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Jeu de Taquin and Diamond Cone for $\mathfrak{so}(2n+1,\mathbb{C})$

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Abstract. The diamond cone is a combinatorial description for a basis of a natural indecomposable $\mathfrak{n}$-module, where $\mathfrak{n}$ is the nilpotent factor of a complex semisimple Lie algebra $\mathfrak{g}$. After N. J. Wildberger who introduced this notion, this description was achieved for $\mathfrak{g} = \mathfrak{sl}(n)$, the rank 2 semisimple Lie algebras and $\mathfrak{g} = \mathfrak{sp}(2n)$.
In this work, we generalize these constructions to the Lie algebra $\mathfrak{g} = \mathfrak{so}(2n+1)$. The orthogonal semistandard Young tableaux were defined by M. Kashiwara and T. Nakashima, they index a basis for the shape algebra of $\mathfrak{so}(2n+1)$. Defining the notion of orthogonal quasistandard Young tableaux, we prove that these tableaux describe a basis for a quotient of the shape algebra, the reduced shape algebra of $\mathfrak{so}(2n+1)$.

Mathematics Subject Classification: 20G05, 05A15, 17B10.

Key Words: Shape algebra, semistandard Young tableau, quasistandard Young tableau, jeu de taquin.

1. Introduction

Semistandard Young tableaux were introduced since the nineteenth century, for the study of representations of $GL(n)$ and the symmetric group. There are many operations on Young tableaux, one of them is the jeu de taquin (jdt). It was defined by Schützenberger in [13] to associate to each skew semistandard Young tableau a semistandard one, by a successive elimination of empty boxes.

There are various classes of Young tableaux, associated to representations of classical simple Lie algebras. For instance, the symplectic (associated to $\mathfrak{sp}(2n)$) and odd orthogonal tableaux (associated to $\mathfrak{so}(2n+1)$) are defined through their splitting, which associate to them a well defined unique semistandard tableau with 2 times more columns. There are also generalizations of the jeu de taquin to these symplectic and orthogonal tableaux. The symplectic jeu de taquin denoted $sjdt$ was defined by Sheats in [12], it uses the splitting of the tableau and a rebuilding of the columns, step by step. The orthogonal jeu de taquin, $ojdt$, was defined by Lecouvey in [10], by a double application of the symplectic jeu de taquin on the splitting of the skew orthogonal tableau. In [4], a step by step description of this jeu de taquin is presented.

On the other hand, to describe finite dimensional, nilpotent modules of the nilpotent Lie algebra $\mathfrak{t}(n)$ of strictly upper triangular matrices, a commutative algebra was

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defined in [2]. This algebra, the reduced shape algebra, is a quotient of the shape algebra of $\mathfrak{sl}(n)$. Since semistandard Young tableaux are elements of a natural basis of the shape algebra of $\mathfrak{sl}(n)$, a basis for the algebra is given by a selection of some semistandard tableaux, the so-called quasistandard tableaux. A quasistandard tableau is a semistandard tableau, from which it is impossible to extract any 'trivial' tableau by using an always horizontal jdt.

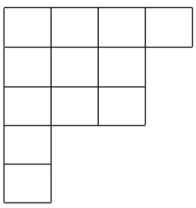
Replacing $\mathfrak{t}(n)$ by the nilpotent factor $\mathfrak{n}$ (the sum of all positive root spaces) in $\mathfrak{sp}(n)$, and jdt by $sjdt$, these results are extended to the symplectic case in [3] where the notion of symplectic reduced shape algebra, quasistandard symplectic tableaux are studied. Our goal in the present article is to generalize these results to the case of the nilpotent factor in $\mathfrak{so}(2n+1)$.

This paper is organized as follows:
In Section 2, we recall the definition of usual and symplectic semistandard tableaux, and the various jeux de taquin. Then, in Section 3, we present orthogonal semistandard tableaux and jeu de taquin ($ojdt$), and its step by step presentation. We use this presentation to prove that the $ojdt$ is horizontal, on the row r, if the split orthogonal tableau is not quasistandard in r. In Section 4, we describe the links between usual, symplectic and orthogonal semistandard one column tableaux and fundamental $\mathfrak{g}$-modules, when $\mathfrak{g} = \mathfrak{sl}(n)$ (resp. $\mathfrak{sp}(2n)$, $\mathfrak{so}(2n+1)$). Then we define the shape algebra of $\mathfrak{g}$ and its natural basis, given by the collection of all semistandard (usual, symplectic or orthogonal) tableaux. In Section 5, we recall classical results on finite dimensional, nilpotent modules of a nilpotent Lie algebra. Then we specialize this to the case of a nilpotent factor $\mathfrak{n}$ of a semisimple Lie algebra $\mathfrak{g}$, showing the role of the reduced shape algebra of $\mathfrak{g}$ to describe these modules. This proves that the family of all (usual, symplectic, and now orthogonal) quasistandard tableaux is a canonical basis for the reduced shape algebra, viewed as $\mathfrak{n}$-module.

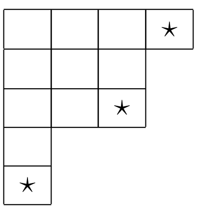
2. Jeu de taquin for semistandard and symplectic tableaux

2.1. Young tableaux

Recall that a *Ferrer diagram* is a collection of columns of empty boxes. The heights of the columns are decreasing from left to right. For instance:



is a Ferrer diagram. A box b in a Ferrer diagram is said a *corner* of the diagram if there is no boxes at the right and under b. In our example, there are:



A *semistandard Young tableau* is the filling of a Ferrer diagram by natural numbers $a_{i,j}$, in such a manner that the entries are strictly increasing along each column, from the top to the bottom, an increasing along each row, from left to right.
From now, we adopt the following convention:

$$\text{If } y \text{ does not exist, then the relation } x < y \text{ holds.} \quad (1)$$

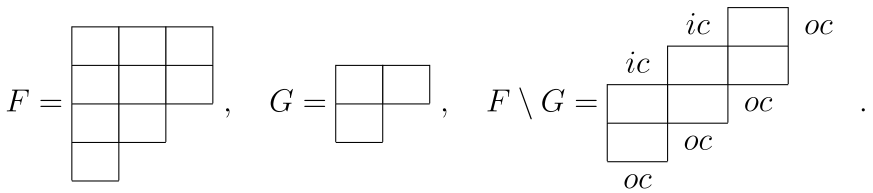
For instance the relation $a_{i,j} < a_{i+1,j}$ for any i and j means: for any i, j, either there is no $a_{i+1,j}$, or the entries $a_{i,j}$ and $a_{i+1,j}$ exist and $a_{i,j} < a_{i+1,j}$. A semistandard tableau is a tableau such that, for any existing $a_{i,j}$ entry, $a_{i,j} < a_{i+1,j}$, $a_{i,j} \leq a_{i,j+1}$ hold. Such a semistandard tableau T is associated to $GL(n)$ if $a_{i,j} \leq n$ for any i and j. Here is an example ($n \geq 6$):

$T =$

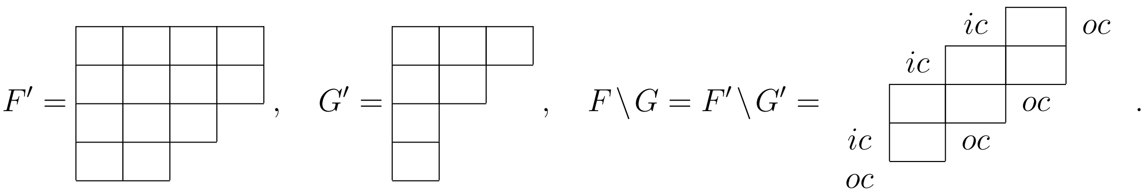
1	2	2	2
2	3	4	
3	5	5	
5			
6			

2.2. Jeu de taquin

Suppose now F and G are two Ferrer diagrams such that for each j, the height of the column j in G is not larger than the height of the column j in F. The *skew Ferrer diagram* $H = F \setminus G$ is the collection of boxes which are in F but not in G. The corners of G are *inner corners* of H, denoted ic. Similarly, the boxes b such that $b \notin F$, $F \cup \{b\}$ is a Ferrer diagram and b is a corner of $F \cup \{b\}$ are outer corners oc of $H = F \setminus G$. Here is an example:



Observe that it can happen that $H = F \setminus G = F' \setminus G'$; the inner (resp. outer) corners of H is the union of all the set of corners of any G (resp. of any b such that $F \cup \{b\}$ is a Ferrer diagram) such that $H = F \setminus G$:



By definition, a *skew semistandard tableau* is the filling of a skew Ferrer diagram with entries $a_{i,j}$ which are natural numbers such that: $a_{i,j} < a_{i+1,j}$, $a_{i,j} \leq a_{i,j+1}$.

For instance, here is a skew semistandard Young tableau:

$T =$

		2	2
	2	4	
3	4	5	
5			
6			

The inner and outer corners of the corresponding Ferrer diagram will be simply the inner and the outer corners of the tableau T. Denote $In(T)$, $Out(T)$ the set of inner (resp. outer) corners of T.

The *Schützenberger jeu de taquin* is now the following operation: Consider a skew semistandard tableau T, and $ic \in In(T)$.

(1) Add a starred box $\boxed{\star}$ at ic, getting the filling $T \cup \boxed{\star}$ of a Ferrer diagram F (start of jdt), after some steps, the star is in a box b of F, in the column j and the row i.

(2) If $a_{i,j+1} < a_{i+1,j}$: exchange the star and the entry $a_{i,j+1}$ (horizontal sliding), for instance:

$$\begin{array}{|c|c|}\hline \star & 2 \\ \hline 3 & 4 \\ \hline \end{array} \mapsto \begin{array}{|c|c|}\hline 2 & \star \\ \hline 3 & 4 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \star & 3 \\ \hline \end{array} \mapsto \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \star \\ \hline \end{array}$$

(3) If $a_{i+1,j} \leq a_{i,j+1}$: exchange the star and the entry $a_{i+1,j}$ (vertical sliding), for instance:

$$\begin{array}{|c|c|}\hline \star & 3 \\ \hline 2 & 4 \\ \hline \end{array} \mapsto \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \star & 4 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|}\hline 1 & \star \\ \hline 2 & 3 \\ \hline \end{array} \mapsto \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \star \\ \hline \end{array}$$

(4) If $a_{i+1,j}$ and $a_{i,j+1}$ do not exist, the star is in a corner c of F, suppress the box in c, getting the semistandard filling T', with a star in an outer corner oc of $F \setminus b$ (end of jdt).

By definition the jeu de taquin is the map: $jdt : (T, ic) \to (T', oc)$.

Here is an example:

$$\begin{array}{cccc} & & 2 & 2 \\ \star & 2 & 4 & \\ 3 & 4 & 5 & \\ 5 & & & \\ 6 & & & \end{array} \mapsto \begin{array}{cccc} & & 2 & 2 \\ \star & 2 & 4 & \\ 3 & 4 & 5 & \\ 5 & & & \\ 6 & & & \end{array} \mapsto \begin{array}{cccc} & & 2 & 2 \\ 2 & \star & 4 & \\ 3 & 4 & 5 & \\ 5 & & & \\ 6 & & & \end{array} \mapsto \begin{array}{cccc} & & 2 & 2 \\ 2 & 4 & 4 & \\ 3 & \star & 5 & \\ 5 & & & \\ 6 & & & \end{array} \mapsto \begin{array}{cccc} & & 2 & 2 \\ 2 & 4 & 4 & \\ 3 & 5 & \star & \\ 5 & & & \\ 6 & & & \end{array}$$

Repeating this operation to T', it is possible to suppress an other inner corner, and so on, at the end the total jeu de taquin transforms the skew semistandard tableau T into a semistandard tableau $totaljdt(T)$.

The following properties hold for jdt (see [13, 11, 5]):

Proposition 2.1. *jdt is a bijective mapping from the set of pairs (T, ic) consisting by a skew semistandard tableau and one of its inner corner to the set of pairs (T', oc) consisting of a skew semistandard tableau and one of its outer corner.*
The inverse map can be defined by performing a rotation of $180°$ of the tableau T and if $\sup(a_{i,j}) \leq n$, by replacing each entry $a_{i,j}$ by $n+1-a_{i,j}$ and $\star$ by $\star$. Let σ be this operation, then:

$$jdt^{-1} = \sigma \circ jdt \circ \sigma.$$

The map $totaljdt$ does not depend on the choice of the successive punctured inner corners.

Denote by $A_j = \{a_1, \dots, a_h\}$ the set of the entries in the column j of a semistandard tableau, and $\mathcal{C}_j = f\,[A_j]$ the column itself. In the case of a skew tableau, filling of the skew diagram $F \setminus G$, we index the set A_j by the number of the row in the skew tableau, writing $A_j = \{a_\ell, a_{\ell+1}, \dots a_h\}$ if the height of the j column in G is $\ell - 1$. We shall write $\mathcal{C}_j^{\star_r}$ or $f\,[A_j \cup \{\star_r\}]$ to say that we are considering an instant of a jdt where the star is in the column j, on the row r.
With this notation, a vertical sliding is simply:

$$f\,[A_j \cup \{\star_r\}] \;\mapsto\; f\,[A_j \cup \{\star_{r+1}\}]\,.$$

And the horizontal sliding can be written:

$$f\,[A_j \cup \{\star_r\}]\, f\,[A_{j+1}] \;\mapsto\; f\,[A_j \cup \{u_r\}]\, f\,[(A_{j+1} \setminus \{u_r\}) \cup \{\star_r\}]\,,$$

if u_r is an element of A_{j+1} with index r.

2.3. Symplectic columns and tableaux

Symplectic semistandard tableaux are used in different versions by various authors. Here, we use the presentation of [10], in the form of [3].

Fix a natural number n. We first define the ordered set:

$$\mathcal{I} = \left\{1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1}\right\}.$$

The entries of each column in a symplectic tableau are elements of $\mathcal{I}$. In each column, the entries increase strictly from the top to the bottom.

Fix now $Y \subset [1,n] = \{1, \dots, n\}$ and say that a subset $J = \{y_1 < \cdots < y_s\}$ *admits left subsets* in Y^c if there is subsets $X = \{a_1 < \cdots < a_s\}$ such that:

$$\#X = \#J, \quad X \cap Y = \emptyset, \quad \text{ and } a_j \leq y_j \;\; (1 \leq j \leq s).$$

If J admits left subsets in Y^c, then there is a unique greatest such subset, namely $I = \{x_1, \dots, x_s\}$ defined by:

$$x_s = \max\{t \notin Y:\; t \leq y_s\}, \quad x_j = \max\{t \notin Y:\; t \leq y_j,\; t < x_{j+1}\}, \quad (s > j \geq 1).$$

If $X = \{a_1, \dots, a_s\}$ is any left subset for J in Y^c, then $a_j \leq x_j$ for any j. We shall write $I = \gamma_Y(J)$.

Similarly, if $I = \{x_1, \dots, x_s\}$, we say that I *admits right subsets* in Y^c if there is subsets $X = \{a_1, \dots, a_s\}$ such that:

$$\#X = \#I, \quad X \cap Y = \emptyset, \quad \text{ and } a_j \geq x_j \;\; (1 \leq j \leq s).$$

If I admits right subsets in Y^c, then there is a unique smallest such subset, namely $J = \{y_1, \dots, y_s\}$ defined by:

$$y_1 = \min\{t \notin Y:\; t \geq x_1\}, \quad y_j = \min\{t \notin Y:\; t \geq x_j,\; t > y_{j-1}\}, \quad (1 < j \leq s).$$

If $X = \{a_1, \dots, a_s\}$ is any right subset for I in Y, then $a_j \geq y_j$ for any j. We shall use the notation $J = \delta_Y(I)$.

Remark 2.1. In [1] the definition of γ_Y, δ_Y was slightly different. We used the maps:

$$\gamma'_Y(J) = \gamma_{Y\cup J}(J), \qquad \delta'_Y(I) = \delta_{Y\cup I}(I).$$

It is easy to prove that:
$$\begin{cases} \gamma'_Y(\delta'_Y(I)) = \gamma_{Y\cup\delta_{Y\cup I}(I)}(\delta_{Y\cup I}(I)) = I, \\ \delta'_Y(\gamma'_Y(J)) = \delta_{Y\cup\gamma_{Y\cup J}(J)}(\gamma_{Y\cup J}(J)) = J. \end{cases}$$

Let $\mathcal{C}$ be the column $\mathcal{C} = \frac{A}{D}$ with entries in $\mathcal{I}$ strictly increasing from the top to the bottom, where A and D are two subsets of $[1,n]$. Define

$$I = A \cap D = \{x_1 < \cdots < x_s\}.$$

Put $A\triangle D = (A\cup D)\setminus(A\cap D)$ and $A\wedge D = [1,n]\setminus(A\triangle D)$. The column $\mathcal{C}$ is a *symplectic column* if I admits right subsets in $(A\cup D)^c$. Put then

$$J = \delta_{A\cup D}(I).$$

(Remark that this implies that the height of the column is at most n).
For later use, let us also put, for any $x \in [1,n]$:

$$\gamma_Y(x) = \max\{t \notin Y:\ t \le x\}, \qquad (\text{resp. } \delta_Y(x) = \min\{t \notin Y:\ t \ge x\}\ ).$$

Example 2.2. Suppose $n = 4$. Then the column:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \overline{3} \\ \hline \overline{2} \\ \hline \end{array}$$

is symplectic. Indeed $A = \{1,2\}$, $D = \{2,3\}$, $I = \{2\}$, $J = \{4\}$.

If $\mathcal{C} = \frac{A}{D}$ is a symplectic column, the *splitting of* $\mathcal{C}$ is the 2-columns tableau:

$$spl(\mathcal{C}) = \frac{A}{C}\ \frac{B}{D},$$

where, with our notation: $\quad B = (A\setminus I)\cup J, \qquad C = (D\setminus I)\cup J.$
Notice that this tableau is semistandard.

Example 2.3. Suppose $n = 4$, then:

$$spl\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \overline{3} \\ \hline \overline{2} \\ \hline \end{array}\right) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 4 \\ \hline \overline{4} & \overline{3} \\ \hline \overline{3} & \overline{2} \\ \hline \end{array}.$$

Now, if the sets B and C are known, it is possible to rebuild A and D. Indeed, consider $J = B\cap C$, and put $I = \gamma_{B\cup C}(J)$. Then $A = (B\setminus J)\cup I$, $D = (C\setminus J)\cup I$.

Denote also:
$$\mathcal{C} = f\begin{bmatrix} A \\ D \end{bmatrix} = g\begin{bmatrix} B \\ C \end{bmatrix}.$$

Now the splitting of a Young tableau T whose each column $\mathcal{C}_j$ $(j = 1, \dots, q)$ is symplectic is the tableau:

$$spl(T) = spl(\mathcal{C}_1)spl(\mathcal{C}_2)\dots spl(\mathcal{C}_q) = \frac{A_1}{C_1}\ \frac{B_1}{D_1}\ \begin{matrix}\dots\\ \dots\end{matrix}\ \frac{A_q}{C_q}\ \frac{B_q}{D_q}.$$

Definition 2.4. A symplectic tableau is a Young tableau T with entries in $\mathcal{I}$, such that each column of T is symplectic and the tableau $spl(T)$ is semistandard.

2.4. Symplectic jeu de taquin

In [12], J. T. Sheats defined the symplectic jeu de taquin $(sjdt)$ on skew symplectic tableaux: *i.e.* skew tableaux T whose each column is symplectic and such that $spl(T)$ is skew semistandard.

With our presentation, the $sjdt$ is defined as follows: T is a skew symplectic tableau, pick $ic \in In(T)$.

(1) Add a starred box $\boxed{\star}$ at ic, getting the filling $T \cup \boxed{\star}$ of a Ferrer diagram F (start of $sjdt$),

after some steps, the tableau becomes $T_1^\star$, the star is in a box b of F, in the column j and the row i,

(2) Split $T_1^\star$, *i.e.* split each column of T_1 and insert two stars in $(i, 2j-1)$, $(i, 2j)$ in $spl(T_1)$, getting a skew semistandard tableau, with two stars on the same line (splitting),

(3) If, in the usual jdt for $spl(T_1^\star)$, the right star is sliding vertically, then slide vertically the star in $T_1^\star$ (vertical sliding),

(4) If, in the usual jdt for $spl(T_1^\star)$, the right star is sliding horizontally, then perform a 'colored' horizontal sliding in $T_1^\star$ as follows:

if in $spl(T^\star)$ there is $\boxed{\star}\boxed{\star}\boxed{a}$, then the sliding is unbarred, it is:

$$g\begin{bmatrix}B_j\\C_j\end{bmatrix} \cup \{\star_r\} \mapsto g\begin{bmatrix}B_j \cup \{a\}\\C_j\end{bmatrix},\ f\begin{bmatrix}A_{j+1}\\D_{j+1}\end{bmatrix} \mapsto f\begin{bmatrix}A_{j+1}\setminus\{a\}\\D_{j+1}\end{bmatrix} \cup \{\star_r\}\quad (u\text{-sliding}),$$

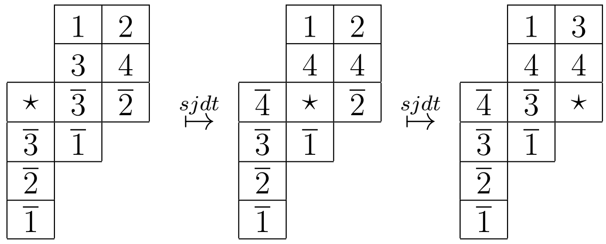
if in $spl(T^\star)$ there is $\boxed{\star}\boxed{\star}\boxed{\overline{c}}$, then the sliding is barred, it is:

$$f\begin{bmatrix}A_j\\D_j\end{bmatrix} \mapsto f\begin{bmatrix}A_j\\D_j \cup \{c\}\end{bmatrix},\quad g\begin{bmatrix}B_{j+1}\\C_{j+1}\end{bmatrix} \mapsto g\begin{bmatrix}B_{j+1}\\C_{j+1} \setminus \{c\}\end{bmatrix} \cup \{\star_r\}\quad (b\text{-sliding}).$$

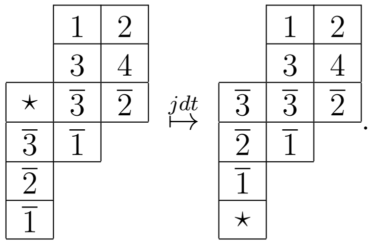
(5) If the star is in a corner c of F, suppress the box in c, getting the symplectic filling T' of $F \setminus b$, with a star in an outer corner (end of $sjdt$).

It is clear that the symplectic jeu de taquin differs completely from the usual jeu de taquin. Here is an example.

Example 2.5. Suppose $n = 4$. Then:



On the other hand,



In [1] (see also [4]), it is proved:

Lemma 2.2. *In the sjdt, the resulting columns are symplectic, more precisely: suppose that the star is in the column j, at the row r:*

$$\mathcal{C}_j^\star = f\begin{bmatrix}A\\D\end{bmatrix} \cup \{\star_r\} = g\begin{bmatrix}B\\C\end{bmatrix} \cup \{\star_r\},$$

and the sliding is horizontal. Let a (resp. $\overline{c}$) be the element in the row r and column $j+1$. Then after the sliding, the column j becomes:

$$g\begin{bmatrix}B\cup\{a\}\\C\end{bmatrix} = f\begin{bmatrix}A\cup\{\gamma_{A\cup C}(a)\}\\(D\cup\{\gamma_{A\cup C}(a)\})\backslash\{a\}\end{bmatrix}, \quad \textit{resp. } f\begin{bmatrix}A\\D\cup\{c\}\end{bmatrix} = g\begin{bmatrix}(B\cup\{\delta_{A\cup C}(c)\})\backslash\{c\}\\C\cup\{\delta_{A\cup C}(c)\}\end{bmatrix}.$$

Finally, Proposition 2.1 holds for the *sjdt*: the inverse of *sjdt* is:

$$sjdt^{-1} = \sigma \circ sjdt \circ \sigma,$$

where σ is the $180°$ rotation of the tableau $T^\star$, and the transform of each entry is as follows:

$$x \mapsto \overline{x}, \qquad \overline{x} \mapsto x, \qquad \text{and} \qquad \star \mapsto \star.$$

Moreover, applying repetitively the *sjdt* on a skew symplectic tableau T, we get a symplectic tableau $totalsjdt(T)$. This tableau does not depend on the choice of the successive used inner corners (see [9]).

3. Tableaux and jeu de taquin for odd orthogonal Lie algebra

3.1. Admissible and orthogonal columns

The definition of admissible column given in this section is equivalent but not identical to the definition given by Cedric Lecouvey in [10].

With the ordering $1 < 2 < \cdots < n < 0 < \overline{n} < \cdots < \overline{1}$ of its entries, a column is said to be admissible if it satisfies the following properties:

(1) The entries are increasing from the top to the bottom and if an entry is not 0, it appears at most one time.

(2) Let $\mathcal{C} = \begin{matrix} A \\ O \\ \overline{D} \end{matrix}$ be such a column. In O all the entries are 0 and A, D are subsets of $[1,n]$, and $\frac{A}{D}$ is a symplectic column.

(3) Put $\frac{A}{D} = f\begin{bmatrix} A \\ D \end{bmatrix} = g\begin{bmatrix} B \\ C \end{bmatrix}$, then $\#(A\cup C) + \#O \leq n$.

Put $k = \#O$, and as above $I = A\cap D$, $J = \delta_{A\cup D}(I)$, $B = (A\setminus I)\cup J$, $C = (D\setminus I)\cup J$ there exist subsets in $[1,n]\setminus(A\cup D\cup J) = A\cup C$ having k elements. Denote by K the greatest of these subsets: explicitly $K = \{z_1,\dots,z_k\}$ with

$$z_k = \max\{t\notin A\cup C:\ t\leq n\},\quad z_j = \max\{t\notin A\cup C:\ t< z_{j+1}\},\quad (k>j\geq 1).$$

Let us denote such a column by: $\mathcal{C} = \begin{matrix} A \\ O \\ \overline{D} \end{matrix} = f\begin{bmatrix} A \\ O \\ D \end{bmatrix} = g\begin{bmatrix} B \\ O \\ C \end{bmatrix}$.

In addition to the admissible columns we have the spin columns denoted:

$$\mathfrak{C} = \frac{A}{D}_{sp} = \mathfrak{f}\begin{bmatrix} A \\ D \end{bmatrix},$$

where $\#A + \#D = n$, $A\cap D = \emptyset$ and the entries increase strictly.

Definition 3.1. A column is *orthogonal* if it is an admissible column or a spin column.

The splitting of an orthogonal column is the two columns tableau:

$$spl(\mathcal{C}) = spl\begin{pmatrix} A \\ O \\ \overline{D} \end{pmatrix} = \begin{matrix} A & B \\ K & \overline{K}, \\ \overline{C} & \overline{D} \end{matrix}\quad spl(\mathfrak{C}) = spl\begin{pmatrix} \frac{A}{D}_{sp} \end{pmatrix} = \begin{matrix} 1 & A \\ \vdots & \\ n & \overline{D}_{sp} \end{matrix},$$

where it is understood that $A\cup K$ and $D\cup K$ are reordered to be written in a strictly increasing way. Clearly the splitting of a spin column characterizes this column. Moreover, as in the symplectic case, the splitting of an admissible column $\mathcal{C}$ characterizes also $\mathcal{C}$. Indeed if this splitting is $\begin{matrix} E & B \\ \overline{C} & \overline{F} \end{matrix}$, then A and D are defined by $f\begin{bmatrix} A \\ D \end{bmatrix} = g\begin{bmatrix} B \\ C \end{bmatrix}$ and the list O has $k = \#K = \#(E\cap F)$ elements.

3.2. Relation with Lecouvey admissibility

Recall that the admissible columns in the sense of Lecouvey (the Lecouvey-admissible columns) are those such that:

(1) The entries are increasing from the top to the bottom and if an entry t is not 0, then it appears at most one time,

(2) Let $\mathcal{C}_L$ such a column. We denote it by $\mathcal{C}_L = \begin{matrix} B \\ O \\ \overline{C} \end{matrix}$. In O all entries are 0 and there is no zero in B and $\overline{C}$.

(3) Let $k = \#O$, and $J^1 = \{y^1_1, \dots, y^1_{k+s}\}$ be the set $(B \cap C) \cup [n+1, n+k]$, define $n^1 = n + k$ and $(Y^1)^c = [1, n+k] \setminus (B \cup C)$. Then J^1 admits left subsets in $(Y^1)^c$.

Put now $I^1 = \gamma_{Y^1}(J^1)$, $E = (B \setminus J^1) \cup I^1$, $F = (C \setminus J^1) \cup I^1$ and define the Lecouvey split of the column $\mathcal{C}_L$ as:

$$spl_L(\mathcal{C}_L) = \begin{matrix} E & B \\ \overline{C} & \overline{F} \end{matrix}.$$

Consider a Lecouvey-admissible column $\mathcal{C}_L = \begin{matrix} B \\ O \\ \overline{C} \end{matrix}$. Let $J = B \cap C = \{y_1, \dots, y_s\}$.

Recall $J^1 = \{y^1_1, \dots, y^1_{k+s}\} = (B \cap C) \cup [n+1, n+k]$.

Put $Y = B \cup C$, by definition, $J = \{y^1_1, \dots, y^1_s\} \subset J^1$. Therefore J admits left subsets in Y^c, for instance $\{x^1_1, \dots, x^1_s\}$. Let $I = \{x_1, \dots, x_s\} = \gamma_Y(J)$. Similarly, $\{x^1_1, \dots, x^1_s\}$ is a left subset for J in $I^1 \subset [1, n]$. Put

$$I' = \{x'_1, \dots, x'_s\} = \gamma_{(I^1)^c}(J).$$

Lemma 3.2. *The subsets I and I' do coincide: $x'_j = x_j$ for any j.*

Proof. Let $j \in [1, s]$. Suppose that for any $i > j$, $x'_i = x_i$. Since x'_j is such that $x'_j \in Y^c$, $x'_j \notin J$, $x'_j < x_{j+1}$ and $x'_j \le y_j$, then $x'_j \le x_j$.
Suppose now $x'_j < x_j$, then $x_j \notin I^1$. Indeed, if x_j is in I^1, then

$$\{x'_1, \dots, x'_{j-1}, x_j, x'_{j+1}, \dots, x'_s\}$$

would be a subset of I^1, which is on the left of J. This would imply $x_j \le x'_j$. Since x'_j, x'_{j+1} are in I^1, there is $i_\ell \ge j$ such that $x'_j \le x^1_{i_\ell} < x_j < x^1_{i_\ell+1} \le x'_{j+1}$. Since x_j is in $(Y^1)^c \setminus J^1$, the subset

$$\{x^1_1, \dots, x^1_{i_\ell-1}, x_j, x^1_{i_\ell+1}, \dots, x^1_{s+k}\} \subset Y^1$$

is greater than I^1, thus not in the left of J^1. This implies $x_j \ge y^1_{i_\ell} \ge y_j$. This is impossible. This proves that $x'_j = x_j$ and $I' = I$. ■

Let us now prove the equivalence between the two notions of admissible column.

Proposition 3.1. *The map $\mathcal{C} = \begin{matrix} A \\ O \\ \overline{D} \end{matrix} \mapsto \begin{matrix} B \\ O \\ \overline{C} \end{matrix} = \mathcal{C}_L$ is a bijection from the set of admissible columns to the set of Lecouvey admissible columns. Moreover this map preserves the splitting: $spl(\mathcal{C}) = spl_L(\mathcal{C}_L)$.*

Proof. Define now $K = I^1 \setminus I$. Remark that if $I^1 = \{x^1_1 < \cdots < x^1_{s+k}\}$, and $K = \{z_1 < \cdots < z_k\}$, we do not necessarily have $x_i = x^1_i$ and $z_j = x^1_{s+j}$ for any i, j.

For instance, if $n=3$, the following column is Lecouvey-admissible:

$$\mathcal{C}_L = \begin{array}{|c|}\hline 3 \\ \hline 0 \\ \hline \overline{3} \\ \hline \end{array}.$$

Indeed, we have here $n=3$, $k=1$, $B=C=\{3\}$, $J=\{3\}$, $J^1=\{3,4\}$, $I^1=\{1,2\}=\{x_1^1,x_2^1\}$, and $I^1\cup J^1=\{1,2,3,4\}$. Then $I=\{2\}=\{x_1\}$ and $K=\{1\}=\{z_1\}$, or $x_1=x_2^1$ and $z_1=x_1^1\neq x_{1+1}^1$.
Put now $A=E\setminus K$, $D=F\setminus K$. We have $A=(B\setminus J)\cup I$, $D=(C\setminus J)\cup I$, and:

$$spl_L(\mathcal{C}_L) = \begin{array}{cc} A & B \\ K & \overline{K} \\ \overline{C} & \overline{D} \end{array}.$$

On the other hand, the column $\mathcal{C} = \begin{array}{c} A \\ O \\ \overline{D} \end{array}$ is orthogonal. In fact its splitting is the Lecouvey split of $\mathcal{C}_L$:

$$spl(\mathcal{C}) = spl_L(\mathcal{C}_L).$$

The preceding construction defines a map Φ from the set of Lecouvey-admissible columns to the set of admissible columns. Conversely, if

$$\mathcal{C} = f\begin{bmatrix} A \\ O \\ D \end{bmatrix} = g\begin{bmatrix} B \\ O \\ C \end{bmatrix} \text{ is admissible, then } \mathcal{C}_L = \Psi(\mathcal{C}) = \begin{array}{c} B \\ O \\ \overline{C} \end{array}$$

is Lecouvey-admissible. By construction the mapping Φ is the inverse of Ψ. This gives an equivalence between the two notions of admissibility. Moreover, the splitting of the corresponding admissible columns coincide. ■

3.3. Admissible tableaux and jeu de taquin

Definition 3.3. Let T be a Young tableau, whose each column is orthogonal and only the first can be a spin column. Denote by $spl(T)$ the Young tableau obtained by splitting each column of T and deleting the eventual subscript sp. We say that T is an *orthogonal tableau* if $spl(T)$ is semistandard.
If T is orthogonal without any spin column, we shall say that T is an *admissible tableau*.

Especially, each row in an orthogonal tableau T contains at most one 0.

Example 3.4. For instance, if $n=5$, the following tableau is admissible:

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 0 \\ \hline 2 & 2 & 2 & 0 \\ \hline 3 & 5 & \overline{3} & \overline{3} \\ \hline 4 & 0 & \overline{2} \\ \cline{1-3} \overline{4} & \overline{3} & \overline{1} \\ \cline{1-3} \end{array}$$

Indeed each column is admissible and its splitting is:

$spl(T) =$

1	1	1	1	1	4	4	$\overline{5}$
2	2	2	2	2	5	5	$\overline{4}$
3	3	4	5	$\overline{5}$	$\overline{3}$	$\overline{3}$	$\overline{3}$
4	5	5	$\overline{4}$	$\overline{4}$	$\overline{2}$		
$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{1}$		

The (odd) *orthogonal jeu de taquin*, denoted $ojdt$ was defined by C. Lecouvey in [10] for admissible tableaux. As for the $sjdt$, its definition uses the splitting of tableaux. As above, a skew tableau T is admissible if each column of T is admissible and $spl(T)$ is skew semistandard.

With our presentation, the $ojdt$ is defined as follows:

(1) Start with a skew admissible tableau T, put a star in an inner corner $ic = (i, j)$ of T, getting a punctured tableau $T^\star$.

(2) Split $T^\star$, that means split each column of T and put two stars in the position $(i, 2j-1)$, $(i, 2j)$ in $spl(T)$. We get a skew symplectic tableau $spl(T^\star) = spl(T)^{\star^2,\star^1}$, with two stars on the same row, the right one $\star^1$ in an inner corner of $spl(T)$.

(3) Perform the $sjdt$ for $\star^1$, until an outer corner, getting a symplectic tableau with one star $sjdt(spl(T)^{\star^2,\cdot}) = sjdt(spl(T))^{\star^2}$. Now the star $\star^2$ is in an inner corner of $sjdt(spl(T))$.

(4) Perform the $sjdt$ for $\star^2$ and we get a skew tableau

$$sjdt(sjdt(spl(T))^{\star^2}) = sjdt^2(spl(T)).$$

This tableau is the splitting of an admissible tableau, denoted $ojdt(T)$.
By definition, $sjdt^2(spl(T)) = spl(ojdt(T))$.

Example 3.5. Put $n = 5$ and consider the punctured orthogonal tableau:

$T^\star =$

	1	2	2
$\star$	2	3	3
3	3	4	4
4	4	5	0
5	5	0	$\overline{1}$

, $spl(T^\star) =$

		1	1	1	2	2	2
$\star^2$	$\star^1$	2	2	2	3	3	3
3	3	3	3	3	4	4	4
4	4	4	4	4	5	5	$\overline{5}$
5	5	5	5	5	$\overline{1}$	$\overline{1}$	$\overline{1}$

Then the $sjdt$ for $\star^1$ on $spl(T)$ is the following:

		1	1	1	2	2	2
$\star^2$	$\star^1$	2	2	2	3	3	3
3	3	3	3	3	4	4	4
4	4	4	4	4	5	5	$\overline{5}$
5	5	5	5	5	$\overline{1}$	$\overline{1}$	$\overline{1}$

$\mapsto$

		1	1	1	2	2	2
$\star^2$	2	$\star^1$	2	2	3	3	3
3	3	3	3	3	4	4	4
4	4	4	4	4	5	5	$\overline{5}$
5	5	5	5	5	$\overline{1}$	$\overline{1}$	$\overline{1}$

$\mapsto$

		1	1	1	2	2	2
$\star^2$	2	2	$\star^1$	2	3	3	3
3	3	3	3	3	4	4	4
4	4	4	4	4	5	5	$\overline{5}$
5	5	5	5	5	$\overline{1}$	$\overline{1}$	$\overline{1}$

$\mapsto$

		1	1	1	2	2	2
$\star^2$	2	2	2	$\star^1$	3	3	3
3	3	3	3	3	4	4	4
4	4	4	4	4	5	5	$\overline{5}$
5	5	5	5	5	$\overline{1}$	$\overline{1}$	$\overline{1}$

$\mapsto$

		1	1	1	2	2	2
$\star^2$	2	2	2	3	3	3	3
3	3	3	3	$\star^1$	4	4	4
4	4	4	4	4	5	5	$\overline{5}$
5	5	5	5	5	$\overline{1}$	$\overline{1}$	$\overline{1}$

$\mapsto$

		1	1	1	2	2	2
$\star^2$	2	2	2	3	3	3	3
3	3	3	3	4	4	4	4
4	4	4	4	$\star^1$	5	5	$\overline{5}$
5	5	5	5	5	$\overline{1}$	$\overline{1}$	$\overline{1}$

$\mapsto$

		1	1	1	2	2	2
$\star^2$	2	2	2	3	3	3	3
3	3	3	3	4	4	4	4
4	4	4	4	5	5	5	$\overline{5}$
5	5	5	5	$\star^1$	$\overline{1}$	$\overline{1}$	$\overline{1}$

$\mapsto$

		1	1	1	2	2	2
$\star^2$	2	2	2	3	3	3	3
3	3	3	3	4	4	4	4
4	4	4	4	5	5	5	$\overline{5}$
5	5	5	5	$\overline{1}$	$\star^1$	$\overline{1}$	$\overline{1}$

$\mapsto$

		1	1	1	2	2	2
$\star^2$	2	2	2	3	3	3	3
3	3	3	3	4	4	4	4
4	4	4	4	5	5	5	$\overline{5}$
5	5	5	5	$\overline{1}$	$\overline{1}$	$\star^1$	$\overline{1}$

$\mapsto$

		1	1	1	2	2	2
$\star^2$	2	2	2	3	3	3	3
3	3	3	3	4	4	4	4
4	4	4	4	5	5	5	$\overline{5}$
5	5	5	5	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\star^1$

$= sjdt(spl(T))^{\star^2}$.

Computing now the $sjdt$ for $\star^2$, we quickly get:

$\mapsto \cdots \mapsto$

		1	1	1	2	2	2
2	2	2	$\star^2$	3	3	3	3
3	3	3	3	4	4	4	4
4	4	4	4	5	5	5	$\overline{5}$
5	5	5	5	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\star^1$

$\mapsto \cdots \mapsto$

		1	1	1	2	2	2
2	2	2	3	3	3	3	3
3	3	3	4	4	4	4	4
4	4	4	5	5	5	5	$\overline{5}$
5	5	5	$\star^2$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\star^1$

$\mapsto$

		1	1	2	2	2	2
2	2	2	3	3	3	3	3
3	3	3	4	4	4	4	4
4	4	4	5	5	5	5	$\overline{5}$
5	5	5	$\overline{2}$	$\star^2$	$\overline{1}$	$\overline{1}$	$\star^1$

$\mapsto \cdots \mapsto$

		1	1	2	2	2	2
2	2	2	3	3	3	3	3
3	3	3	4	4	4	4	4
4	4	4	5	5	5	5	$\overline{5}$
5	5	5	$\overline{2}$	$\overline{1}$	$\overline{1}$	$\star^2$	$\star^1$

$= spl\Bigg($

	1	2	2
2	3	3	3
3	4	4	4
4	5	5	0
5	0	$\overline{1}$	$\star$

$\Bigg)$.

Therefore: $ojdt(T^\star) =$

	1	2	2
2	3	3	3
3	4	4	4
4	5	5	0
5	0	$\overline{1}$	

.

As for the usual and the symplectic jeu de taquin, there exists a direct presentation of the orthogonal jeu de taquin, as a succession of steps, of the form:

$$\mathcal{C}_j \cup \{\star_r\}\mathcal{C}_{j+1} \mapsto \mathcal{C}_j \cup \{\star_{r+1}\}\mathcal{C}_{j+1}, \text{ or } \mathcal{C}_j \cup \{\star_r\}\mathcal{C}_{j+1} \mapsto \mathcal{C}_j \cup \{x\}(\mathcal{C}_{j+1} \setminus \{x\}) \cup \{\star_{r'}\}.$$

Notice that it is however impossible to use only horizontal and vertical sliding. It happens that the star is going simultaneously to the right and the bottom. Here is an example:

Example 3.6. Fix $n = 7$ and consider:

$$
\begin{array}{|c|c|}
\multicolumn{1}{c|}{\star} & 2 \\ \hline
2 & 3 \\ \hline
3 & 4 \\ \hline
4 & 5 \\ \hline
5 & 0 \\ \hline
\overline{7} & \overline{7} \\ \hline
\overline{6} & \overline{6} \\ \hline
\end{array}
\overset{spl}{\to}
\begin{array}{|c|c|c|c|}
\multicolumn{1}{c}{\star^2} & \multicolumn{1}{c|}{\star^1} & 1 & 2 \\ \hline
2 & 2 & 2 & 3 \\ \hline
3 & 3 & 3 & 4 \\ \hline
4 & 4 & 4 & 5 \\ \hline
5 & 5 & 5 & \overline{7} \\ \hline
\overline{7} & \overline{7} & \overline{7} & \overline{6} \\ \hline
\overline{6} & \overline{6} & \overline{6} & \overline{1} \\ \hline
\end{array}
\mapsto
\begin{array}{|c|c|c|c|}
\multicolumn{1}{c|}{\star^2} & 1 & 2 & 2 \\ \hline
2 & 2 & 3 & 3 \\ \hline
3 & 3 & 4 & 4 \\ \hline
4 & 4 & 5 & 5 \\ \hline
5 & 5 & \overline{7} & \overline{7} \\ \hline
\overline{7} & \overline{7} & \overline{6} & \overline{6} \\ \hline
\overline{6} & \overline{6} & \star^1 & \overline{1} \\ \hline
\end{array}
\mapsto
\begin{array}{|c|c|c|c|}
\multicolumn{1}{c|}{\star_2} & 1 & 2 & 2 \\ \hline
2 & 2 & 3 & 3 \\ \hline
3 & 3 & 4 & 4 \\ \hline
4 & 4 & 5 & 5 \\ \hline
5 & 5 & \overline{7} & \overline{7} \\ \hline
\overline{7} & \overline{7} & \overline{6} & \overline{6} \\ \hline
\overline{6} & \overline{6} & \overline{1} & \multicolumn{1}{c}{\star^1} \\ \cline{1-3}
\end{array}
$$

$$
\mapsto
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 2 & 2 \\ \hline
2 & 3 & 3 & 3 \\ \hline
3 & 4 & 4 & 4 \\ \hline
4 & 5 & 5 & 5 \\ \hline
5 & \overline{7} & \overline{7} & \overline{7} \\ \hline
\overline{7} & \overline{6} & \overline{6} & \overline{6} \\ \hline
\overline{6} & \star^2 & \overline{1} & \star^1 \\ \hline
\end{array}
\mapsto
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 2 & 2 \\ \hline
2 & 3 & 3 & 3 \\ \hline
3 & 4 & 4 & 4 \\ \hline
4 & 5 & 5 & 5 \\ \hline
5 & \overline{7} & \overline{7} & \overline{7} \\ \hline
\overline{7} & \overline{6} & \overline{6} & \overline{6} \\ \hline
\overline{6} & \overline{1} & \multicolumn{1}{c}{\star^2} & \multicolumn{1}{c}{\star^1} \\ \cline{1-2}
\end{array}
\overset{spl^{-1}}{\to}
\begin{array}{|c|c|}
\hline
2 & 2 \\ \hline
3 & 3 \\ \hline
4 & 4 \\ \hline
5 & 5 \\ \hline
0 & \overline{7} \\ \hline
\overline{7} & \overline{6} \\ \hline
\overline{6} & \multicolumn{1}{c}{\star} \\ \cline{1-1}
\end{array}
$$

This description step-by-step of the *ojdt* is given in [4]. Let us recall it :

Proposition 3.2. *Let $(\mathcal{C}_j \cup \{\star_r\})\mathcal{C}_{j+1}$ be an instant of the ojdt. Suppose that in the splitting of the orthogonal tableau, the stars are as follows (x_r, y_r, z_{r+1} are barred or unbarred):*

$$
\begin{array}{|c|c|c|c|}
\hline
\star^2 & \star^1 & x_r & y_r \\ \hline
 & z_{r+1} \\ \cline{1-2}
\end{array}
$$

Then the next sliding is as follows:

Vertical sliding: if $z_{r+1} \leq x_r$, then the sliding is vertical:

$$(\mathcal{C}_j \cup \{\star_r\})\mathcal{C}_{j+1} \ \mapsto \ (\mathcal{C}_j \cup \{\star_{r+1}\})\mathcal{C}_{j+1}.$$

Non-vertical sliding: if $z_{r+1} > x_r$, the sliding is not vertical, more precisely, writing

$$spl(\mathcal{C}_{j+1}) = \frac{E_{j+1}}{C_{j+1}} \ \frac{B_{j+1}}{D_{j+1}} = \begin{array}{cc} x_1 & y_1 \\ \vdots & \vdots \\ x_p & y_p \end{array},$$

there are 3 cases:

(1) *u-sliding: if $\{x_1, \ldots, x_r\} \subset A_{j+1}$, then the sliding is horizontal:*

$$(g \begin{bmatrix} B_j \\ O_j \\ C_j \end{bmatrix} \cup \{\star_r\}) f \begin{bmatrix} A_{j+1} \\ O_{j+1} \\ D_{j+1} \end{bmatrix} \ \mapsto \ g \begin{bmatrix} B_j \cup \{x_r\} \\ O_j \\ C_j \end{bmatrix} (f \begin{bmatrix} A_{j+1} \setminus \{x_r\} \\ O_{j+1} \\ D_{j+1} \end{bmatrix} \cup \{\star_r\})$$

$$= f \begin{bmatrix} A_j \cup \{\gamma_{A_j \cup C_j}(x_r)\} \\ O_j \\ (D_j \cup \{\gamma_{A_j \cup C_j}(x_r)\}) \setminus \{x_r\} \end{bmatrix} (g \begin{bmatrix} B_{j+1} \setminus \{y_r\} \\ O_{j+1} \\ (C_{j+1} \cup \{x_r\}) \setminus \{y_r\} \end{bmatrix} \cup \{\star_r\}).$$

(2) *0-sliding: if* $x_r \in E_{j+1}$, *but* $\{x_1, \dots, x_r\} \not\subset A_{j+1}$, *then define the row index* $\overline{r}$, *as the index of the row of* $\overline{x}_r$ *in the column*

$$\begin{matrix} E_{j+1} \setminus \{x_r\} \\ \overline{C}_{j+1} \cup \{\overline{x}_r\} \end{matrix}.$$

The sliding is simultaneously horizontal and vertical:

$$\begin{aligned} (f \begin{bmatrix} A_j \\ O_j \\ D_j \end{bmatrix} \cup \{\star_r\}) f \begin{bmatrix} A_{j+1} \\ O_{j+1} \\ D_{j+1} \end{bmatrix} &\mapsto f \begin{bmatrix} A_j \\ O_j \cup \{0\} \\ D_j \end{bmatrix} (f \begin{bmatrix} A_{j+1} \\ O_{j+1} \setminus \{0\} \\ D_{j+1} \end{bmatrix} \cup \{\star_{\overline{r}}\}) \\ &= g \begin{bmatrix} B_j \\ O_j \cup \{0\} \\ C_j \end{bmatrix} (g \begin{bmatrix} B_{j+1} \\ O_{j+1} \setminus \{0\} \\ C_{j+1} \end{bmatrix} \cup \{\star_{\overline{r}}\}). \end{aligned}$$

(3) *b-sliding: if* $x_r = \overline{u}_r \in \overline{C}_{j+1}$, $y_r = \overline{v}_r \in \overline{F}_{j+1}$, *the sliding is horizontal:*

$$(f \begin{bmatrix} A_j \\ O_j \\ D_j \end{bmatrix} \cup \{\star_r\}) g \begin{bmatrix} B_{j+1} \\ O_{j+1} \\ C_{j+1} \end{bmatrix} \mapsto f \begin{bmatrix} A_j \\ O_j \\ D_j \cup \{u_r\} \end{bmatrix} (g \begin{bmatrix} B_{j+1} \\ O_{j+1} \\ C_{j+1} \setminus \{u_r\} \end{bmatrix} \cup \{\star_r\})$$

Let us now describe the inverse map $ojdt^{-1}$.

Let $T \setminus S$ be an admissible skew tableau. Denote $\sigma^0(T \setminus S)$ the tableau obtained from $T \setminus S$ by rotating $T \setminus S$ half a tour, and replace each entry as follows

$$x \mapsto \overline{x}, \quad \overline{x} \mapsto x, \quad 0 \mapsto 0, \quad \text{and} \quad \star \mapsto \star.$$

Lemma 3.3. *If* σ *is the map defined on symplectic skew tableaux in the preceding section, the following holds:*

$$\sigma\left(spl(T \setminus S)\right) = spl\left(\sigma^o(T \setminus S)\right).$$

Proof. If $T \setminus S$ has k columns and the column $\mathcal{C}_j$ is $\mathcal{C}_j = f \begin{bmatrix} A_j \\ O_j \\ D_j \end{bmatrix}$, then $\sigma^o(T \setminus S)$ has k columns and its $k+1-j$ column is $\mathcal{C}'_{k+1-j} = f \begin{bmatrix} D_j \\ O_j \\ A_j \end{bmatrix}$.

The columns number $2j-1$, and $2j$ in $spl(T \setminus S)$ are $f \begin{bmatrix} A_j \cup K_j \\ C_j \end{bmatrix} f \begin{bmatrix} B_j \\ D_j \cup K_j \end{bmatrix}$.

Splitting the column $\mathcal{C}'_{k+1-j}$, we immediately see that $K'_{k+1-j} = K_j$. Therefore in $spl(\sigma^o(T \setminus S))$, the columns $2(k-j)+3$, and $2(k-j)+4$ are:

$$f \begin{bmatrix} D_j \cup K_j \\ B_j \end{bmatrix} f \begin{bmatrix} C_j \\ A_j \cup K_j \end{bmatrix}.$$

This proves the lemma. ∎

The Lemma proves that $ojdt$ is invertible and

$$(ojdt)^{-1} = spl^{-1} \circ (sjdt)^{-2} \circ spl = spl^{-1} \circ \sigma \circ (sjdt)^2 \circ \sigma \circ spl = \sigma^o \circ (ojdt) \circ \sigma^o.$$

As for the jdt and the $sjdt$, starting from a skew admissible tableau, and applying the $ojdt$ successively, we get an admissible tableau $totalojdt(T)$ which is independent of the chosen sequence of successive used inner corners.

3.4. Quasistandard admissible tableaux

Recall now that a semistandard Young tableau T, with entries $t_{i,j}$ is non quasistandard on the row r, if the following holds:

(1) The entry $t_{r,1}$ is r,

(2) There is a column in T with height r,

(3) The entries on the rows r, $r+1$ satisfy: for any j, $t_{r,j+1} < t_{r+1,j}$.

Suppose T is non quasistandard on the row r, if we delete the r first entries of the first column of T, and their boxes, we get a skew semistandard tableau with exactly one inner corner (in the box $r,1$), and, in the jdt, each sliding is horizontal.

In [3], it is proved that the same holds for symplectic tableaux and jeu de taquin: if the splitting of a symplectic tableau is not quasistandard at the row r, then deleting the first r entries of the first column and their boxes, that is the empty tableau $\mathcal{C}_1^r$ we get a symplectic skew tableau $T \setminus \mathcal{C}_1^r$ with one inner corner, and a $sjdt$ always horizontal. As an application of Proposition 3.2, let us show that the same is holding for the $ojdt$.

Definition 3.7. An admissible tableau T, with entries $t_{i,j}$, is non quasistandard on the row r, if the following holds:

(1) The entry $t_{r,1}$ is r,

(2) There is a column in T with height r,

(3) In $spl(T)$, the entries on the rows r, $r+1$ satisfy: for any j, $st_{r,j+1} < st_{r+1,j}$.

If one of the above condition does not hold, we say that T is quasistandard on the row r. If T is quasistandard on each row, we say that T is a quasistandard admissible (QA) tableau.

Proposition 3.4. *Let T be an admissible tableau, with first column $\mathcal{C}_1$. Suppose T non quasistandard on the row r, delete the r first entries in the first column of T and their boxes, getting a skew admissible tableau $T_r = T \setminus \mathcal{C}_1^r$, with only one inner corner. Then any sliding of the ojdt on T_r is horizontal.*
If $r > 1$, complete the top of the first column in $ojdt(T_r)$ by the column $\mathcal{C}_1^{r-1} = f\big[\{1, ..., r-1\}\big]$, getting a tableau T'. Then T' is non quasistandard on the row $r-1$.

Proof. We use argument similar to proofs of Lemmas 5.1 - 5.3 in [3]. Consider a column $\mathcal{C}$ of T, suppose the rows of $\mathcal{C}$ are the rows $1, 2, \dots, p$ and $1 \leq r \leq p$.

Write: $$\mathcal{C} = f\begin{bmatrix} A \\ O \\ D \end{bmatrix} = g\begin{bmatrix} B \\ O \\ C \end{bmatrix}, \quad \text{and} \quad spl(\mathcal{C}) = \begin{matrix} A & B \\ K & \overline{K} \\ \overline{C} & \overline{D} \end{matrix} = \begin{matrix} E & B \\ \overline{C} & \overline{F} \end{matrix}.$$

As usual, an element in A, B, C, D, E, or F, which appears on the row t is denoted respectively by a_t, b_t, c_t, d_t, e_t or f_t. There are 3 possibilities for the row r in $spl(\mathcal{C})$.

(1) **u-case:** The two entries are unbarred. The row is $\boxed{e_r \mid b_r}$.

Put $B^- = \{b_1, \ldots, b_r\} = (B^- \cap C) \cup (B^- \setminus C)$. By definition, $B^- \setminus C \subset A$, and if $e_{r+1} \in E$ exists on the row $r+1$, the relation $b_r < e_{r+1}$ implies $B^- \setminus C \subset A^- = \{a \in A : \ a < e_{r+1}\}$. On the other hand, with our usual notation: $B^- \cap C = J^- = \{y_1, \ldots, y_t\}$ is formed by the t first elements in $J = B \cap C$, and there are $x_1, \ldots, x_t$ in A^-, the t first elements in $I = A \cap D$.

This implies $\#A^- \geq r$, thus $A^- = \{e_1, \ldots, e_r\}$. More precisely, if $e_r = a_r \notin D$, then $b_r = a_r = e_r$, if $a_r \in D$, then $e_r = x_t$ and $b_r = y_t$.

(2) **0-case:** The first entry is unbarred, the second one is barred. The row is $\boxed{e_r \mid \overline{f}_r}$.

Suppose that n is not in E, thus $n \notin K$, by definition of K, $n \in A \cup C$ and $n \notin A$, thus $n \in C$, $\overline{n}$ is the smallest element of $\overline{C}$, thus it is the first entry $\overline{c}_{r+1}$ in the row $r+1$. The relation $\overline{f}_r < \overline{c}_{r+1} = \overline{n}$ is impossible. Thus n is the greatest element of E, $e_r = n$. Since $K \neq \emptyset$. This implies $\{e_1, \ldots e_r\} \not\subset A$.

Suppose now there is $f_u \in D$, with $u \leq r$, thus either $f_u \notin A$, this would imply $f_u \in C$, $\overline{f}_u \leq \overline{f}_r < \overline{c}_{r+1}$, this is impossible, or $f_u \in I = A \cap D$, put $f_u = x_s$, consider the corresponding element $y_s \in J \subset C$, then $\overline{y}_s < \overline{x}_s = \overline{f}_u \leq \overline{f}_r < \overline{c}_{r+1}$, this is still impossible. Thus K contains all the elements in any row $u \leq r$ in the second column, since $\#K$ is the number of these rows,

$$K = \{f \in F : \ f \geq f_r\}, \qquad D = \{f \in F : \ f < f_r\}.$$

(3) **b-case:** The two entries are barred. The row is $\boxed{\overline{c}_r \mid \overline{f}_r}$.

The same argument as in the u-case proves that if $F = \{f_p, f_{p-1}, \ldots, f_t\}$, then $\{f_p, f_{p-1}, \ldots, f_r\} \subset D$.

Suppose that $\mathcal{C}$ is the first column in T, then $e_r = r$, and in T_r the first column is:

$$\mathcal{C} = f \begin{bmatrix} A \setminus [1, r] \\ O \\ D \end{bmatrix} = f \begin{bmatrix} A' \\ O \\ D \end{bmatrix} = g \begin{bmatrix} B' \\ O \\ C' \end{bmatrix}, \quad \text{denote} \quad spl(\mathcal{C}') = \begin{matrix} A' & B' \\ K' & \overline{K'}. \\ \overline{C'} & \overline{D} \end{matrix}$$

In the u-case, O does not change $A' = \{a \in A : \ a > r\}$, by construction $B' = \{b \in B : \ b > b_r\}$, thus $K' = K$, and the two first columns in $spl(T_r)$ are simply the elements of the two first columns of $spl(T)$, which are on the rows $r+1, \ldots, p$. Therefore the entries $st_{i,j}$ of $spl(T_r)$ verify $st_{r,3} < st_{r+1,2}$, and the first sliding in the $ojdt$ is horizontal.

In the 0-case, $r = n$, and the two first columns disappear (except for the stars on last row) and T_r and in te $ojdt$, each sliding is horizontal.

Suppose now $\mathcal{C} = \mathcal{C}_j$ is any other column of T_r, we are at a step of $ojdt$ where the star is on the row r and the column $j-1$ and the next sliding is not vertical: in the corresponding splitting, we have in the columns $2j-3$ to $2j$:

$\star^2$	$\star^1$	x_r	y_r
	z_{r+1}	u_{r+1}	v_{r+1}

and $x_r < z_{r+1}$.

In the u-case, $x_r = a_r$, $y_r = b_r$ the sliding is a horizontal u-sliding, after this sliding, the column j becomes

$$\begin{aligned}(\mathcal{C}'_j \cup \{\star_r\}) &= (f\begin{bmatrix}A\setminus\{a_r\}\\ O\\ D\end{bmatrix}\cup\{\star_r\}) = (g\begin{bmatrix}B\setminus\{b_r\}\\ O\\ (C\cup\{a_r\})\setminus\{b_r\}\end{bmatrix}\cup\{\star_r\}),\\ spl(\mathcal{C}'_j) &= \begin{matrix}A' & B'\\ K' & \overline{K'}\\ \overline{C'} & \overline{D'}\end{matrix} = \begin{matrix}E' & B'\\ \overline{C'} & \overline{F'}\end{matrix}.\end{aligned}$$

If $\#A > r$, then in the splitting of this column, the row $r+1$ is $\boxed{e'_{r+1} \mid b_{r+1}}$, in the next step, the new entry $v'_{r+1} = b_{r+1} = st_{r+1,2j} = v_{r+1}$.

If $\#A = r$ the row $r+1$ in this splitting is $\boxed{e'_{r+1} \mid \overline{f'_{r+1}}}$, or $\boxed{\overline{c'_{r+1}} \mid \overline{f'_{r+1}}}$. But we saw that $K \subset \{e \in E, e \geq e_{r+1}\}$, and $e_{r+1} > b_r$. Since $A' \cup C' = (A\cup C)\setminus\{b_r\}$, and $\#K' = \#K$, this implies $K' = K$ and $F' = (F\setminus\{b_r\})\cup\{a_r\}$. In this case, $f'_{r+1} = \min(F') \leq \min(F) = f_{r+1}$. Therefore $v'_{r+1} = \overline{f'_{r+1}} \geq \overline{f_{r+1}} = v_{r+1}$.

In any case, $v'_{r+1} > st_{r,2j+1}$ and the next sliding will be not vertical.

In the 0-case, since $e_r = n$, the conjugate entry $\overline{e}_r$ is $\overline{n}$, it is on the row r in the column

$$\begin{matrix}E\setminus\{n\}\\ \overline{C}\cup\{\overline{n}\}\end{matrix}.$$

This means $\overline{r} = r$, the sliding is a horizontal 0-sliding, the column j becomes:

$$\begin{aligned}(\mathcal{C}'_j \cup \{\star_r\}) &= (f\begin{bmatrix}A\\ O\setminus\{0\}\\ D\end{bmatrix}\cup\{\star_r\}) = (g\begin{bmatrix}B\\ O\setminus\{0\}\\ C\end{bmatrix}\cup\{\star_r\}),\\ spl(\mathcal{C}'_j) &= \begin{matrix}A & B\\ K' & \overline{K'}\\ \overline{C} & \overline{D}\end{matrix} = \begin{matrix}E' & B\\ \overline{C} & \overline{F'}\end{matrix}.\end{aligned}$$

By definition, $K' = K\setminus\{\min(K)\} = K\setminus\{f_r\}$. Here too, $v'_{r+1} = \overline{f'_{r+1}} = \overline{f_{r+1}} = v_{r+1}$ and the next sliding is still not vertical.

Finally, in the b-case, the sliding is horizontal, $x_r = \overline{c_r}$, $y_r = \overline{d_r}$, after this sliding, the column j becomes

$$\begin{aligned}(\mathcal{C}'_j \cup \{\star_r\}) &= (g\begin{bmatrix}B\\ O\\ C\setminus\{c_r\}\end{bmatrix}\cup\{\star_r\}) = (f\begin{bmatrix}(A\cup\{d_r\})\setminus\{c_r\}\\ O\\ D\setminus\{d_r\}\end{bmatrix}\cup\{\star_r\}),\\ spl(\mathcal{C}'_j) &= \begin{matrix}A' & B\\ K' & \overline{K'}\\ \overline{C'} & \overline{D'}\end{matrix} = \begin{matrix}E' & B\\ \overline{C'} & \overline{F'}\end{matrix}.\end{aligned}$$

As in the u-case, $K \subset \{f \in F,\ f > f_r = d_r\}$, therefore $K' = K$, and $F' = F\setminus\{f_r\}$, the new entry v'_{r+1}, on the row $r+1$ in the column $2j$ is $v'_{r+} = v_{r+1}$, the next sliding will be not vertical.

This proves that in the $ojdt$ each sliding is horizontal, therefore for the corresponding two $sjdt$ on $spl(T_r)$, each sliding is also horizontal. Considering the rows $r-1$ and r during these two $sjdt$, in the columns $2j-2$, $2j-1$ and $2j$, we can write this:

$$\begin{array}{|c|c|c|c|}\hline & z_{r-1} & u_{r-1} & v_{r-1} \\ \hline \star^2 & \star^1 & x_r & y_r \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|}\hline & z'_{r-1} & u'_{r-1} & v'_{r-1} \\ \hline \star^2 & x_r & x'_r & y'_r \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|}\hline & z''_{r-1} & u''_{r-1} & v''_{r-1} \\ \hline & w_r & x''_r & y''_r \\ \hline \end{array}$$

Now in such a horizontal sliding the following inequalities are proved in [3], Proposition 5.1:

$$u'_{r-1} < x_r, \quad v'_{r-1} < x'_r \quad \text{then} \quad u''_{r-1} < w_r, \quad v''_{r-1} < x''_r.$$

This achieves the proof of the Proposition. ∎

Let $\lambda = (a_1, \ldots, a_n)$ be a n-tuple of natural numbers, an admissible tableau T has shape λ if it has exactly a_j columns with height j $(1 \leq j \leq n)$. Denote by $\mathcal{A}^{[\lambda]}$ the set of admissible tableaux with shape λ.

If $\mu = (b_1, \ldots, b_n)$ in the shape, we say that $\mu \leq \lambda$ if $b_j \leq a_j$ for each j.

Proposition 3.4 allows us to define a map p from the set $\mathcal{A}^{[\lambda]}$ onto the set $\cup_{\mu\leq\lambda}\mathcal{QA}^{[\mu]}$ of quasistandard admissible tableaux with a shape smaller than λ.

Let T be in $\mathcal{A}^{[\lambda]}$. If T is quasistandard, the algorithm stops, we put $p(T) = T$.

If T is non quasistandard, there is r such that T is non quasistandard on the row r, and T is quasistandard on each row $t > r$. Using Proposition 3.4, we perform the $ojdt$ on the tableau $T_r = T \setminus \mathcal{C}_1^r$, if $r > 1$ $T'_{r-1} = ojdt(T_r) = T' \setminus \mathcal{C}_1^{r-1}$ has an unique inner corner, and is non quasi-stadard on the row $r-1$. We repeat this operation, getting successive admissible skew tableaux

$$T''_{r-2} = ojdt(T'_{r-1}), \ldots, S = T_0^{(r)} = ojdt(T_1^{(r-1)}).$$

By construction S is an admissible tableau, and its shape is $\lambda' = \lambda - (0, \ldots, 1, \ldots, 0)$, with 1 in the r position.

Lemma 3.5. *For each $t > r$, S is quasistandard in the row t.*

Proof. In [3] it is proved that the action of the $sjdt$ on a symplectic semistandard tableau which is non quasistandard on the row r, but quasistandard on any row $t > r$ is still quasistandard in each row $t > r$. The result follows immediately. ∎

Repeat now the above operation on $S = ojdt^r(T_r)$.

After a finite number of steps, this algorithm stops either on an empty tableau U (a tableau with shape $(0, \ldots, 0)$) or on a quasistandard tableau $U \in \mathcal{QA}^{[\mu]}$, with $\mu \leq \lambda$. Put: $U = p(T) = (ojdt)^{max}(T)$.

The above algorithm defines the map $p : \mathcal{A}^{[\lambda]} \to \bigsqcup_{\mu\leq\lambda} \mathcal{QA}^{[\mu]}$.

Theorem 3.6. *The orthogonal jeu de taquin defines a bijection $p = (ojdt)^{max}$ from the set $\mathcal{A}^{[\lambda]}$ of admissible tableaux with shape λ onto the disjoint union $\bigsqcup_{\mu\leq\lambda} \mathcal{QA}^{[\mu]}$ of the set of admissible quasistandard tableaux with shape $\mu \leq \lambda$.*

Proof. Let us build the inverse mapping of p. For each shape μ, let us denote by F_μ the Ferrer diagram with shape μ. Let now $\mu \leq \lambda$ and $U \in \mathcal{QA}^{[\mu]}$.

Apply σ^o to U, one gets an admissible skew tableau, filling of the skew diagram $\sigma(F_\mu)$ this diagram is a subdiagram of $\sigma(F_\lambda)$.

Put a star in the lowest inner corner in $\sigma(F_\lambda)\setminus\sigma(F_\mu)$, and apply the $ojdt$, getting a new skew tableau filling of $\sigma(F_{\mu'})\subset\sigma(F_\lambda)$. Repeat this operation as far as there is some inner corner in $\sigma(F_\lambda)\setminus\sigma(F_{\mu'})$. Write $T'\setminus S'=(ojdt)^{|\lambda-\mu|}(U)$, which is the filling of the skew Ferrer diagram $\sigma(F_{\mu(|\lambda-\mu|)})$ such that $\sigma(F_\lambda)\setminus\sigma(F_{\mu(|\lambda-\mu|)})$ has no inner corner. This implies that $F_\lambda\setminus F_{\mu(|\lambda-\mu|)}$ is the Ferrer diagram $F_{\lambda-\mu}$.

Now the admissible skew tableau $\sigma^o(T'\setminus S')$ can be written in an unique way $T\setminus S$ where T is admissible, has shape λ, S shape $\lambda-\mu$ and S is trivial, meaning that its entries $s_{i,j}$ satisfy $s_{i,j}=i$ for any i and j.

Since $\sigma^o\circ(ojdt)^{|\lambda-\mu|}\circ\sigma^o(T)=ojdt^{max}(T)$, we have $(ojdt)^{max}(T)=U$. This proves that p is a surjective map.

On the other hand, let T_1 and T_2 be two admissible tableaux of shape λ such that $p(T_1)=p(T_2)=U\in\mathcal{QA}^{[\mu]}$. Suppose T_i is not quasistandard on the row r, and quasistandard on any row $t>r$. Then $r=\min\{j:\ \lambda_j-\mu_j>0\}$. This means that the algorithm defining p is starting on the row r for T_1 and T_2. After the first r $ojdt$, we get two tableaux T_1' and T_2' with shape $\lambda-(0,\dots,1,\dots,0)$, such that $p(T_1')=p(T_2')=U$. By induction this proves that $T_1'=ojdt^r(T_{1r})=T_2'=ojdt^r(T_{2r})$, thus $T_1=T_2$. The map p is a bijection and the above map its inverse mapping. ■

4. Shape and reduced shape algebra of a semisimple Lie algebra

4.1. Shape algebra

Let $\mathfrak{g}$ be a semisimple complex Lie algebra with a Cartan subalgebra $\mathfrak{h}$. The theory of finite dimensional $\mathfrak{g}$-module is well known (see for instance [7, 14]. Each module is a weight modules, which means that the action of $\mathfrak{h}$ is diagonal on a base of weight vectors (common eigenvectors). Each module is semisimple and the simple modules are characterized by their highest weights. More precisely, fix a simple roots system $\Pi=\{\alpha_1,\dots,\alpha_\ell\}$, consider the corresponding basis (H_i) of $\mathfrak{h}$, defined by $\alpha_j(H_i)=A_{i,j}$. If $|A_{i,j}]$ is the Cartan matrix of $\mathfrak{g}$, then the fundamental weights ω_i are defined by $\omega_i(H_j)=\delta_{ij}$. The set of all possible weights is the lattice generated by the ω_i: $P=\sum\mathbb{Z}\omega_i$. Denote Λ the semigroup generated by the ω_i:

$$\Lambda=\sum_i\mathbb{Z}^+\omega_i.$$

There is a natural ordering on P: $\mu\le\nu$ if and only if $\nu-\mu\in\Lambda$. For each simple module M, there is a unique greatest weight λ, called the highest weight of M, $\lambda\in\Lambda$, the corresponding eigenvector space is one-dimensional and the map $M\mapsto\lambda$ is a bijective map between the set of equivalence classes of simple modules and the set Λ. Especially, the contragredient module $M^\star$ is simple, denote ${}^t\lambda$ its highest weight.

To describe the simple module M with highest weight $\lambda=\sum_i a_i\omega_i$, fix a highest weight vector v_λ in M. Associated to the choice of Π, the Lie algebra $\mathfrak{g}$ splits into $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}$. On v_λ, $\mathfrak{h}+\mathfrak{n}$ acts by dilation:

$$(H+X)v_\lambda=\lambda(H)v_\lambda\qquad(H\in\mathfrak{h},\ X\in\mathfrak{n}).$$

If X_i (resp. Y_i) are the α_i (resp. the $-\alpha_i$) root vectors ($[H,X_i]=\alpha_i(H)X_i$), then as a Lie algebra $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) is generated as a Lie algebra by the X_i (the Y_i). The module M is isomorphic with the quotient $\mathbb{S}^\lambda=\mathfrak{A}(\mathfrak{g})/(X_i,H_i-a_i,Y_i^{a_i+1})$ of the enveloping algebra of $\mathfrak{g}$ by the ideal generated by the elements X_i, H_i-a_i and $Y_i^{a_i+1}$.

Therefore as a vector space and a $\mathfrak{n}^-$-module, $\mathbb{S}^\lambda$ is isomorphic to the quotient $\mathfrak{A}(\mathfrak{n}^-)/(Y_i^{a_i+1})$ of the enveloping algebra of $\mathfrak{n}^-$ by the ideal generated by $Y_i^{a_i+1}$.

By duality, for each lowest weight vector $u_\lambda\in\mathbb{S}^\lambda$, we denote the $\mathfrak{n}$-module $\mathbb{S}_\lambda$ by:

$$\mathbb{S}^\lambda|_{\mathfrak{n}}=\mathfrak{A}(\mathfrak{n})u_\lambda=\mathfrak{A}(\mathfrak{n})/(X_i^{a'_i+1})$$

where the natural numbers are such that the weight of u_λ is $-{}^t\lambda=-\sum_i a'_i\omega_i$.

The theory of simple $\mathfrak{g}$-modules is therefore described by the space:

$$\mathbb{S}=\bigoplus_{\lambda\in\Lambda}\mathbb{S}^\lambda.$$

This space has a commutative algebra structure, it is called *the shape algebra of* $\mathfrak{g}$. The multiplication is induced by the transpose of the equivariant maps $\left(\mathbb{S}^{\lambda+\mu}\right)^\star\to\left(\mathbb{S}^\lambda\right)^\star\otimes(\mathbb{S}^\mu)^\star$.

Choose a highest weight vector $v_{\omega_i}\in\mathbb{S}^{\omega_i}$ $(1\le i\le\ell)$. The shape algebra $\mathbb{S}$ is generated (as an algebra and a $\mathfrak{g}$ module) by the vectors v_{ω_i}. More precisely, let $\lambda=\sum_i a_i\omega_i\in\Lambda$. A natural highest weight vector in $\mathbb{S}^\lambda$ is

$$v_\lambda=\prod_{i=1}^{\ell}(v_{\omega_i})^{a_i}.$$

Then, for any λ and μ in Λ, $v_\lambda\cdot v_\mu=v_{\lambda+\mu}$. The multiplication map

$$\cdot\ :\ \mathbb{S}\otimes\mathbb{S}\ \longrightarrow\ \mathbb{S},$$

is $\mathfrak{g}$-equivariant. These properties define completely the multiplication.

4.2. Reduced shape algebra

A natural class of modules for a nilpotent Lie algebra $\mathfrak{n}$ is the finite dimensional nilpotent modules M, such that any element X in $\mathfrak{n}$ has a nilpotent action on M. Such a module is generally not semisimple, it can be decomposed into a sum of monogenic modules (module generated by a unique vector v).

Suppose that $M=\mathfrak{A}(\mathfrak{n})v$ is monogenic, then it is indecomposable (see [1]). Suppose (X_i) is a system of vectors generating $\mathfrak{n}$ as a Lie algebra. Then there are natural numbers a'_i such that:

$$X_i^{a'_i}v\neq 0,\quad X_i^{a'_i+1}v=0.$$

In other words, M is a quotient of the module $\mathfrak{A}(\mathfrak{n})/(X_i^{a'_i+1})$. Now if $b'_i<a'_i$ for any i, $\mathfrak{A}(\mathfrak{n})/(X_i^{b'_i+1})$ is a quotient of $\mathfrak{A}(\mathfrak{n})/(X_i^{a'_i+1})$.

From now on, we suppose that $\mathfrak{n}$ is the nilpotent subalgebra of a complex semisimple Lie algebra split into $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}$, and X_i are the root vectors associated to the simple root system Π. In this case, the above modules are the $\mathbb{S}^\lambda|_\mathfrak{n}$, and, if $\mu < \lambda$, the map

$$f_{\mu,\lambda} : \mathbb{S}^\mu|_\mathfrak{n} \to \mathbb{S}^\lambda|_\mathfrak{n}, \qquad w \mapsto v_{\lambda-\mu} \cdot w$$

is $\mathfrak{n}$-invariant and one-to-one: if $f_{\mu,\lambda}(w) = 0$ and $w \neq 0$, since there is $U \in \mathfrak{A}(\mathfrak{n})$ such that $Uw = v_\mu$, this yields to the impossible relation $v_\lambda = Uf_{\mu,\lambda}(w) = 0$.

Due to this observation, for the nilpotent Lie algebra $\mathfrak{n}$, it is natural to replace the sum of the $\mathbb{S}^\lambda$ by the union of these spaces. This is equivalent to consider a quotient of the shape algebra of $\mathfrak{g}$.

Define the reduced shape algebra $\mathbb{S}_{red}$ as the quotient $\mathbb{S}_{red} = \mathbb{S}/\mathcal{I}$ of the shape algebra $\mathbb{S}$ by the ideal $\mathcal{I}$ generated by the $1 - v_{\omega_i}$. Denote π the canonical projection from $\mathbb{S}$ onto $\mathbb{S}_{red}$. The reduced shape algebra is a commutative algebra, with unity $1 = \pi(1)$, and a $\mathfrak{n}$-module.

Proposition 4.1. [1] *Denote by π the canonical projection from $\mathbb{S}$ onto $\mathbb{S}_{red}$. Then*

(i) *For any λ, the $\mathfrak{n}$-modules $\mathbb{S}^\lambda|_\mathfrak{n}$ and $\pi(\mathbb{S}^\lambda|_\mathfrak{n})$ are isomorphic.*

(ii) *For any $\mu \leq \lambda$, $\pi\left(\mathbb{S}^\mu|_\mathfrak{n}\right)$ is a submodule of $\pi\left(\mathbb{S}^\lambda|_\mathfrak{n}\right)$.*

(iii) *The space of vectors $w \in \mathbb{S}_{red}$ such that $\mathfrak{n} \cdot w = 0$ is $\mathbb{C}1$.*

(iv) *$\mathbb{S}_{red}$ is an indecomposable $\mathfrak{n}$-module.*

As an $\mathfrak{n}$-module, $\mathbb{S}_{red}$ is the union of all 'maximal' monogenic $\mathfrak{n}$-modules $\mathbb{S}^\lambda|_\mathfrak{n} = \mathfrak{A}(\mathfrak{n})/(X_i^{a'_i+1})$, with the stratification $\mathbb{S}^\mu|_\mathfrak{n} \subset \mathbb{S}^\lambda|_\mathfrak{n}$ if $\mu \leq \lambda$.

5. Young tableaux and shape algebras

5.1. The $\mathfrak{sl}(n)$ and $\mathfrak{sp}(2n)$ cases

Young tableaux were introduced in the 19th century to describe a well known decomposition of the tensor algebra $T(V)$ of a finite dimensional vector space into $\mathfrak{gl}(n)$-invariant spaces (see [7] for instance). Let us recall this decomposition.

Say that the filling of a Ferrer diagram with d boxes is a standard Young tableau ($S \in St(d)$) if the entries are $\{1, \dots, d\}$ and strictly increasing in each column from the top to the bottom and in each row from the left to the right. Let S be such a standard tableau. We associate to S the set P_S respectively Q_S of permutations $\sigma \in S_d$ preserving each row (resp. each column) of S, then the a Young symmetrizer:

$$c_S = (\sum_{\sigma \in P_S} \sigma)(\sum_{\sigma \in Q_S} \varepsilon(\sigma)\sigma).$$

Up to a constant, c_S is an idempotent element Y_S of the group algebra (S_d). Now S_d thus $\mathbb{C}(S_d)$ are acting on $\otimes^d V$ on the right by:

$$(v_1 \otimes \cdots \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)},$$

and the decompostion of $\otimes^d V$ into irreducible S_d-invariant subspaces is given by:

$$\otimes^d V = \bigoplus_{S \in St(d)} V \cdot Y_S.$$

The group $GL(V)$ (or $SL(V)$) is acting on V, thus on $\otimes^d V$ on the left, this action is commuting with the S_d-action on the right and the spaces invariant under the pair $(SL(V), S_d)$ are irreducible. More precisely, fix a basis $(e_1, \dots, e_n)$ for the space V, identifying the Lie algebra of $SL(V)$ to $\mathfrak{sl}(n)$. For each shape $\lambda = (a_1, \dots, a_n)$ with $d = \sum_i i a_i$ boxes, pick a pair (T, S) of Young tableaux where T is semistandard with entries in $\{1, \dots, n\}$ and shape λ ($T \in SS^\lambda$) and S is standard with shape λ ($S \in St^\lambda$). Denote t^j the entry of T which is in the box with entry j in S $(1 \le j \le d)$, put:

$$e_{(T,S)} = (e_{t^1} \otimes \cdots \otimes e_{t^d}) \cdot Y_S.$$

Then the set $\mathcal{B} = \{e_{(T,S)} :\ T \in SS^\lambda,\ S \in St^\lambda,\ \lambda \text{ a shape with } d \text{ boxes}\}$ is a basis of $\otimes^d V$, and if S is fixed, with shape $\lambda = (a_1, \dots, a_{n-1}, 0)$, the set $\{e_{(T,S)} :\ T \in SS^\lambda\}$ is a basis of a simple $\mathfrak{sl}(n)$-module, with highest weight $\sum_i a_i \omega_i$.

Now, for each λ, define the standard Young tableau $S(\lambda) = s_{i,j}$ with shape λ by numerating the boxes of the Ferrer diagram with shape λ column by column from the left to the right and from the top to the bottom or: $j < j'$ implies $s_{i,j} < s_{i',j'}$ for any i, i'.

Proposition 5.1. *As a vector space, the shape algebra $\mathbb{S}^\bullet$ of $\mathfrak{sl}(n)$ is a quotient of $T(V) = \sum_d \otimes^d V$, by the invariant subspace $W = \sum_\lambda \sum_{S \in St^\lambda,\ S \neq S(\lambda)} V \cdot Y_S$. Let $P : T(V) \to \mathbb{S}^\bullet$ be the quotient map.*
The set $\mathcal{B}_0 = \{e_{(T,S(\lambda))} :\ T \in SS^\lambda,\ \lambda = (a_1, \dots, a_{n-1}, 0)\}$ is a basis of a supplementary space to W, thus defines a basis of $\mathbb{S}^\bullet$, indexed by the collection of semistandard tableaux with entries in $\{1, \dots, n\}$ without any column of height n. For each fixed shape $\lambda = (a_1, \dots, a_{n-1}, 0)$, $\{P(e_{(T,S(\lambda))}) : T \in SS^\lambda\}$ is a basis of $\mathbb{S}^\lambda$, the simple module with highest weight $\lambda = \sum_i a_i \omega_i$.

Since the reduced shape algebra $\mathbb{S}^\bullet_{red}$ is a quotient of $\mathbb{S}^\bullet$, with quotient map π, we have to select, in the above basis, a basis for a supplementary space of the ideal $\mathcal{I}$. Denote QS^λ the set of quasistandard tableaux with shape $\lambda = (a_1, \dots, a_{n-1}, 0)$, it is proved in [2] that:

Proposition 5.2. *The set $\{P(e_{(T,S(\lambda))}) :\ T \in QS^\lambda,\ \lambda = (a_1, \dots, a_{n-1}, 0)\}$ is a basis of a supplementary space to $\mathcal{I}$ in $\mathbb{S}^\bullet$, thus defines a basis of $\mathbb{S}^\bullet_{red}$, indexed by the collection of quasistandard tableaux with entries in $\{1, \dots, n\}$ without any column of height n. For each fixed shape $\lambda = (a_1, \dots, a_{n-1}, 0)$, in $\mathbb{S}^\bullet/\mathcal{I}$, $\{\pi(P(e_{(T,S(\mu))})) :\ T \in QS^\mu,\ \mu \le \lambda\}$ is a basis of $\pi(\mathbb{S}^\lambda)$.*

The same presentation is working for $\mathfrak{sp}(2n)$. Now V is $2n$-dimensional, with basis $\{e_1, \dots, e_n,\ e_{\overline{n}}, \dots, e_{\overline{1}}\}$, as a vector space, the corresponding shape algebra $\mathbb{S}^{\langle\bullet\rangle}$ is a quotient of the shape algebra $\mathbb{S}^\bullet$ of $\mathfrak{sl}(2n)$. If $\lambda = (a_1, \dots, a_n)$, denote $SS^{\langle\lambda\rangle}$ the set of symplectic tableaux with shape λ. The following is proved in [6]:

Proposition 5.3. *As a vector space, the shape algebra $\mathbb{S}^{\langle\bullet\rangle}$ of $\mathfrak{sp}(2n)$ is a quotient of $T(V) = \sum_d \otimes^d V$, by an invariant subspace W'. Let $P' : T(V) \to \mathbb{S}^{\langle\bullet\rangle}$ the quotient map. Then the set $\{e_{(T,S(\lambda))} :\ T \in SS^{\langle\lambda\rangle},\ \lambda\}$ is a basis of a supplementary space to W', thus defines a basis of $\mathbb{S}^{\langle\bullet\rangle}$, indexed by the collection of symplectic tableaux with entries in $\{1, \dots, n, \overline{n}, \dots, \overline{1}\}$. For each fixed shape $\lambda = (a_1, \dots, a_n)$, $\{P'(e_{(T,S(\lambda))}) :\ T \in SS^{\langle\lambda\rangle}\}$ is a basis of $\mathbb{S}^{\langle\lambda\rangle}$, the simple $\mathfrak{sp}(2n)$-module with highest weight $\lambda = \sum_i a_i \omega_i$.*

Using the symplectic jeu de taquin, the result of Proposition 5.2 was proved in [3] in the symplectic case.

Proposition 5.4. *Denote $QS^{\langle\lambda\rangle}$ the set of symplectic quasistandard tableaux with shape $\lambda = (a_1, \dots, a_n)$. The set $\{P'(e_{(T,S(\lambda))}) : T \in QS^{\langle\lambda\rangle},\ \lambda\}$ is a basis of a supplementary space to $\mathcal{I}$ in $\mathbb{S}^{\langle\bullet\rangle}$, thus defines a basis of $\mathbb{S}^{\langle\bullet\rangle}_{red}$, indexed by the collection of symplectic quasistandard tableaux with entries in $\{1, \dots, n, \overline{n}, \dots, \overline{1}\}$. For each fixed shape $\lambda = (a_1, \dots, a_n)$, in $\mathbb{S}^{\langle\bullet\rangle}/\mathcal{I}$, $\{\pi(P'(e_{(T,S(\mu))})) : T \in QS^{\langle\mu\rangle},\ \mu \leq \lambda\}$ is a basis of $\pi(\mathbb{S}^{\langle\lambda\rangle})$.*

5.2. The $\mathfrak{g} = \mathfrak{so}(2n+1)$ case

Let us now describe the combinatoric associated to the shape algebras of $\mathfrak{so}(2n+1)$. First recall the presentation of the simple modules of $\mathfrak{so}(2n+1)$, see for instance [7, 14].

Let V be a $2n{+}1$-dimensional complex vector space with basis $(e_1, ..., e_n, e_0, e_{\overline{n}}, ..., e_{\overline{1}})$ (recall the ordering $1 < \dots < n < 0 < \overline{n} < \dots < \overline{1}$). Put $\overline{\overline{i}} = i$ and $\overline{0} = 0$. On V, define the non degenerated symmetric bilinear form $Q = \langle\ ,\ \rangle$ by

$$\langle e_i, e_{\overline{j}}\rangle = \delta_{ij}, \quad \forall\ i,\ j.$$

Let $\mathfrak{so}(2n+1)$ be the Lie algebra of matrices preserving Q. It is the set of all $(2n+1) \times (2n+1)$ matrices X such that ${}^sX = -X$, where s is the symmetry with respect to the second diagonal, or:

$$X = \begin{bmatrix} A & u & B \\ -\,{}^sv & 0 & -\,{}^su \\ C & v & -\,{}^sA \end{bmatrix}$$

with $n \times n$ matrices A, B and C such that ${}^sB = -B$, ${}^sC = -C$, $n \times 1$ matrices u and v and where ${}^su = [u_{n1}, ..., u_{11}]$. The Lie algebra $\mathfrak{so}(2n{+}1)$ is simple, of type (B_n).

The space $\mathfrak{h}$ of diagonal matrices H in $\mathfrak{so}(2n{+}1)$ is a Cartan subalgebra of $\mathfrak{so}(2n{+}1)$. For any $1 \leq j \leq n$, define ε_j by $He_j = \varepsilon_j(H)e_j$. The root system of $\mathfrak{so}(2n+1)$ is thus $\Delta = \{\pm(\varepsilon_i \pm \varepsilon_j),\ \ 1 \leq i < j \leq n\} \cup \{\pm\varepsilon_i,\ \ 1 \leq i \leq n\}$.

Choose the simple root system

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n,\ \varepsilon_n\}.$$

Then the positive roots are $\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j$ $(1 \leq i < j \leq n)$, and ε_i $(1 \leq i \leq n)$. The nilpotent factor $\mathfrak{n}$ is the set of upper triangular matrices in $\mathfrak{so}(2n+1)$.

The fundamental weights for $\mathfrak{so}(2n+1)$ are:

$$\omega_1 = \varepsilon_1,\ \omega_2 = \varepsilon_1 + \varepsilon_2, \dots,\ \omega_{n-1} = \varepsilon_1 + \dots + \varepsilon_{n-1},\ \omega_n = \frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_n).$$

The set of highest weights is $\Lambda = \sum_{i=1}^n \mathbb{Z}^+\omega_i$. For any λ in Λ, denote $\mathbb{S}^{[\lambda]}$ the corresponding simple module.

For the Lie algebras $\mathfrak{sl}(n)$ and $\mathfrak{sp}(2n)$, their simple modules are submodules of $T(V)$, this is no longuer true for $\mathfrak{so}(2n+1)$. As for $\mathfrak{sl}(2n+1)$, the fundamental modules

$\mathbb{S}^{[\omega_k]}$, for $k=1,\dots,n-1$ are the submodules $\wedge^k V$. However, the fundamental module $\mathbb{S}^{[\omega_n]}$ does not appear in the decomposition of $T(V)$. This module is 2^n-dimensional, and called the spin representation. Remark that $\wedge^n V \simeq \mathbb{S}^{[2\omega_n]}$ and $\mathbb{S}^{[\omega_n]} \otimes \mathbb{S}^{[\omega_n]} \simeq \oplus_{k=0}^n \wedge^k V$.

Let $\lambda = \sum_{k=1}^n a_k\omega_k$ be in Λ. If a_n is even, we say that λ is admissible or $\lambda \in \Lambda_{adm}$, the module $\mathbb{S}^{[\lambda]}$ is the submodule of

$$S_{a_1,\dots,a_n} = Sym^{a_1}(V) \otimes Sym^{a_2}(\wedge^2 V) \otimes \cdots \otimes Sym^{a_{n-1}}(\wedge^{n-1}V) \otimes Sym^{\frac{a_n}{2}}(\wedge^n V)$$

generated by $e_1^{a_1} \otimes (e_1\wedge e_2)^{a_2} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_n)^{a_n/2}$. We say that $\mathbb{S}^{[\lambda]}$ is a tensor module and λ an admissible weight.

However, if a_n is odd $\mathbb{S}^{[\lambda]}$ is not a submodule of $T(V)$, it is a submodule of

$$S_{a_1,\dots,a_n} = Sym^{a_1}(V) \otimes Sym^{a_2}(\wedge^2 V) \otimes \cdots \otimes Sym^{a_{n-1}}(\wedge^{n-1}V) \otimes Sym^{\frac{a_n-1}{2}}(\wedge^n V) \otimes \mathbb{S}^{[\omega_n]}.$$

Therefore, the sum $\mathbb{S}^{[\bullet]}_{adm} = \sum_{\lambda\in\Lambda_{adm}} \mathbb{S}^{[\lambda]}$, called the tensor shape algebra of $\mathfrak{so}(2n+1)$ is a subalgebra of the shape algebra $\mathbb{S}^{[\bullet]}$ of $\mathfrak{so}(2n+1)$.

Each simple module $\wedge^k V$ $(k \leq n)$ has a basis which is labeled with the admissible columns of height k. The number of admissible columns $f\begin{bmatrix} A \\ O \\ D \end{bmatrix}$ with height k is exactly $\begin{pmatrix} n \\ k \end{pmatrix}$. As usual, if $I = \{i_1 < i_2 < \cdots < i_k\} \subset \{1,\dots,n,0,\overline{n},\dots,\overline{1}\}$, denote by e_I the vector $e_{i_1} \wedge \cdots \wedge e_{i_k}$. If $\#O \leq 1$ the admissible column $\mathcal{C} = f\begin{bmatrix} A \\ O \\ D \end{bmatrix}$ labels the vector $e_{A\cup O\cup\overline{D}} = \varphi(\mathcal{C})$.

Order now the set of admissible columns $f\begin{bmatrix} A \\ O \\ D \end{bmatrix}$ such that $\#O \geq 2$ by using the lexicographic ordering: this set is $\{\mathcal{C}_1,\dots,\mathcal{C}_N\}$, similarly order the set of the vectors $e_I = e_{A\cup\overline{D}}$ and $e_I = e_{A\cup\{0\}\cup\overline{D}}$ such that the corresponding column $\begin{matrix} A \\ O \\ \overline{D} \end{matrix}$ is not admissible by using lexicographic ordering on the sets I, this set of vectors is $\{e_{I_1},\dots,e_{I_n}\}$. Put $e_{I_j} = \varphi(\mathcal{C}_j)$. The admissible columns of height $k<n$ label a basis of $\mathbb{S}^{[\omega_k]}$, and the admissible columns of height n a basis of $\mathbb{S}^{[2\omega_n]}$.

On the other hand, since the dimension of $\mathbb{S}^{[\omega_n]}$ is 2^n, we can use the spin column to label a basis of the spin representation. We do not present here such an explicit procedure.

Thus the orthogonal columns are labeling the basis of the fundamental modules of $\mathfrak{so}(2n+1)$. Consider now the tableaux T whose columns are admissible and shape is $(a_1,\dots,a_{n-1},\frac{a_n}{2})$ (a_n even). They label a basis of $S_{a_1,\dots,a_n}$ and the tableaux $\mathfrak{C}T$, where $\mathfrak{C}$ is a spin column label a basis of $S_{a_1,\dots,a_n+1}$.

As a vector space, the tensor shape algebra $\mathbb{S}^{[\bullet]}_{adm}$ is a quotient of the sum of the spaces $S_{a_1,\dots,a_n}$ with a_n even, and the shape algebra $\mathbb{S}^{[\bullet]}$ is a quotient of the sum of all the spaces $S_{a_1,\dots,a_n}$, selecting a subset of tableaux, we get basis in these quotients.

Proposition 5.5. [10] *Let $\lambda = \sum_i a_i\omega_i$. If a_n is even, then the set $AS^{[\lambda]} = \mathcal{A}^{\tilde{\lambda}}$ of admissible tableaux with shape $\tilde{\lambda} = (a_1, \dots, a_{n-1}, \frac{a_n}{2})$ is a basis for the simple module $\mathbb{S}^{[\lambda]}$. If a_n is odd, then the set $SS^{[\lambda]}$ of orthogonal tableaux of the form $\mathfrak{C}T$ where $\mathfrak{C}$ is a spin column and $T \in AS^{[\lambda-\omega_n]}$ is a basis for the simple module $\mathbb{S}^{[\lambda]}$. The set $SS^{[\bullet]}$ of all orthogonal tableaux is a basis for the shape algebra $\mathbb{S}^{[\bullet]}$ of $\mathfrak{so}(2n+1)$.*

Consider now the reduced shape algebra $\mathbb{S}^{[\bullet]}_{red}$ of $\mathfrak{so}(2n+1)$. Recall the quotient map:

$$\pi : \mathbb{S}^{[\bullet]} \to \mathbb{S}^{[\bullet]}_{red} = \mathbb{S}^{[\bullet]}/\mathcal{I},$$

where $\mathcal{I}$ is the ideal generated by the $v_{\omega_k} - 1 = f\,[\{1,\dots,k\}] - 1$ $(k < n)$ and $v_{\omega_n} - 1 = \mathfrak{f}\,[\{1,\dots,n\}] - 1$. This algebra contains the admissible reduced algebra

$$\mathbb{S}^{[\bullet]}_{adm,red} = \mathbb{S}^{[\bullet]}_{adm}/(\mathcal{I} \cap \mathbb{S}^{[\bullet]}_{adm}).$$

Since the vectors x in $\mathcal{I} \cap \mathbb{S}^{[\bullet]}_{adm}$ verifying $\mathfrak{n}x = 0$ are linear combination of the v_λ with $\lambda \in \Lambda_{adm}$, we have:

$$\mathbb{S}^{[\bullet]}_{adm,red} = \bigsqcup_{\lambda\in\Lambda_{adm}} \pi(\mathbb{S}^{[\lambda]})$$

with the relation $\pi(\mathbb{S}^{[\mu]}) \subset \pi(\mathbb{S}^{[\lambda]})$ if $\mu \le \lambda$. Since, $\pi(\mathbb{S}^{[\lambda-\omega_n]}) \subset \pi(\mathbb{S}^{[\lambda]})$, we have $\mathbb{S}^{[\bullet]}_{adm,red} = \mathbb{S}^{[\bullet]}_{red}$. Thus, Theorem 3.6 gives:

Corollary 5.6. *The set of admissible quasistandard tableaux $\mathcal{QA}^\bullet = \bigsqcup_{\lambda\in\Lambda_{adm}} \mathcal{QA}^\lambda$ is labeling a basis of $\mathbb{S}^{[\bullet]}_{adm,red}$, the disjoint union $\bigsqcup_{\mu\le\lambda} \mathcal{QA}^\mu$ giving a basis of the submodule $\pi(\mathbb{S}^{[\lambda']})$, where $\lambda' = \sum_{k<n} a_k\omega_k + 2a_n\omega_n$ if $\lambda = (a_1, \dots, a_n)$.*

To extend this construction to the reduced shape algebra of $\mathfrak{so}(n)$, we have to modify the definition of quasistandard tableaux.

Definition 5.1. Let $T \in \mathcal{QA}^\lambda$ be an admissible quasistandard tableau with shape $\lambda = (a_1, \dots, a_n)$.

If $a_n = 0$ we say that T is *orthogonal quasistandard* and write $T \in QS^{[\lambda]}$, $P(T) = T$.

If $a_n > 0$ and its first column $\mathcal{C}_1$ is such that $spl(\mathcal{C}_1) = \begin{matrix} E & B \\ C & F \end{matrix}$ with $C \neq \emptyset$, we say that T is*orthogonal quasistandard* and write $T \in QS^{[\lambda']}$ where $\lambda' = \sum_{k<n} a_k\omega_k + 2a_n\omega_n$, $P(T) = T$.

If $a_n > 0$ and $T = \mathcal{C}_1 S$ with $spl(\mathcal{C}_1) = E \ \begin{matrix} B \\ F \end{matrix}$ $(E = \{1, \dots n\})$, we say that T is *not orthogonal quasistandard*, we put $T' = P(T) = \mathfrak{f}\begin{bmatrix} A \\ D \end{bmatrix} S$, where $D = \{1, \dots, n\} \setminus A$ and say that T' is orthogonal quasistandard and write $T' \in QS^{[\lambda-\omega_n]}$.

By construction, P is a bijection from the set of admissible quasistandard tableaux into the set of orthogonal quasistandard tableaux. Thus:

Corollary 5.7. *The set of orthogonal quasistandard tableaux $QS^{[\bullet]} = \bigsqcup_{\lambda\in\Lambda} QS^{[\lambda]}$ is labeling a basis of $\mathbb{S}^{[\bullet]}_{red}$, the disjoint union $\bigsqcup_{\mu\le\lambda} QS^{[\mu]}$ giving a basis of the submodule $\pi(\mathbb{S}^{[\lambda]})$.*

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