Journal of Lie Theory
Volume **30** (2020) 371–405

On the Areas of Level Sets in Compact Connected Sublattices of Three-Dimensional Euclidean Space

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Communicated by G. Ólafsson

Abstract. As is well-known the three-dimensional Euclidean space $\Re^3$, equipped with the order relation $(x_1, x_2, x_3) \leq (x_1', x_2', x_3')$ if $x_i \leq x_i'$ for $i = 1, 2, 3$, is a distributive, topological lattice. Let L be a compact, connected sublattice of $\Re^3$. For $(x_1, x_2, x_3) \in L$ we define $\lambda(x_1, x_2, x_3) = x_1 + x_2 + x_3$ and for $r \in \Re$ we let $L_r = \{(x_1, x_2, x_3) \in L : \lambda(x_1, x_2, x_3) = r\}$. If $\mu_L(r)$ denotes the surface area of L_r, then we show that the function $r \mapsto \mu_L(r)$ is continuously differentiable, and that the value of $\mu_L'(r)$ can be computed in two different ways: Either as an integral of a certain function over the boundary of L_r, or as the value of the expression $\sqrt{3}\,(\lambda(\sup L_r) + \lambda(\inf L_r) - 2r)$.

Mathematics Subject Classification: Primary 06B30, 26B20; secondary 54F05, 26B15.

Key Words: Level sets and rank functions, sublattices of $\Re^3$, integral formulas.

1. Introduction

With the Cartesian product ordering and its usual topology, finite dimensional Euclidean space is a connected, distributive topological lattice. In order to study compact sublattices of $\Re^n$ and their geometry, it follows from R. Wille's description scheme [16] that the compact sublattices of three-space are important building blocks for those sublattices (see also [11]).

In general, lattices utilize the functions of taking least upper bounds and greatest lower bounds. In $\Re^3$, if $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, then the greatest lower and least upper bounds of x and y are given by

$$x \wedge y = glb\,\{x, y\} = (\min(x_1, y_1), \min(x_2, y_2), \min(x_3, y_3))$$

and
$$x \vee y = lub\,\{x, y\} = (\max(x_1, y_1), \max(x_2, y_2), \max(x_3, y_3))$$

respectively, and corresponding formulas are valid in all dimensions. Compact, connected sublattices of Euclidean space are not "smooth": As the cubes $[0, 1]^n$ indicate, sublattices can and will have lots of vertices and edges. In spite of this, they still admit tangent hyperplanes almost everywhere (see [12]). In this paper, a study of level sets in compact, connected sublattices $L \subseteq \Re^3$ of three-space will be presented. Level sets arise from the level function $\lambda : (x, y, z) \mapsto x + y + z;\ L \to \Re$ as the pre-images of points: $L_r = \{(x, y, z) \in L : x + y + z = r\}$.

ISSN 0949–5932 / \$2.50

We will show that the boundary of L_r, when viewed as a subset of the plane

$$E_r = \{(x, y, z) \in \Re^3 : x + y + z = r\}$$

is a curve, which is differentiable almost everywhere and has a derivative that is measurable - a property that reminds of analogous statements for convex sets.

Compact subsets $A \subseteq E_r$ have surface areas, which will be denoted by $|A|$. In particular, each level set L_r of a compact sublattice $L \subseteq \Re^3$ has a surface area, and we obtain a function $\mu_L : \Re \to \Re$ defined by

$$\mu_L(r) = |L_r|$$

It will be shown that even though L is in general not "smooth", the function $r \mapsto \mu_L(r)$ is still continuously differentiable. Moreover, it turns out that the derivative $\mu_L'(r)$ can be computed as a boundary integral in the following way: Let $(x, y, z) \in \partial L_r$ be a point on the boundary of the level set L_r, and let $n(x, y, z)$ be the outer normal to L_r in the plane E_r. Together with the derivative of the boundary curve, this outer normal also exists almost everywhere and is measurable. Then there is a function φ so that

$$\mu_L'(r) = \int_{\partial L_r} \varphi(n(x, y, z))\, ds$$

This function φ does not depend on the lattice L nor on the level L_r, and is given by

$$\varphi(x, y, z) = \begin{cases} x & \text{if } y \leq x \leq z \text{ or } z \leq x \leq y \\ y & \text{if } x \leq y \leq z \text{ or } z \leq y \leq x \\ z & \text{if } y \leq z \leq x \text{ or } x \leq z \leq y \end{cases}$$

This result is surprising: In general, the level set L_{r_0} at a fixed value of r_0 does not determine the lattice L. So there are quite different lattices L with the same level set L_{r_0}, which may give raise to quite different functions $\mu_L(r)$. However, each of those lattices and each of those functions will have same derivative at r_0. So the derivative $\mu_L'(r_0)$ is really an invariant of the level set L_{r_0}.

It is possible to evaluate the integral $\int_{\partial L_r} \varphi(n(x, y, z))\, ds$, and one obtains the formula

$$\mu_L'(r) = \sqrt{3}\,(\lambda(\inf L_r) + \lambda(\sup L_r) - 2r)$$

Level sets are important in the study of maximal antichains in partially ordered sets. For example, in products of finite chains, maximal antichains are given by level sets (see [1], [4], [5]). Furthermore, the bandwidth of distributive lattices is also closely related to maximal level sets (see [6] and [9]). So one might hope that the results of this paper can be generalized to higher dimension in order to study the width and the bandwidth in large distributive lattices.

It should also be noted that this paper has two "discrete" precursors (see [7] and [9]). More information on notations and results concerning lattices and their topology can be found in [3], [13], [14] as well as [8].

2. First examples

Example 2.1. The first example shows that level sets do not uniquely determine the lattice L : If $L = [0,1]^3$, then $L_1 = \{x \in L : \lambda(x) = 1\}$ is the triangle spanned by the three canonical unit vectors, and the same triangle can be obtained as level set of the lattice $M = [0,2]^3$.

Example 2.2. We consider the cube $L = [0, r_0]^3$, where $r_0 > 0$ is a fixed real number. For $r \le r_0$ the level set

$$L_r = \{x \in L : \lambda(x) = r\}$$

is an equilateral triangle with vertices $(r,0,0)$, $(0,r,0)$ and $(0,0,r)$. For $r_0 < r < 2r_0$, the level set L_r is a hexagon with vertices $(r_0, r - r_0, 0)$, $(r_0, 0, r - r_0)$, $(r - r_0, r_0, 0)$, $(0, r_0, r - r_0)$, $(r - r_0, 0, r_0)$ and $(0, r - r_0, r)$. Hence

$$\mu_L(r) = \left\{ \begin{array}{cc} \frac{\sqrt{3}}{2} r^2 & 0 \le r \le r_0 \\ \frac{\sqrt{3}}{2}\left(r^2 - 3(r - r_0)^2\right) & r_0 < r \le 2r_0 \\ \frac{\sqrt{3}}{2}(3r_0 - r)^2 & 2r_0 \le r \le 3r_0 \end{array} \right\}$$

We observe that the level set L_{r_0} and L_{2r_0} are both triangles. However, even though the situation is symmetric in the sense that the map

$$(x_1, x_2, x_3) \mapsto (r_0 - x_1, r_0 - x_2, r_0 - x_3) : L \to L$$

is an anti-isomorphism mapping L_{r_0} to L_{2r_0}, one has to be careful with "duality": What is immediately below level r_0 "looks like" what is immediately above level $2r_0$, but is different from the levels immediately above r_0. If we know the levels below r_0, we cannot just invoke "duality" to draw inferences for the levels above r_0.

It follows that the function $\mu_L(r)$ is continuously differentiable and that

$$\mu_L'(r_0) = \sqrt{3} r_0$$

If x is on the line segment between $(r_0,0,0)$ and $(0,r_0,0)$, then the outer normal to $A = L_{r_0}$ is given by

$$\frac{(-r_0, r_0, 0) \times (1,1,1)}{\|(-r_0, r_0, 0) \times (1,1,1)\|} = \frac{1}{\sqrt{6}}(1,1,-2)$$

and it follows that $\varphi(n(x)) = \frac{1}{\sqrt{6}}$. Integrating along the straight line from $(r_0,0,0)$ to $(0,r_0,0)$ gives

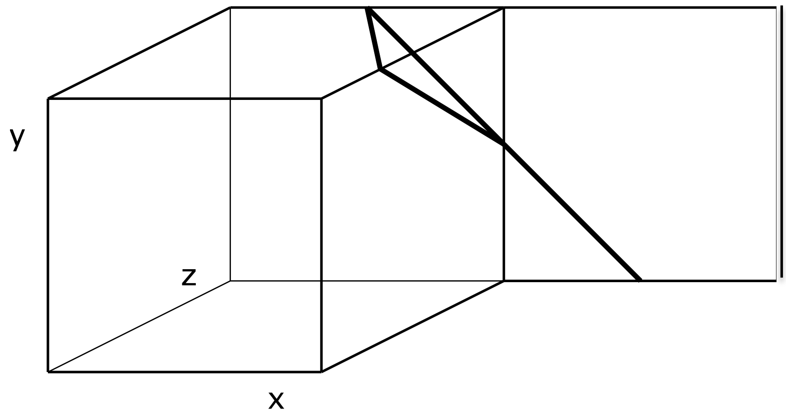
$$\int_{(r_0,0,0)}^{(0,r_0,0)} \varphi(n(s))\, ds = \frac{1}{\sqrt{3}} r_0 \quad \text{and therefore} \quad \int_{\partial A} \varphi(n(s))\, ds = \sqrt{3} r_0$$

Hence, at least in this example,

$$\int_{\partial A} \varphi(n(s)) ds = \mu_L'(r_0).$$

Example 2.3. In contrast to Example 2.2, level sets of sublattices in $\Re^3$ can have "spikes". If $L = [0,1]^3 \cup \{(x,y,1) : 1 \le x \le 2, 0 \le y \le 1\}$ then the level set at $\frac{3}{2}$ is equal to

$$L_{\frac{3}{2}} = \left\{ (x,y,z) : 0 \le x,y,z \le 1 \text{ and } x+y+z = \frac{3}{2} \right\} \cup \\ \cup \left\{ (x,y,1) : \frac{1}{2} \le x \le \frac{3}{2} \text{ and } y = \frac{1}{2} - x \right\}$$



Example 2.4. The formula $\int_{\partial L_{r_0}} \varphi(n_A(s))\,ds = \mu_L'(r_0)$ is not valid for bodies in $\Re^3$ that are not sublattices. For example, if $B = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$, then for $-1 \le r \le 1$ the level set $B_r = \{(x,y,z) : x+y+z = r \text{ and } x^2+y^2+z^2\}$ is a disk, and hence symmetry yields that

$$\int_{\partial B_r} \varphi(n(s))\,ds = 0.$$

Of course, the surface areas of B_r are not constant, and so $\mu_B'(r) \ne 0$.__
Further examples can be found at the end of section 4

3. Characterization of level sets of sublattices of $\Re^3$

label2
In this section, we will give a complete geometrical characterization of level sets of connected sublattices of $\Re^3$. If $x \in \Re^3$, then we let x_i be the i^{th} coordinate of x, i.e. $x = (x_1, x_2, x_3)$. For each fixed real number $r \in \Re$, we will frequently make use of the plane

$$E_r = \left\{ x \in \Re^3 : x_1 + x_2 + x_3 = r \right\}$$

Let L be a sublattice of $\Re^3$. We define the level function

$$\lambda : L \to \Re, \quad x \mapsto x_1 + x_2 + x_3$$

For each $r \in \Re$ we let $L_r = \{x \in L : \lambda(x) = r\} = L \cap E_r$. Moreover, we equip L with the distance induced by a slightly modified ℓ_1-norm on $\Re^3$:

$$d(x,y) = \frac{1}{2}(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|) = \frac{1}{2}(\lambda(x \vee y) - \lambda(x \wedge y))$$

If $x, y \in L$, and if $x \leq y$, then we denote the interval between x and y in L by

$$[x, y] = \{z \in L : x \leq z \leq y\}$$

The proof of the following proposition is an easy calculation:

Proposition 3.1. *For all $x, y \in L \subseteq \Re^3$ we have*

$$\lambda(x) + \lambda(y) = \lambda(x \wedge y) + \lambda(x \vee y).$$

Further, if $x, y \in L_r$ and if $d(x, y) = s$, then $x \wedge y \in L_{r-s}$ and $x \vee y \in L_{r+s}$.

Lemma 3.2. *If $L \subseteq \Re^3$ is a connected sublattice, if $r < s$, and if $L_r \neq \varnothing \neq L_s$, then for every $x \in L_s$ there is an element $y \in L_r$ so that $y < x$, and vice versa.*

Proof. Let $x \in L_s$ be given. Since $L_r \neq \varnothing$, there is an element $a \in L_r$. Let $z = a \wedge x$. Then $z \leq a$ and hence $\lambda(z) \leq r$. The interval $[z, x] \subseteq L$ is a retract of L under the map $u \mapsto (z \vee u) \wedge x$ and hence connected. Hence the image of $[z, x]$ under the map λ is connected in $\Re$. Since $\lambda(z) \leq r < s = \lambda(x)$, there is an element $y \in [z, x]$ so that $\lambda(y) = r$. ■

Proposition 3.3. *If $L \subseteq \Re^3$ is connected, then L_r is connected.*

Proof. Let $x, y \in L_r$ and assume that $x \neq y$. Either x or y agrees with $x \wedge y$ in exactly 2 coordinates; let us assume that x is that point. Then $x_1 \leq y_1$, $x_2 \leq y_2$, and $x_3 \geq y_3$, i.e. $x \wedge y = (x_1, x_2, y_3)$. Moreover, the intervals $[x \wedge y, x] \subseteq L$ and $[x \wedge y, y] \subseteq L$ are retracts of L, hence are connected. It follows that the interval $[x \wedge y, x]$ is a connected chain, and therefore

$$[x \wedge y, x] = \{(x_1, x_2, s) : y_3 \leq s \leq x_3\}.$$

We now define a map

$$\varphi : [x \wedge y, y] \to \Re^3, \quad z \mapsto (z_1, z_2, r - (z_1 + z_2)).$$

Then φ maps $[x \wedge y, y]$ into L_r: Clearly, by definition, the level of $\varphi(z)$ is equal to r. Moreover, $x_1 \leq z_1 \leq y_1$, $x_2 \leq z_2 \leq y_2$, and $z_3 = y_3$. Since $z \leq y$, the level of z is less than or equal to r and therefore $z_1 + z_2 + z_3 \leq r$. We conclude that $y_3 = z_3 \leq r - (z_1 + z_2) \leq r - (x_1 + x_2) = x_3$. Hence

$$(x_1, x_2, r - (z_1 + z_2)) \in [x \wedge y, x] \subseteq L,$$

and therefore $\varphi(z) = z \vee (x_1, x_2, r - (z_1 + z_2)) \in L$. Since $\varphi(y) = y$, since $\varphi(x \wedge y) = x$, and since the map φ is continuous, x and y belong to the connected set $\varphi([x \wedge y, y])$, hence to the same connected component. It follows that L_r is connected. ■

In order to formulate the next proposition, we will need the following definition:

Definition 3.4. A subset $A \subseteq E_r$ is called an *l-set*, if for every pair $x, y \in A$ such that $x_i = y_i$ for some $i \in \{1, 2, 3\}$, the line segment between x and y is contained in A.

For later application, we record

Proposition 3.5. *If $A \subseteq E_r$ is an l-set, then the relative interior of A in E_r is also an l-set.Let B be the relative interior of A in E_r, and let $x, y \in B$ so that $x_1 = y_1$. We have to show that the line segment between x and y also belongs to B.*

Proof. Let B be the relative interior of A in E_r, and let $x, y \in B$ so that $x_1 = y_1$. We have to show that the line segment between x and y also belongs to B.
First, we can find a number $\varepsilon > 0$ so that the sets

$$U = \{z \in E_r : \lambda (x - z) < \varepsilon\} \quad \text{and} \quad V = \{z \in E_r : \quad \lambda (y - z) < \varepsilon\}$$

both are contained in A. Let a be on the line segment between x and y. Then there is a number $0 \leq r \leq 1$ so that $a = rx + (1 - r) y$. The set

$$W = \{z \in E_r : \quad \lambda (a - z) < \varepsilon\}$$

is relatively open, and we would like to show that this set is contained in A. So let $b \in W$. If $c = x + (b - a)$ and $d = y + (b - a)$ then $\lambda (x - c) < \varepsilon$ and $\lambda (y - d) < \varepsilon$, hence $c \in U$ and $d \in V$, and therefore $c, d \in A$. Moreover, since $c_1 = d_1$, the line segment between c and d is contained in A. Finally,

$$rc + (1 - r) d = rx + (1 - r) y + (b - a) = a + b - a = b.$$

Therefore, $b \in A$. ■

Proposition 3.6. *If $L \subseteq \Re^3$ is connected, then each level set L_r is an l-set.*

Proof. We begin with the observation that for a connected sublattice $M \subseteq \Re^2$ each level set $M_r = \{(x_1, x_2) \in M : x_1 + x_2 = r\}$ is a convex set. Indeed, if $x, y \in M$, then each of the intervals $[x \wedge y, x], [x \wedge y, z] \subseteq M$ are chains parallel to Cartesian coordinate axes. As retracts of the connected lattice M, these chains are also connected, and hence generate the rectangle

$$R = \left\{z \in \Re^2 : x \wedge y \leq z \leq x \vee y\right\}$$

So $R \subseteq M$, and R of course contains the line segment between x and y.
Now let $x, y \in L_r$ be given, and assume without loss of generality that $x_1 < y_1$, $x_2 > y_2$ and $x_3 = y_3$. Then

$$M = \{z \in L : z_3 = x_3\}$$

is a union of an increasing family of intervals in L, each of which is connected as a retract of L. Hence M is connected. Since M is a sublattice of the plane $E = \{z \in \Re^3 : z_3 = x_3\}$, all level sets of M are convex, hence M contains the line segment between x and y, which is then also contained in L_r. Hence L_r is an l-set. ■

We are now going to show that each subset $A \subseteq \left\{x \in [0,1]^3 : \lambda (x) = r\right\}$ that is a connected l-set is indeed the level set of a connected sublattice. In order to prove this result, we need the following version of Bergman's Double Projection Theorem (see [2]).

If $L \subseteq \Re^n$ is a subset, and if $1 \le i, j \le n$, then we define

$$\pi_{i,j} : \Re^n \to \Re \times \Re, \quad x \mapsto (x_i, x_j), \quad \text{and} \quad L_{i,j} = \pi_{i,j}(L).$$

Theorem 3.7. (Bergman) *If $L \subseteq \Re^n$ is a sublattice, then*

$$L = \bigcap_{1 \le i < j \le n} \pi_{i,j}^{-1}(L_{i,j})$$

Of course, this theorem has the following consequence:

Corollary 3.8. *If $A \subseteq \Re^n$ is a subset, then the sublattice $\langle A \rangle$ generated by A can be calculated as $\langle A \rangle = \bigcap_{1 \le i < j \le n} \pi_{i,j}^{-1}(\langle A_{ij} \rangle)$.*

If $A \subseteq E_r$ is a connected l-set, then the last corollary suggest that in order to find the sublattice generated by A, we should first study the sublattices generated by $A_{i,j} = \{(x_i, x_j) : x \in A\}$.

Lemma 3.9. *Assume that $A \subseteq E_r$ is a connected l-set. Then each $A_{i,j}$ is connected. Moreover, let $a, b \in A_{i,j} \subseteq \Re^2$. If either $a_1 + a_2 = b_1 + b_2$ or $a_1 = b_1$ or $a_2 = b_2$, then the line segment between a and b belongs to $A_{i,j}$.*

Proof. Since $A_{i,j}$ is a continuous image of A, the connectivity of $A_{i,j}$ follows from the connectivity of A. Now assume that $a, b \in A_{i,j}$. Then we can find $x, y \in A$ so that $a_1 = x_i$, $a_2 = x_j$, $b_1 = y_i$, and $b_2 = y_j$. Let $i \ne k \ne j$. Then, if $a_1 + a_2 = b_1 + b_2$, it follows from $\lambda(x) = \lambda(y)$ that $x_k = y_k$. Hence the line segment between x and y belongs to A. Since the line segment between a and b is the image of the line segment between x and y, the line segment between a and b belongs to $A_{i,j}$. The other cases are treated similarly. ■

Lemma 3.10. *Let $A \subseteq \Re^2$ be a connected subset. Then the sublattice generated by A can be written as*

$$\langle A \rangle = A^+ \cup A^-,$$

where $A^+ = \{x \vee y : x, y \in A\}$ *and* $A^- = \{x \wedge y : x, y \in A\}$.

Proof. Clearly, both A^+ and A^- are contained in $\langle A \rangle$. Hence it is enough to verify that $A^+ \cup A^-$ is a sublattice.
Let $a, b \in A^+ \cup A^-$. By symmetry, it is enough to show that $a \vee b \in A^+ \cup A^-$. We may assume that a and b form an antichain, say $a_1 \le b_1$ and $b_2 \le a_2$. It follows that

$$a \vee b = (b_1, a_2).$$

There are three different cases to consider:

Case 1. $a, b \in A^+$. Pick $x, y, u, v \in A$ so that $a = x \vee y$ and $b = u \vee v$. Then $a \vee b = x \vee y \vee u \vee v$. Since $\Re^2$ has breadth 2, there are two elements $c, d \in \{x, y, u, v\}$ so that $a \vee b = c \vee d$. Hence $a \vee b \in A^+$.

Case 2. $a \in A^+$ and $b \in A^-$. In this case, pick $x, y, u, v \in A$ so that $a = x \vee y$ and $b = u \wedge v$. We may arrange it so that

$$x_1 \leq a_1 \leq b_1, \quad x_2 = a_2 \geq b_2, \quad y_1 = a_1 \leq b_1, \quad y_2 \leq a_2$$

and
$$u_1 = b_1 \geq a_1 \quad u_2 \geq b_2 \quad v_1 \geq b_1 \geq a_1 \quad v_2 = b_2 \leq a_2.$$

If $u_2 \leq a_2$, then $a \vee b = u \vee x \in A^+$. Hence we may assume that $a_2 < u_2$.

Similarly, if $b = v$, then $a \vee b = x \vee v \in A^+$, hence we may assume that $b_1 < v_1$.

Consider the sets $\mathcal{A} = \{(b_1, t) : t \leq a_2\}$ and $\mathcal{B} = \{(s, a_2) : b_1 \leq s\}$.

Then $\mathcal{A} \cup \mathcal{B}$ cuts $\Re^2$ into two components:

One component is equal to $\{(s,t) : b_1 < s \text{ and } t < a_2\}$ and therefore contains v. The other component equals $\{(s,t) : b_1 > s \text{ or } t > a_2\}$ and hence contains x. Since both x and v belong to A and since A is connected, either $\mathcal{A}$ or $\mathcal{B}$ contains a point $w \in A$. If $w \in \mathcal{A}$, then $a \vee b = x \vee w \in A^+$, and if $w \in \mathcal{B}$, then $a \vee b = u \wedge w \in A^-$.

Case 3. $a, b \in A^-$. Pick $x, y, u, v \in A$ so that $a = x \wedge y$ and $b = u \wedge v$. Again, we may rename the elements in such a way that

$$a_1 = x_1, \quad a_2 \leq x_2, \quad a_1 \leq y_1, \quad a_2 = y_2,$$
$$b_1 = u_1, \quad b_2 \leq u_2, \quad b_1 \leq v_1, \quad b_2 = v_2.$$

If $y_1 \leq b_1$ and $u_2 \leq a_2$, then $a \vee b = y \vee u \in A^+$. Similarly, if $y_1 \geq b_1$ and $u_2 \geq a_2$, then $a \vee b = y \wedge u \in A^-$. Hence we may assume that either $y_1 \leq b_1$ and $u_2 \geq a_2$ or that $y_1 \geq b_1$ and $u_2 \leq a_2$. Since both cases are similar, we will only argue the case in which

$$y_1 \leq b_1 \text{ and } u_2 \geq a_2.$$

As before, we form the sets $\mathcal{A} = \{(b_1, t) : t \leq a_2\}$ and $\mathcal{B} = \{(s, a_2) : b_1 \leq s\}$ that separate $\Re^2$ into two components, one of which contains x and the other contains v. Again, since A is connected, either $\mathcal{A}$ or $\mathcal{B}$ contains a point $w \in A$. If $w \in \mathcal{A}$, then $a \vee b = y \vee w \in A^+$, and if $w \in \mathcal{B}$, then $a \vee b = u \wedge w \in A^-$.

This last lemma gives a method to describe the lattice generated by a connected l-set $A \subseteq E_r$. This lattice is given as

$$\langle A \rangle = \bigcap_{1 \leq i < j \leq 3} \pi_{i,j}^{-1} \left(\pi_{i,j}(A)^+ \cup \pi_{i,j}(A)^- \right). \qquad \blacksquare$$

Lemma 3.11. *Let $A \subseteq E_r$ be a connected l-set, and let $L = \langle A \rangle$ be the sublattice generated by L. Then $\{x \in L : \lambda(x) = r\} = A$.*

Proof. Clearly, $A \subseteq \{x \in L : \lambda(x) = r\}$. Conversely, assume that $x \in L$ is given so that $\lambda(x) = r$. Then, by the above characterization of $\langle A \rangle$ and since there are only three different two-element subsets of $\{1, 2, 3\}$, either we can find an index i so that $\pi_{i,k}(x) \in \pi_{i,k}(A)^+$ for all $k \neq i$, or we can find an index i so that $\pi_{i,k}(x) \in \pi_{i,k}(A)^-$ for all $k \neq i$. We will argue the first case only, since both cases are dual to each other. Hence, after renumbering the indices, we may assume that $\pi_{1,2}(x) \in \pi_{1,2}(A)^+$ and $\pi_{1,3}(x) \in \pi_{1,3}(A)^+$. Therefore, there are elements $u, v, a, b \in A$ so that

$$\pi_{1,2}(x) = \pi_{1,2}(u) \vee \pi_{1,2}(v) \quad \text{and} \quad \pi_{1,3}(x) = \pi_{1,3}(a) \vee \pi_{1,3}(b)$$

And, after interchanging u and v or a and b, respectively, we may assume that

$$x_1 = u_1, \quad x_2 \geq u_2, \quad x_1 = a_1, \quad x_3 \geq a_3.$$

Since $x_1 + x_2 + x_3 = u_1 + u_2 + u_3 = a_1 + a_2 + a_3$, it follows that

$$x_2 + x_3 = u_2 + u_3 = a_2 + a_3, \quad a_2 \leq x_2 \leq u_2, \quad a_3 \leq x_3 \leq u_3$$

If we let $r = \dfrac{u_2 - x_2}{u_2 - a_2}$, $ra_2 + (1-r)\,u_2 = u_2 + r\,(a_2 - u_2)$, then

$$\begin{aligned} ra_1 + (1-r)\,u_1 &= rx_1 + (1-r)\,x_1 = x_1 \\ ra_2 + (1-r)\,u_2 &= u_2 + r\,(a_2 - u_2) = x_2 \\ ra_3 + (1-r)\,u_3 &= r\,(x_2 + x_3 - a_2) + (1-r)\,(x_2 + x_3 - u_2) = x_3. \end{aligned}$$

This implies that x is on the line segment between a and u, and therefore $x \in A$. It might be interesting to note that this is the first time where we use that A is an l-set. ■

Lemma 3.12. *If A is connected, then the sublattice generated by A is connected.*

Proof. Let L be the sublattice generated by A, and let L_0 be a connected component of L so that $L_0 \cap A \neq \varnothing$. Then, since A is connected, it follows that $A \subseteq L_0$. Moreover, L_0 is a sublattice, hence $L_0 = L$. It follows that L is connected. ■

We summarize these results in the following

Theorem 3.13. *Let $L \subseteq \Re^3$ be a connected sublattice, and let $r \in \Re$. Then $L_r = \{x \in L : \lambda(x) = r\}$ is a connected l-set. Conversely, if $A \subseteq E_r$ is a connected l-set, then the lattice L_A generated by A is connected, and $A = \{x \in L_A : \lambda(x) = r\}$.*

In the remainder of this section, we shall give some additional information on the structure of a lattice generated by a level set. Hence, let $r_0 > 0$, let $A \subseteq E_{r_0}$ be a connected l-set, and let L be the sublattice generated by A. Our goal is to show that every $x \in L$ with $\lambda(x) < r_0$ can be written as the infimum of three elements $a, b, c \in A$, and, dually, every $y \in L$ with $\lambda(y) > r_0$ can be obtained as the supremum of three elements $u, v, w \in A$.

The cube $L = [0, r_0]^3$ may serve as an illustration of this statement: The level set L_{r_0} generates L, and indeed every $x \in L$ is either the infimum or the supremum of three elements of L_{r_0}, depending on whether $\lambda(x) < r_0$ or $\lambda(x) > r_0$.

Lemma 3.14. *Let $A \subseteq E_r$ be a connected l-set, let $a_1, b_1, a_2, b_2, a_3, b_3 \in A$ be given and let $x = (a_1 \wedge a_2 \wedge a_3) \vee (b_1 \wedge b_2 \wedge b_3)$. If $\lambda(x) < r$, then there are elements $c_1, c_2, c_3 \in A$ so that $x = c_1 \wedge c_2 \wedge c_3$.*

Proof. Since $\Re^3$ is distributive, we have

$$\begin{aligned} x = {} & (a_1 \vee b_1) \wedge (a_1 \vee b_2) \wedge (a_1 \vee b_3) \wedge \\ & \wedge (a_2 \vee b_1) \wedge (a_2 \vee b_2) \wedge (a_2 \vee b_3) \wedge \\ & \wedge (a_3 \vee b_1) \wedge (a_3 \vee b_2) \wedge (a_3 \vee b_3). \end{aligned}$$

For each pair of indices $i, j \in \{1,2,3\}$ we have $\lambda(a_i \vee b_j) \geq r$. Since the intervals $[x, a_i \vee b_j]$ are connected, and since λ is continuous, there has to be an element $x_{i,j} \in [x, a_i \vee b_j]$ with $\lambda(x_{i,j}) = r$. We conclude from Theorem 3.13 that $x_{i,j} \in A$. Obviously $x = \inf_{1 \leq i,j \leq 3} x_{i,j}$, and since $L = \langle A \rangle$ has breadth 3, there are $c_1, c_2, c_3 \in \{x_{i,j} : 1 \leq i, j \leq 3\}$ so that $x = c_1 \wedge c_2 \wedge c_3$. ■

Lemma 3.15. *Assume that $A \subseteq E_r$ is a connected l-set, let $a_1, b_1, a_2, b_2, a_3, b_3 \in A$ be given and let $x = (a_1 \wedge a_2 \wedge a_3) \vee (b_1 \wedge b_2 \wedge b_3)$. If $\lambda(x) > r$, then there are elements $c_1, c_2 \in A$ so that $x = c_1 \vee c_2$.*

Proof. Again, since $\lambda(a_1 \wedge a_2 \wedge a_3), \lambda(b_1 \wedge b_2 \wedge b_3) \leq r$ and since $L = \langle A \rangle$ is connected, there are elements $c_1 \in [a_1 \wedge a_2 \wedge a_3, x]$ and $c_2 \in [b_1 \wedge b_2 \wedge b_3, x]$ such that $\lambda(c_1) = \lambda(c_2) = r$. From Theorem 3.13 we conclude that $c_1, c_2 \in A$. Clearly, $x = c_1 \vee c_2$. ■

Lemma 3.16. *Assume that $A \subseteq E_r$ is a connected l-set, let $a_1, b_1, a_2, b_2, a_3, b_3 \in A$ be given and let $x = (a_1 \wedge a_2 \wedge a_3) \vee (b_1 \vee b_2 \vee b_3)$. Then there are elements $c_1, c_2, c_3 \in A$ so that $x = c_1 \vee c_2 \vee c_3$.*

Proof. Applying our previous argumentation to the interval $[a_1 \wedge a_2 \wedge a_3, x]$ yields an element $b \in A$ so that $a_1 \wedge a_2 \wedge a_3 \leq b \leq x$. The four elements $b, b_1, b_2, b_3 \in A$ then give $x = b \vee b_1 \vee b_2 \vee b_3$. Since $L = \langle A \rangle$ has breadth 3, we can find three elements $c_1, c_2, c_3 \in \{b, b_1, b_2, b_3\}$ so that $x = c_1 \vee c_2 \vee c_3$. ■

It follows from Lemmas 3.14, 3.15, 3.16 and their duals that $\{a \wedge b \wedge c : a, b, c \in A\} \cup \{u \vee v \vee w : u, v, w \in A\}$ is a sublattice containing A. Hence this set is equal to L. This yields

Theorem 3.17. *Let $A \subseteq E_r$ be a connected l-set, let $L = \langle A \rangle$ be the sublattice generated by A, and let $x \in L$ be given. If $\lambda(x) \leq r$, then there are three elements $a, b, c \in A$ so that $x = a \wedge b \wedge c$. Dually, if $\lambda(x) \geq r$, then there are three elements $u, v, w \in A$ so that $x = u \vee v \vee w$.*

As a corollary to Theorem 3.17 we obtain

Corollary 3.18. *Let $A \subseteq E_{r_0}$ be a connected l-set and let $L = \langle A \rangle$ be the lattice generated by A. If $r \leq s \leq r_0$, then L_r is contained in the lattice generated by L_s.*

Proof. If $x \in L_r$. Then we can find $a_1, a_2, a_3 \in L_{r_0} = A$ so that $x = a_1 \wedge a_2 \wedge a_3$. As before, the intervals $[x, a_i]$ intersect L_s. Pick $b_i \in [x, a_i] \cap L_s$. Then $x = b_1 \wedge b_2 \wedge b_3$. ■

Theorem 3.19. *Let $A \subseteq E_r$ be a connected l-set. If A is relatively open in E_r, then the sublattice generated by A is open.*

Proof. By Bergman's double projection Theorem 3.7 and Lemma 3.10 it is enough to show that $\langle \pi_{i,j}(A) \rangle = \pi_{i,j}(A)^+ \cup \pi_{i,j}(A)^-$ is open for every pair i, j of indices. First, note that $\pi_{i,j}(A)$ is an open subset of $\Re^2$, since the map $(x_1, x_2, x_3) \mapsto (x_i, x_j)$ is a homeomorphism between the hyperplane E_r and $\Re^2$. Since the operations $\vee, \wedge : \Re^2 \to \Re^2$ are open maps, $\langle \pi_{i,j}(A) \rangle = \pi_{i,j}(A)^+ \cup \pi_{i,j}(A)^-$ is open. ■

For later application, we record the following interesting separation theorem:

Theorem 3.20. *Let $A_1, A_2 \subseteq E_r$ be two connected l-sets and let $L_1 = \langle A_1 \rangle$ and $L_2 = \langle A_2 \rangle$ be the lattices generated by A_1 and A_2, respectively. If there is a vector $n \in \Re^3$ which is not a multiple of $(1,1,1)$ such that $a \cdot n \geq 0$ for all $a \in A_1$ and $a \cdot n \leq 0$ for $a \in A_2$, then there exists a plane (not necessarily containing the origin) that separates L_1 and L_2. Moreover, we can choose this plane in such a way that both half spaces determined by this plane are sublattices of $\Re^3$.*

Proof. After renumbering the coordinates, we may assume that $n_1 \leq n_2 \leq n_3$. Since n is not a multiple of $(1,1,1)$, we have $n_1 < n_2$ or $n_2 < n_3$. Let $\mathcal{L} = \{x \in E_r : \quad n \cdot x = 0\}$. Then $\mathcal{L}$ is a line in $\Re^3$. Since the projection $\pi_{1,3} : \Re^3 \to \Re^2$ maps E_r bijectively onto $\Re^2$, the line $\pi_{1,3}(\mathcal{L})$ separates $\pi_{1,3}(A_1)$ and $\pi_{1,3}(A_2)$. If $(x_1, x_3) \in \pi_{1,3}(\mathcal{L})$, then there is an $x_2 \in \Re$ so that $(x_1, x_2, x_3) \in \mathcal{L}$. It follows that

$$x_1 + x_2 + x_3 = r, \quad n_1 x_1 + n_2 x_2 + n_3 x_3 = 0,$$

and therefore

$$(n_1 - n_2) x_1 + (n_3 - n_2) x_3 = n_1 x_1 + n_2 x_2 + n_3 x_3 - n_2 (x_1 + x_2 + x_3) = -r n_2.$$

Hence the vector $(n_1 - n_2, n_3 - n_2)$ is orthogonal to $\pi_{1,3}(\mathcal{L})$, and since $n_1 - n_2 \leq 0$ and $n_3 - n_2 \geq 0$, both half spaces determined by $\pi_{1,3}(\mathcal{L})$ are sublattices, and the plane $\pi_{1,3}^{-1} \pi_{1,3}(\mathcal{L})$ separates L_1 and L_2 by corollary 3.8. ■

4. Level sets in full sublattices of $\Re^3$

In this section, we will be concerned with full sublattices $L \subseteq \Re^3$ and their level sets. A compact sublattice $L \subseteq \Re^n$ is called a full sublattice, if the topological interior of L is connected and dense in L. In this case, the interior L^o of L is a sublattice of L (see [10] or [11]). We will use the notations

$$\bot = \inf(L), \qquad \top = \sup(L)$$

Lemma 4.1. *Let $L \subseteq \Re^3$ be a full sublattice, let $x \in L^\circ$ belong to the interior of L, and assume that $\lambda(x) < r < \lambda(\top)$. Then there is an element $y \in L^\circ$ such that $x < y$ and $\lambda(y) = r$.*

Proof. Since the interior of L is dense and connected, there is an element $z \in L^\circ$ such that $\lambda(z) = r$. If $x \leq z$, then pick $y = z$. Else, if $x \not\leq z$, then $z < x \vee z$ and hence $\lambda(x \vee z) > r$. Since the set $\{y \in L^\circ : x \leq y \leq x \vee z\}$ is a retract of L° (under the map $y \mapsto (x \vee y) \wedge (x \vee z)$), it follows that this set is also connected, and hence contains a point y so that $\lambda(y) = r$. ■

Lemma 4.2. *If $L \subseteq \Re^3$ is a full sublattice and if $\lambda(\bot) < r < \lambda(\top)$, then the set $\{x \in L^\circ : \lambda(x) = r\}$ is a connected l-set, and this set is equal to the relative interior of L_r in E_r.*

Proof. L° is a connected sublattice; so the first assertion follows from Theorem 3.13. For the proof of the second part of the lemma, we observe that $\{x \in L^\circ : \lambda(x) = r\}$ is contained in the relative interior of L_r in E_r.

Conversely, assume that y belongs to the relative interior of L_r in the plane E_r. Then, since

$$\bigcap_{\varepsilon>0}\{x \in E_r : \quad x_i \geq y_i - \varepsilon \text{ for } 1 \leq i \leq 3\} = \{y\}$$

there is a number $\varepsilon_0 > 0$ so that $\{x \in E_r : \quad x_i > y_i - \varepsilon_0 \text{ for } 1 \leq i \leq 3\}$ is contained in the relative interior of L_r in E_r. Hence

$$D = \{x \in E_r : \quad x_i > y_i - \varepsilon_0 \text{ for } 1 \leq i \leq 3\} \subseteq L_r$$

Since D is a connected l-set that is relatively open in E_r, Theorem 3.19 implies that the sublattice $\langle D \rangle$ is open and contained L. It follows that y belongs to L°. ■

Lemma 4.3. *If $L \subseteq \Re^3$ be a full sublattice and if $\lambda(\bot) < r < \lambda(\top)$, then the set $\{x \in L^\circ : \lambda(x) = r\}$ is dense in $\{x \in L : \lambda(x) = r\}$.*

Proof. Let $x \in L$ be given, and assume that $\lambda(x) = r$. Since the interior of L is dense in L, there is a sequence $(x_n)_n$ in L°such that $\lim_{n\to\infty} x_n = x$. By selecting a subsequence, we may assume that either $\lambda(x_n) \leq r$ for all n or $\lambda(x_n) \geq r$ for all n. By duality, we may assume the first case. Then, by Lemma 4.1, for each n there is a y_n so that $x_n \leq y_n$, $y_n \in L^\circ$, and $\lambda(y_n) = r$. After selecting another subsequence, we may assume that $\lim_{n\to\infty} y_n$ exists. Clearly, $\lambda(y) = r$ and $x = \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$. It follows that $x = \lim_{n\to\infty} y_n$. ■

The next theorem provides a complete characterization of level sets of full sublattices of $\Re^3$:

Theorem 4.4. *Let $L \subseteq \Re^3$ be a full sublattice, and let $\lambda(\bot) < r < \lambda(\top)$. Then $L_r \cap L^o$ is a connected l-set that is dense in L_r, and equal to the relative interior of L_r in E_r.*
Conversely, if $A \subseteq E_r$ is a connected l-set such that the interior of A in E_r is connected and dense in A, then the sublattice L generated by A is full, and the relative interior of A is equal to $A \cap L^\circ$.

Proof. The first part follows from Lemmas 4.2 and 4.3.
Conversely, assume that $A \subseteq E_r$ is a connected l-set such that the interior of A in E_r is connected and dense in A. Let $int(A)$ be the relative interior of A. It follows from Proposition 3.5 that $int(A)$ is also an l-set. Let L be the sublattice generated by A and let V be the sublattice generated by $int(A)$. Then, by Theorems 3.13 and 3.19, V is open and connected, and the closure of V contains A, hence L. It follows from Theorem 4.5 of [11] that $L^\circ = V$. We conclude that L is a full sublattice. The last statement follows from $int(A) = \{x \in V : \lambda(x) = r\} = \{x \in L^\circ : \lambda(x) = r\} = A \cap L^\circ$. ■

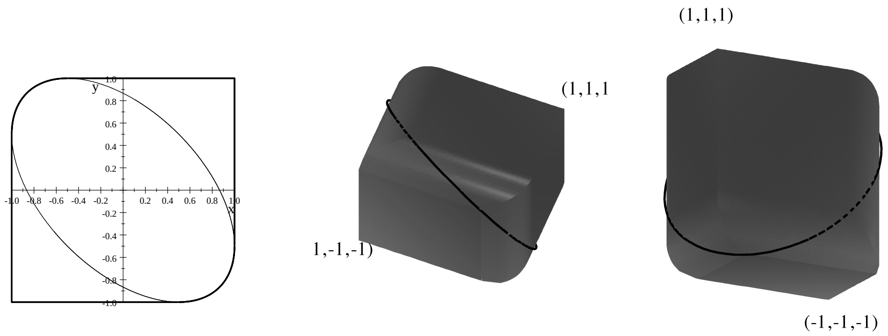
The following two examples illustrate the previous results:

Example 4.5. Since every convex subset $A \subseteq E_r$ is an l-set, every such convex subset can be obtained as the level set in a sublattice $L \subseteq \Re^3$. Moreover, if A is not a line segment, then we can realize A as a level set in a full sublattice. Especially, this has the somewhat surprising consequence that a disk is a level set in a full sublattice, although lattices are normally thought to have lots of corners and are typically not "differentiable".

The following illustrations show the lattice generated by the disk

$$\{(x, y, z) : \ x + y + z = 0 \ \text{and} \ x^2 + y^2 + z^2 \leq 1\}$$

together with its double projections into the planes.



Example 4.6. The results of this paper cannot easily generalized to other dimensions. For example, in $\Re^4$ the convex set

$$A = \{(w, x, y, z) : w + x + y + z = 0 \text{ and } w^2 + x^2 + y^2 + z^2 \leq 2\}$$

is not a level set of a sublattice of $\Re^4$. Let L be the sublattice generated by A. Since $(1, 0, 0, -1), (0, 1, 0, -1) \in L,$ it follows that

$$(1, 0, 0, -1) \vee (0, 1, 0, -1) = (1, 1, 0, -1) \in L.$$

Also, $(1, 1, -1, 0) \in L,$ and hence $(1, 1, 0, -1) \wedge (1, 1, -1, 0) = (1, 1, -1, -1) \in L.$ So L contains an element at level 0 which does not belong to A.

Without proof we remark that for full sublattices $L \subseteq \Re^n$, $n \geq 4$, the outer normals to a level set are – up to permutations of coordinates – of the form $\vec{n} = (a, b, \ldots, b, c)$ where $a \leq b \leq c$.

5. The boundary of level sets

In this section, we will show that the boundary of a level set is a piecewise differentiable closed curve. More precisely, we will find that the boundary of a level set is the union of twelve differentiable curves. We will need the following proposition:

Proposition 5.1. *Let $f, g : [a, b] \to \Re$ be two monotone increasing functions such that $f + g$ is a continuously differentiable function. Then*

(1) f and g are continuous.

(2) f and g are differentiable almost everywhere.

(3) The functions f' and g' are Lebesgue measurable and

(4) $\int_a^b f'(t)\, dt = f(b) - f(a)$ as well as $\int_a^b g'(t)\, dt = g(b) - g(a)$.

In particular, if f and g are two monotone decreasing functions with the property that $f(x) + g(x) + x = c$ for a certain constant c, then f and g are continuous, f' and g' exist almost everywhere and are Lebesgue measurable, and $\int_a^b f'(t)dt = f(b) - f(a)$ as well as $\int_a^b g'(t) = g(b) - g(a)$.

Proof. (1) Let $M \subseteq [a, b]$. Since the function $h = f + g$ is continuous, it follows that $\inf h(M) = h(\inf M)$ and $\sup h(M) = h(\sup M)$. Also, since f and

g are monotone increasing, we have $f(\inf M) \leq \inf f(M)$, $g(\inf M) \leq \inf g(M)$, $\sup f(M) \leq f(\sup M)$ and $\sup g(M) \leq g(\sup M)$. None of the last four inequalities can be strict: If one of f or g had a jump, then $f+g$ also has a jump, contradicting the continuity of $f+g$. Hence both functions f and g are continuous.

(2) and (3) follow immediately from (1) (see Theorem 5.3 in [15])

(4) If f is any continuous, monotone increasing map, then $\int_a^b f'(t)\,dt \leq f(b) - f(a)$. If we had strict inequality, then we would arrive at the contradiction

$$\begin{aligned} h(b) - h(a) &= \int_a^b h'(t)\,dt = \int_a^b f'(t)\,dt + \int_a^b g'(t)\,dt \\ &< f(b) - f(a) + g(b) - g(a) = h(b) - h(a). \end{aligned}$$ ■

Let $A \subseteq E_{r_0}$ be a compact, connected l-set. The following points and numbers will be of interest:

$$\begin{aligned} \xi_{\min} &= \min\{x_1 : x \in A\} & \xi_{\max} &= \max\{x_1 : x \in A\} \\ \eta_{\min} &= \min\{x_2 : x \in A\} & \eta_{\max} &= \max\{x_2 : x \in A\} \\ \zeta_{\min} &= \min\{x_3 : x \in A\} & \zeta_{\max} &= \max\{x_3 : x \in A\} \end{aligned}$$

$$\begin{aligned}
x^y_{\max} &= (\xi_{\max}, \min\{x_2 : x \in A, x_1 = \xi_{\max}\}, \max\{x_3 : x \in A, x_1 = \xi_{\max}\}) \\
x^z_{\max} &= (\xi_{\max}, \max\{x_2 : x \in A, x_1 = \xi_{\max}\}, \min\{x_3 : x \in A, x_1 = \xi_{\max}\}) \\
x^y_{\min} &= (\xi_{\min}, \max\{x_2 : x \in A, x_1 = \xi_{\min}\}, \min\{x_3 : x \in A, x_1 = \xi_{\min}\}) \\
x^z_{\min} &= (\xi_{\min}, \min\{x_2 : x \in A, x_1 = \xi_{\min}\}, \max\{x_3 : x \in A, x_1 = \xi_{\min}\}) \\
y^x_{\max} &= (\min\{x_1 : x \in A, x_2 = \eta_{\max}\}, \eta_{\max}, \max\{x_3 : x \in A, x_2 = \eta_{\max}\}) \\
y^z_{\max} &= (\max\{x_1 : x \in A, x_2 = \eta_{\max}\}, \eta_{\max}, \min\{x_3 : x \in A, x_2 = \eta_{\max}\}) \\
y^x_{\min} &= (\max\{x_1 : x \in A, x_2 = \eta_{\min}\}, \eta_{\min}, \min\{x_3 : x \in A, x_2 = \eta_{\min}\}) \\
y^z_{\min} &= (\min\{x_1 : x \in A, x_2 = \eta_{\min}\}, \eta_{\min}, \max\{x_3 : x \in A, x_2 = \eta_{\min}\}) \\
z^x_{\max} &= (\min\{x_1 : x \in A, x_3 = \zeta_{\max}\}, \max\{x_2 : x \in A, x_3 = \zeta_{\max}\}, \zeta_{\max}) \\
z^y_{\max} &= (\max\{x_1 : x \in A, x_3 = \zeta_{\max}\}, \min\{x_2 : x \in A, x_3 = \zeta_{\max}\}, \zeta_{\max}) \\
z^x_{\min} &= (\max\{x_1 : x \in A, x_3 = \zeta_{\min}\}, \min\{x_2 : x \in A, x_3 = \zeta_{\min}\}, \zeta_{\min}) \\
z^y_{\min} &= (\min\{x_1 : x \in A, x_3 = \zeta_{\min}\}, \max\{x_2 : x \in A, x_3 = \zeta_{\min}\}, \zeta_{\min})
\end{aligned}$$

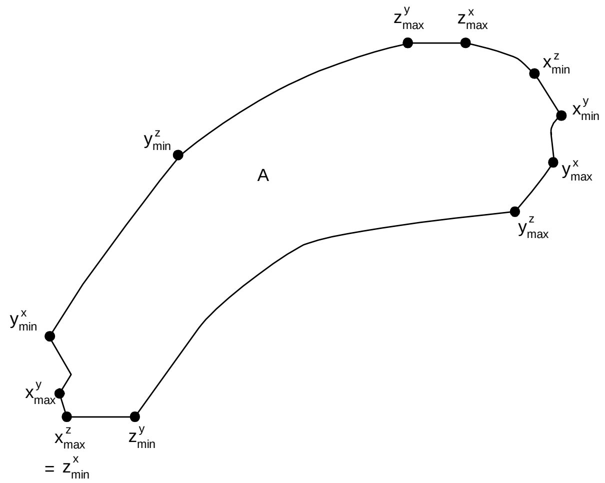
This leads to 6 line segments ($0 \leq t \leq 1$):

$$\begin{aligned}
X_{\max}(t) &= (1-t)\,x^y_{\max} + t x^z_{\max}; \; X_{\min}(t) = (1-t)\,x^y_{\min} + t x^z_{\min} \\
Y_{\max}(t) &= (1-t)\,y^z_{\max} + t y^x_{\max}; \; Y_{\min}(t) = (1-t)\,y^z_{\min} + t y^x_{\min} \\
Z_{\max}(t) &= (1-t)\,z^x_{\max} + t z^y_{\max}; \; Z_{\min}(t) = (1-t)\,z^x_{\min} + t z^y_{\min}.
\end{aligned}$$

The following figure illustrates the situation. Of course, several of the arcs shown in this picture can be degenerate; several of the points could coincide. The line segments between $x^z_{\min}$ and $x^y_{\min}$, as well as the line segments between the other pairs of corresponding points, are on the boundary of A, because they all have maximal or minimal first, second, or third coordinates. In order to describe the boundary points 'between' $x^y_{\min}$ and $y^x_{\max}$, 'between' $z^y_{\min}$ and $y^z_{\max}$, etc., we introduce the following functions: First, note that

$$\begin{aligned}(x^y_{\min})_3 &= r_0 - \xi_{\min} - \max\{x_2 : x \in A,\ x_1 = \xi_{\min}\} \geq r_0 - \xi_{\min} - \eta_{\max} \\ &\geq r_0 - \min\{x_1 : x \in A, x_2 = \eta_{\max}\} - \eta_{\max} = (y^x_{\max})_3\,,\end{aligned}$$

and corresponding inequalities hold for the other pairs of 'adjacent' points on the boundary of A.



Further, define

$$\begin{aligned}
\gamma^1_2(\xi) &= \min\{x_2 : x \in A, x_1 = \xi\} \text{ for } \xi_{\min} \leq \xi \leq \xi_{\max} \\
\gamma^1_3(\xi) &= \max\{x_3 : x \in A, x_1 = \xi\} \text{ for } \xi_{\min} \leq \xi \leq \xi_{\max} \\
\sigma^1_2(\xi) &= \max\{x_2 : x \in A, x_1 = \xi\} \text{ for } \xi_{\min} \leq \xi \leq \xi_{\max} \\
\sigma^1_3(\xi) &= \min\{x_3 : x \in A, x_1 = \xi\} \text{ for } \xi_{\min} \leq \xi \leq \xi_{\max} \\
\gamma^2_1(\eta) &= \max\{x_1 : x \in A, x_2 = \eta\} \text{ for } \eta_{\min} \leq \eta \leq \eta_{\max} \\
\gamma^2_3(\eta) &= \min\{x_3 : x \in A, x_2 = \eta\} \text{ for } \eta_{\min} \leq \eta \leq \eta_{\max} \\
\sigma^2_1(\eta) &= \min\{x_1 : x \in A, x_2 = \eta\} \text{ for } \eta_{\min} \leq \eta \leq \eta_{\max} \\
\sigma^2_3(\eta) &= \max\{x_3 : x \in A, x_2 = \eta\} \text{ for } \eta_{\min} \leq \eta \leq \eta_{\max} \\
\gamma^3_1(\zeta) &= \min\{x_1 : x \in A, x_3 = \zeta\} \text{ for } \zeta_{\min} \leq \zeta \leq \zeta_{\max} \\
\gamma^3_2(\zeta) &= \max\{x_2 : x \in A, x_3 = \zeta\} \text{ for } \zeta_{\min} \leq \zeta \leq \zeta_{\max} \\
\sigma^3_1(\zeta) &= \max\{x_1 : x \in A, x_3 = \zeta\} \text{ for } \zeta_{\min} \leq \zeta \leq \zeta_{\max} \\
\sigma^3_2(\zeta) &= \min\{x_2 : x \in A, x_3 = \zeta\} \text{ for } \zeta_{\min} \leq \zeta \leq \zeta_{\max}
\end{aligned}$$

This leads to 6 more curves. In the following, note that the direction of the curves σ_i has to be reversed to keep the positive orientation around the boundary of A.

$$\gamma_1(\xi) = \left(\xi, \gamma^1_2(\xi), \gamma^1_3(\xi)\right) \quad \text{for } (z^y_{\max})_1 \leq \xi \leq (y^z_{\min})_1$$

$$\begin{aligned}
\sigma_3(\zeta) &= \left(\sigma_1^3(-\zeta), \sigma_2^3(-\zeta), -\zeta\right) \text{ for } -(y^x_{\min})_3 \le \zeta \le -(x^y_{\max})_3 \\
\gamma_2(\eta) &= \left(\gamma_1^2(\eta), \eta, \gamma_3^2(\eta)\right) \quad \text{for } (x^z_{\max})_2 \le \eta \le (z^x_{\min})_2 \\
\sigma_1(\xi) &= \left(-\xi, \sigma_2^1(-\xi), \sigma_3^1(-\xi)\right) \quad \text{for } -(z^y_{\min})_1 \le \xi \le -(y^z_{\max})_1 \\
\gamma_3(\zeta) &= \left(\gamma_1^3(\zeta), \gamma_2^3(\zeta), \zeta\right) \quad \text{for } (y^x_{\max})_3 \le \zeta \le (x^y_{\min})_3 \\
\sigma_2(\eta) &= \left(\sigma_1^2(-\eta), -\eta, \sigma_3^2(-\eta)\right) \quad \text{for } -(x^z_{\min})_2 \le \eta \le -(z^x_{\max})_2
\end{aligned}$$

Let δ be the concatenation of the 12 curves obtained so far. We obtain a closed curve:

$$\begin{aligned}
\delta = X_{\max} \to \gamma_2 \to Z_{\min} \to \sigma_1 \to Y_{\max} \to \gamma_3 \to \\
\to X_{\min} \to \sigma_2 \to Z_{\max} \to \gamma_1 \to Y_{\min} \to \sigma_3
\end{aligned}$$

We will now show that δ is continuous and differentiable almost everywhere, and that all coordinate functions are integrable. Moreover, we will see that the image of δ is the topological boundary of A in the plane E_{r_0}.

Lemma 5.2. *All the functions σ_j^i and γ_j^i are monotone decreasing. Moreover, if $\{i,j,k\} = \{1,2,3\}$, then $\gamma_j^i(\alpha) + \gamma_k^i(\alpha) + \alpha = r_0$ and $\sigma_j^i(\alpha) + \sigma_k^i(\alpha) + \alpha = r_0$. Especially, all the functions σ_j^i and γ_j^i are continuous, differentiable almost everywhere functions whose derivatives are Lebesgue measurable with*

$$\int_a^b \frac{d}{dt}\left(\sigma_j^i(t)\right) dt = \sigma_j^i(b) - \sigma_j^i(a) \quad \text{and} \quad \int_a^b \frac{d}{dt}\left(\gamma_j^i(t)\right) dt = \gamma_j^i(b) - \gamma_j^i(a).$$

Proof. The only non-trivial statement of this lemma says that all the functions are monotone decreasing. Since all the definitions are similar, it suffices to verify that γ_1^3 and γ_2^3 are decreasing. First, note that for every $(y^x_{\max}) \le \zeta \le (x^y_{\min})_3$ the set $\{x \in A : x_3 = \zeta\}$ is non-empty, since otherwise the line $\{x \in E_{r_0} : x_3 = \zeta\}$ would separate A into two closed sets $A_1 = \{x \in A : x_3 < \zeta\}$ and $A_2 = \{x \in A : x_3 > \zeta\}$, neither of which is empty since $y^z_{\max} \in A_1$ and $x^y_{\min} \in A_2$, contradicting the connectivity of A. Especially, for every ζ in the domain of γ_1^3 we have $(\gamma_1^3(\zeta), \gamma_2^3(\zeta), \zeta) \in A$.

Now let
$$(y^x_{\max})_3 \le \zeta_1 < \zeta_2 \le (x^y_{\min})_3.$$

Assume that none of the points $(u, r_0 - u - \zeta_2, \zeta_2)$ with $u \le \gamma_1^3(\zeta_1)$ belonged to A. Then especially $(\gamma_1^3(\zeta_1), r_0 - \gamma_1^3(\zeta_1) - \zeta_2, \zeta_2) \notin A$. Since A is an l-set and since $(\gamma_1^3(\zeta_1), \gamma_2^3(\zeta_1), \zeta_1) \in A$, none of the points $(\gamma_1^3(\zeta_1), v, r_0 - \gamma_1^3(\zeta_1) - v)$ belongs to A for $v \le r_0 - \gamma_1^3(\zeta_1) - \zeta_2$. It follows that A can be written as the disjoint union of the closed sets

$$A_1 = \left\{x \in A : x_1 < \gamma_1^3(\zeta_1) \text{ and } x_3 > \zeta_2\right\} = \left\{x \in A : x_1 \le \gamma_1^3(\zeta_1) \text{ and } x_3 \ge \zeta_2\right\}$$

and $A_2 = A \setminus A_1$. By construction, $(\gamma_1^3(\zeta_1), \gamma_2^3(\zeta_1), \zeta_1) \in A_2$, hence $A_2 \ne \varnothing$. Moreover, since $(x^y_{\min})_1 = \xi_{\min} \le \gamma_1^3(\zeta_1)$and $(x^y_{\min})_3 \ge \zeta_2$, $x^y_{\min} \in A_1 \ne \varnothing$; contradicting the connectivity of A.
Hence there has to be an element $(u, r_0 - u - \zeta_2, \zeta_2) \in A$ with $u \le \gamma_1^3(\zeta_1)$, therefore $\gamma_1^3(\zeta_2) \le \gamma_1^3(\zeta_1)$.
It remains to show that $\gamma_2^3(\zeta_2) \le \gamma_2^3(\zeta_1)$. Assume that the claimed inequality were not true. Then none of the points of the form (u, v, ζ_1) with $v \ge \gamma_2^3(\zeta_2)$ belongs to

A, and hence, using the facts that A is an l-set and that $(\gamma_1^3(\zeta_2), \gamma_2^3(\zeta_2), \zeta_2) \in A$, none of the points of the $(u, \gamma_2^3(\zeta_2), r_0 - u - \gamma_2^3(\zeta_2))$ with $u \geq r_0 - \gamma_2^3(\zeta_2) - \zeta_1$ belongs to A. Again, we can write A as the disjoint union of the two non-empty closed sets $A_1 = \{x \in A : x_2 > \gamma_2^3(\zeta_2)$ and $x_3 < \zeta_1\}$ and $A_2 = A \setminus A_1$, contradicting the connectivity of A. ■

Lemma 5.3. *The image of δ coincides with the boundary $\partial A = A \backslash A^o$ of A in E_{r_0}.*

Proof. Let $x \in A$. Then the triangular convexity of A implies that each of the three lines $\ell_i = \{y \in E_{r_0}; y_i = x_i\}$ intersects the compact set A in an interval J_i. Clearly, the endpoints of each J_i belong to the boundary ∂A of A.

First, assume that x belongs to the image of the curve δ. If x belongs to the image of $X_{\max}$, then clearly x belongs to the boundary ∂A. If x belongs to the image of γ_1, then $x = \gamma_1(\xi) = (\xi, \gamma_2^1(\xi), \gamma_3^1(\xi))$ for $(z_{\max}^y)_1 \leq \xi \leq (y_{\min}^z)_1$.So $x_1 = \xi$ and $x_2 = \gamma_2^1(x_1) = \max\{y_2 : y \in A, y_1 = x_1\}$, which implies that x is one of the endpoints of the interval J_1, and so x belongs to the boundary of A. - The remaining 10 cases are handled accordingly.

Conversely, assume that x belongs to the boundary of A. If x is not one of the endpoints of the intervals J_i, then x would be the midpoint of open 3 open intervals I_i, each of which is contained in $J_i \subseteq A$. The triangular convexity of A would imply that x is the center of an open hexagon which is contained in A, and so $x \in A^o$, contradicting the fact that $x \in \partial A$.

So by symmetry, we may assume that x is an endpoint of $J_1 = \{y \in A : y_1 = x_1\}$, so $x_2 = \max\{y_2 : y \in J_1\}$ or $x_2 = \min\{y_2 : y \in J_1\}$. Again by symmetry, we may assume that $x_2 = \min\{y_2 : y \in A$ and $y_1 = x_1\}$ and so

$$x = \left(x_1, \gamma_2^1(x_1), \gamma_3^1(x_1)\right).$$

Case 1. If $x_1 = \xi_{\min}$, then $x = x_{\min}^z$, so x belongs to the image of δ.

Case 2. If $(x_{\min}^z)_1 = \xi_{\min} < x_1 < (z_{\max}^x)_1$, then we would like to show that x belongs to the image of σ_2.The line ℓ_1 divides the plane E_{r_0} into the two half-plane $\{y \in E_{r_0} : y_1 < x_1\}$ and $\{y \in E_{r_0} : y_1 > x_1\}$, one of which contains $z_{\max}^x$ and the other contains $x_{\min}^z$. The continuous curve σ_2 contains $z_{\max}^x$ and $x_{\min}^z$, and therefore intersects ℓ_1 in at least one point. Let η be the smallest number so that $\sigma_2(\eta) = (\sigma_1^2(\eta), \eta, \sigma_3^2(\eta)) \in \ell_1$. Then $x_1 = \sigma_1^2(\eta)$, $\gamma_2^1(x_1) \leq \eta \leq (x_{\min}^z)_2$ and

$$(z_{\max}^x)_2 = r_0 - (z_{\max}^x)_1 - \xi_{\max} \leq r_0 - x_1 - \gamma_3^1(x_1) = \gamma_2^1(x_1) \leq (x_{\min}^z)_2$$

So $(z_{\max}^x)_2 \leq \gamma_2^1(x_1) \leq (x_{\min}^z)_2$ and hence

$$\sigma_2\left(\gamma_2^1(x_1)\right) = \left(\sigma_1^2\left(\gamma_2^1(x_1)\right), \gamma_2^1(x_1), \gamma_2^1(x_1), \sigma_3^2\left(\gamma_2^1(x_1)\right)\right)$$

belongs to the image of σ_2. Since $x = (x_1, \gamma_2^1(x_1), \gamma_3^1(x_1)) \in A$, the definition of σ_3^2 implies that $\gamma_3^1(x_1) \leq \sigma_3^2(\gamma_2^1(x_1))$. Hence

$$x_1 + \gamma_2^1(x_1) + \gamma_3^1(x_1) = r_0 = \sigma_1^2(\gamma_2^1(x_1)) + \gamma_2^1(x_1) + \sigma_3^2(\gamma_2^1(x_1))$$

implies $x_1 \geq \sigma_1^2(\gamma_2^1(x_1))$. Conversely, since σ_1^2 is monotone decreasing, $\gamma_2^1(x_1) \leq \eta$, we have $x_1 = \sigma_1^2(\eta) \leq \sigma_1^2(\gamma_2^1(x_1))$, i.e. $x_1 = \sigma_1^2(\gamma_2^1(x_1))$ and consequently

$$x = \left(x_1, \gamma_2^1(x_1), \gamma_3^1(x_1)\right) = \left(\sigma_1^2\left(\gamma_2^1(x_1)\right), \sigma_3^2\left(\gamma_2^1(x_1)\right)\right)$$

belongs to the image of the curve δ.

Case 3. If $(z^x_{\max})_1 \leq x_1 \leq (z^y_{\max})_1$, then it is easy to see that $x_3 = \zeta_{\max}$, and hence x is on the line segment $Z_{\max}$ and therefore belongs to the image of δ.

Case 4. If $(z^y_{\max})_1 \leq x_1 \leq (y^z_{\min})_1$, then it follows immediately from the definitions that $x = (x_1, \gamma^1_2(x_1), \gamma^1_3(x_1))$ belongs to the image of γ_1, and therefore also to the image of δ.

The cases where $(y^z_{\min})_1 \leq x_1 \leq \xi_{\max}$ are analogous to case 1 and 2 after interchanging the roles of the second and third coordinate. ■

We now can orient the boundary of A by following the closed curve δ. Standing on the plane E_{r_0} with the direction of $(1,1,1)$ pointing upward, this orientation of ∂A follows the curve δ counterclockwise. Using Lemma 5.2, we may assume that δ is has a parametrization $\delta = \delta(t) = (\delta_1(t), \delta_2(t), \delta_3(t))$, $a \leq t \leq b$ that is continuous, so that all coordinate functions $\delta_i(t)$ have integrable derivatives with $\int_c^d \frac{d}{dt}\delta_i(t)\,dt = \delta_i\ (d) - \delta_i(c)$.

Let $f(x)$ be an integrable function defined on the boundary of A. We define

$$\int_{\partial A} f(s)\,ds = \int_a^b f(\delta(t))\,\|\delta'(t)\|\,dt.$$

Whenever necessary, we may change the parameter t with a continuously differentiable function without changing the value of the intergral.

Definition 5.4. Let $x \in \partial A$ and let t be a parameter so that $x = \delta(t)$. The vector

$$n_A(x) = \frac{1}{\sqrt{3}\,\|\delta'(t)\|}\begin{cases} \delta'(t) \times (1,1,1) & \text{if } \delta'(t) \text{ exists and is different from 0} \\ 0 & \text{else} \end{cases}$$

is called an *outer normal to A at t.*

As the example of line segments in E_r show, points on the boundary of ∂A might have two outer normals, which necessarily point in opposite directions. Note that $n_A(x)$ is perpendicular to the tangent line to A at x and also perpendicular to the vector $(1,1,1)$. The outer normals exist almost everywhere, and the coordinate functions of n_A are integrable.

Definition 5.5. The function $\varphi\colon \Re^3 \to \Re$ is defined by

$$\varphi(x,y,z) = \begin{cases} x & \text{if } y \leq x \leq z \text{ or } z \leq x \leq y \\ y & \text{if } x \leq y \leq z \text{ or } z \leq y \leq x \\ z & \text{if } y \leq z \leq x \text{ or } x \leq z \leq y \end{cases}$$

Note that $\varphi(x,y,z) = \max\{\min\{x,y\}, \min\{x,z\}, \min\{y,z\}\}$, and hence φ is a continuous function, with $\varphi(n_A(s))$ being integrable on the boundary of A.

We now start to evaluate $\int_{\partial A} \varphi(n_A(s))\,ds$.

Proposition 5.6. *If $\gamma_1(\zeta)$ and $\gamma_2(\zeta)$ are monotone decreasing functions, defined for $a \leq \zeta \leq b$, so that $\gamma_1(\zeta) + \gamma_2(\zeta) + \zeta = r_0$, and if $\sigma(\zeta) = (\gamma_1(\zeta), \gamma_2(\zeta), \zeta)$, then*

$$\int_a^b \varphi\left(\frac{1}{\sqrt{3}\,\|\sigma'(\zeta)\|}\sigma'(\zeta) \times (1,1,1)\right)\|\sigma'(\zeta)\|\,d\zeta = \frac{1}{\sqrt{3}}(\gamma_1(b) - \gamma_1(a) - (\gamma_2(b) - \gamma_2(a)))$$

Proof. Since γ_1 and γ_2 are monotone decreasing, they are differentiable almost everywhere. Hence, for a value of t where both γ_1 and γ_2 are differentiable, we have

$$\begin{aligned}\frac{1}{\sqrt{3}\left\|\sigma'(\zeta)\right\|}\sigma'(\zeta)\times(1,1,1) &= \frac{1}{\sqrt{3}\sqrt{\gamma_1'(\zeta)^2+\gamma_2'(\zeta)^2+1}}\left(\gamma_1'(\zeta),\gamma_2'(\zeta),1\right)\times(1,1,1)\\ &= \frac{1}{\sqrt{3}\sqrt{\gamma_1'(\zeta)^2+\gamma_2'(\zeta)^2+1}}\cdot\left(\gamma_2'(\zeta)-1,1-\gamma_1'(\zeta),\gamma_1'(\zeta)-\gamma_2'(\zeta)\right).\end{aligned}$$

Since $\gamma_1(\zeta)+\gamma_2(\zeta)+\zeta = r_0$, it follows that $\gamma_1'(\zeta)+\gamma_2'(\zeta)+1 = 0$. Since also $\gamma_1'(\zeta),\gamma_2'(\zeta)\leq 0$, we conclude that $\gamma_2'(\zeta)+1 = -\gamma_1'(\zeta)\geq 0\geq 2\gamma_1'(\zeta)$ and hence $1-\gamma_1'(\zeta)\geq\gamma_1'(\zeta)-\gamma_2'(\zeta)$. Similarly, we see that $1-\gamma_2'(\zeta)\geq\gamma_2'(\zeta)-\gamma_1'(\zeta)$ and so

$$\gamma_2'(\zeta)-1\leq\gamma_1'(\zeta)-\gamma_2'(\zeta)\leq 1-\gamma_1'(\zeta).$$

Therefore

$$\varphi\left(\frac{1}{\sqrt{3}\left\|\sigma'(\zeta)\right\|}\sigma'(\zeta)\times(1,1,1)\right) = \frac{1}{\sqrt{3}\sqrt{\gamma_1'(\zeta)^2+\gamma_2'(\zeta)^2+1}}\left(\gamma_1'(\zeta)-\gamma_2'(\zeta)\right).$$

Proposition 5.1 implies that

$$\begin{aligned}\int_a^b\varphi\left(\frac{1}{\sqrt{3}\left\|\sigma'(\zeta)\right\|}\sigma'(\zeta)\times(1,1,1)\right)\left\|\sigma'(\zeta)\right\|d\zeta &= \frac{1}{\sqrt{3}}\int_a^b\left(\gamma_1'(\zeta)-\gamma_2'(\zeta)\right)d\zeta\\ &= \frac{1}{\sqrt{3}}\left(\gamma_1(b)-\gamma_1(a)-\left(\gamma_2(b)-\gamma_2(a)\right)\right)\end{aligned}$$

and this number depends only on the endpoints of the arc $\mathcal{C}_1$, but otherwise is independent on the special choice of the functions γ_1 and γ_2. ■

Theorem 5.7. *If $A\subseteq E_{r_0}$ is a compact, connected l-convex set, then*

$$\int_{\partial A}\varphi\left(n_A(s)\right)ds = \sqrt{3}\left(\lambda(\inf A)+\lambda(\sup A)-2r_0\right)$$

Proof. Elementary calculations show

$$\begin{aligned}\int_{X_{\min}}\varphi\left(n_A(s)\right)ds &= \frac{1}{\sqrt{3}}\left((x^z_{\min})_3-(x^y_{\min})_3\right)\\ \int_{Z_{\max}}\varphi\left(n_A(s)\right)ds &= \frac{1}{\sqrt{3}}\left((z^y_{\max})_2-(z^x_{\max})_2\right)\\ \int_{Y_{\min}}\varphi\left(n_A(s)\right)ds &= \frac{1}{\sqrt{3}}\left((y^x_{\min})_1-(y^z_{\min})_1\right)\\ \int_{X\max}\varphi\left(n_A(s)\right)ds &= \frac{1}{\sqrt{3}}\left((x^z_{\max})_3-(x^y_{\max})_3\right)\\ \int_{Z\min}(n_A(s))ds &= \frac{1}{\sqrt{3}}\left((z^y_{\min})_2-(z^x_{\min})_2\right)\\ \int_{Y_{\max}}\varphi\left(n_A(s)\right)ds &= \frac{1}{\sqrt{3}}\left((y^x_{\max})_1-(y^z_{\max})_1\right)\end{aligned}$$

For the other curves that are part of ∂A, it is a consequence of 5.6 that

$$\int_{\gamma_3} \varphi(n_A(s))\,ds = \frac{1}{\sqrt{3}}\left((x^y_{\min})_1 - (y^x_{\max})_1 - (x^y_{\min})_2 + (y^x_{\max})_2\right)$$
$$\int_{\sigma_2} \varphi(n_A(s))\,ds = \frac{1}{\sqrt{3}}\left((z^x_{\max})_3 - (x^z_{\min})_3 - (z^x_{\max})_1 + (x^z_{\min})_1\right)$$
$$\int_{\gamma_1} (n_A(s))\,ds = \frac{1}{\sqrt{3}}\left((y^z_{\min})_2 - (z^y_{\max})_2 - (y^z_{\min})_3 + (z^y_{\max})_3\right)$$
$$\int_{\sigma_3} \varphi(n_A(s))ds = \frac{1}{\sqrt{3}}\left((x^y_{\max})_1 - (y^x_{\min})_1 - (x^y_{\max})_2 + (y^x_{\min})_2\right)$$
$$\int_{\gamma_2} \varphi(n_A(s))\,ds = \frac{1}{\sqrt{3}}\left((z^x_{\min})_3 - (x^z_{\max})_3 - (z^x_{\min})_1 + (x^z_{\max})_1\right)$$
$$\int_{\sigma_1} \varphi(n_A(s))\,ds = \frac{1}{\sqrt{3}}\left((y^z_{\max})_2 - (z^y_{\min})_2 - (y^z_{\max})_3 + (z^y_{\min})_3)\right)$$

Adding all equations for the integrals over subcurves together yields

$$\int_{\partial A} \varphi(n_A(s))\,ds = \frac{1}{\sqrt{3}}\Big((x^z_{\min})_1 + (x^y_{\min})_1 - (x^y_{\min})_2 - (x^y_{\min})_3 + (x^z_{\max})_1 + \\ + (x^y_{\max})_1 - (x^y_{\max})_2 - (x^y_{\max})_3 + (y^x_{\min})_2 + (y^z_{\min})_2 - (y^z_{\min})_1 - (y^z_{\min})_3 + \\ + (y^x_{\max})_2 + (y^z_{\max})_2 - (y^z_{\max})_1 - (y^z_{\max})_3 + (z^y_{\min})_3 + (z^x_{\min})_3 - (z^x_{\min})_1 - \\ - (z^x_{\min})_3 + + (z^y_{\max})_3 + (z^x_{\max})_3 - (z^x_{\max})_1 - (z^x_{\max})_3 \Big).$$

Since $(x^y_{\min})_1 + (x^y_{\min})_2 - (x^y_{\min})_3 = r_0$ and since $(x^z_{\min})_1 = \xi_{\min} = (x^y_{\min})_1$, we find that $(x^z_{\min})_1 + (x^y_{\min})_1 - (x^y_{\min})_2 - (x^y_{\min})_3 = 3\xi_{\min} - r_0$. Similar equalities hold for the other sub-terms. Hence

$$\int_{\partial A} \varphi(n_A(x))\,ds = \sqrt{3}\left(\xi_{\min} + \eta_{\min} + \zeta_{\min} + \xi_{\max} + \eta_{\max} + \zeta_{\max} - 2r_0\right).$$

Finally, note that $\inf A = (\xi_{\min}, \eta_{\min}, \zeta_{\min})$ and $\sup A = (\xi_{\max}, \eta_{\max}, \zeta_{\max})$. Hence $\int_{\partial A} \varphi(n_A(x))\,ds = \sqrt{3}\left(\lambda(\inf A) + \lambda(\sup A) - 2r_0\right)$. ■

6. The derivative of the area function: two examples

In this section, we deal with two special cases of level sets: Equilateral triangles and lense-shaped level sets.

Proposition 6.1. *Fix a vector $h \in \Re^3$ and a number $s \geq 0$. Let $r_0 = s + \lambda(h)$, and let $A = \{x \in E_{r_0}: x_1 \geq h_1, x_2 \geq h_2, x_3 \geq h_3\}$ be the equilateral triangle with vertices $(h_1, h_2, h_3 + s)$, $(h_1, h_2 + s, h_3)$ and $(h_1 + s, h_2, h_3)$. Let L be any compact connected sublattice of $\Re^3$ such that $L_{r_0} = A$. Then the functions μ_L is differentiable at r_0 and*

$$\mu'_L(r_0) = \sqrt{3}\left(\lambda(\inf A) + \lambda(\sup A) - 2r_0\right)$$

Especially, $\mu'_L(r_0)$ only depends on L_{r_0} and not on the particular lattice L.

Proof. Note that a translation of L and A by $-h$ does not change the differentiability of μ nor the value of the derivative, but changes r_0 to $r_0 - \lambda(h) = r_0 - (h_1 + h_2 + h_3) = s$ and transports h to 0. Hence we may assume without loss of generality that $h = 0$ and that A is a triangle with vertices $(r_0, 0, 0)$, $(0, r_0, 0)$ and $(0, 0, r_0)$. Hence we have to verify that

$$\mu_L'(r_0) = \sqrt{3}r_0$$

for every compact and connected lattice L such that L_{r_0} is equal to the triangle A.

First, we show that for $r \geq r_0$ we have that $\mu_L(r) \leq \frac{\sqrt{3}}{2} \cdot r^2$.

Indeed, by Lemma 3.2, for every $x \in L_r$ there is an element $x' \in L_{r_0}$ such that $x' \leq x$. Hence $L_r \subseteq \uparrow \inf L_r = \uparrow 0$. Since the surface area of $\{x \in E_r : 0 \leq x\}$ is equal to $\frac{\sqrt{3}}{2} \cdot r^2$, the assertion follows. For $r \leq r_0$ we claim that

$$\mu_L(r) \leq \frac{\sqrt{3}}{2} \cdot r^2 + 6 \cdot (r_0 - r)^2.$$

To verify this, we observe again that for every $x \in L_r$ there is an $x' \in L_{r_0}$ such that $x \leq x'$. Hence, for every coordinate we have $x_i \geq (r - r_0)$, because for example $x_1 < (r - r_0)$ would imply that $x_2 + x_3 > r_0$, and no $x' \in L_{r_0}$ could dominate x.

It follows that $L_r \subseteq \{x \in \Re^3 : x_1, x_2, x_3 \geq r - r_0\}$.

For every $i \in \{1, 2, 3\}$ let $F_i = \{x \in L_r : x_i < 0\}$.

We would like to show that $|F_i| \leq 2 \cdot (r_0 - r)^2$.

By symmetry, it is enough to verify this inequality for $i = 1$. Fix a number $s < 0$ and consider the set $F_{1,s} = \{x \in L_r : x_1 = s\}$. Since L_r is an l-set, $F_{1,s}$ is a line segment. The length of this line segment is less than $\sqrt{2} \cdot (r_0 - r)$, because otherwise we could find two points $(s, b, c), (s, b + (r_0 - r), c - (r_0 - r)) \in F_{i,s}$. The element $(s, b + (r_0 - r), c) = (s, b, c) \vee (s, b + (r_0 - r), c - (r_0 - r))$ then belongs to L_{r_0} and has a negative number as its first coordinate, a contradiction.

All the line segments $F_{1,s}$ are parallel to each other, and parallel to $F_{1,0}$. Since $F_{1,s} = \varnothing$ for $s < r - r_0$ and since the shortest distance between the line through $F_{1,0}$ and a point on $F_{1,r-r_0}$ is equal to $\sqrt{2} \cdot (r_0 - r)$, Fubini's Theorem gives the inequality $|F_1| \leq 2 \cdot (r_0 - r)^2$. The inequality $\mu_L(r) \leq \frac{\sqrt{3}}{2} \cdot r^2 + 6 \cdot (r_0 - r)^2$ now follows from

$$L_r \subseteq \{x \in E_r : x \geq 0\} \cup F_1 \cup F_2 \cup F_3.$$

Next let V be the lattice generated by $A = L_{r_0}$. Then we have

$$\mu_V(r) \leq \mu_L(r) \leq \frac{\sqrt{3}}{2} \cdot r^2 + 6 \cdot (r_0 - r)^2$$

for each $r \in \Re$. Since $\mu_V'(r_0) = \sqrt{3} \cdot r_0$ by Example 2.2, and since this value agrees with the derivative of $\frac{\sqrt{3}}{2} \cdot r^2 + 6 \cdot (r_0 - r)^2$ at r_0, we conclude that

$$\mu_L'(r_0) = \sqrt{3} \cdot r_0,$$

and this value is independent on the particular choice of L as long as $L_{r_0} = A$. ■

We now come to our second type of examples:

Proposition 6.2. *Let $\sigma_1, \sigma_2, \gamma_1$, and γ_2 be four monotone decreasing functions of ζ such that $\sigma_1(\zeta)+\sigma_2(\zeta)+\zeta = r_0$ and $\gamma_1(\zeta)+\gamma_2(\zeta)+\zeta = r_0$, where $\zeta \in [\zeta_{\min}, \zeta_{\max}]$. Additionally, assume that*

$$\sigma_2(\zeta) \le \gamma_2(\zeta) \text{ for } \zeta_{\min} \le \zeta \le \zeta_{\max}, \quad \sigma_2(\zeta_{\min}) = \gamma_2(\zeta_{\min}), \quad \sigma_2(\zeta_{\max}) = \gamma_2(\zeta_{\max}),$$

or, equivalently,

$$\sigma_1(\zeta) \ge \gamma_1(\zeta) \text{ for } \zeta_{\min} \le \zeta \le \zeta_{\max}, \quad \sigma_1(\zeta_{\min}) = \gamma_1(\zeta_{\min}), \quad \sigma_1(\zeta_{\max}) = \gamma_1(\zeta_{\max}).$$

If
$$\begin{aligned} A &= \{x : \lambda(x) = r_0, \zeta_{\min} \le x_3 \le \zeta_{\max}, \gamma_1(x_3) \le x_1 \le \sigma_1(x_3)\} \\ &= \{x : \lambda(x) = r_0, \zeta_{\min} \le x_3 \le \zeta_{\max}, \sigma_2(x_3) \le x_2 \le \gamma_2(x_3)\}. \end{aligned}$$

then A is an l-set, and for every compact, connected sublattice $L \subseteq \Re^3$ with $L_{r_0} = A$ we have

$$\mu'_A(A) = 0 = \sqrt{3}(\lambda(\sup A) + \lambda(\inf A) - 2r_0)$$

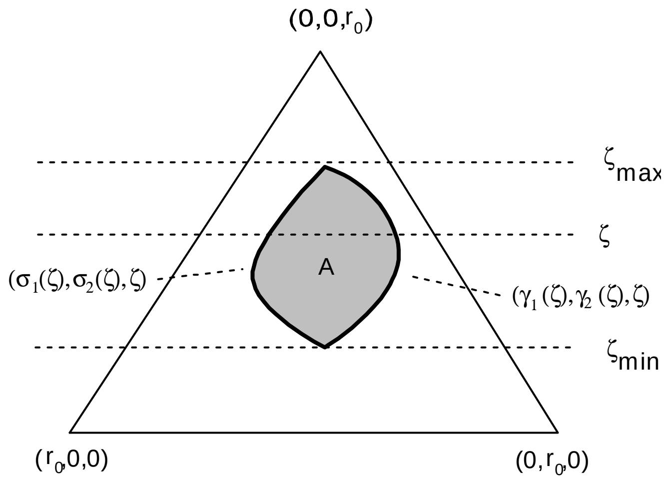
Proof. The proof goes through several steps.
Step 1. We show that A is an l-set: From Proposition 5.1 we conclude that the conditions $\sigma_1(\zeta) + \sigma_2(\zeta) + \zeta = r_0$ and $\gamma_1(\zeta) + \gamma_2(\zeta) + \zeta = r_0$ imply that the decreasing functions σ_1, σ_2, γ_1 and γ_2 are automatically continuous. Hence the set

$$\begin{aligned} A &= \{x : \lambda(x) = r_0, \zeta_{\min} \le x_3 \le \zeta_{\max}, \gamma_1(x_3) \le x_1 \le \sigma_1(x_3)\} \\ &= \{x : \lambda(x) = r_0, \zeta_{\min} \le x_3 \le \zeta_{\max}, \sigma_2(x_3) \le x_2 \le \gamma_2(x_3)\}. \end{aligned}$$

is compact. The boundary of A consists of the two curves

$$\gamma_3(\zeta) = (\gamma_1(\zeta), \gamma_2(\zeta), \zeta) \quad \text{and} \quad \sigma_3(\zeta) = (\sigma_1(\zeta), \sigma_2(\zeta), \zeta)$$

Let $\perp = \inf A$. Since the translation $x \mapsto x + \perp$ is a lattice automorphism that preserves level sets, we may assume in the following considerations that $A \subseteq [0, \infty)^3$ without changing any of the results. This implies that we may assume that $[\zeta_{\min}, \zeta_{\max}] \subseteq [0, r_0]$.



In order to verify that A is an l-set, we first show that the set

$$B_1 = \{x : \lambda(x) = r_0, \gamma_1(x_3) \le x_1\} = \{x : \lambda(x) = r_0, x_2 \le \gamma_2(x_3)\}$$

is an l-set. Let $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ be two elements of B_1 that agree in at least one coordinate, and let S be the line segment with endpoints p and q. We have to show that $S \subseteq B_1$. Let $s \in S$. We have to show that $s \in B_1$. This is obvious in the case where $p_3 = q_3$. Assume that $p_3 < q_3$. Then $p_3 \le s_3 \le q_3$. There are two cases possible: If $p_1 = q_1$, then $s_1 = p_1$ and since γ_1 is decreasing, we obtain $s_1 = p_1 \ge \gamma_1(p_3) \ge \gamma_1(s_3)$, hence $s \in B_1$. If $p_2 = q_2$, then we use the second expression for B_1 and the fact that γ_2 is also decreasing to arrive at $\gamma_2(s_3) \ge \gamma_2(q_3) \ge q_2 = s_2$, thus $s \in B_1$. Hence B_1 is an l-set. Similarly, the set

$$B_2 = \{x : \lambda(x) = r_0, x_2 \le \sigma_2(x_3)\} = (x : \lambda(x) = r_0, \sigma_1(x_3) \le x_1)$$

is an l-set. The fact that A is an l-set now follows from $A = B_1 \cap B_2$.
Since every pair of decreasing functions γ_1 and γ_2 satisfying $\zeta + \gamma_1(\zeta) + \gamma_2(\zeta) = r_0$ is actually a pair of continuous functions, it follows that A is a closed, connected l-set. Hence there is a lattice L such that the level set L_{r_0} is equal to A.

Step 2. Next, we will show that

$$\sqrt{3}\left(\lambda(\sup A) + \lambda(\inf A) - 2r_0\right) = 0.$$

By Theorem 5.7, we have to show that

$$\int_{\partial A} \varphi(n_A(s))ds = 0.$$

First, the boundary of A consists of the curves γ_3 and σ_3 , and Proposition 5.6 implies,

$$\int_{\gamma_3} \varphi(n_A(s))\,ds = \frac{1}{\sqrt{3}}\left(\gamma_1(\zeta_{\max}) - \gamma_1(\zeta_{\min}) - \gamma_2(\zeta_{\max}) + \gamma_2(\zeta_{\min})\right)$$

and similarly, since the outer normals point "the other way", i.e. are given by the direction of $(1,1,1) \times (\sigma_1'(\zeta), \sigma_2'(\zeta), 1)$:

$$\int_{\sigma_3} \varphi(n_A(s))\,ds = \frac{1}{\sqrt{3}}\left(\sigma_1(\zeta_{\min}) - \sigma_1(\zeta_{\max}) - \gamma_2(\xi_{\min}) + \gamma_2(\zeta_{\max})\right)$$

The $\sigma's$ and $\gamma's$ agree at the endpoints, hence $\int_{\partial A} .\varphi(n_A(s))ds = 0$.

Step 3. Let V be the lattice generated by A, and assume that $(0,0,0) \in V$. We would like to show that $\mu_V(r)$ is differentiable at r_0 and that $\mu_V'(r_0) = 0$. First, we shall describe the set V_r for $r \le r_0$. We claim that

$$V_r = \{x : \lambda(x) = r, \zeta_{\min} \le x_3 \text{ and } x + (0,0,r_0 - r) \in A\}.$$

Indeed, let $a \in V_r$. Then, since all the elements y in the generating set A satisfy $y_3 \ge \zeta_{\min}$, we conclude that $a_3 \ge \zeta_{\min}$. Let $h = r_0 - r$. We now would like to show that $a + (0,0,h) \in A$. First, using Theorem 3.17, we can find elements $x, y, z \in A \cap \uparrow a$ such that $x_1 = a_1$, $y_2 = a_2$ and $z_3 = a_3$.

Assume, if possible, that $(a_1, a_2, a_3 + h) \notin A$. Since $x_1 = a_1, x_2 \geq a_2$, it follows that $x_3 = r_0 - x_1 - x_2 \leq r_0 - a_1 - a_2 = a_3 + h$. Since A is an l-set, we conclude that none of the points in

$$C_1 = \{u \in E_{r_0} : u_1 = a_1 \text{ and } u_3 \geq a_3 + h\}$$

belong to A. Similarly, the intersection of A and

$$C_2 = \{u \in E_{r_0} : u_2 = a_2 \text{ and } u_3 \geq a_3 + h\}$$

is empty. It follows that A is the disjoint union of the two closed sets

$$A_1 = \{u \in A : u_1 < a_1 \text{ and } u_2 < a_2\} = \{u \in A : u_1 \leq a_1 \text{ and } u_2 \leq a_2\}$$

and $A_2 = A \setminus A_1$. Since $z \in A_2$, the set A_2 is non-empty.

Since $a_1 = x_1 \geq \gamma_1(x_3) \geq \gamma_1(\zeta_{\max})$ and $a_2 = x_2 \geq \gamma_1(x_3) \geq \gamma_2(\zeta_{\max})$, it follows that

$$(\gamma_1(\zeta_{\max}), \gamma_2(\zeta_{\max}), \zeta_{\max}) \in A_1 \neq \varnothing.$$

This contradicts the connectivity of A.

Conversely, assume that $a = (a_1, a_2, a_3)$ satisfies the conditions $a_1 + a_2 + a_3 = r$, $a_3 \geq \zeta_{\min}$ and $a + (0, 0, r_0 - r) \in A$. We would like to show that $a \in V$. In order to verify this, we have to find an element $z \in A \cap \uparrow a$ so that $z_3 = a_3$, since then $a = (a + (0, 0, r_0 - r)) \wedge z \in V$.

Consider the point $(\gamma_1(\zeta_{\min}), \gamma_2(\zeta_{\min}), \zeta_{\min})$. If actually $a_3 = \zeta_{\min}$, then

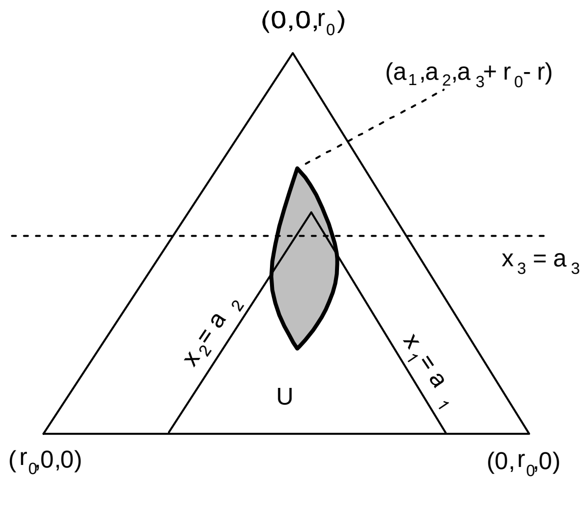
$$\begin{aligned} a + (0, 0, r_0 - r) \in A &= \{x : \lambda(x) = r_0, \zeta_{\min} \leq x_3 \leq \zeta_{\max}, \gamma_1(x_3) \leq x_1 \leq \sigma_1(x_3)\} \\ &= \{x : \lambda(x) = r_0, \zeta_{\min} \leq x_3 \leq \zeta_{\max}, \sigma_2(x_3) \leq x_2 \leq \gamma_2(x_3)\} \end{aligned}$$

implies

$$\begin{aligned} (a_1, a_2, a_3) \leq (\sigma_1(a_3), \gamma_2(a_3), a_3) &= (\sigma_1(\zeta_{\min}), \gamma_2(\zeta_{\min}), \zeta_{\min}) \\ &= (\gamma_1(\zeta_{\min}), \gamma_2(\zeta_{\min}), \zeta_{\min}) \in A \end{aligned}$$

and we may choose $z = (\gamma_1(\zeta_{\min}), \gamma_2(\zeta_{\min}), \zeta_{\min})$.

Hence we may assume that $a_3 > \zeta_{\min}$. If $\gamma_1(\zeta_{\min}) = a_1$, then the fact that A is an l-set implies that every point on the line segment between $(a_1, a_2, a_3 + r_0 - r)$ and $((\gamma_1(\zeta_{\min}), \gamma_2(\zeta_{\min}), \zeta_{\min})) = (a_1, \gamma_2(\zeta_{\min}), \zeta_{\min})$ belongs to A. Since we also have $\zeta_{\min} < a_3 < a_3 + r_0 - r$, we conclude that $z = (a_1, r_0 - a_1 - a_3, a_3) \in \cap A \cap \uparrow a$. Hence we also may assume that $a_1 < \gamma_1(\zeta_{\min})$.



Similarly, $a_2 = \gamma_2(\zeta_{\min})$ would allow us to choose $z = (r_0 - a_2 - a_3, a_2, a_3) \in A \cap \uparrow a$. Hence we may assume that

$$a_1 < \gamma_1(\zeta_{\min}), \quad a_2 < \gamma_2(\zeta_{\min}), \quad \text{and} \quad a_3 > \zeta_{\min}.$$

In E_{r_0} we consider the open set $\quad U = \{x \in \Re^3 : a_1 < x_1, a_2 < x_2, a_3 > x_3\}$.
The point $(\gamma_1(\zeta_{\min}), \gamma_2(\zeta_{\min}), \zeta_{\min})$ belongs to U, and the point $(a_1, a_2, a_3 + r_0 - r)$ does not belong to the closure of U.
Since A is connected, there is a point $z' \in A \cap \overline{U} \setminus U$, hence we have

$$a_1 \leq z_1', \quad a_2 \leq z_2', \quad \text{and} \quad a_3 \geq z_3',$$

with equality holding in at least one case. If actually $a_3 = z_3'$, then $z = z'$ is the point we were looking for. Else we have either $a_1 = z_1'$ or $a_2 = z_2'$ and $z_3' < a_3 \leq a_3 + r_0 - r$. In the first case, the fact that A is an l-set yields a point $z \in A$ with $z_1 = a_1$ and $z_3 = a_3$. Since $z_2 = r_0 - a_1 - a_3 \geq r - a_1 - a_3 = a_2$, we have $a \leq z$, as desired. The second case is handled accordingly.

Since translations preserve surface area, we now can compute the area of V_r as

$$\begin{aligned} |V_r| &= |\{x : \lambda(x) = r, \zeta_{\min} \leq x_3 \text{ and } x + (0,0,r_0 - r) \in A\}| \\ &= |\{x \in A : \zeta_{\min} + r_0 - r \leq x_3\}| \end{aligned}$$

and it follows that for $r \leq r_0$ we have $0 \leq |V_{r_0}| - |V_r| = |\{x \in A : x_3 \leq \zeta_{\min} + (r_0 - r)\}|$. Since $x \in A$ and $x_3 \leq \zeta_{\min} + (r_0 - r)$ imply that

$$\gamma_1(\zeta_{\min} + (r_0 - r)) \leq \gamma_1(x_3) \leq x_1 \leq \sigma_1(x_3) \leq \sigma_1(\zeta_{\min}) = \gamma_1(\zeta_{\min})$$

we find that $\{x \in A : x_3 \leq \zeta_{\min} + (r_0 - r)\} \subseteq \{x \in E_{r_0} : \zeta_{\min} \leq x_3 \leq \zeta_{\min} + (r_0 - r)$ and $\gamma_1(\zeta_{\min} + (r_0 - r)) \leq x_1 \leq \gamma_1(\zeta_{\min})$. Hence

$$|\{x \in A : x_3 \leq \zeta_{\min} + (r_0 - r)\}| \leq \frac{\sqrt{3}}{2}(r_0 - r)(\gamma_1(\zeta_{\min}) - \gamma_1(\zeta_{\min} + (r_0 - r)))$$

and therefore $\;0 \leq \dfrac{|V_{r_0}| - |V_r|}{r_0 - r} \leq \dfrac{\sqrt{3}}{2}(\gamma_1(\zeta_{\min}) - \gamma_1(\zeta_{\min} + (r_0 - r)))$.

The continuity of γ_1 yields $\;\lim\limits_{r \to r_0^-} \dfrac{|V_{r_0}| - |V_r|}{r_0 - r} = 0.$

Utilizing the fact that the map $x \mapsto -x$ is an anti-automorphism of $\Re^3$ that preserves level sets, we find that also

$$\lim_{r \to r_0^+} \frac{|V_{r_0}| - |V_r|}{r_0 - r} = 0,$$

and hence, in this particular case, $\;\mu_V'(r_0) = \lim\limits_{r \to r_0} \dfrac{|V_{r_0}| - |V_r|}{r_0 - r} = 0.$

Step 4. Now let L be any compact connected lattice such that

$$L_{r_0} = A = \{x \in E_{r_0} : \zeta_{\min} \leq x_3 \leq \zeta_{\max}, \gamma_1(x_3) \leq x_1 \leq \sigma_1(x_3)\}.$$

We have to show that $\mu_L'(r_0)$ exists and $\mu_L'(r_0) = 0$.
In the following, we will show that for all r we have

$$\mu_L(r) \le \mu_L(r_0) + \left(\frac{\sqrt{3}}{2} + 4\right)(r_0 - r)^2.$$

Then, if V denotes to lattice generated by A, we obtain

$$\mu_V(r) \le \mu_L(r) \le \mu_L(r_0) + \frac{\sqrt{3}}{2}(r_0 - r)^2,$$

and since the functions $\mu_V(r)$ and $\mu_L(r_0) + \frac{\sqrt{3}}{2}(r_0 - r)^2$ have derivative 0 at r_0 (see step 3), we arrive at the desired equality $\mu_L'(r_0) = 0$.
Let $r < r_0$ and let $x \in L_r$. First, note that $(x_1, x_2) \le (\sigma_1(\zeta_{\min}), \sigma_2(\zeta_{\min}))$.
To verify this assertion, pick any $y \in A$ such that $x < y$. Then

$$(x_1, x_2) \le (y_1, y_2) \le (\sigma_1(y_3), \gamma_2(y_3)) \le (\sigma_1(\zeta_{\min}), \gamma_2(\zeta_{\min})) = (\sigma_1(\zeta_{\min}), \sigma_2(\zeta_{\min})).$$

We decompose L_r as the union of four sets F_i, $1 \le i \le 4$, where

$$\begin{aligned}
F_1 &= \{x \in L_r : (x_1, x_2) \le (\sigma_1(\zeta_{\max}), \sigma_2(\zeta_{\max}))\} \\
F_2 &= \{x \in L_r : (x_1, x_2) \ge (\sigma_1(\zeta_{\max}), \sigma_2(\zeta_{\max}))\} \\
F_3 &= \{x \in L_r : x_1 \ge \sigma_1(\zeta_{\max}), x_2 < \sigma_2(\zeta_{\max})\} \\
F_4 &= \{x \in L_r : x_1 < \sigma_1(\zeta_{\max}), x_2 \ge \sigma_2(\zeta_{\max})\}
\end{aligned}$$

If $x \in F_1$, pick an element $y \in A$ such that $x \le y$. It follows that $x_3 \le y_3 \le \zeta_{\max}$, and hence $x \le (\sigma_1(\zeta_{\max}), \sigma_2(\zeta_{\max}), \zeta_{\max})$. Hence

$$F_1 \subseteq \{u \in E_r : \quad u \le (\sigma_1(\zeta_{\max}), \sigma_2(\zeta_{\max}), \zeta_{\max})\}.$$

The latter set is an equilateral triangle with surface area $\frac{\sqrt{3}}{2}(r_0 - r)^2$, hence

$$|F_1| \le \frac{\sqrt{3}}{2}(r_0 - r)^2.$$

Assume that $(x_1, x_2) \ge (\sigma_1(\zeta_{\max}), \sigma_2(\zeta_{\max}))$. We claim that this implies that $(x_1, x_2, r_0 - x_1 - x_2) = x + (r_0 - r)(0, 0, 1) \in A$. Indeed, the element

$$\begin{aligned}
&(\sigma_1(\zeta_{\max}), \sigma_2(\zeta_{\max}), \zeta_{\min}) = \\
&\qquad = (\sigma_1(\zeta_{\max}), \sigma_2(\zeta_{\max}), \zeta_{\max}) \wedge (\sigma_1(\zeta_{\min}), \sigma_2(\zeta_{\min}), \zeta_{\min})
\end{aligned}$$

belongs to L. Since the interval

$$[(\sigma_1(\zeta_{\max}), \sigma_2(\zeta_{\max}), \zeta_{\min}), (\sigma_1(\zeta_{\max}), \sigma_2(\zeta_{\max}), \zeta_{\max})]$$

is a chain and since L is connected, this chain belongs completely to L. Hence $(\sigma_1(\zeta_{\max}), \sigma_2(\zeta_{\max}), t) \in L$ whenever $\zeta_{\min} \le t \le \zeta_{\max}$. Let $t = r_0 - x_1 - x_2$.
An elementary calculation using the inequalities $(x_1, x_2) \ge (\sigma_1(\zeta_{\max}), \sigma_2(\zeta_{\max}))$ and $(x_1, x_2) \le (\sigma_1(\zeta_{\min}), \sigma_2(\zeta_{\min}))$ shows that $\zeta_{\min} \le t \le \zeta_{\max}$.

From $x_3 = r - x_1 - x_2 \le r_0 - x_1 - x_2 = t$, we obtain

$$(x_1, x_2, r_0 - x_1 - x_2) = x \vee (\sigma_1(\zeta_{\max}), \sigma_2(\zeta_{\max}), t) \in L_{r_0} = A.$$

Since translations preserve surface area, we conclude that $|F_2| \le |A| = \mu_L(r_0)$.
Next, let $x \in F_3$. Again, let $x \le y$, where $y \in A$. It follows that

$$\begin{aligned}\sigma_2(\zeta_{\max}) > x_2 = r - x_1 - x_3 &\ge r_0 - y_1 - y_3 - (r_0 - r) = y_2 - (r_0 - r)\\ &\ge \sigma_2(\zeta_{\max}) - (r_0 - r).\end{aligned}$$

We now repeat some of the argumentation of the proof of Proposition 6.2:
Every line segment of the form $\{x \in F_3 : x_2 = s\}$ for $s < \sigma_2(\zeta_{\max})$ has length less than $\sqrt{2}(r_0 - r)$, because otherwise we could find two points (a, s, c) and $(a + (r_0 - r), s, c - (r_0 - r)) \in F_3 \subseteq L$. The element

$$(a + (r_0 - r), s, c) = (a, s, c) \vee (a + (r_0 - r), s, c - (r_0 - r))$$

then belongs to L_{r_0}, leading to the contradiction $\sigma_2(\zeta_{\max}) > s \ge \sigma_2(c) \ge \sigma_2(\zeta_{\max})$.
Since the distance of the lines

$$\ell_1 = \{x \in E_r : x_2 = \sigma_2(\zeta_{\max})\} \quad \text{and} \quad \ell_2 = \{x \in E_r : x_2 = \sigma_2(\zeta_{\max}) - (r_0 - r)\}$$

is equal to $\sqrt{2} \cdot (r_0 - r)$ and since the subset $\{x \in F_3 : x_2 = s\}$ is empty for $s < \sigma_2(\zeta_{\max}) - (r_0 - r)$, Fubini's Theorem gives the inequality

$$|F_3| \le 2 \cdot (r_0 - r)^2.$$

Similarly, we obtain $|F_3| \le 2 \cdot (r_0 - r)^2$. Adding all four inequalities yields

$$\mu_L(r) \le \mu_L(r_0) + \left(\frac{\sqrt{3}}{2} + 4\right)(r_0 - r)^2. \qquad \blacksquare$$

7. The cutting theorem

In this section, we show that the area function of level sets behaves in the expected way when cutting a compact connect lattice $L \subset \Re^3$ into two sublattices L_1 and L_2. Indeed, if we use a suitable plane to perform the cut, then $\mu_L(r) = \mu_{L_1}(r) + \mu_{L_2}(r)$.

Lemma 7.1. *Let $A \subseteq E_r$ be a compact, connected l-set. Let $H \subseteq \Re^3$ be a plane that is not parallel to E_r and let H_1 and H_2 be the two half spaces determined by H. If $A_1 = H_1 \cap A$ and $A_2 = H_2 \cap A$ are connected, then $A_1 \cap A_2$ is a line segment.*

Proof. We may assume w.l.o.g. that $A \subseteq [0, \infty)^3$. Since H is not parallel to E_r, H intersects E_r in a line ℓ, and we may also assume that H is spanned by ℓ and $(0, 0, 0)$. Hence there is a vector $a \in \Re^3$ so that $H = \{x \in \Re^3 : a \cdot x = 0\}$, and we may assume that

$$A_1 = \{x \in A : a \cdot x \ge 0\} \quad \text{and} \quad A_2 = \{x \in A : a \cdot x \le 0\}$$

We have to show that $A_1 \cap A_2$ is a line segment.

The statement is obvious if $A_1 \cap A_2$ contains only one element. Hence, after renumbering the coordinates, if necessary, we may assume that there are elements $x, y \in A_1 \cap A_2$ such that $x_1 < y_1$. We also may assume that

$$x_1 = \min\{z_1 : z \in A_1 \cap A_2\} \quad \text{and} \quad y_1 = \max\{z_1 : z \in A_1 \cap A_2\}.$$

By definition, $A_1 \cap A_2$ is contained in the line $H \cap E_r$, hence $A_1 \cap A_2$ is contained in the line segment spanned by x and y. Conversely, we have to show that every element of this line segment belongs to $A_1 \cap A_2$. Thus, let z be an arbitrary element of this line segment. Then $x_1 \le z_1 \le y_1$. Since $x, y \in A_1$ and since A_1 is connected, there is an element $u \in A_1$ with $u_1 = z_1$ (otherwise the non-empty disjoint open sets $\{u \in A_1 : u_1 < z_1\}$and $\{u \in A_1 : u_1 > z_1\}$ would be a cover of A_1). Similarly, there is an element $v \in A_2$ so that $v_1 = z_1$. Since A is an l-set, the line segment between u and v belongs to A.
If $a \cdot u = 0$, then $u \in A_1 \cap A_2$, and hence u would be a convex combination of x and y with $u_1 = z_1$, and we would be able to conclude that $u = z \in A_1 \cap A_2$. Similarly, $a \cdot v = 0$ would imply that also $z \in A_1 \cap A_2$.

Hence we may assume that $a \cdot u > 0 > a \cdot v$. The line segment between u and v intersects H in exactly one point, and therefore there is a unique element $w \in A_1 \cap A_2$ so that $w_1 = z_1$. Since the line segment between x and y contains only one point w with $w_1 = z_1$, namely z, we conclude that $z = w \in A_1 \cap A_2$. ■

Theorem 7.2. *Let $A \subseteq E_r$ be a closed, connected l-set. Let $H = \{x \in \Re^3 : n \cdot x = 0\}$ be the plane perpendicular to $n \in \Re^3$, and assume that*

$$A_1 = \{x \in A : n \cdot x \ge 0\} \quad \textit{and} \quad A_2 = \{x \in A : n \cdot x \le 0\}$$

are connected. Further, let L, L_1, and L_2 be the lattices such that $L_r = A$, $(L_1)_r = A_1$, and $(L_2)_r = A_2$, respectively. If the functions μ_{L_1} and μ_{L_2} are differentiable at r, and if the derivative of those two functions only depends on the level set and not on the choice of lattices L_1 and L_2 leading to this level set, then the same statement is true for the third function μ_L, and $\mu'_L(r) = \mu'_{L_1}(r) + \mu'_{L_2}(r)$. Moreover, if

$$\mu'_{L_1}(r) = \sqrt{3}\,(\lambda(\inf A_1) + \lambda(\sup A_2) - 2r)$$

$$\mu'_{L_2}(r) = \sqrt{3}\,(\lambda(\inf A_2) + \lambda(\sup A_2) - 2r)$$

then also

$$\mu'_L(r) = \sqrt{3}\,(\lambda(\inf A) + \lambda(\sup A) - 2r)$$

Proof. First, let us assume that we already know that $\mu'_{L_i}(r)$ exists and does not depend on L_i for $i = 1, 2$. We then would like to show that $\mu'_L(r)$ exists and does not depend on L. It follows from Theorem 3.20 that there is a plane G (not necessarily containing the origin) such that G separates A_1 and A_2, and such that the half spaces H_1 and H_2 determined by H are sublattices. Let $V_1 = L \cap H_1$ and $V_2 = L \cap H_2$.

Hence for every real number t the set $\{x \in E_t : x \in V_1 \cap V_2\} = \{x \in E_t : x \in L \cap H\}$ has surface area 0. Thus $\mu_{V_1}(t) + \mu_{V_2}(t) = \mu_L(t)$. Since both μ_{V_1} and μ_{V_2} are differentiable at r and since $\mu'_{L_i}(r) = \mu'_{V_i}(r)$, we conclude that $\mu'_L(r)$ exists, that $\mu'_L(r) = \mu'_{L_1}(r) + \mu'_{L_2}(r)$, and that $\mu'_L(r)$ depends only L_r only and not on L.

By Lemma 7.1, the set $A_1 \cap A_2$ is a line segment that belongs to the boundaries of A_1 and A_2, respectively. On the line segment $A_1 \cap A_2$, the outer normals to A_1 and A_2, respectively, have opposite signs. Hence, in the expression $\int_{\partial A_1} \varphi(n_{A_1}(s))ds + \int_{\partial A_2} \varphi(n_{A_2}(s))ds$, the integrals along $A_1 \cap A_2$ cancel out, and therefore $\int_{\partial A} \varphi(n_A(s))ds =$ We now use Theorem 5.7 to compute

$$\begin{aligned}\mu'_L(r) &= \mu'_{L_1}(r) + \mu'_{L_2}(r) = \sqrt{3}\left(\lambda(\inf A_1) + \lambda(\sup A_2) - 2r\right) \\ &= \int \partial A\varphi(nA(s))ds = \sqrt{3}\left(\lambda(\inf A) + \lambda(\sup A) - 2r\right).\end{aligned}$$ ■

In order to use the previous theorem, one has to find lines in E_r that cut A into two connected components. Examples show that not every line necessarily does that. However, the next result gives some special cases which guarantee this.

Proposition 7.3. *Let $A \subseteq E_r$ be a compact, connected l-set, and let*

$$\ell_{i,h} = \{x \in E_r : x_i = h\}.$$

If $\ell_{i,h}$ intersects A, then $\ell_{i,h}$ divides A into two connected components, each of which is an l-set.

Proof. In order to simplify notations, we will only consider the case $i = 1$. In this case, $\ell = \ell_{1,h}$ and ℓ divides A into

$$A_1 = \{x \in A : x_1 \geq h\} \quad \text{and} \quad A_2 = \{x \in A : x_1 \leq h\}.$$

Clearly, both A_1 and A_2 are l-sets. Hence, we only have to show that A_1 and A_2 are connected. Assume that A_1 were not connected. Then we could find two non-empty, disjoint closed subsets $C, D \subseteq A_1$ so that $A_1 = C \cup D$. If $C \cap \ell = \varnothing$, then C and $D \cup A_2$ would be two non-empty, disjoint closed subsets of A with $C \cup (D \cup A_2) = A$, which contradicts the assumption that A is connected. Hence $C \cap \ell \neq \varnothing$. In the same way, we can show that $D \cap \ell \neq \varnothing$. Hence, we can find $x \in C \cap \ell$ and $y \in D \cap \ell$. Then, since A is an l-set, the line segment S between x and y belongs to A. It follows that $S \subseteq A_1 = C \cup D$, $S \cap C \neq \varnothing$, $S \cap D \neq \varnothing$, contradicting the fact that line segments are connected. ■

8. Polygonal l-sets

In this section, we will show that every l-set A for which the boundary ∂A is a finite union of line segments is indeed satisfies the equation

$$\mu'_L(r) = \sqrt{3}\left(\lambda(\inf A) + \lambda(\sup A) - 2r\right).$$

So, let us start with showing that level sets that are triangles are admissible. Even this easy case has to be divided in a series of sub-cases, most of which can be treated with elementary Euclidean geometry. In order to avoid repetition, we call an l-set A

admissible, if the equation $\mu_L'(r) = \sqrt{3}(\lambda(\inf A) + \lambda(\sup A) - 2r)$ holds for every compact, connected lattice $L \subseteq \Re^3$ with $L_r = A$.

In the following lemmas, we let $A \subseteq E_{r_0}$ be a triangle with vertices

$$P = (p_1, p_2, p_3), \quad Q = (q_1, q_2, q_3), \quad \text{and} \quad R = (r_1, r_2, r_3).$$

Lemma 8.1. *If $p_1 = q_1$, $q_2 = r_2$ and $r_3 = p_3$, then A is admissible.*

Proof. First, assume that $r_3 \le q_3$, and let $s = q_3 - r_3 \ge 0$. Then $q_1 + s = r_1$ and $q_2 + s = p_2$, and hence $(q_1 + s, q_2, r_3) = (r_1, r_2, r_3)$, $(q_1, q_2 + s, r_3) = (p_1, p_2, p_3)$ and $(q_1, q_2, r_3 + s) = (q_1, q_2, q_3)$. Hence it follows from Proposition 6.1 that A is admissible.

The anti-isomorphism $x \mapsto -x$ can be uses to reduce the case $s \le 0$ to the case where $s \ge 0$. ∎

Lemma 8.2. *Let $A \subseteq E_{r_0}$ be a triangle with vertices r, p and q. If*

$$r_1 \le p_1 = q_1, \quad r_2 \le q_2 \le p_2 \quad \textit{and} \quad p_3 \le q_3 < r_3,$$

then the triangle A is admissible.

Proof. We claim that the triangle A is one of the regions described in Example 6.2: We let

$$\sigma_1(\zeta) = \begin{cases} p_1 & p_3 \le \zeta \le q_3 \\ p_1 + \frac{r_1 - p_1}{r_3 - q_3}(\zeta - q_3) & q_3 \le \zeta \le r_3 \end{cases}$$

$$\sigma_2(\zeta) = \begin{cases} p_2 + \frac{q_2 - p_2}{q_3 - p_3}(\zeta - p_3) & p_3 \le \zeta \le q_3 \\ q_2 + \frac{r_2 - q_2}{r_3 - q_3}(\zeta - q_3) & q_3 \le \zeta \le r_3 \end{cases}$$

$$\gamma_1(\zeta) = p_1 + \frac{r_1 - p_1}{r_3 - p_3}(\zeta - p_3) \quad \text{and} \quad \gamma_2(\zeta) = p_2 + \frac{r_2 - p_2}{r_3 - p_3}(\zeta - p_3),$$

and note that A is bounded by the curves $(\sigma_1(\zeta), \sigma_2(\zeta), \zeta)$ and $(\gamma_1(\zeta), \gamma_2(\zeta), \zeta)$. All functions are monotone decreasing for $p_3 \le \zeta \le q_3$ by our assumptions. Hence A is admissible by Proposition 6.2. ∎

Lemma 8.3. *If at least two of the pairs $\{P, Q\}$, $\{P, R\}$, or $\{Q, R\}$ agree in at least one coordinate, then the triangle A is admissible.*

Proof. If $P = Q = R$, then the assertion follows from Lemma 8.1.

If all points agree in one of the coordinates, for instance the first coordinate, then

$$r_1 = p_1 = q_1.$$

After using the anti-automorphism $x \mapsto -x$, if necessary, we may assume that

$$p_3 \le q_3 < r_3$$

and it follows that $r_2 \leq q_2 \leq p_2$. Hence A (which is actually a line segment) is admissible by Lemma 8.2. Hence we may assume $p_1 = q_1$ and $q_3 = r_3$.

If $p_2 = r_2$, then A is an equilateral triangle of the type that was discussed in Lemma 8.1 and hence admissible.

If $p_2 \neq r_2$, then after applying the anti-automorphism $x \mapsto -x$ if necessary, we may assume that $p_3 < q_3$,

If $r_2 < q_2$, then after a permutation of the coordinates (use the triangle R, Q, P instead of the triangle P, Q, R), the triangle A is of the type discussed in Lemma 8.2.

If $q_2 \leq r_2 < p_2$, then we cut A along the line $\{x \in E_{r_0} : x_2 = r_2\}$ and obtain two triangles. The first triangle has vertices

$$Q = (q_1, q_2, q_3), \quad R = (r_1, r_2, r_3), \quad \text{and} \quad S = (q_1, r_2, r_0 - q_1 - r_2),$$

which falls in the category of a triangle discussed in Lemma 8.1. The second triangle has vertices

$$P = (p_1, p_2, p_3), \quad S = (q_1, r_2, r_0 - q_1 - r_2), \quad \text{and} \quad R = (r_1, r_2, r_3),$$

which was discussed in Lemma 8.2. A similar argument can be used, if $q_2 \leq p_2 \leq r_2$. We then we cut A along the line $\{x \in E_{r_0} : x_2 = p_2\}$. ∎

Lemma 8.4. *If at least one of the pairs $\{P, Q\}$, $\{P, R\}$, or $\{Q, R\}$ agrees in at least one coordinate, then the triangle A is admissible.*

Proof. After renaming and renumbering and employing the anti-isomorphism $x \mapsto -x$, if necessary, we may assume that $p_3 = q_3 < r_3$ and $p_1 > q_1$. Consider the line $\ell = \{x \in E_{r_0} : x_1 = r_1\}$ and let S be the point of intersection of ℓ and $\{x \in \Re^3 : x_3 = p_3\}$. Then

$$S = (r_1, r_0 - r_1 - p_3, p_3).$$

If $s_1 = r_1 > p_1$, then A is a triangle of the form that was discussed in Lemma 8.2 (with the third coordinate playing the role of the first coordinate in Lemma 8.2). Hence A is admissible.

If $q_1 \leq s_1 \leq p_1$ then S is on the line segment PQ, then ℓ cuts the triangle A into two triangles: A_1 with vertices P, S, R and A_2 with vertices S, Q, R. Both triangles are admissible but the previous Lemma 8.3, hence A is admissible by Theorem 7.2.

Hence we may assume that $s_1 = r_1 < q_1 < p_1$. Likewise, using a similar argument, we may assume that $r_2 < p_2 < q_2$. Let $k = \{x \in E_{r_0} : x_1 = q_1\}$.

This line will intersect the line segment PR in a point T. The triangle A_1 with vertices PQ, T will be of the type discussed in Lemma 8.3, and the triangle A_2 with vertices Q, R, T will be of the type discussed in in Lemma 8.2. Hence A_1 and A_2 are both admissible, and therefore A is admissible by Theorem 7.2. ∎

Proposition 8.5. *Every triangle is admissible.*

Proof. Let $A \subseteq E_{r_0}$ be a triangle with vertices $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3)$, and $R = (r_1, r_2, r_3)$. We may assume that $p_1 \leq q_1 \leq r_1$. If $p_1 = q_1$ or $q_1 = r_1$, then

A is admissible by Lemma 8.4. Hence we may assume that $p_1 < q_1 < r_1$. Consider the line $\ell = \{x \in E_{r_0} : x_1 = q_1\}$. This triangle cuts A into two triangles A_1 and A_2, both of which are admissible by the previous Lemma 8.4. Hence A is admissible by Theorem 7.2. ■

Lemma 8.6. *Let $A \subseteq E_{r_0}$ be a connected l-set. Assume that ∂A is a union of at most 4 line segments. Then A is admissible.*

Proof. If ∂A consists of one line segment, then A itself is a line segment, which may be viewed as a degenerate triangle.
If ∂A consists of two line segments, then we can find a line ℓ that cuts A into two line segments, both of which are admissible. Hence A itself is admissible.
If ∂A consists of three line segments, then either A is a triangle, or else we can find a line that cuts A into a line segment and a union of two line segments, both of which are admissible.
Finally, we assume that A is a union of four line segments. If the interior of A is not dense in A, we can find a line ℓ that cuts A into a line segment and a part A_1 for which ∂A_1 is a union of 3 line segments. Hence A is admissible. Otherwise, if the interior of A is dense in A, it follows from elementary geometric considerations that A is a quadrilateral, hence one of the diagonals cuts A into two triangles, both of which are admissible. Hence A is also admissible. ■

We now come to the main result of this section:

Theorem 8.7. *Let $A \subseteq E_{r_0}$ be a connected l-set. Assume that ∂A is a finite union of line segments. Then A is admissible.*

Proof. Let $P_1, \ldots, P_n$ be the end points of the line segments of ∂A, where

$$P_i = (p_{i1}, p_{i2}, p_{i3}) \text{ for } 1 \leq i \leq n$$

and let
$$E_A = \{p_{i3} : 1 \leq i \leq n\}$$

We prove the theorem by induction on the number of elements in E.

If E_A contains only one element z, then A is contained in $\{x \in E_{r_0} : x_3 = z\}$, i.e. A is a line segment, which may be viewed as a degenerate triangle. Hence E is admissible by Proposition 8.5.

If E_A contains exactly two element w_1 and w_2 with $w_1 < w_2$, then the fact that A is an l-set implies that the sets $E_l = \{x \in A : x_3 = w_1\}$ and $E_u = \{x \in A : x_3 = w_2\}$ are line segments.

Let $F_A = \{p_{i1} : 1 \leq i \leq n\}$. We can write F_A in the form $F_A = \{u_j : 1 \leq j \leq m\}$ with $u_1 < u_2 < \ldots < u_m$.

We proceed by induction on m:

If $m = 1$, then $\{P_1, \ldots, P_n\}$ consists of the elements $(u_1, r_0 - u_1 - z_1, z_1)$ and $(u_1, r_0 - u_1 - z_2, z_2)$. In this case, ∂A and therefore A is a line segment.

If $m = 2$, then

$$\{P_1, \ldots, P_n\} \subseteq \{(u_1, r_0 - u_1 - z_1, z_1), (u_2, r_0 - u_2 - z_1, z_1), \\ (u_1, r_0 - u_1 - z_2, z_2), (u_2, r_0 - u_2 - z_2, z_2)\}.$$

The latter four points form a parallelogram, and it follows that ∂A is a union of at most four line segments. Hence A is admissible by Lemma 8.6.

If $m > 2$, then we cut A along the line $\ell = \{x : \lambda(x) = r_0 \text{ and } x_1 = u_2\}$, and obtain two l-sets

$$A_1 = \{x \in A : x_1 \leq u_2\} \quad \text{and} \quad A_2 = \{x \in A : x_1 \geq u_2\}.$$

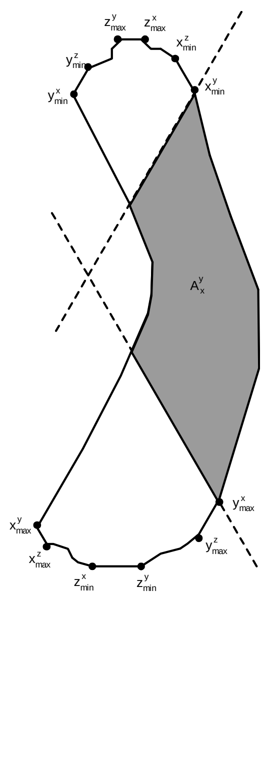
The boundary of A_1 and A_2 is still a union of line segments. For each piece, we have $E_{A_i} \subseteq E_A$, hence E_{A_i} contains either one or two elements. Moreover $F_{A_1} \subseteq \{u_1, u_2\}$ and $F_{A_2} \subseteq \{u_2, \ldots, u_m\}$. Hence we are allowed to conclude by induction that A_1 and A_2 are admissible, and therefore A is admissible. Now assume that E_A contains 3 or more elements, and let w be any element of E_A that is neither the maximum nor the minimum of E_A. Then cutting A along the line $\ell = \{x : \lambda(x) = r_0 \text{ and } x_3 = w\}$ will yield two connected l-sets $A_1 = \{x \in A : x_3 \geq w\}$ and $A_2 = \{x \in A : x_3 \leq w\}$, for which ∂A_i is a union of line segments, and for which the sets E_{A_i} contain fewer than n elements. Hence A_1 and A_2 are admissible, and we conclude that A is also admissible. ∎

9. The derivative of surface areas of level sets

In this section we shall show that every level set of a connected and compact sublattice $L \subseteq \Re^3$ is admissible. Of course, Theorem 7.2 will be again of central importance.

The following six subsets of A contain all parts of the boundary that are not straight lines:

$$A_y^x = \{x \in A : x_1 \geq (y_{\min}^x)_1 \text{ and } x_2 \leq (x_{\max}^y)_2\}$$
$$A_x^y = \{x \in A : x_2 \geq (x_{\min}^y)_2 \text{ and } x_1 \leq (y_{\max}^x)_1\}$$
$$A_z^x = \{x \in A : x_1 \geq (z_{\min}^x)_1 \text{ and } x_3 \leq (x_{\max}^z)_3\}$$
$$A_x^z = \{x \in A : x_3 \geq (x_{\min}^z)_3 \text{ and } x_1 \leq (z_{\max}^x)_1\}$$
$$A_z^y = \{x \in A : x_2 \geq (z_{\min}^y)_2 \text{ and } x_3 \leq (y_{\max}^z)_3\}$$
$$A_y^z = \{x \in A : x_3 \geq (y_{\min}^z)_3 \text{ and } x_2 \leq (z_{\max}^y)_2\}$$



Although the set $B = \overline{A \setminus \left(A_y^x \cup A_x^y \cup A_z^x \cup A_x^z \cup A_z^y \cup A_y^z\right)}$ need not be connected (see the previous illustration), it is clear that the boundary of B is a union of line segments, hence B is a union of admissible sets. The following lemmas allow us to reduce A to B using Theorem 7.2 together with Proposition 7.3, and hence prove the admissibility of A.

Lemma 9.1. *Each of the six sets A_y^x, A_x^y, A_z^x, A_x^z, A_z^y, and A_y^z is admissible.*

Proof. Because all six sets are obtained from A by cutting twice along lines of the form $L_{i,h} = \{x \in \Re^3 : x_i = h\}$, it follows easily from Proposition 7.3 that each of the six sets is a connected l-set, . All the six sets are symmetric to each other via a permutation of the coordinates. Hence it suffices to show that A_x^y is admissible. Note that the set A_x^y is equal to one of the sets discussed in Proposition 6.2, where the functions γ_1, γ_2, σ_1, and σ_2 are given as follows:

$$\gamma_1(\zeta) = \gamma_1^3(\zeta), \quad \gamma_2(\zeta) = \gamma_2^3(\zeta)$$

$$\sigma_1(\zeta) = \begin{cases} \max\left\{\sigma_1^3(\zeta), r_0 - (x_{\min}^y)_2 - \zeta, (y_{\max}^x)_1\right\} & \text{if } \sigma_1^3(\zeta) \text{ is defined,} \\ \max\left\{r_0 - (x_{\min}^y)_2 - \zeta, (y_{\max}^x)_1\right\} & \text{else;} \end{cases}$$

$$\sigma_2(\zeta) = \begin{cases} \min\left\{\sigma_2^3(\zeta), r_0 - (y_{\max}^x)_1 - \zeta, (x_{\min}^y)_2\right\} & \text{if } \sigma_2^3(\zeta) \text{ is defined,} \\ \min\left\{r_0 - (y_{\max}^x)_1 - \zeta, (x_{\min}^y)_2\right\} & \text{else;} \end{cases}$$

and each of those functions is decreasing by Lemma 5.2. It follows that A_x^y is admissible. ■

A quick look at the above illustration shows that $A \setminus A_x^y$ need not be connected. However it follows from Proposition 7.3 that each connected component of $A \setminus A_x^y$ is a connected l-set. The following lemma follows immediately from Theorem 7.2, and Lemma 9.1:

Lemma 9.2. *If each connected component of $\overline{A \setminus A_x^y}$ is admissible, then A is admissible.*

Finally, we come to the main result:

Theorem 9.3. *If $A \subseteq E_{r_0}$ is a compact, connected l-set, and if L is any compact, connected lattice so that $A = L_{r_0}$, then*

$$\int_{\partial A} \varphi(n_A(s))ds = \mu_L'(r_0) = 3(\lambda(infA) + \lambda(supA) - 2r0)$$

Proof. Note that if D is a component of $\overline{A \setminus A_x^y}$, then $D \setminus D_y^x = D \setminus A_y^x$, and corresponding equations hold for the other sets in $\left\{A_z^x, A_x^z, A_z^y, A_y^z\right\}$. Six successive applications of Lemma 9.2. leaves us with the task to show that each connected component of $B = \overline{A \setminus \left(A_y^x \cup A_x^y \cup A_z^x \cup A_x^z \cup A_z^y \cup A_y^z\right)}$ is admissible. But since the boundary of each connected component of B is a union of line segments, the admissibility of the connected components of B follows from Theorem 8.7 ■

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Received April 27, 2019
and in final form October 21, 2019