3, e_4]_B = e_1, & [e_1, e_3, e_4]_B = e_2, & [e_4, e_3, e_2^*]_B = e_1^*, & [e_1, e_4, e_2^*]_B = e_3^*,\\
[e_3, e_1, e_2^*]_B = e_4^*, & [e_3, e_2, e_1^*]_B = e_4^*, & [e_4, e_3, e_1^*]_B = e_2^*, & [e_2, e_4, e_1^*]_B = e_3^*,\\
[e_1, e_3, \mathfrak{e}_4]_B = \mathfrak{e}_2, & [e_3, e_4, \mathfrak{e}_1]_B = \mathfrak{e}_2, & [e_4, e_1, \mathfrak{e}_3]_B = \mathfrak{e}_2, & [e_2, e_3, \mathfrak{e}_4]_B = \mathfrak{e}_1,\\
[e_3, e_4, \mathfrak{e}_2]_B = \mathfrak{e}_1, & [e_4, e_2, \mathfrak{e}_3]_B = \mathfrak{e}_1, & [e_3, e_4, \mathfrak{e}_2^*]_B = \mathfrak{e}_1^*, & [e_1, e_4, \mathfrak{e}_2^*]_B = \mathfrak{e}_3^*,\\
[e_3, e_1, \mathfrak{e}_2^*]_B = \mathfrak{e}_4^*, & [e_3, e_2, \mathfrak{e}_1^*]_B = \mathfrak{e}_4^*, & [e_4, e_3, \mathfrak{e}_1^*]_B = \mathfrak{e}_2^*, & [e_2, e_4, \mathfrak{e}_1^*]_B = \mathfrak{e}_3^*,\\
[\mathfrak{e}_2^*, \mathfrak{e}_3, e_1]_B = \mathfrak{e}_4^*, & [\mathfrak{e}_2^*, e_1, \mathfrak{e}_4]_B = \mathfrak{e}_3^*, & [\mathfrak{e}_2^*, \mathfrak{e}_4, e_3]_B = \mathfrak{e}_1^*, & [\mathfrak{e}_2^*, e_3, \mathfrak{e}_1]_B = \mathfrak{e}_4^*,\\
[\mathfrak{e}_2^*, \mathfrak{e}_1, e_4]_B = \mathfrak{e}_3^*, & [\mathfrak{e}_2^*, e_4, \mathfrak{e}_3]_B = \mathfrak{e}_1^*, & [\mathfrak{e}_1^*, \mathfrak{e}_3, e_2]_B = \mathfrak{e}_4^*, & [\mathfrak{e}_1^*, e_2, \mathfrak{e}_4]_B = \mathfrak{e}_3^*,\\
[\mathfrak{e}_1^*, e_3, \mathfrak{e}_2]_B = \mathfrak{e}_4^*, & [\mathfrak{e}_1^*, \mathfrak{e}_4, e_3]_B = \mathfrak{e}_2^*, & [\mathfrak{e}_1^*, \mathfrak{e}_2, e_4]_B = \mathfrak{e}_3^*, & [\mathfrak{e}_1^*, e_4, \mathfrak{e}_3]_B = \mathfrak{e}_2^*,\\
[\mathfrak{e}_3, \mathfrak{e}_2, \mathfrak{e}_4]_B = \mathfrak{e}_1, & [\mathfrak{e}_3, \mathfrak{e}_1, \mathfrak{e}_4]_B = \mathfrak{e}_2, & [\mathfrak{e}_3, \mathfrak{e}_4, \mathfrak{e}_2^*]_B = \mathfrak{e}_1^*, & [\mathfrak{e}_4, \mathfrak{e}_1, \mathfrak{e}_2^*]_B = \mathfrak{e}_3^*,\\
[\mathfrak{e}_1, \mathfrak{e}_3, \mathfrak{e}_2^*]_B = \mathfrak{e}_4^*, & [\mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_1^*]_B = \mathfrak{e}_4^*, & [\mathfrak{e}_3, \mathfrak{e}_4, \mathfrak{e}_1^*]_B = \mathfrak{e}_3^*, & [\mathfrak{e}_4, \mathfrak{e}_2, \mathfrak{e}_1^*]_B = \mathfrak{e}_3^*,\\
[\mathfrak{e}_3, \mathfrak{e}_1, e_4]_B = e_2, & [\mathfrak{e}_4, \mathfrak{e}_3, e_1]_B = e_2, & [\mathfrak{e}_1, \mathfrak{e}_4, e_3]_B = e_2, & [\mathfrak{e}_3, \mathfrak{e}_2, e_4]_B = e_2,\\
[\mathfrak{e}_4, \mathfrak{e}_3, e_2]_B = e_1, & [\mathfrak{e}_2, \mathfrak{e}_4, e_3]_B = e_1, & [\mathfrak{e}_3, \mathfrak{e}_4, e_2^*]_B = e_1^*, & [\mathfrak{e}_4, \mathfrak{e}_1, e_2^*]_B = e_3^*,\\
[\mathfrak{e}_1, \mathfrak{e}_3, e_2^*]_B = e_4^*, & [\mathfrak{e}_2, \mathfrak{e}_3, e_1^*]_B = e_4^*, & [\mathfrak{e}_3, \mathfrak{e}_4, e_1^*]_B = e_2^*, & [\mathfrak{e}_4, \mathfrak{e}_2, e_1^*]_B = e_3^*,\\
[\mathfrak{e}_2^*, \mathfrak{e}_1, e_3]_B = e_4^*, & [\mathfrak{e}_2^*, \mathfrak{e}_4, e_1]_B = e_3^*, & [\mathfrak{e}_2^*, \mathfrak{e}_3, e_4]_B = e_1^*, & [\mathfrak{e}_2^*, e_1, \mathfrak{e}_3]_B = e_4^*,\\
[\mathfrak{e}_2^*, \mathfrak{e}_4, e_1]_B = e_3^*, & [\mathfrak{e}_2^*, e_3, \mathfrak{e}_4]_B = e_1^*, & [\mathfrak{e}_1^*, \mathfrak{e}_2, e_3]_B = e_4^*, & [\mathfrak{e}_1^*, e_4, \mathfrak{e}_2]_B = e_3^*,\\
[\mathfrak{e}_1^*, e_2, \mathfrak{e}_3]_B = e_4^*, & [\mathfrak{e}_1^*, \mathfrak{e}_3, e_4]_B = e_2^*, & [\mathfrak{e}_1^*, \mathfrak{e}_4, e_2]_B = e_3^*, & [\mathfrak{e}_1^*, e_3, \mathfrak{e}_4]_B = e_2^*.
\end{array}$$

Next, we analyse the structure of Manin triples of 3-Lie algebras given in Example 5.1, Example 5.2 and Example 5.3 from the solvability and the nilpotency. First we recall some definitions.

Let A be a 3-Lie algebra. Denote by $A^1 = [A, A, A]$, which is called *the derived algebra* of A, $Z(A) = \{x \in A \mid [x, A, A] = 0\}$ is the *center* of A.

Define $A^{(0)} = A^0 = A$, $A^{(i)} = [A^{(i-1)}, A^{(i-1)}, A^{(i-1)}]$, $A^i = [A^{i-1}, A, A]$, $i \ge 1$.

If there is an r such that $A^{(r)} = 0$, then A is called *solvable*. If there is an r such that $A^r = 0$, then A is called *nilpotent*. It is clear that if A is nilpotent, then A is solvable.

Corollary 5.4. *The* 12*-dimensional Manin triple* B *of* 3*-Lie algebras in Example 5.1 is* 3*-step-solvable but non-nilpotent, and* $\dim B^1 = 8$, $\dim Z(B) = 4$.

Proof. It is obvious that $B^1 = B^{(1)} = [B, B, B]_B = \langle e_1, e_1^*, e_2^*, e_3^*, \mathfrak{e}_1, \mathfrak{e}_1^*, \mathfrak{e}_2^*, \mathfrak{e}_3^*\rangle$, and $Z(B) = \langle e_2^*, e_3^*, \mathfrak{e}_2^*, \mathfrak{e}_3^*\rangle$. Therefore, $\dim B^1 = 8$, $\dim Z(B) = 4$.

We have

$$B^{(2)} = [B^{(1)}, B^{(1)}, B^{(1)}]_B = \langle e_2^*, e_3^*, \mathfrak{e}_2^*, \mathfrak{e}_3^* \rangle, \quad B^{(3)} = [B^{(2)}, B^{(2)}, B^{(2)}]_B = 0,$$

$$B^2 = [B^1, B, B]_B = \langle e_1, e_1^*, \mathfrak{e}_1, \mathfrak{e}_1^* \rangle = B^r \neq 0, \quad r \geq 2.$$

Therefore, B is a 3-step-solvable 3-Lie algebra but non-nilpotent. ■

Corollary 5.5. *The* 16*-dimensional Manin triple* B *of* 3*-Lie algebras in Example 5.2 is nilpotent with* $\dim B^1 = 8$*, and* $B^1 = Z(B)$*.*

Proof. By a direct computation, we have

$$B^1 = \langle e_1, e_2^*, e_3^*, e_4^*, \mathfrak{e}_1, \mathfrak{e}_2^*, \mathfrak{e}_3^*, \mathfrak{e}_4^* \rangle, \quad B^2 = [B^1, B, B]_B = 0.$$

It follows the result. ■

Corollary 5.6. *The* 16*-dimensional* 3*-Lie algebra* B *in Example 5.3 is* 2*-step-solvable but non-nilpotent, and* $\dim B^1 = 12$*,* $\dim Z(B) = 4$*.*

Proof. It is obvious that

$$B^1 = B^{(1)} = [B, B, B]_B = \langle e_1, e_2, e_1^*, e_2^*, e_3^*, e_4^*, \mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_1^*, \mathfrak{e}_2^*, \mathfrak{e}_3^*, \mathfrak{e}_4^* \rangle,$$

and

$$Z(B) = \langle e_3^*, e_4^*, \mathfrak{e}_3^*, \mathfrak{e}_4^* \rangle.$$

Therefore, $\dim B^1 = 12$ and $\dim Z(B) = 4$. Since

$$B^{(2)} = [B^{(1)}, B^{(1)}, B^{(1)}]_B = 0, \quad B^2 = [B^1, B, B]_B = B^1 = B^r \neq 0, \quad r \geq 3,$$

B is 2-step-solvable, but non-nilpotent. ■

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