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Nonabelian Tensor Product of n-Lie Algebras

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Abstract. Let L and P be two n-Lie algebras over a field $\mathbb{F}$. We define the notion of nonabelian tensor products of L and P, which is denoted by $L \otimes P$. We obtain some properties of nonabelian tensor products, and finally, we aim to study the abelianess of n-Lie algebras as well as their dimensions.

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Key Words: n-Lie algebra, nonabelian tensor product, nonabelian tensor square.

1. Introduction and history

The nonabelian tensor products of groups were introduced by Brown and Loday [2]. In [10, 11], by using the nonabelian tensor products of groups and their nonabelian derived functors, the nonabelian homology of groups were constructed and studied, which generalize the classical Eilenberg-MacLane homology of groups and extend the nonabelian homology introduced in [8]. Conduche and Rodriguez-Fernandez [3] introduced the nonabelian tensor product modulo an integer q of groups, which is a generalization of definitions due to Brown [1], Ellis and Rodriguez [6]. This construction is the mod q version of the nonabelian tensor product of groups of Brown and Loday [2].

Ellis [5] developed an analogous theory of nonabelian tensor products for Lie algebras (see also [4]). Using tensor (exterior) products of Lie algebras, Ellis described the universal central extension of Lie algebras. The importance of this product is the constructing of the nonabelian homology of Lie algebras in low dimensions, which has applications in the cyclic homology; see [9]. Also, Inassaridze, Khmaladzade, and Ladra [12] constructed a nonabelian homology of Lie algebras with coefficients in any Lie algebra in any dimension as a nonabelian derived functor of the tensor product of Lie algebras, which generalize the classical homology of Lie algebras and extend Guin's nonabelian homology of Lie algebras. Moreover, they established some properties of the nonabelian tensor products and nonabelian homology of Lie algebras. Khmaladze [13] introduced the nonabelian tensor (exterior) products modulo q, denoted as $M \otimes^q N$ ($M \wedge^q N$), where M and N are two crossed P-modules, in the context of Lie algebras, as the mod q version of Ellis' tensor (exterior) product of Lie algebras and investigated its properties.

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In 1986, Filippov [7] introduced the notion of n-Lie algebra. An *n-Lie algebra* over a field $\mathbb{F}$ is a vector space L over $\mathbb{F}$ along with an anti-symmetric n-linear form $[x_1,\ldots,x_n]$ satisfying the Jacobi identity:

$$[[x_1,\ldots,x_n],y_2,\ldots,y_n]=\sum_{i=1}^{n}[x_1,\ldots,x_{i-1},[x_i,y_2,\ldots,y_n],x_{i+1},\ldots,x_n].$$

Clearly, n-Lie algebras are nothing but ordinary Lie algebras when $n=2$. Studying n-Lie algebras is important for their applications in physics and geometry.

Let $L_1,L_2,\ldots,L_n$ be subalgebras of an n-Lie algebra L. Denote by $[L_1,L_2,\ldots,L_n]$ the subalgebra of L generated by all vectors $[x_1,x_2,\ldots,x_n]$, where $x_i\in L_i$, $i=1,2,\ldots,n$. The subalgebra $[L,L,\ldots,L]$ is called the *derived algebra* of L, and it is denoted by L^2. If $L^2=0$, then L is called an abelian algebra. An *ideal* I of an n-Lie algebra L is a subspace of L such that $[I,L,\ldots,L]\subseteq I$. If $[I,I,L,\ldots,L]=0$, then I is called an *abelian ideal*.

The theory of tensor products of Lie algebras has been developed by Ellis in [5, 4]. In [14], Salemkar et al. studied some common properties between Lie algebras and their tensor products and presented some bounds on the nilpotency class and solvability length of $L\otimes K$, provided some information is given on L or K. Also, they gave some upper and lower bounds for the dimension of $L\otimes K$, if L and K are finite-dimensional nilpotent Lie algebras as well as ideals of a Lie algebras.

Throughout this paper, we use the following notation:

Notation

$\mathbb{F}$	$\longrightarrow$	Field
$\otimes^n_{\text{mod}}$	$\longrightarrow$	Modular tensor product
$\otimes$	$\longrightarrow$	Nonabelian tensor product
$L(V,W)$	$\longrightarrow$	The space of all linear functions from a vector space V into a vector space W
$L(V_1\times V_2\times\cdots\times V_n,W)$	$\longrightarrow$	The space of all multilinear functions from a vector space $V_1\times V_2\times\cdots\times V_n$ into a vector space W
L^2	$\longrightarrow$	The derived ideal of L, i.e., $L^2=[L,L,\ldots,L]$
L^{ab}	$\longrightarrow$	$\frac{L}{L^2}$

2. Fundamental extended concepts

In this section, we extend some definitions and state necessary concepts.

Free n-Lie Algebras

Let X be a set. We construct an n-Lie algebra generated by X, satisfying no relations other than these:

$$[x_1,\ldots,x_{i-1},x_i,\ldots,x_{j-1},x_i,\ldots,x_n]=0,\qquad \text{for all } x_r\in X, \tag{1}$$

$$\sum_{i=1}^{n}[x_1,\ldots,x_{i-1},[x_i,x'_1,\ldots,x'_{n-1}],x_{i+1},\ldots,x_n]=0, \tag{2}$$

$$\text{for all } \ x_i,x'_j\in X,\quad 1\le i\le n,\ 1\le j\le n-1.$$

Let $M(X)$ be the set inductively defined as follows:

(1) $X \subseteq M(X)$;

(2) If $x_1, \ldots, x_n \in M(X)$, then also $(x_1, \ldots, x_n) \in M(X)$.

The set $M(X)$ is called the *free magma* on X, which consists of all bracketed expressions on the elements of X. We define an n-ary operation on $M(X)$ by

$$\begin{array}{rcl} \bullet : M(X) \times \cdots \times M(X) & \longrightarrow & M(X) \\ x_1 \bullet \ \cdots \ \bullet x_n & \longmapsto & (x_1, \ldots, x_n). \end{array}$$

For $m \in M(X)$, we define its degree recursively as follows: $\deg(m) = 1$ if $m \in X$ and $\deg(m) = \sum_{i=1}^{n} \deg(m_i)$ if $m = (m_1, \ldots, m_n)$. So the degree of an element $m \in M(X)$ is just the number of elements of X that occur in m (counted with multiplicities). For an integer $d \geq 1$, we let $M_d(X)$ be the subset of $M(X)$ consisting of all $m \in M(X)$ of degree d. Then

$$M(X) = \bigcup_{d \geq 1} M_d(X).$$

Let $\mathbb{F}$ be a field and let $A(X)$ be the vector space over $\mathbb{F}$ spanned by $M(X)$. If we n-linearly extend the n-ary operation on $M(X)$ to $A(X)$, then $A(X)$ becomes an (nonassociative) algebra; it is called the free n-algebra over $\mathbb{F}$ on X. Let $f \in A(X)$; if also $f \in M(X)$, then f is said to be a monomial. For $f \in A(X)$, we define $deg(f)$ to be the maximum of the degrees of m, where m runs over all $m \in M(X)$ that occur in f with nonzero coefficient.

Let I_0 be the ideal of $A(X)$ generated by all the following elements:

$$(m_1, \ldots, m_{i-1}, m_i, \ldots, m_{j-1}, m_i, \ldots, m_n), \quad \text{for all } m_r \in M(X),$$

$$(m_1, \ldots, m_i, \ldots, m_j, \ldots, m_n) + (m_1, \ldots, m_j, \ldots, m_i, \ldots, m_n),$$

$$\text{for all } m_i \in M(X),\ 1 \leq i, j \leq n,$$

$$\sum_{i=1}^{n} (m_1, \ldots, m_{i-1}, (m_i, m'_1, \ldots, m'_{n-1}), m_{i+1}, \ldots, m_n), \quad \text{for all } m_i, m'_i \in M(X).$$

Set $L(X) = A(X)/I_o$. Let $\mathcal{B}$ be a basis of $L(X)$ consisting of (images of) elements of $M(X)$. Then it is immediate that

$$(m_1, \ldots, m_{i-1}, m_i, \ldots, m_{j-1}, m_i, \ldots, m_n) = 0,$$

$$(m_1, \ldots, m_i, \ldots, m_j, \ldots, m_n) + (m_1, \ldots, m_j, \ldots, m_i, \ldots, m_n) = 0,$$

and

$$\sum_{i=1}^{n} (m_1, \ldots, m_{i-1}, (m_i, m'_1, \ldots, m'_{n-1}), m_{i+1}, \ldots, m_n) = 0,$$

for all $m, m_i, m'_i \in \mathcal{B}$. It is easy to check that the relations (1) and (2) hold for all elements of $L(X)$, so that $L(X)$ is an n-Lie algebra. Therefore, we use the bracket to denote the product in $L(X)$ and say $L(X)$ is the free n-Lie algebra on X.

n-Tensor spaces

Multilinear algebra begins with the study of tensor spaces. In the most general settings, this involves dealing with n vector spaces, each with its own basis.

Let $V_1, V_2, ..., V_n$ be vector spaces over the field $\mathbb{F}$ with finite dimensions $d_1, d_2, ..., d_n$, respectively. Their Cartesian product under the componentwise addition and scalar multiplication is a vector space of dimension $\sum_{i=1}^{n} d_i$.

The function f from $V_1 \times V_2 \times \cdots \times V_n$ into the vector space W is called *multilinear* (or *n-linear*), if it is linear with respect to each component of $V_1 \times V_2 \times \cdots \times V_n$. We want to rewrite the same linear extension process for multilinear functions. Let $\{e_{ij}\}_{j=1}^{d_i}$ be a basis of vector space V_i, $1 \leq i \leq n$. Then there exists exactly one multilinear function as $f : V_1 \times V_2 \times \cdots \times V_n \longrightarrow W$ that takes prescribed values on the elements of the set

$$\{(e_{1j_1}, e_{2j_2}, \ldots, e_{nj_n}) : \ 1 \leq i \leq n, \ \ 1 \leq j_i \leq d_i\}. \tag{3}$$

Note that the above set contains $d_1 \times d_2 \times \cdots \times d_n$ members, and usually the number of members is more than the number of members of the basis set.

The process of extending a function defined on the members of the above set to a multilinear function is called the multilinear extension. We now compare the linear and multilinear functions and discuss their differences. The image of linear function $f : V \longrightarrow W$ is a subspace of W, but this is not necessarily true for multilinear functions. The linear closure of the image of multilinear function f is called the reach of f.

The pair $(\mathbb{T}, \Phi)$ (where $\mathbb{T}$ is a vector space and Φ is a multilinear function from $V_1 \times V_2 \times \cdots \times V_n$ into $\mathbb{T}$) is said to satisfy the *universal factorization property* if for every vector space W and every n-linear function $f : V_1 \times V_2 \times \cdots \times V_n \longrightarrow W$ there exists a linear function $h : \mathbb{T} \longrightarrow W$ such that $f = h\Phi$. It is easy to prove the existence of the universal pairs.

The two pairs $(\mathbb{T}, \Phi)$ and $(\mathbb{S}, \Psi)$ satisfying the universal factorization property for $V_1, V_2, \ldots, V_n$, are said to be *isomorphic*, if there exists an invertible linear transformation $T\colon \mathbb{S} \longrightarrow \mathbb{T}$ such that $\Phi = T\Psi$. If the reach of Ψ (respectively, Φ) is all of $\mathbb{S}$ (respectively, $\mathbb{T}$), then $(\mathbb{S}, \Psi)$ and $(\mathbb{T}, \Phi)$ are isomorphic and hence, up to isomorphism, there exists a unique pair $(\mathbb{T}, \Phi)$ satisfying the universal factorization property, where the reach of Φ is all of $\mathbb{T}$. Therefore $\mathbb{T}$ is the tensor product of $V_1, V_2, \ldots, V_n$ and we denote it by $\mathbb{T} = V_1 \otimes V_2 \otimes \cdots \otimes V_n$, and also $\Phi(v_1, v_2, \ldots, v_n)$ is a decomposable tensor. Since the choice of vector spaces V_i $(1 \leq i \leq n)$ is arbitrary, put $V_1 = V_2 = \cdots = V_j = V$ and $V_{j+1} = V_{j+2} = \cdots = V_n = W$, for some $1 \leq j \leq n-1$. In this case, we denote the tensor space

$$\underbrace{V \otimes V \otimes \cdots \otimes V}_{j-times} \otimes \underbrace{W \otimes \cdots \otimes W}_{(n-j)-times}$$

(or briefly, $V^{\otimes j} \otimes W^{\otimes (n-j)}$) by $\mathbb{T}_j$ (modular j-tensor product), for all $1 \leq j \leq n-1$.

Now, let $$V \otimes_{\text{mod}}^{n} W = \text{span}\{\mathbb{T}_j; \ \ 1 \leq j \leq n-1\}.$$

The vector space $V \otimes_{\text{mod}}^{n} W$ is called *modular tensor product* of V and W.

The following equivalent method can be used to construct the space of the modular tensor product. Let V and W be two $\mathbb{F}$-modules. Suppose that F is the free abelian group on all of the elements of the sets $V^{\times i_1} \times W^{\times i_2} \times \cdots \times V^{\times i_k}$, where $\sum_{j=1}^{k} i_j = n$. Thus, F is an $\mathbb{F}$-module and all of the sets $V^{\times i_1} \times W^{\times i_2} \times \cdots \times V^{\times i_k}$ are subsets of F.

Now, let K be the subgroup of F generated by all of the elements (for all elements $a_i, a_i' \in V \cup W$):

$$(a_1, a_2, \ldots, a_i + a_i', \ldots, a_n) - (a_1, a_2, \ldots, a_i, \ldots, a_n) - (a_1, a_2, \ldots, a_i', \ldots, a_n), \quad \text{for all } 1 \leq i \leq n,$$

$$(a_1, a_2, \ldots, ra_i, \ldots, a_j, \ldots, a_n) - (a_1, a_2, \ldots, a_i, \ldots, ra_j, \ldots, a_n), \qquad \text{for all } r \in \mathbb{F}.$$

The factor group F/K is the modular tensor product of M and N and it is denoted by $V \otimes^n_{\text{mod}} W$.

3. Nonabelian tensor product of n-Lie algebras

In this section, our aim is to define the nonabelian tensor products of n-Lie algebras. For this, we need to introduce some notation and definitions.
The first important concept of this section is the action of n-Lie algebras.

Definition 3.1. Let L and P be two n-Lie algebras over the field $\mathbb{F}$ with the brackets $[-, \ldots, -]_L$ and $[-, \ldots, -]_P$, respectively. We say that L *acts on* P if there exists a family of n-linear functions $\{f_i\}_{1 \leq i \leq n-1}$ with

$$f_i : \underbrace{L \times L \times \cdots \times L \times L}_{i-times} \times \underbrace{P \times \cdots \times P}_{(n-i)-times} \longrightarrow P$$

for all $1 \leq i \leq n-1$, such that the following identities hold for all $1 \leq i, j, k \leq n-1$:

$$\begin{aligned} &f_i([l_1, \ldots, l_n]_L, l_{n+1}, \ldots, l_{n+s}, p_1, \ldots, p_{n-s}) \\ &\quad = (-1)^{n-1} f_{n-1}(l_2, \ldots, l_n, f_i(l_1, l_{n+1}, \ldots, l_{n+s}, p_1, \ldots, p_{n-s})) \\ &\qquad + \sum_{j=2}^{n} (-1)^{n-j+1} f_{n-1}(l_1, \ldots, l_{j-1}, l_{j+1}, \ldots, l_n, f_i(l_j, l_{n+1}, \ldots, l_{n+s}, p_1, \ldots, p_{n-s})); \end{aligned} \tag{4}$$

$$\begin{aligned} &f_i(l_1, \ldots, l_i, p_1, \ldots, p_{n-i-1}, [p_{n-i}, p_2', \ldots, p_n']_P) \\ &\quad = [f_i(l_1, \ldots, l_i, p_1, \ldots, p_{n-i}), p_2', \ldots, p_n']_P \\ &\qquad - \sum_{j=1}^{i} (-1)^{n-j} f_{i-1}(l_1, \ldots, l_{j-1}, l_{j+1}, \ldots, l_i, p_1, \ldots, p_{n-i}, f_1(l_j, p_2', \ldots, p_n')) \\ &\qquad - \sum_{k=1}^{n-i} (-1)^{n-k} f_i(l_1, \ldots, l_i, p_1, \ldots, p_{k-1}, [p_k, p_2', \ldots, p_n']_P, p_{k+1}, \ldots, p_{n-i-1}); \end{aligned} \tag{5}$$

$$\begin{aligned} &f_i(l_1, \ldots, l_i, p_1, \ldots, p_{n-i-1}, f_j(l_{i+1}, l_2', \ldots, l_j', p_1', \ldots, p_{n-j}')) \\ &\quad = (-1)^{2n-i-2} f_{j-1}(l_2', \ldots, l_j', p_1', \ldots, p_{n-j}', f_{i+1}(l_1, \ldots, l_{i+1}, p_1, \ldots, p_{n-i-1})) \\ &\qquad - \sum_{k=1}^{i} (-1)^{2n-i-2} f_i\Big(l_2, \ldots, l_{k-1}, f_j(l_k, l_2', \ldots, l_j', p_1', \ldots, p_{n-j}'), \\ &\qquad\qquad l_{k+1}, \ldots, l_i, l_{i+1}, p_1, \ldots, p_{n-i-1}\Big) \\ &\qquad - \sum_{r=1}^{n-i-1} (-1)^{n-i-r+j-1} f_i(l_1, ..., l_i, l_{i+1}, p_1, ..., p_{r-1}, f_{j-1}(l_2', ..., l_j', p_r, p_1', ..., p_{n-j}')). \end{aligned} \tag{6}$$

The Lie multiplication in an n-Lie algebra L can induce an action on itself via

$$f_i(l_1, l_2, \dots, l_n) = [l_1, l_2, \dots, l_n], \quad \text{for all } 1 \leq i \leq n-1.$$

In [5], Ellis introduced the notion of Lie pairs, which we aim to generalize to n-Lie algebras as follows.

Definition 3.2. Let L, P, and Q be n-Lie algebras such that L and P act on each other by n-linear functions $\{f_i\}_{1\leq i\leq n-1}$ and $\{g_i\}_{1\leq i\leq n-1}$, respectively, and act on themselves by their brackets. The family of n-linear functions $\{\phi_i\}_{1\leq i\leq n-1}$ as

$$\phi_i : \underbrace{L \times \cdots \times L}_{i-times} \times \underbrace{P \times \cdots \times P}_{(n-i)-times} \longrightarrow Q,$$

is said *n-multiplying*, if

$$\begin{aligned}
&\phi_i([l_1, \dots, l_n], l'_2, \dots, l'_i, p_1, \dots, p_{n-i}) \qquad (7)\\
&= \sum_{j=1}^{n} (-1)^{n-j} \phi_{n-1}(l_1, \dots, l_{j-1}, l_{j+1}, \dots, l_n, f_i(l_j, l'_2, \dots, l'_i, p_1, \dots, p_{n-i})),
\end{aligned}$$

$$\begin{aligned}
&\phi_i(l_1, \dots, l_i, p_1, \dots, p_{n-i-1}, [p'_1, \dots, p'_n]) \qquad (8)\\
&= \sum_{j=1}^{n} (-1)^{n-j-2} \phi_1(g_i(p'_j, l_1, \dots, l_i, p_1, \dots, p_{n-i-1}), p'_1, \dots, p'_{j-1}, p'_{j+1}, \dots, p'_n),
\end{aligned}$$

$$\begin{aligned}
&\Big[\phi_{i_1}(l^1_1, \dots, l^1_{i_1}, p^1_1, \dots, p^1_{n-i_1}), \phi_{i_2}(l^2_1, \dots, l^2_{i_2}, p^2_1, \dots, p^2_{n-i_2}), \dots,\\
&\phi_{i_{n-1}}(l^{n-1}_1, \dots, l^{n-1}_{i_{n-1}}, p^{n-1}_1, \dots, p^{n-1}_{n-i_{n-1}}), \phi_{i_n}(l^n_1, \dots, l^n_{i_n}, p^n_1, \dots, p^n_{n-i_n})\Big] \qquad (9)\\
&= \frac{1}{2^{n-1}} \Big\{ (-1)^{\sum_{s=1}^{n-1} i_s(n-i_s)} \phi_1\Big(g_{n-i_1}(p^1_1, \dots, p^1_{n-i_1}, l^1_1, \dots, l^1_{i_1}),\\
&\quad f_{i_2}(l^2_1, \dots, l^2_{i_2}, p^2_1, \dots, p^2_{n-i_2}), \dots, f_{i_{n-1}}(l^{n-1}_1, \dots, l^{n-1}_{i_{n-1}}, p^{n-1}_1, \dots, p^{n-1}_{n-i_{n-1}}),\\
&\qquad f_{i_n}(l^n_1, \dots, l^n_{i_n}, p^n_1, \dots, p^n_{n-i_n})\Big)\\
&\quad + \sum_{r=2}^{n-1} (-1)^{\sum_{s=1, s\neq r}^{n-1} i_s(n-i_s)} \phi_2\Big(g_{n-i_1}(p^1_1, \dots, p^1_{n-i_1}, l^1_1, \dots, l^1_{i_1}),\\
&\quad g_{n-i_r}(p^r_1, \dots, p^r_{n-i_r}, l^r_1, \dots, l^r_{i_1}), f_{i_2}(l^2_1, \dots, l^2_{i_2}, p^2_1, \dots, p^2_{n-i_2}), \dots,\\
&\quad f_{i_{r-1}}(l^{r-1}_1, ..., l^{r-1}_{i_{r-1}}, p^{r-1}_1, ..., p^{r-1}_{n-i_{r-1}}), f_{i_{r+1}}(l^{r+1}_1, ..., l^{r+1}_{i_{r+1}}, p^{r+1}_1, ..., p^{r+1}_{n-i_{r+1}}), ...,\\
&\quad f_{i_{n-1}}(l^{n-1}_1, \dots, l^{n-1}_{n-1}, p^{n-1}_1, \dots, p^{n-1}_{n-i_{n-1}}), f_{i_n}(l^n_1, \dots, l^n_{i_n}, p^n_1, \dots, p^n_{n-i_n})\Big)\\
&\quad + \cdots\\
&\quad + (-1)^{i_1(n-i_1)} \phi_{n-1}\Big(g_{n-i_1}(p^1_1, ..., p^1_{n-i_1}, l^1_1, ..., l^1_{i_1}), g_{n-i_2}(p^2_1, ..., p^2_{n-i_2}, l^2_1, ..., l^2_{i_2}), ...,\\
&\quad g_{n-i_{n-1}}(p^{n-1}_1, \dots, p^{n-1}_{n-i_{n-1}}, l^{n-1}_1, \dots, l^{n-1}_{i_{n-1}}), f_{i_n}(l^n_1, \dots, l^n_{i_n}, p^n_1, \dots, p^n_{n-i_n})\Big)\Big\}.
\end{aligned}$$

for all $l_k, l'_k, l^w_k \in L,\ p_s, p'_s, p^w_s \in P,\ w = 1, \dots, n$.

The notion of n-multiplying plays a key role in the definition of nonabelian tensor products of n-Lie algebras.

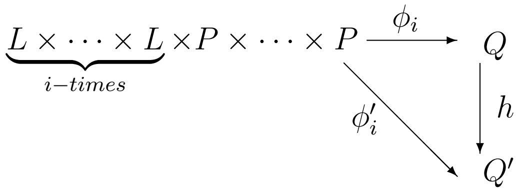
Definition 3.3. Let L, P, and Q be n-Lie algebras. The pair $(Q, \{\phi_i\}_{1\leq i\leq n-1})$ (or briefly Q) is said a nonabelian tensor product of L and P if there exists a family of n-multiplying maps as

$$\{\phi_i;\ \phi_i : \underbrace{L\times\cdots\times L}_{i-times}\times\underbrace{P\times\cdots\times P}_{(n-i)-times}\longrightarrow Q\}_{1\leq i\leq n-1}$$

such that for any other n-Lie algebra Q' and family of n-multiplying maps as $\{\phi'_i\}_{1\leq i\leq n-1}$, where

$$\phi'_i : \underbrace{L\times\cdots\times L}_{i-times}\times\underbrace{P\times\cdots\times P}_{(n-i)-times}\longrightarrow Q' \quad \text{for all } 1\leq i\leq n-1,$$

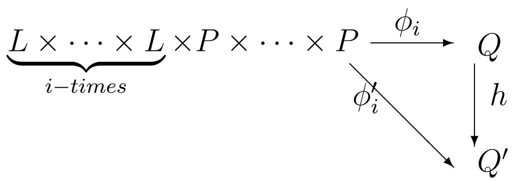
there is a unique homomorphism of n-Lie algebras $h : Q \longrightarrow Q'$ such that the following diagrams are commutative:

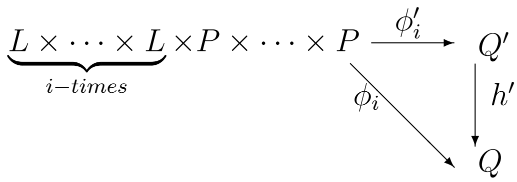


In the next theorem, we prove that a nonabelian tensor product of two n-Lie algebras is unique, up to isomorphism.

Theorem 3.4. *Let L and P be two n-Lie algebras and let $(Q, \{\phi_i\})$ and $(Q', \{\phi'_i\})$ be two nonabelian tensor products of them. Then $Q \cong Q'$.*

Proof. By the definition of nonabelian tensor products of n-Lie algebras, there exist unique homomorphisms of n-Lie algebras $h : Q \longrightarrow Q'$ and $h' : Q \longrightarrow Q$ such that the following diagrams are commutative (i.e., $h\phi_i = \phi'_i$ and $h'\phi'_i = \phi_i$):





for all $1 \leq i \leq n-1$. Therefore,

$$hh'\phi'_i = \phi'_i, \qquad \text{for all } 1\leq i\leq n-1 \Longrightarrow hh' = I,$$
$$h'h\phi_i = \phi_i, \qquad \text{for all } 1\leq i\leq n-1 \Longrightarrow h'h = I'.$$

Hence $(hh')^{-1} = h'h$ if and only if $(hh')o(h'h) = I$ and $(h'h)o(hh') = I'$. Thus $T := hh'$ is an isomorphism from Q' onto Q. ■

The following theorem shows that nonabelian tensor products can always be defined and exist for any two n-Lie algebras. It also describes how to build them.

Theorem 3.5. *Let L and P be two n-Lie algebras. Then the nonabelian tensor product of L and P exists.*

Proof. Let F be the free n-Lie algebra on the set

$$X = \bigcup_{1\leq i\leq n-1} \underbrace{L\times\cdots\times L}_{i-times}\times P\times\cdots\times P,$$

and assume that K is an ideal generated by the following elements of F:

$$
\begin{aligned}
&(l_1,\ldots,l_j+rl'_j,\ldots,l_i,p_1,\ldots,p_{n-i})-(l_1,\ldots,l_j,\ldots,l_i,p_1,\ldots,p_{n-i})\\
&\quad -r(l_1,\ldots,l'_j,\ldots,l_i,p_1,\ldots,p_{n-i}),\\
&(l_1,\ldots,l_i,p_1,\ldots,p_k+rp'_k,\ldots,p_{n-i})-(l_1,\ldots,l_i,p_1,\ldots,p_k,\ldots,p_{n-i})\\
&\quad -r(l_1,\ldots,l_i,p_1,\ldots,p'_k,\ldots,p_{n-i}),\ f_i([l_1,\ldots,l_n],l_{n+1},\ldots,l_{n+s},p_1,\ldots,p_{n-s})\\
&-(-1)^{n-1}f_{n-1}(l_2,\ldots,l_n,f_i(l_1,l_{n+1},\ldots,l_{n+s},p_1,\ldots,p_{n-s}))\\
&-\sum_{j=2}^{n}(-1)^{n-j+1}f_{n-1}(l_1,\ldots,l_{j-1},l_{j+1},\ldots,l_n,f_i(l_j,l_{n+1},\ldots,l_{n+s},p_1,\ldots,p_{n-s})),\\
&f_i(l_1,\ldots,l_i,p_1,\ldots,p_{n-i-1},[p_{n-i},p'_2,\ldots,p'_n])-[f_i(l_1,\ldots,l_i,p_1,\ldots,p_{n-i}),p'_2,\ldots,p'_n]\\
&+\sum_{j=1}^{i}(-1)^{n-j}f_{i-1}(l_1,\ldots,l_{j-1},l_{j+1},\ldots,l_i,p_1,\ldots,p_{n-i},f_1(l_j,p'_2,\ldots,p'_n))\\
&-\sum_{k=1}^{n-i}(-1)^{n-k}f_i(l_1,\ldots,l_i,p_1,\ldots,p_{k-1},[p_k,p'_2,\ldots,p'_n],p_{k+1},\ldots,p_{n-i-1}),\\
&f_i(l_1,\ldots,l_i,p_1,\ldots,p_{n-i-1},f_j(l_{i+1},l'_2,\ldots,l'_j,p'_1,\ldots,p'_{n-j}))\\
&-(-1)^{2n-i-2}f_{j-1}(l'_2,\ldots,l'_j,p'_1,\ldots,p'_{n-j},f_{i+1}(l_1,\ldots,l_{i+1},p_1,\ldots,p_{n-i-1}))\\
&+\sum_{k=1}^{i}(-1)^{2n-i-2}f_i\Big(l_2,\ldots,l_{k-1},f_j(l_k,l'_2,\ldots,l'_j,p'_1,\ldots,p'_{n-j}),\\
&\qquad\qquad l_{k+1},\ldots,l_i,l_{i+1},p_1,\ldots,p_{n-i-1}\Big)\\
&+\sum_{r=1}^{n-i-1}(-1)^{n-i-r+j-1}f_i\Big(l_1,\ldots,l_i,l_{i+1},p_1,\ldots,p_{r-1},\\
&\qquad\qquad f_{j-1}(l'_2,\ldots,l'_j,p_r,p'_1,\ldots,p'_{n-j})\Big),\\
&\Big[(l^1_1,\ldots,l^1_{i_1},p^1_1,\ldots,p^1_{n-i_1}),(l^2_1,\ldots,l^2_{i_2},p^2_1,\ldots,p^2_{n-i_2}),\ldots,\\
&\qquad (l^{n-1}_1,\ldots,l^{n-1}_{i_{n-1}},p^{n-1}_1,\ldots,p^{n-1}_{n-i_{n-1}}),(l^n_1,\ldots,l^n_{i_n},p^n_1,\ldots,p^n_{n-i_n})\Big]\\
&-\frac{1}{2^{n-1}}\Big\{(-1)^{\sum_{s=1}^{n-1}i_s(n-i_s)}\Big(g_{n-i_1}(p^1_1,\ldots,p^1_{n-i_1},l^1_1,\ldots,l^1_{i_1}),\\
&\quad f_{i_2}(l^2_1,\ldots,l^2_{i_2},p^2_1,\ldots,p^2_{n-i_2}),\ldots,f_{i_{n-1}}(l^{n-1}_1,\ldots,l^{n-1}_{i_{n-1}},p^{n-1}_1,\ldots,p^{n-1}_{n-i_{n-1}}),\\
&\qquad f_{i_n}(l^n_1,\ldots,l^n_{i_n},p^n_1,\ldots,p^n_{n-i_n})\Big)\\
&-\sum_{r=2}^{n-1}(-1)^{\sum_{\substack{s=1\\ s\neq r}}^{n-1}i_s(n-i_s)}\Big(g_{n-i_1}(p^1_1,\ldots,p^1_{n-i_1},l^1_1,\ldots,l^1_{i_1}),\\
&\quad g_{n-i_r}(p^r_1,\ldots,p^r_{n-i_r},l^r_1,\ldots,l^r_{i_1}),f_{i_2}(l^2_1,\ldots,l^2_{i_2},p^2_1,\ldots,p^2_{n-i_2}),\ldots,\\
&\quad f_{i_{r-1}}(l^{r-1}_1,\ldots,l^{r-1}_{i_{r-1}},p^{r-1}_1,\ldots,p^{r-1}_{n-i_{r-1}}),f_{i_{r+1}}(l^{r+1}_1,\ldots,l^{r+1}_{i_{r+1}},p^{r+1}_1,\ldots,p^{r+1}_{n-i_{r+1}}),\ldots,\\
&\quad f_{i_{n-1}}(l^{n-1}_1,\ldots,l^{n-1}_{i_{n-1}},p^{n-1}_1,\ldots,p^{n-1}_{n-i_{n-1}}),f_{i_n}(l^n_1,\ldots,l^n_{i_n},p^n_1,\ldots,p^n_{n-i_n})\Big)\\
&-\cdots\\
&-(-1)^{i_1(n-i_1)}\Big(g_{n-i_1}(p^1_1,\ldots,p^1_{n-i_1},l^1_1,\ldots,l^1_{i_1}),g_{n-i_2}(p^2_1,\ldots,p^2_{n-i_2},l^2_1,\ldots,l^2_{i_2}),\ldots,\\
&g_{n-i_{n-1}}(p^{n-1}_1,\ldots,p^{n-1}_{n-i_{n-1}},l^{n-1}_1,\ldots,l^{n-1}_{i_{n-1}}),f_{i_n}(l^n_1,\ldots,l^n_{i_n},p^n_1,\ldots,p^n_{n-i_n})\Big)\Big\}.
\end{aligned}
$$

The quotient n-Lie algebra F/K and its elements $(l_1, \dots, l_i, p_1, \dots, p_{n-i}) + K$ are denoted by $L \otimes P$ and $l_1 \otimes \cdots \otimes l_i \otimes p_1 \otimes \cdots \otimes p_{n-i}$, respectively.
Now we define the following maps:

$$\begin{cases} \bigotimes_i : \underbrace{L \times \cdots \times L}_{i-times} \times P \times \cdots \times P & \longrightarrow \quad L \otimes P \\ \qquad (l_1, \dots, l_i, p_1, \dots, p_{n-i}) & \longmapsto \quad l_1 \otimes \ldots \otimes l_i \otimes p_1 \otimes \ldots \otimes p_{n-i} \end{cases}$$

for all $1 \leq i \leq n-1$. It is easy to check that $\{\bigotimes_i\}_{1 \leq i \leq n-1}$ is the family of n-multiplying maps.

Now, let $(Q, \{\phi_i\}_{1 \leq i \leq n-1})$ be another nonabelian tensor product of L and P. We prove that there exists a unique homomorphism $h : L \otimes P \longrightarrow Q$ such that the following diagram commutes:

$$\begin{array}{ccc} \underbrace{L \times \cdots \times L}_{i-times} \times P \times \cdots \times P & \xrightarrow{\ \bigotimes_i\ } & L \otimes P \\ & \overset{\phi_i}{\searrow} & \downarrow h \\ & & Q \end{array}$$

Since $L \otimes P = F/K$, where F is a free n-Lie algebra over

$$X = \bigcup_{1 \leq i \leq n-1} \underbrace{L \times \cdots \times L}_{i-times} \times P \times \cdots \times P,$$

there exists a unique homomorphism $h' : F \longrightarrow Q$ such that the following diagram commutes:

$$\begin{array}{ccc} X = \bigcup_{1 \leq i \leq n-1} \underbrace{L \times \cdots \times L}_{i-times} \times P \times \cdots \times P & \xrightarrow{\ inc.\ } & F \\ & \overset{\{\phi_i\}_{1 \leq i \leq n-1}}{\searrow} & \downarrow h' \\ & & Q \end{array}$$

$$h'(inc.|_{A_i}) = \phi_i, \qquad \text{for all } 1 \leq i \leq n-1,$$

where $inc.$ is the inclusion map and $A_i = \underbrace{L \times \cdots \times L}_{i-times} \times P \times \cdots \times P$, so that we can write $X = \bigcup_{1 \leq i \leq n-1} A_i$. Moreover, h' on the every generator of the ideal K is zero; e.g.

$$\begin{aligned} & h'\big(inc.|_{A_j}((l_1, \dots, l_j + l'_j, \dots, l_i, p_1, \dots, p_{n-i}) - (l_1, \dots, l_j, \dots, l_i, p_1, \dots, p_{n-i}) \\ & \qquad - (l_1, \dots, l'_j, \dots, l_i, p_1, \dots, p_{n-i}))\big) \\ & = h'\big((l_1, \dots, l_j + l'_j, \dots, l_i, p_1, \dots, p_{n-i}) - (l_1, \dots, l_j, \dots, l_i, p_1, \dots, p_{n-i}) \\ & \qquad - (l_1, \dots, l'_j, \dots, l_i, p_1, \dots, p_{n-i})\big) \\ & = \phi_j\big((l_1, \dots, l_j + l'_j, \dots, l_i, p_1, \dots, p_{n-i}) - (l_1, \dots, l_j, \dots, l_i, p_1, \dots, p_{n-i}) \\ & \qquad - (l_1, \dots, l'_j, \dots, l_i, p_1, \dots, p_{n-i})\big) \\ & = \phi_j((l_1, \dots, l_j + l'_j, \dots, l_i, p_1, \dots, p_{n-i})) \\ & \qquad - \phi_j((l_1, ..., l_j, ..., l_i, p_1, ..., p_{n-i})) - \phi_j((l_1, \dots, l'_j, ..., l_i, p_1, ..., p_{n-i})) = 0. \end{aligned}$$

Therefore, $K \leq \ker h'$ and so the following homomorphism can be defined by

$$\begin{aligned} h'' : L \otimes P &\longrightarrow Q \\ \sum_{i=1}^{n-1} (l_1^i, \ldots, l_i^i, p_1^i, \ldots, p_{n-i}^i) &\longmapsto \sum_{i=1}^{n-1} (l_1^i \otimes \ldots \otimes l_i^i \otimes p_1^i \otimes \ldots \otimes p_{n-i}^i). \end{aligned}$$

Since $K \leq \ker h'$ and h' is a homomorphism of n-Lie algebras, h'' is a well-defined homomorphism of n-Lie algebras. Also, the above diagram commutes, since

$$\begin{aligned} &h'' \bigotimes\nolimits_i (l_1, \ldots, l_i, p_1, \ldots, p_{n-i}) = h''(l_1 \otimes \ldots \otimes l_i \otimes p_1 \otimes \ldots \otimes p_{n-i}) \\ &= h'(inc.|_{A_i}(l_1, \ldots, l_i, p_1, ..., p_{n-i})) = \phi_i(l_1, ..., l_i, p_1, \ldots, p_{n-i}), \quad \text{for all } 1 \leq i \leq n-1. \end{aligned}$$

Now, if $h''' : L \otimes P \longrightarrow Q$ is an another homomorphism such that $h''' \bigotimes_i = \phi_i$ for $1 \leq i \leq n-1$, then

$$\begin{aligned} &h'''(l_1 \otimes \ldots \otimes l_i \otimes p_1 \otimes \ldots \otimes p_{n-i}) = h''' \bigotimes\nolimits_i (l_1, \ldots, l_i, p_1, \ldots, p_{n-i}) \\ &= \phi_i(l_1, \ldots, l_i, p_1, \ldots, p_{n-i}) = h'' \bigotimes\nolimits_i (l_1, \ldots, l_i, p_1, \ldots, p_{n-i}) \\ &= h''(l_1 \otimes \ldots \otimes l_i \otimes p_1 \otimes \ldots \otimes p_{n-i}) \end{aligned}$$

and hence the proof is complete. ■

By the above theorems, we can accept the following definition for the nonabelian tensor product of n-Lie algebras.

Let L and P be n-Lie algebras over a field $\mathbb{F}$ such that L and P act on each other by n-linear functions $\{f_i\}_{1 \leq i \leq n-1}$ and $\{g_i\}_{1 \leq i \leq n-1}$, respectively, and act on themselves by their brackets. Then $L \otimes P$ is generated by the symbols $l_1 \otimes \cdots \otimes l_i \otimes p_{i+1} \otimes \cdots \otimes p_n$ subject to the following relations:

$$\begin{aligned} &l_1 \otimes \cdots \otimes \alpha l_i + l_i' \otimes \cdots \otimes l_j \otimes p_{j+1} \otimes \cdots \otimes p_n \\ &= \alpha(l_1 \otimes \cdots \otimes l_i \otimes \cdots \otimes l_j \otimes p_{j+1} \otimes \cdots \otimes p_n) \\ &\quad + (l_1 \otimes \cdots \otimes l_i' \otimes \cdots \otimes l_j \otimes p_{j+1} \otimes \cdots \otimes p_n), \end{aligned} \tag{10}$$

$$\begin{aligned} &l_1 \otimes \cdots \otimes l_j \otimes p_{j+1} \otimes \cdots \otimes \alpha p_{j+i} + p_{j+i}' \otimes \cdots \otimes p_n \\ &= \alpha(l_1 \otimes \cdots \otimes l_j \otimes p_{j+1} \otimes \cdots \otimes p_{j+i} \otimes \cdots \otimes p_n) \\ &\quad + (l_1 \otimes \cdots \otimes l_j \otimes p_{j+1} \otimes \cdots \otimes p_{j+i}' \otimes \cdots \otimes p_n), \end{aligned} \tag{11}$$

$$\begin{aligned} &[l_1, \ldots, l_n] \otimes l_2' \otimes \ldots \otimes l_i' \otimes p_1 \otimes \ldots \otimes p_{n-i} \\ &= \sum_{j=1}^{n} (-1)^{n-j} l_1 \otimes \ldots \otimes l_{j-1} \otimes l_{j+1} \otimes \ldots \otimes l_n \otimes f_i(l_j, l_2', \ldots, l_i', p_1, \ldots, p_{n-i}), \end{aligned} \tag{12}$$

$$\begin{aligned} &l_1 \otimes \ldots \otimes l_i \otimes p_1 \otimes \ldots \otimes p_{n-i-1} \otimes [p_1', \ldots, p_n'] \\ &= \sum_{j=1}^{n} (-1)^{n-j-2} g_i(p_j', l_1, \ldots, l_i, p_1, \ldots, p_{n-i-1}) \otimes p_1' \otimes \ldots \otimes p_{j-1}' \otimes p_{j+1}' \otimes \ldots \otimes p_n', \end{aligned} \tag{13}$$

$$\Big[(l_1^1 \otimes \cdots \otimes l_{i_1}^1 \otimes p_1^1 \otimes \cdots \otimes p_{n-i_1}^1), (l_1^2 \otimes \cdots \otimes l_{i_2}^2 \otimes p_1^2 \otimes \cdots \otimes p_{n-i_2}^2), \ldots,$$

$$(l_1^{n-1} \otimes \cdots \otimes l_{i_{n-1}}^{n-1} \otimes p_1^{n-1} \otimes \cdots \otimes p_{n-i_{n-1}}^{n-1}), (l_1^n \otimes \cdots \otimes l_{i_n}^n \otimes p_1^n \otimes \cdots \otimes p_{n-i_n}^n) \Big]$$

$$= \frac{1}{2^{n-1}}\Big\{(-1)^{\sum_{s=1}^{n-1} i_s(n-i_s)}\Big(g_{n-i_1}(p_1^1,\dots,p_{n-i_1}^1,l_1^1,\dots,l_{i_1}^1)\otimes f_{i_2}(l_1^2,\dots,l_{i_2}^2,p_1^2,\dots,p_{n-i_2}^2)\otimes\dots\otimes f_{i_{n-1}}(l_1^{n-1},\dots,l_{i_{n-1}}^{n-1},p_1^{n-1},\dots,p_{n-i_{n-1}}^{n-1})\otimes f_{i_n}(l_1^n,\dots,l_{i_n}^n,p_1^n,\dots,p_{n-i_n}^n)\Big)$$

$$+\sum_{r=2}^{n-1}(-1)^{\sum_{\substack{s=1\\ s\neq r}}^{n-1} i_s(n-i_s)}\Big(g_{n-i_1}(p_1^1,\dots,p_{n-i_1}^1,l_1^1,\dots,l_{i_1}^1)\otimes g_{n-i_r}(p_1^r,\dots,p_{n-i_1}^r,l_1^r,\dots,l_{i_1}^r)\otimes f_{i_2}(l_1^2,\dots,l_{i_2}^2,p_1^2,\dots,p_{n-i_2}^2)\otimes\dots\otimes f_{i_{r-1}}(l_1^{r-1},...,l_{i_{r-1}}^{r-1},p_1^{r-1},...,p_{n-i_{r-1}}^{r-1})\otimes f_{i_{r+1}}(l_1^{r+1},...,l_{i_{r+1}}^{r+1},p_1^{r+1},...,p_{n-i_{r+1}}^{r+1})\otimes...\otimes f_{i_{n-1}}(l_1^{n-1},\dots,l_{i_{n-1}}^{n-1},p_1^{n-1},\dots,p_{n-i_{n-1}}^{n-1})\otimes f_{i_n}(l_1^n,\dots,l_{i_n}^n,p_1^n,\dots,p_{n-i_n}^n)\Big)$$

$$+\cdots$$

$$+(-1)^{i_1(n-i_1)}\Big(g_{n-i_1}(p_1^1,...,p_{n-i_1}^1,l_1^1,...,l_{i_1}^1)\otimes g_{n-i_2}(p_1^2,...,p_{n-i_2}^2,l_1^2,...,l_{i_2}^2)\otimes...\otimes g_{n-i_{n-1}}(p_1^{n-1},\dots,p_{n-i_{n-1}}^{n-1},l_1^{n-1},\dots,l_{i_{n-1}}^{n-1})\otimes f_{i_n}(l_1^n,\dots,l_{i_n}^n,p_1^n,\dots,p_{n-i_n}^n)\Big)\Big\}. \quad (14)$$

for each $l_k, l'_k, l_k^w \in L,\ p_s, p'_s, p_s^w \in P,\ w = 1,\dots,n$, and $\alpha \in \mathbb{F}$.

If $L = P$, then their nonabelian tensor product is called *nonabelian tensor square* and is denoted by $L\otimes L$. In this case, L acts with bracket of n-Lie algebra on itself.

Proposition 3.6. *Let L and P be two n-Lie algebras such that L and P act on each other by $\{f_i\}_{1\le i\le n-1}$ and $\{g_i\}_{1\le i\le n-1}$, respectively. Then there exist the following homomorphisms:*

$$\begin{aligned} \mu : L\otimes P &\longrightarrow L\\ \sum_{i=1}^{n-1}(l_1^i\otimes\cdots\otimes l_i^i\otimes p_{i+1}^i\otimes\cdots\otimes p_n^i) &\longmapsto \sum_{i=1}^{n-1}(-1)^{(n-i)i}g_{n-i}(p_{i+1}^i,\dots,p_n^i,l_1^i,\dots,l_i^i),\\ \textit{and}\qquad \nu : L\otimes P &\longrightarrow P\\ \sum_{i=1}^{n-1}(l_1^i\otimes\cdots\otimes l_i^i\otimes p_{i+1}^i\otimes\cdots\otimes p_n^i) &\longmapsto \sum_{i=1}^{n-1} f_i(l_1^i,\dots,l_i^i,p_{i+1}^i,\dots,p_n^i). \end{aligned}$$

Proof. Since f_i's, g_j's, and the tensor n-multiplying are n-linear maps, so are μ and ν. Moreover, by (14), μ and ν preserve the bracket. ∎

Ellis [5] proved some properties of the nonabelian tensor product of Lie algebras, which require a concept of compatibility actions. In the following definition, we define this concept for n-Lie algebras and then prove some analogous results.

Definition 3.7. Let L and P be two n-Lie algebras. Suppose that L and P act on each other by $\{f_i\}_{1\le i\le n-1}$ and $\{g_i\}_{1\le i\le n-1}$, respectively, and each of them acts on itself by the Lie multiplication. Then these actions are called compatible if

$$f_i(g_j(p'_1,\dots,p'_j,l'_1,\dots,l'_{n-j}),l_2,\dots,l_i,p_1,\dots,p_{n-i}) = (-1)^{j(n-j)+n-1}f_i(l_2,\dots,l_i,p_1,\dots,p_{n-i},f_{n-j}(l'_1,\dots,l'_{n-j},p'_1,\dots,p'_j)), \quad (15)$$

$$g_i(f_k(l_1,\dots,l_k,p_1,\dots,p_{n-k}),p_2,\dots,p_i,l_{i+1},\dots,l_n) = (-1)^{k(n-k)+n-1}g_{n-i}(p_2,\dots,p_i,l_{i+1},\dots,l_n,g_{n-k}(p_1,\dots,p_k,l_1,\dots,l_{n-k})), \quad (16)$$

$$f_1(g_k(p_1,\ldots,p_k,l_1,\ldots,l_{n-k}),p_2,\ldots,p_n) \tag{17}$$
$$= (-1)^{k(n-k)+n-1}[p_2,\ldots,p_n,f_{n-k}(l_1,\ldots,l_{n-k},p_1,\ldots,p_k)]_P,$$

$$g_1(f_k(l_1,\ldots,l_k,p_1,\ldots,p_{n-k}),l'_2,\ldots,l'_n) \tag{18}$$
$$= (-1)^{k(n-k)+n-1}[l'_2,\ldots,l'_n,g_{n-k}(p_1,\ldots,p_{n-k},l_1,\ldots,l_k)]_L,$$

for all $2 \leq i,j \leq n-1$ and $1 \leq k \leq n-1$.

Let N be a central ideal in the n-Lie algebra L. Then clearly L and N act trivially on each other. Also, the n-Lie algebras L and L/N act compatibly on each other by $\{f_i\}_{1\leq i\leq n-1}$ and $\{g_i\}_{1\leq i\leq n-1}$, where f_i and g_i are defined as

$$f_i(l_1,\ldots,l_i,\bar{l'}_1,\ldots,\bar{l'}_{n-i}) = [l_1,\ldots,l_i,l'_1,\ldots,l'_{n-i}]+N, \qquad \text{with } \bar{l'}_s = l'+N,$$

and
$$g_i(\bar{l'}_1,\ldots,\bar{l'}_{n-i},l_1,\ldots,l_i) = [l'_1,\ldots,l'_{n-i},l_1,\ldots,l_i],$$

for all $l'_s, l_t \in L$ ($1 \leq s \leq n-i$ and $1 \leq t \leq i$).

4. Preparatory results

In what follows, we prove some properties of nonabelian tensor products of n-Lie algebras.

Proposition 4.1. *Let L and P be two n-Lie algebras such that L and P act on each other in a compatible way by $\{f_i\}_{1\leq i\leq n-1}$ and $\{g_i\}_{1\leq i\leq n-1}$, respectively. Then L and P act on $L\otimes P$.*

Proof. Define the following maps

$$\phi_i : \underbrace{L\times\cdots\times L}_{i-times}\times(L\otimes P)\times\cdots\times(L\otimes P)\longrightarrow L\otimes P$$

$$\phi_i\left(\left(l_1,\ldots,l_i,\sum_{j=1}^{n-1}\left(l_1^{i+1}\otimes\cdots\otimes l_j^{i+1}\otimes p_{j+1}^{i+1}\otimes\cdots\otimes p_n^{i+1}\right),\ldots,\right.\right.$$
$$\left.\left.\sum_{j=1}^{n-1}\left(l_1^{n}\otimes\cdots\otimes l_j^{n}\otimes p_{j+1}^{n}\otimes\cdots\otimes p_n^{n}\right)\right)\right) = \sum_{k=i+1}^{n}\sum_{j=1}^{n} f_i^*(l_1,\ldots,l_i,a_j^k,\ldots,a_j^k)$$

where $a_j^k = l_j^k$ or $a_j^k = p_j^k$ with $f_i^* = [-,\ldots,-]_L$ if $a_j^k = l_j^k$, and $f_i^* = f_i$ if $a_j^k = p_j^k$. Thus $\{\phi_i\}_{1\leq i\leq n-1}$ is the action of L on $L\otimes P$. We can also define the functions

$$\psi_i : \underbrace{P\times\cdots\times P}_{i-times}\times(L\otimes P)\times\cdots\times(L\otimes P)\longrightarrow L\otimes P,$$

(for all $1 \leq i \leq n-1$) in a similar way. ■

The two concepts of nonabelian tensor products and modular n-tensor products of n-Lie algebras as two vector spaces are interrelated, and the following theorem explains the relation between them.

Proposition 4.2. *Let L and P be two n-Lie algebras such that L and P compatibly act on each other by $\{f_i\}_{1\leq i\leq n-1}$ and $\{g_i\}_{1\leq i\leq n-1}$, respectively. Then the following map is an epimorphism:*

$$\psi : L \otimes^n_{\mathrm{mod}} P \longrightarrow L \otimes P$$

$$\sum_{i=1}^{n-1} l^i_1 \otimes \cdots \otimes l^i_i \otimes p^i_{i+1} \otimes \cdots \otimes p^i_n \longmapsto \sum_{i=1}^{n-1} l^i_1 \otimes \cdots \otimes l^i_i \otimes p^i_{i+1} \otimes \cdots \otimes p^i_n.$$

Proof. Let $L\otimes^n_{\mathrm{mod}} P = \dfrac{F^*}{K^*}$, $L\otimes P = \dfrac{F}{K}$, $\alpha \in \mathbb{F}$, and $a, b \in L\otimes^n_{\mathrm{mod}} P$ be arbitrary elements. Clearly,

$$a = \sum_{i=1}^{n-1} l^i_1 \otimes \cdots \otimes l^i_i \otimes p^i_{i+1} \otimes \cdots \otimes p^i_n, \quad \text{and} \quad b = \sum_{i=1}^{n-1} l'^i_1 \otimes \cdots \otimes l'^i_i \otimes p'^i_{i+1} \otimes \cdots \otimes p'^i_n.$$

It is easy to prove that ψ is a well-defined modular homomorphism and surjective, by the definitions and since K has a subset similar to K^* (we can assume that $K^* \subseteq K$). ■

The following proposition proposes the conditions that the nonabelian tensor products and modular n-tensor products of n-Lie algebras as two vector spaces are isomorphic.

Proposition 4.3. *Let L and P be two n-Lie algebras such that they act trivially on each other. Then $L \otimes P \cong L^{ab} \otimes^n_{\mathrm{mod}} P^{ab}$. Moreover, if L and P are disjoint nonabelian, then $L \otimes P$ is of dimension $\binom{a+b}{n} - \binom{a}{n} - \binom{b}{n}$, where $L^{ab} = L/L^2$ and $P^{ab} = P/P^2$, such that $a = \dim L^{ab}$ and $b = \dim P^{ab}$.*

Proof. By Proposition 4.2, We know that the following epimorphism is available:

$$\psi : L \otimes^n_{\mathrm{mod}} P \longrightarrow L \otimes P$$

$$\sum_{i=1}^{n-1} l^i_1 \otimes \cdots \otimes l^i_i \otimes p^i_{i+1} \otimes \cdots \otimes p^i_n \longmapsto \sum_{i=1}^{n-1} l^i_1 \otimes \cdots \otimes l^i_i \otimes p^i_{i+1} \otimes \cdots \otimes p^i_n.$$

Since L and P act trivially on each other, so $L \otimes P$ is an n-Lie algebra with trivial bracket and also, the elements in the both sides of equations (12), (13), and (14) are equal to zero. Hence $K = K^*$ and consequently ψ are one-to-one.

Moreover, if $L^2 \neq 0$ and $P^2 \neq 0$, then by equations (12) and (13), one can check that $l^* \otimes l_2 \otimes \cdots \otimes l_i \otimes p_{i+1} \otimes \cdots \otimes p_n = 0$, and $l_1 \otimes \cdots \otimes l_i \otimes p_{i+1} \otimes \cdots \otimes p_{n-1} \otimes p^* = 0$, where $l^* \in L^2$, $l_j \in L$, $p_r \in P$, and $p^* \in P^2$, for $2 \leq j \leq i$, $i+1 \leq r \leq n$, and $1 \leq i \leq n-1$. Hence we obtain

$$l_1 \otimes \cdots \otimes l_i \otimes p_{i+1} \otimes \cdots \otimes p \otimes \cdots \otimes p \otimes \cdots \otimes p_n = 0,$$

$$l_1 \otimes \cdots \otimes l \otimes \cdots \otimes l \otimes \cdots \otimes l_i \otimes p_{i+1} \otimes \cdots \otimes p_n = 0.$$

Also,

$$l_1 \otimes \cdots \otimes l_i \otimes p_{i+1} \otimes \cdots \otimes p_r \otimes \cdots \otimes p_s \otimes \cdots \otimes p_n = l_1 \otimes \cdots \otimes l_i \otimes p_{i+1} \otimes \cdots \otimes p_s \otimes \cdots \otimes p_r \otimes \cdots \otimes p_n,$$

$$l_1 \otimes \cdots \otimes l_r \otimes \cdots \otimes l_s \otimes \cdots \otimes l_i \otimes p_{i+1} \otimes \cdots \otimes p_n = l_1 \otimes \cdots \otimes l_s \otimes \cdots \otimes l_r \otimes \cdots \otimes l_i \otimes p_{i+1} \otimes \cdots \otimes p_n.$$

Thus $\dim\left(L^{ab} \otimes^n_{\mathrm{mod}} P^{ab}\right) = \binom{a+b}{n} - \binom{a}{n} - \binom{b}{n}$. ■

The following proposition states that although $L\otimes P$ and $P\otimes L$ are different in appearance, they are isomorphic.

Proposition 4.4. *Let L and P be two n-Lie algebras such that L and P act compatibly on each other by $\{f_i\}_{1\leq i\leq n-1}$ and $\{g_i\}_{1\leq i\leq n-1}$, respectively. Then $L\otimes P\cong P\otimes L$.*

Proof. Let $L\otimes P=\dfrac{F}{K}$ and $P\otimes L=\dfrac{F'}{K'}$. Define

$$\psi : L\otimes P \longrightarrow P\otimes L$$
$$\sum_{i=1}^{n-1} l_1^i\otimes\cdots\otimes l_i^i\otimes p_{i+1}^i\otimes\cdots\otimes p_n^i \longmapsto \sum_{i=1}^{n-1}(-1)^{i(n-i)}p_{i+1}^i\otimes\cdots\otimes p_n^i\otimes l_1^i\otimes\cdots\otimes l_i^i.$$

One can define a function $\phi : P\otimes L\longrightarrow L\otimes P$ analogously. It is easy to check that ψ and ϕ are n-Lie homomorphisms and $\psi\phi=I$ satisfying $\phi\psi=I$. ∎

In the following statement, we show how to construct a new operator as a tensor of functions with the help of two n-Lie homomorphisms.

Proposition 4.5. *Let L, P, K, and Q be n-Lie algebras such that L and K act compatibly on each other by $\{f_i\}_{1\leq i\leq n-1}$ and $\{g_i\}_{1\leq i\leq n-1}$, and so they act P and Q by $\{h_i\}_{1\leq i\leq n-1}$ and $\{s_i\}_{1\leq i\leq n-1}$, respectively. Moreover, suppose that $\sigma_1 : L\longrightarrow P$ and $\sigma_2 : K\longrightarrow Q$ are n-Lie homomorphisms preserving the actions in the sense that*

$$\sigma_1(g_i(a_1,\ldots,a_n))=s_i(\sigma_r(a_1),\ldots,\sigma_r(a_n)),$$
$$\sigma_2(f_i(a_1,\ldots,a_n))=h_i(\sigma_r(a_1),\ldots,\sigma_r(a_n)),$$

for all $a_1,\ldots,a_n\in L\cup K$, where $r=1$ if $a_i\in L$, and $r=2$ if $a_i\in K$. Then there is a unique homomorphism (up to sign)

$$\begin{array}{rcl}\sigma_1\otimes\sigma_2 : L\otimes K & \longrightarrow & P\otimes Q\\ \sum_{i=1}^{n-1} l_1^i\otimes\cdots\otimes l_i^i\otimes k_{i+1}^i\otimes\cdots\otimes k_n^i & \longmapsto & \sum_{i=1}^{n-1}\sigma_1(l_1^i)\otimes\cdots\otimes\sigma_1(l_i^i)\otimes\sigma_2(k_{i+1}^i)\otimes\cdots\otimes\sigma_2(k_n^i).\end{array}$$

Proof. Let $P\otimes Q=\dfrac{F'}{K'}$ and $a,b\in L\otimes K=\dfrac{F}{K}$ be arbitrary as follows:

$$a=\sum_{i=1}^{n-1} l_1^i\otimes\cdots\otimes l_i^i\otimes k_{i+1}^i\otimes\cdots\otimes k_n^i,\qquad b=\sum_{i=1}^{n-1} l_1'^i\otimes\cdots\otimes l_i'^i\otimes k_{i+1}'^i\otimes\cdots\otimes k_n'^i.$$

If $a=b$, then $\sum_{i=1}^{n-1}\big((l_1^i,\ldots,l_i^i,k_{i+1}^i,\ldots,k_n^i)-(l_1'^i,\ldots,l_i'^i,k_{i+1}'^i,\ldots,k_n'^i)\big)\in K$. Since σ_1 and σ_2 are well-defined and preserve the actions, we have $\sigma_1\otimes\sigma_2(a)=\sigma_1\otimes\sigma_2(b)$. It is clear that the function $\sigma_1\otimes\sigma_2$ is an n-Lie homomorphism, because σ_1 and σ_2 are homomorphisms, and by Theorem 3.5, $\sigma_1\otimes\sigma_2$ is a unique homomorphism. ∎

Let L and P be two n-Lie algebras such that L acts on P by $\{f_i\}_{1\leq i\leq n-1}$. Then $[L,P]^P$ (or $[P,L]^P$) is the ideal of P generated by all $f_i(l_1,\ldots,l_i,p_{i+1},\ldots,p_n)$ (for all $1\leq i\leq n-1$). It is clear that if L acts trivial on P, then $[L,P]^P=0$.

Lemma 4.6. *Let L_1, L_2, and P be n-Lie algebras such that they satisfy the following conditions:*

1. *L_1 and P act on each other in a compatible way by $\{f_i^1\}_{1\le i\le n-1}$ and $\{g_i^1\}_{1\le i\le n-1}$, and L_2 and P act compatibly on each other by $\{f_i^2\}_{1\le i\le n-1}$ and $\{g_i^2\}_{1\le i\le n-1}$, respectively.*
2. *For all $1\le i\le n-1$ and $j,s=1,2$ we have*
$$f_i^j\left(l_1^j,\dots,l_i^j,f_i^s(l_1^s,\dots,l_i^s,p_{i+1},\dots,p_n),p'_{i+2},\dots,p'_n\right)$$
$$= f_i^s\left(l_1^s,\dots,l_i^s,f_i^j(l_1^j,\dots,l_i^j,p_{i+1},\dots,p_n),p'_{i+2},\dots,p'_n\right),$$
for all $p_{i+1},\dots,p_n,p'_{i+2},\dots,p'_n\in P$ and $l_1^j,\dots,l_i^j\in L$.
3. *The canonical homomorphisms $[L_j,P]^P\otimes L_s \longrightarrow P\otimes L_s$ are trivial, for $j,s=1,2$ $(j\neq s)$.*

Then the n-Lie algebras $L_1\oplus L_2$ and P act on each other in a compatible way by $\{h_i\}_{1\le i\le n-1}$ and $\{h'_i\}_{1\le i\le n-1}$, respectively, in which

$$h_i:(L_1\oplus L_2)^{\times i}\times P^{\times(n-i)}\longrightarrow P$$
$$\left((l_1^1,l_1^2),\dots,(l_i^1,l_i^2),p_{i+1},\dots,p_n\right)\longmapsto f_i^1(l_1^1,\dots,l_i^1,p_{i+1},\dots,p_n) + f_i^2(l_1^2,\dots,l_i^2,p_{i+1},\dots,p_n)$$

and
$$h'_i:P^{\times i}\times(L_1\oplus L_2)^{\times(n-i)}\longrightarrow(L_1\oplus L_2)$$
$$\left(p_1,\dots,p_i,(l_{i+1}^1,l_{i+1}^2),\dots,(l_n^1,l_n^2),\right)\longmapsto\Big(g_i^1(p_1,\dots,p_i,l_{i+1}^1,\dots,l_n^1), g_i^2(p_1,\dots,p_i,l_{i+1}^2,\dots,l_n^2)\Big).$$

Proof. The functions h_i's and h'_i's are well-defined and n-linear since f_i^j's and g_i^j's are well-defined and n-linear maps, for all $1\le i\le n-1$ and $j=1,2$. Also, h_i's and h'_i's satisfy the conditions of compatible actions, because f_i^j's and g_i^j's are compatible. ■

In the following, we prove that the nonabelian tensor product, as an operator, is distributive over the direct sum of two n-Lie algebras.

Lemma 4.7. *By the assumptions of Lemma 4.6, we have the following two homomorphisms of n-Lie algebras*

$$\alpha:(L_1\oplus L_2)\otimes P\longrightarrow(L_1\otimes P)\oplus(L_2\otimes P)$$
$$\alpha\left(\sum_{i=1}^{n-1}(l_{i,1}^1,l_{i,1}^2)\otimes\cdots\otimes(l_{i,i}^1,l_{i,i}^2)\otimes p_{i,i+1}\otimes\cdots\otimes p_{i,n}\right)$$
$$=\left(\sum_{i=1}^{n-1}\left(l_{i,1}^1\otimes\cdots\otimes l_{i,i}^1\otimes p_{i,i+1}\otimes\cdots\otimes p_{i,n}\right),\sum_{i=1}^{n-1}\left(l_{i,1}^2\otimes\cdots\otimes l_{i,i}^2\otimes p_{i,i+1}\otimes\cdots\otimes p_{i,n}\right)\right)$$

and
$$\beta:(L_1\otimes P)\oplus(L_2\otimes P)\longrightarrow(L_1\oplus L_2)\otimes P$$

$$\beta\left(\sum_{i=1}^{n-1}\left(l_{i,1}^1\otimes\cdots\otimes l_{i,i}^1\otimes p_{i,i+1}\otimes\cdots\otimes p_{i,n}\right),\sum_{i=1}^{n-1}\left(l_{i,1}^2\otimes\cdots\otimes l_{i,i}^2\otimes p_{i,i+1}\otimes\cdots\otimes p_{i,n}\right)\right)$$
$$=\sum_{i=1}^{n-1}(l_{i,1}^1,l_{i,1}^2)\otimes\cdots\otimes(l_{i,i}^1,l_{i,i}^2)\otimes p_{i,i+1}\otimes\cdots\otimes p_{i,n}$$

such that $\alpha\beta=I$ *and* $\beta\alpha=I$, *and hence* $(L_1\oplus L_2)\otimes P\cong(L_1\otimes P)\oplus(L_2\otimes P)$.

Proof. The functions α and β are induced by $\{h_i\}_{1\le i\le n-1}$ and $\{h'_i\}_{1\le i\le n-1}$ defined in Lemma 4.6. ■

Using the last two lemmas, we prove the following theorem, which is our main result in this section.

Theorem 4.8 (Main Theorem)**.** *By the assumptions of Lemma 4.6, there is the following isomorphism:*

$$(L\oplus P)\otimes(L\oplus P)\cong(L\otimes L)\oplus(L\otimes P)\oplus(P\otimes L)\oplus(P\otimes P).$$

Proof. It is easily obtained by Lemma 4.7. ■

5. Some results and properties

In this section, we aim to study under which conditions an n-Lie algebra is abelian or has a finite dimension.
The first theorem of this section proves that tensor product of two n-Lie algebras is abelian, when one of them acts trivially on the other.

Theorem 5.1. *Let L and P be two n-Lie algebras such that L and P act on each other in a compatible way by $\{f_i\}_{1\le i\le n-1}$ and $\{g_i\}_{1\le i\le n-1}$, respectively. Then $L\otimes P$ is abelian if at least one of $\{f_i\}_{1\le i\le n-1}$ or $\{g_i\}_{1\le i\le n-1}$ is trivial.*

Proof. Let L acts trivially on P. Then it follows from equation (14) that every element $a=\sum_{i=1}^{n-1}l_1^i\otimes\cdots\otimes l_i^i\otimes k_{i+1}^i\otimes\cdots\otimes k_n^i\in L\otimes P$ belongs to the center of $L\otimes P$ and thus $Z(L\otimes P)=L\otimes P$, that is, $L\otimes P$ is abelian. ■

The following corollary, which is a special case of Theorem 5.1, examines the commutativity condition for nonabelian tensor product of two ideals of an n-Lie algebra.

Corollary 5.2. *Let I and J be two ideals of the n-Lie algebras M and $J\subseteq Z(M)$. Then $I\otimes J$ is abelian.*

Proof. By the assumption, I and J are two ideals of the n-Lie algebras M, and hence they act on each other in a compatible way by brackets. Since J is central, its action on I is trivial. Thus $I\otimes J$ is abelian. ■

Using Corollary 5.2, the following result is straightforward.

Corollary 5.3. *Let L be an abelian n-Lie algebra. Then $L\otimes L$ is abelian.*

Proof. Since L is abelian, we have $Z(L)=L$. Now, according to Corollary 5.2, the result follows. ■

The following theorem guarantees that the dimension of abelian tensor product of two finite-dimensional n-Lie algebras will never be infinite.

Theorem 5.4. *Let L and P be two finite-dimensional n-Lie algebras. Then the dimension of $L \otimes P$ is finite.*

Proof. Let $\dim L = r$, let $\dim P = s$, and let $B_L = \{e_1, \ldots, e_r\}$, and $B_P = \{e'_1, \ldots, e'_s\}$ be the basis for L and P, respectively. Suppose that $L \otimes P = \dfrac{F}{K}$, where F is the free n-Lie algebra over

$$X = \bigcup_{i=1}^{n-1} \left\{ (e_{j_1}, \ldots, e_{j_i}, e'_{j_{i+1}}, \ldots, e'_{j_n}); \quad \begin{matrix} 1 \le j_k \le r, \ 1 \le j_t \le s, \\ 1 \le k \le i, \ i+1 \le t \le n \end{matrix} \right\}.$$

It is known that $\dim L \otimes P \le \dim F = \mathrm{Card}(X)$. Moreover,

$$\mathrm{Card}(X) = \sum_{i=1}^{n-1} \mathrm{Card} \left(\left\{ (e_{j_1}, ..., e_{j_i}, e'_{j_{i+1}}, ..., e'_{j_n}) \right\} \right) = \sum_{i=1}^{n-1} (i \times r)((n-i) \times s) < \infty.$$

Therefore, $L \otimes P$ is finite-dimensional. ■

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