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Affine-Quadratic Problems on Lie Groups: Tops and Integrable Systems

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Abstract. This paper focuses on the relevance of a certain class of left-invariant Hamiltonians (affine-quadratic) on a reductive semi-simple Lie algebra $\mathfrak{g}$ for the theory of integrable systems and the equations of applied mathematics. Any semi-simple Lie group G that contains a closed subgroup K is reductive, in the sense, that the orthogonal complement $\mathfrak{p}$ in $\mathfrak{g}$ of the Lie algebra $\mathfrak{k}$ of K, relative to the Killing form, satisfies $[\mathfrak{k},\mathfrak{p}]\subseteq\mathfrak{p}$. This implies that K acts (by adjoint action) on $\mathfrak{p}$ and therefore induces the semi-direct product $\mathfrak{p}\rtimes K$. Consequently, $\mathfrak{g}$, as a vector space, carries two Lie algebra structures: semi-simple, and semi-direct. Hence, the dual $\mathfrak{g}^*$ carries two Poisson structures as well. Any affine-quadratic function H on $\mathfrak{g}$ can be simultaneously viewed as a Hamiltonian for either Poisson structure.
We will show that certain coadjoint orbits relative to the semi-direct action are the cotangent bundles of $SO(n)$. This implies that the equations of an n-dimensional top can be represented on such coadjoint orbits. In this situation there is a canonical affine-quadratic Hamiltonian whose Hamiltonian equations on these coadjoint orbits coincide with the equations of the top. This implies that the integrable cases of the top correspond to the the integrable cases of the overseeing affine Hamiltonian. More generally, we will identify a subclass of affine-quadratic Hamiltonians, called isospectral, that provides new insights into the theory of integrable systems based on the contributions of S. V. Manakov, A. T. Fomenko, A. S. Mischenko and O. Bogoyavlensky listed in the references.

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1. Introduction

My last book drew attention to a particular class of variational problems on Lie groups, called *affine-quadratic*, whose Hamiltonians show ubiquitous presence in various problems from geometry and mechanics. Defined over certain "canonical" affine spaces on reductive Lie algebras, these problems share a common base with the theory of semigroups and geometric control theory; as such, they seem a fitting subject for a volume on Lie theory inspired by Jimmy Lawson and his collaborators.

This paper addresses some finer points raised in the aforementioned book. In particular the paper clarifies the assertion made by G. Kirchhoff in 1885 that the equations which describe the equilibrium configurations of an elastic rod in $\mathbb{R}^3$ are the same as the equations of the heavy top, quoted in the literature on elasticity as "the kinetic

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analogue of Kirchhoff" ([14]). More generally, the paper shows how the equations of an n-dimensional heavy top seem the "same" as the equilibrium equations of an n-dimensional elastic rod in all space forms.

To set the paper in motion, recall first the mathematical formalism initiated by G. Kirchhoff in his study of an elastic rod that made his assertion plausible. In Kirchhoff's model each configuration of the rod is described by a curve $(x(t), R(t))$, $t \in [0, T]$, in the orthonormal frame bundle $SE(3)$ of $\mathbb{R}^3$, where $x(t)$ stands for the central line of the rod, and where $R(t) \in SO(3)$ stands for the frame along $x(t)$ that measures the amount of bending and twisting relative to a fixed orthonormal frame in $SO(3)$. This model also assumes that the rod is inextensible, $||\frac{dx}{dt}|| = 1$, so that T is the length of the rod, and that the frame is adapted to the central line so that the first leg of the frame is equal to the tangent vector of $x(t)$, $\frac{dx}{dt} = R(t)e_1$.

Configuration curves $(x(t), R(t))$ are assigned the potential energy

$$E = \frac{1}{2}\int_0^T (c_1u_1^2(t) + c_2u_2^2(t) + c_3u_3^2(t))\, dt,$$

where c_1, c_2, c_3 are constants that reflect the physical characteristics of the rod, and where $u_1(t), u_2(t), u_3(t)$ are the strains defined by the deformations

$$\frac{dR}{dt} = R(t)(u_1(t)A_1 + u_2(t)A_2 + u_3(t)A_3)$$

relative to the standard basis

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Kirchhoff's variational principle states that the equilibrium configurations correspond to the critical values of E among all possible configurations that satisfy the same boundary conditions at the ends of the rod.

Kirchhoff's model for an elastic rod in $\mathbb{R}^3$ admits natural generalizations to the n-dimensional space forms due to the fact that the orthonormal frame bundle coincides with the isometry group on such spaces. To pass to this level, let $M_\epsilon = SO_\epsilon(n+1)/K$ denote the space form, where $\epsilon = \pm 1, 0$ stands for the curvature of M_ϵ, and where $K = 1 \times SO(n)$ stands for the isotropy subgroup $ge_0 = e_0$. In this context $SO_\epsilon(n+1)$ is equal to $SO(1, n)$ when $\epsilon = -1$, $SO(n+1)$ when $\epsilon = 1$, and $SE(n) = \mathbb{R}^n \rtimes SO(n)$ when $\epsilon = 0$.

Any curve $x(t)$ in M_ϵ parametrized by arc length is traced by a framed curve $g(t)$, in the sense that $g(t)e_1, \ldots, g(t)e_n$ is identified with an orthonormal frame along $x(t) = g(t)e_0$. In Kirchhoff's spirit assume that the frame $g(t)e_1, \ldots, g(t)e_n$ is adapted to $x(t)$, except that, instead of Kirchhoff's choice $\frac{dx}{dt} = g(t)e_1$, assume a slightly more general adaptation $\frac{dx}{dt} = \sum_{i=1}^n a_i g(t)e_i$ for some constants $a_1, \ldots, a_n$ such that $\sum_{i=1}^n a_i^2 = 1$. Framed curves $g(t)$ that satisfy this condition are called *Darboux* ([10]).

In the non-Euclidean cases each Darboux curve $g(t)$ is a solution of $\frac{dg}{dt} = g(t)V(t)$ for a curve $V(t)$ in the affine space $\Gamma = \{A + U, U \in \mathfrak{k}\}$ in $\mathfrak{so}_\epsilon(n+1)$, where A is a fixed element in the orthogonal complement $\mathfrak{p}_\epsilon$ of the Lie algebra $\mathfrak{k}$ of K, relative to the bilinear form $\langle A, B\rangle = -\frac{1}{2}Trace(AB)$ in $\mathfrak{so}_\epsilon(n+1)$. In fact, A satisfies $Ae_0 = \sum_{i=1}^n a_i e_i, Ae_i = -\epsilon a_i e_0, i = 1, \ldots, n$. Hence, Darboux curves are the solutions of

$$\frac{dg}{dt} = g(t)(A + U(t)), \tag{1}$$

for some curve $U(t) \in \mathfrak{k}$. One can show that Euclidean Darboux curves can be identified with the solutions of (1) in $SE(n)$ corresponding to $\epsilon = 0$.

Then any positive definite operator $\mathcal{P}$ on $\mathfrak{k}$ relative to the trace form $\langle\,,\,\rangle$ (restricted to $\mathfrak{k}$) induces the energy function $E = \frac{1}{2}\int_0^T \langle \mathcal{P}(t)(U(t)), U(t)\rangle\, dt$ over Darboux curves, and equilibrium configurations correspond to the solutions of (1) that satisfy the given boundary conditions in $SO_\epsilon(n+1)$ along which the energy E is locally minimal.

It then follows from the Maximum Principle of Pontryagin that every equilibrium configuration is the projection of an extremal curve in the cotangent bundle of $SO_\epsilon(n+1)$. Since strictly abnormal extremals do not exist in this context, every equilibrium configuration is the projection of a normal extremal curve, that is the projection of an integral curve of the Hamiltonian vector field $\vec{H}$ associated to the Hamiltonian H, which is most commonly written on the tangent bundle of $SO_\epsilon(n+1)$ as

$$H = \frac{1}{2}\langle \mathcal{P}^{-1}(L_\mathfrak{k}), L_\mathfrak{k}\rangle + \langle A,\ L_\mathfrak{p}\rangle,\ L_\mathfrak{k} \in \mathfrak{k}, L_\mathfrak{p} \in \mathfrak{p}_\epsilon. \tag{2}$$

(see [10]). Inspired by Kirchhoff's "kinetic analogue", we will compare the equations of motion generated by an n-dimensional "top-like" Hamiltonian

$$\hat{H}(R, M) = \frac{1}{2}\langle \mathcal{P}^{-1}M, M\rangle + \sum_{i=1}^n c_i(\alpha_i, a),\ R \in SO(n),\ M \in \mathfrak{so}(n), \tag{3}$$

to the equations of $\vec{H}$. Here, $R^T e_i = \alpha_i, i = 1, \ldots, n,\ c_1, \ldots, c_n$ are constants, and a is a fixed vector in $\mathbb{R}^n$, in practice equal to the coordinate vector of the centre of gravity of the body relative to an orthonormal frame affixed to the body at the stationary point on the body. We will show that the equations of motion generated by $\hat{H}$ coincide with the restriction of the Hamiltonian equations of H to certain coadjoint orbits in $\mathfrak{so}_\epsilon(n+1)$ (Theorem 3.1).

The above theorem transcends the physical context that motivated the comparisons (hence "top-like") since the Hamiltonians associated with mechanical tops conform to the constraint $\mathcal{P}(U) = SU + US$ for some positive matrix S. At the same time the theorem gives a precise meaning to the statement that the equations of an n-dimensional heavy top are the "same" as the equations of an n-dimensional elastic rod (rather than the other way around as postulated by Kirchhoff).

The questions raised above form only a small part of a more general mathematical inquiry centred around a distinguished class of optimal control problems, called *affine-quadratic*, which exist on any semi-simple Lie group G with a finite centre that contains a compact subgroup K, and whose equations are formally the same as the equations of an elastic rod. This class of systems is defined through the following theoretic ingredients.

Since K has finite centre, the Killing form $Kl(A, B) = Tr(adA \circ adB)$ is negative definite on the Lie algebra $\mathfrak{k}$ of K ([7]). Therefore $\langle A, B\rangle = -Kl(A, B)$ is positive on $\mathfrak{k}$, and inherits the essential properties from the Killing form. In particular, it is symmetric and non-degenerate on $\mathfrak{g}$, and additionally satisfies

$$\langle A, [B, C]\rangle = \langle [A, B], C\rangle, \tag{4}$$

for any elements A, B, C in $\mathfrak{g}$. If $\mathfrak{p}$ denotes the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ relative to $\langle\, ,\, \rangle$, then

$$\langle [\mathfrak{p}, \mathfrak{k}], \mathfrak{k}\rangle = \langle \mathfrak{p}, [\mathfrak{k}, \mathfrak{k}]\rangle = \langle \mathfrak{p}, \mathfrak{k}\rangle = 0.$$

Hence, $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}$, and therefore G is reductive in the sense of ([18]). We will make an additional assumption that $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, which is automatically satisfied on semi-simple Lie groups that admit an involutive automorphism (in this situation K is the connected component through the identity of the group of fixed points by the automorphism, and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ relative to the Killing form).

Analogous to the elastic problem, any positive definite operator $\mathcal{P}$ on $\mathfrak{k}$ defines an energy functional $E = \frac{1}{2}\int_o^T \langle \mathcal{P}(U(t)), U(t)\rangle\, dt$ associated with curves $U(t)$ in $\mathfrak{k}$. In this setting any element $A \in \mathfrak{p}$ defines an affine space $\Gamma = \{A + U : U \in \mathfrak{k}\}$ in $\mathfrak{g}$, and this space defines a left invariant differential system

$$\frac{dg}{dt} = g(t)(A + U(t)), g(t) \in G, \tag{5}$$

where $U(t)$ is a bounded and measurable curve in $\mathfrak{k}$. We will now think of curves $U(t)$ in $\mathfrak{k}$ as the control functions for the control system (5) and seek solutions $g(t)$ that satisfy the given boundary conditions $g(0) = g_0$, $g(T) = g_1$ for which the energy of transfer $\frac{1}{2}\int_o^T \langle \mathcal{P}(U(t)), U(t)\rangle\, dt$ is minimal. The above optimal control problem will be referred to as *the affine-quadratic* problem (reminiscent of linear-quadratic problems in control theory literature).

Any reductive semi-simple Lie algebra $\mathfrak{g}$ also carries along a "hidden" semi-direct product Lie algebra $\mathfrak{g}_0 = \mathfrak{p} \rtimes \mathfrak{k}$ for the following reasons. Since $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}$, K acts linearly on $\mathfrak{p}$ by adjoint action $h \to \mathrm{Ad}_h|_{\mathfrak{p}}, h \in K$, and induces the semi-direct product $G_0 = \mathfrak{p} \rtimes K$ with the group operation

$$(A_1, h_1)(A_2, h_2) = (A_1 + \mathrm{Ad}_{h_1}(A_2), h_1 h_2).$$

Then the Lie algebra $\mathfrak{g}_0$ can be represented in $\mathfrak{g}$ with the Lie bracket

$$(A_1, B_1), (A_2, B_2)] = [B_1, A_2] - [B_2, A_1] + [B_1, B_2], (A_i, B_i) \in \mathfrak{p}_\epsilon \times \mathfrak{k},\ \ i = 1, 2.$$

Thus $\mathfrak{g}$, as a vector space, carries two Lie brackets:

$$[A_1 + B_1, A_2 + B_2]_s = [B_1, A_2] - [B_2, A_1] + s[A_1, A_2] + [B_1, B_2],$$

$s = 0$ in the semi-direct case, and $s = 1$ in the semi-simple case.

It follows that every affine space $\Gamma = \{A + U : U \in \mathfrak{k}\}$ that defines an affine left-invariant system on G also defines a corresponding left-invariant affine system on the semi-direct product G_0. Thus behind every affine quadratic optimal problem on G there is a corresponding affine-quadratic "shadow" problem on the semi-direct product G_0.

One can show that every affine-quadratic system (semi-simple, or semi-direct) is controllable, in the sense that for any pair of points (g_0, g_1) in the underlying Lie group there is a solution $g(t)$ of (5) defined on some interval $[0,T]$ such that $g(0)=g_0$ and $g(T)=g_1$, which in turn implies that optimal solutions exist for each pair of boundary points.

For the purposes of this paper it is the associated Hamiltonian

$$H=\frac{1}{2}\langle\mathcal{P}^{-1}(L_{\mathfrak{k}}),L_{\mathfrak{k}}\rangle+\langle A,L_{\mathfrak{p}}\rangle, L_{\mathfrak{k}}\in\mathfrak{k}, L_{\mathfrak{p}}\in\mathfrak{p}, \tag{6}$$

on the tangent bundle of G_s and its Hamiltonian equations

$$\frac{dg}{dt}=g(t)(A+\mathcal{P}^{-1}(L_{\mathfrak{k}}(t))),\quad \frac{dL_{\mathfrak{k}}}{dt}=[\mathcal{P}^{-1}(L_{\mathfrak{k}}(t)),L_{\mathfrak{k}}(t)]+[A,L_{\mathfrak{p}}(t)], \tag{7}$$
$$\frac{dL_{\mathfrak{p}}}{dt}=[\mathcal{P}^{-1}(L_{\mathfrak{k}}(t)),L_{\mathfrak{p}}(t)]+s[A,L_{\mathfrak{k}}(t)],\ s=0,1,$$

that are the principal objects of interest.

Our observation that the top is "like" an elastic rod, and that the integrable cases of the rod dominate the integrable cases of the top is symptomatic of a more general situation in which an affine Hamiltonian system on a semi-simple reductive Lie group stands out as a central ingredient in the theory of integrable systems based on Lie algebraic methods ([19],[22]). We will present several results that justify this claim. Remarkably, it is the semi-direct Hamiltonian that makes crucial links with mechanical tops, rather than its semi-simple antecedent.

2. Symplectic background, Hamiltonian systems

The theoretic foundation upon which the above claims are made is rooted in the modern developments of symplectic geometry. Below is a brief summary of the necessary ingredients required for the main results.

Recall that a manifold M endowed with a non-degenerate and closed 2-form ω is called *symplectic*. The symplectic form induces a correspondence between functions and vector fields: to every function f there corresponds a vector field $\vec{f}$ defined by $\omega(\vec{f},X)=df(X)$ for all vector fields X on M. In this context $\vec{f}$ is called the Hamiltonian vector field generated by f. Every symplectic manifold is even dimensional, and at each point of M there exists a neighbourhood with coordinates $(x_1,\dots,x_n,p_1,\dots,p_n)$ such that the Hamiltonian vector fields are given by

$$\vec{f}=\sum_{i=1}^{n}\frac{\partial f}{\partial p_i}\frac{\partial}{\partial x_i}-\frac{\partial f}{\partial x_i}\frac{\partial}{\partial p_i}. \tag{8}$$

This choice of coordinates in which $\vec{f}$ is represented by (8) is called *symplectic*.

Any cotangent bundle T^*M is a symplectic manifold endowed with its canonical symplectic form, usually written as $\omega=dp\wedge dx$ relative to a choice of symplectic coordinates $\sum_{i=1}^{n}p_idx_i$. As a symplectic manifold the cotangent bundle is somewhat special. For instance, every vector field X on M can be lifted to a unique Hamiltonian vector field $\vec{f}_X$ in T^*M via the function $f_X(\xi)=\xi(X(x))$, $\xi\in T^*_xM$. Vector field $\vec{f}_X$ is called the *Hamiltonian lift of* X. The same procedure is applicable to any time varying vector field, and by extension to any differential system on M. Thus any differential system in M can be lifted to a Hamiltonian system in T^*M. This fact is also important for problems of optimal control where the Maximum Principle singles out the appropriate Hamiltonian lifts that govern the optimal solutions ([11]).

When the base manifold is a Lie group G, and when the underlying differential system is either left or right invariant, then there is privileged system of coordinates based on the realization of T^*G as $G \times \mathfrak{g}^*$, with $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. This coordinate system preserves the left invariant symmetries and elucidates the conservation laws of the associated Hamiltonian system. The passage to these coordinates is explained below.

2.1. Left-invariant trivializations and the symplectic form

Having in mind applications involving left-invariant variational systems, the cotangent bundle T^*G and the tangent bundle TG will be represented as $G\times\mathfrak{g}^*$ and $G\times\mathfrak{g}$ via the left-translations. That is, tangent vectors $v \in T_gG$ will be identified with pairs $(g, X) \in G \times \mathfrak{g}$ via the relation $v = L_{g_*}X$. Similarly, linear functions $\xi \in T_g^*G$ will be identified with pairs $(g, \ell) \in G \times \mathfrak{g}^*$ via $\xi = L_g{}^*\ell$. Then $T(T^*G)$ is naturally identified with $(G\times\mathfrak{g})\times(\mathfrak{g}^*\times\mathfrak{g}^*) \cong (G\times\mathfrak{g}^*)\times(\mathfrak{g}\times\mathfrak{g}^*)$, with the understanding that an element $((g, \ell), (A, a)) \in (G \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*)$ denotes the tangent vector (A, a) at the base point (g, ℓ).

Note that $G \times \mathfrak{g}^*$ is a Lie group in its own right since $\mathfrak{g}^*$ is an abelian Lie group with the group multiplication given by the vector addition. Then left-invariant vector fields in $G \times \mathfrak{g}^*$ are the left-translates of the pairs (A, a) in the Lie algebra $\mathfrak{g} \times \mathfrak{g}^*$ of $G \times \mathfrak{g}^*$. In this formalism, $\exp\big(t(A, a)\big)(g, \ell) = (g \exp(tA), \ell + ta)$ are the corresponding one-parameter groups of diffeomorphisms associated with the left-invariant vector fields on $G \times \mathfrak{g}^*$.

In terms of the left-invariant vector fields defined by $V_1 = (A_1, a_1)$ and $V_2 = (A_2, a_2)$, the canonical symplectic form on T^*G is given by the following formula:

$$\omega_{(g,\ell)}(V_1, V_2) = a_2(A_1) - a_1(A_2) - \ell([A_1, A_2]) \tag{9}$$

The above differential form is invariant under the left-translations in $G \times \mathfrak{g}^*$, and is particularly revealing for the Hamiltonian vector fields generated by the left-invariant functions on $G\times\mathfrak{g}^*$. A function H on $G\times\mathfrak{g}^*$ is said to be left-invariant if $H(hg, \ell) = H(g, \ell)$ for all $g, h \in G$ and all $\ell \in \mathfrak{g}^*$. It follows that the left-invariant functions on $G \times \mathfrak{g}^*$ are in exact correspondence with functions on $\mathfrak{g}^*$. Each left-invariant vector field $X(g) = (L_g)_*X$, $X \in \mathfrak{g}$, lifts to a linear function on $\mathfrak{g}^*$ because

$$h_X(\xi) = \xi(X(g)) = \xi \circ (L_g)_*(X) = \ell(X), \xi \in T_g^*G.$$

Any function H on $\mathfrak{g}^*$ generates a Hamiltonian vector field $\vec{H}$ on $G \times \mathfrak{g}^*$ whose integral curves are the solutions of

$$\frac{dg}{dt}(t) = g(t)dH_{\ell(t)}, \quad \frac{d\ell}{dt}(t) = -\mathrm{ad}^* dH_{\ell(t)}(\ell(t)). \tag{10}$$

For when H is a function on $\mathfrak{g}^*$, then its differential at a point ℓ is a linear function on $\mathfrak{g}^*$, hence an element of $\mathfrak{g}$, because $\mathfrak{g}^*$ is a finite dimensional vector space. If $\vec{H}_{(g,\ell)} = (A(g, \ell), a(g, \ell))$ for some vectors $A(g, \ell) \in \mathfrak{g}$ and $a(g, \ell) \in \mathfrak{g}^*$, then

$$b(dH_\ell) = b(A) - a(B) - \ell[A, B],$$

must hold for any tangent vector (B,b) at (g,ℓ). This implies that $A(g,\ell)=dH_\ell$, and $a=-\mathrm{ad}^*dH_\ell(\ell)$, where $(\mathrm{ad}^*A)(\ell)(B)=\ell[A,B]$ for all $B\in\mathfrak{g}$. This argument validates (10).

In a more general case where H is a function of both g and ℓ, the equations for $\vec{H}$ are given by

$$\frac{dg}{dt}(t)=g(t)dH_{\ell(t)},\quad \frac{d\ell}{dt}(t)=-\mathrm{ad}^*dH_{\ell(t)}(\ell(t))-dH_g\circ L_{g*},\tag{11}$$

as can be easily verified through the relations

$$b(dH_\ell)+dH_g\circ L_{g_*}B=b(A)-a(B)-\ell[A,B].$$

This situation typically occurs in problems of mechanics in the presence of potential functions. For instance, the motion of an n-dimensional rigid body with a potential function $V:SO(n)\to R$ is described by the Hamiltonian

$$H(R,\ell)=H_0(\ell)+V(\alpha_1,\dots,\alpha_n,)$$

where $\alpha_1,\dots,\alpha_n$ denote the columns of the matrix transpose R^T of R.
If $R(t)=Re^{tX}$ is a curve in $SO(n)$ defined by an element $X\in\mathfrak{so}(n)$, then we have $\alpha_i(t)=R(t)^Te_i=e^{-tX}R^Te_i=e^{-tX}\alpha_i(t)$. Therefore,

$$dV(RX)=\sum_{i=1}^n\Big(\frac{\partial V}{\partial\alpha_i},\frac{d\alpha_i}{dt}\Big)|_{t=0}=\sum_{i=1}^n\Big(\frac{\partial V}{\partial\alpha_i},-X\alpha_i\Big)=\sum_{i=1}^n\Big(\frac{\partial V}{\partial\alpha_i}\wedge\alpha_i,X\Big)$$

Thus $dH_g\circ L_{g*}=\sum_{i=1}^n\frac{\partial V}{\partial\alpha_i}\wedge\alpha_i$ interpreted as an element of $\mathfrak{so}^*(n)$. Hence the equations of motion are given by

$$\frac{dg}{dt}(t)=g(t)dH_0(\ell(t)),\quad \frac{d\ell}{dt}(t)=-\mathrm{ad}^*dH_0(\ell(t))(\ell(t))+\sum_{i=1}^n\alpha_i\wedge\frac{\partial V}{\partial\alpha_i}.\tag{12}$$

2.2. Poisson manifolds, coadjoint orbits

We now address the Poisson structure on $\mathfrak{g}^*$ inherited from the symplectic form ω given by (9). Recall that a manifold M together with a bilinear, skew-symmetric form

$$\{\ ,\ \}:C^\infty(M)\times C^\infty(M)\to C^\infty(M)$$

that satisfies

$$\{fg,h\}=f\{g,h\}+g\{f,h\}\ \ \text{(Leibniz's rule), and}$$

$$\{f,\{g,h\}\}+\{h,\{f,g\}\}+\{g,\{h,f\}\}=0\ \ \text{(Jacobi's identity),}$$

for all functions f,g,h on M, is called a *Poisson manifold.*

Every symplectic manifold is a Poisson manifold with the Poisson bracket defined by $\{f,g\}(p)=\omega_p(\vec{f}(p),\vec{g}(p)),p\in M$. However, a Poisson manifold need not be symplectic, because it may happen that the Poisson bracket is degenerate at some points of M. Nevertheless, each function f on M induces a Poisson vector field $\vec{f}$ through the formula $\vec{f}(g)=\{f,g\}$. It is known that every Poisson manifold is foliated by the orbits of its family of Poisson vector fields, and that each orbit is a symplectic submanifold of M with its symplectic form $\omega_p(\vec{f},\vec{h})=\{f,h\}(p)$. (This foliation is known as a *the symplectic foliation of* M).

Proposition 2.1. *The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is a Poisson manifold with the Poisson bracket*

$$\{f,h\}(\ell) = \ell([dh, df])$$

for any functions f and h on $\mathfrak{g}^*$.

Proof. Functions on $\mathfrak{g}^*$ coincide with the left-invariant functions on $G \times \mathfrak{g}^*$. Hence,

$$\omega_{(g,\ell)}(\vec{f}, \vec{h}) = \omega_{(g,\ell)}((df, 0), (dh, 0)) = -\mathrm{ad}^*([df, dh])(\ell) = \ell([dh, df]).$$

It follows that the Poisson bracket on $\mathfrak{g}^*$ is the restriction of the canonical Poisson bracket on $G \times \mathfrak{g}^*$ to the left-invariant functions. As such, it automatically satisfies the properties of a Poisson manifold. ■

In the literature on integrable systems, Poisson bracket $\{f,h\}(\ell) = \ell([df, dh])$ is often referred as *the Lie-Poisson bracket* ([22]). We have taken its negative so that Poisson vector fields agree with the projections of the Hamiltonian vector fields generated by left-invariant functions (and also to agree with the sign convention in [10] and [11]).

It follows that each function H on $\mathfrak{g}^*$ defines a Poisson vector field $\vec{H}$ on $\mathfrak{g}^*$ through the formula $\vec{H}(f)(\ell) = \{H, f\}(\ell) = -\ell([dH, df])$. The integral curves of $\vec{H}$ are the solutions of

$$\frac{d\ell}{dt}(t) = -\mathrm{ad}^* dH_{\ell(t)}(\ell(t)) \tag{13}$$

That is, each function H on $\mathfrak{g}^*$ may be considered both as a Hamiltonian on T^*G, as well as a function on the Poisson space $\mathfrak{g}^*$. It follows that the Poisson equations of the associated Poisson field are the projections of the Hamiltonian equations (10) on $\mathfrak{g}^*$.

Solutions of (13) are intimately linked with the coadjoint orbits of G. We recall that the coadjoint orbit of G through a point $\ell \in \mathfrak{g}^*$ is given by

$$\mathrm{Ad}^*_g(\ell) = \{\ell \circ \mathrm{Ad}_{g^{-1}},\ g \in G\}.$$

The following proposition is a paraphrase of A. A. Kirillov' fundamental contributions to the Poisson structure of $\mathfrak{g}^*$ ([12]).

Proposition 2.2. *Let $\mathcal{F}$ denote the family of Poisson vector fields on $\mathfrak{g}^*$ and let $M = \mathcal{O}_{\mathcal{F}}(\ell_0)$ denote the orbit of $\mathcal{F}$ through a point $\ell_0 \in \mathfrak{g}^*$. Then M is equal to the connected component of the coadjoint orbit of G that contains ℓ_0. Consequently each coadjoint orbit is a symplectic submanifold of $\mathfrak{g}^*$.*

The fact that the Poisson equations evolve on coadjoint orbits implies useful reductions in the theory of Hamiltonian systems with symmetries. Our main results will make use of this fact.

2.3. Representation of coadjoint orbits on Lie algebras – semi-simple vs. semi-direct

On semi-simple Lie groups the Killing form, or any scalar multiple of it $\langle\, ,\, \rangle$, is non-degenerate, and can be used to identify linear functions ℓ on $\mathfrak{g}$ with points $L \in \mathfrak{g}$ via the formula $\langle L, X\rangle = \ell(X)$, $X \in \mathfrak{g}$. Then Poisson equation (13) can be expressed dually on $\mathfrak{g}$ as

$$\frac{dL}{dt} = [dH, L]. \tag{14}$$

The argument is simple:

$$\langle \frac{dL}{dt}, X\rangle = \frac{d\ell}{dt}(X) = -\ell([dH, X]) = \langle L, [X, dH]\rangle = \langle [dH, L], X\rangle.$$

Since X is arbitrary, equation (14) follows.

Under the above indentification coadjoint orbits are identified with the adjoint orbits $\mathcal{O}(L_0) = \{gL_0g^{-1} : g \in G\}$, and the Poisson vector fields $\vec{f}_X(\ell) = -\mathrm{ad}^*X(\ell)$ are identified with vector fields $\vec{X}(L) = [X, L]$. Each vector field $[X, L]$ is tangent to $\mathcal{O}(L_0)$ at L, and $\omega_L([X, L], [Y, L]) = \langle L, [Y, X]\rangle$, X, Y in $\mathfrak{g}$ is the symplectic form on each orbit $\mathcal{O}(L_0)$.

In a reductive semi-simple Lie group G with a subgroup K there is also the semi-direct product $G_0 = \mathfrak{p} \rtimes K$, described earlier in the introduction. Then Poisson equations on $\mathfrak{g}_0^* = (\mathfrak{p} \rtimes \mathfrak{k})^*$ can be also represented on $\mathfrak{g}_0$ via the quadratic form $\langle\, ,\, \rangle$, as in the semi-simple case, but the resulting expression takes on a slightly different form. To see the difference, let $dH = dH_{\mathfrak{p}} + dH_{\mathfrak{k}}$ and $L = L_{\mathfrak{p}} + L_{\mathfrak{k}}$ denote the decompositions of dH and L onto the factors $\mathfrak{p}$ and $\mathfrak{k}$. On the semi-direct product,

$$\begin{aligned}\langle \frac{dL_{\mathfrak{p}}}{dt}, X_{\mathfrak{p}}\rangle + \langle \frac{dL_{\mathfrak{k}}}{dt}, X_{\mathfrak{k}}\rangle = \langle \frac{dL}{dt}, X\rangle &= \frac{d\ell}{dt}(X) = -\ell([dH, X]) = -\langle L, [dH, X]\rangle \\ &= -\langle L, [dH_{\mathfrak{p}}, X_{\mathfrak{k}}] + [dH_{\mathfrak{k}}, X_{\mathfrak{p}}] + [dH_{\mathfrak{k}}, X_{\mathfrak{k}}]\rangle \\ &= -\langle L_{\mathfrak{p}}, [dH_{\mathfrak{p}}, X_{\mathfrak{k}}] + [dH_{\mathfrak{p}}, X_{\mathfrak{k}}]\rangle - \langle L_{\mathfrak{k}}, [dH_{\mathfrak{k}}, X_{\mathfrak{k}}] \\ &= \langle X_{\mathfrak{k}}, [dH_{\mathfrak{k}}, L_{\mathfrak{k}}] + [dH_{\mathfrak{p}}, L_{\mathfrak{p}}]\rangle + \langle X_{\mathfrak{p}}, [dH_{\mathfrak{k}}, L_{\mathfrak{p}}]\rangle.\end{aligned}$$

Hence the Poisson equations are given by

$$\frac{dL_{\mathfrak{k}}}{dt} = [dH_{\mathfrak{k}}, L_{\mathfrak{k}}] + [dH_{\mathfrak{p}}, L_{\mathfrak{p}}], \frac{dL_{\mathfrak{p}}}{dt} = [dH_{\mathfrak{k}}, L_{\mathfrak{p}}]. \tag{15}$$

This equation can be combined with the equations for the semi-simple case in terms of the parameter s with

$$\frac{dL_{\mathfrak{k}}}{dt} = [dH_{\mathfrak{k}}, L_{\mathfrak{k}}] + [dH_{\mathfrak{p}}, L_{\mathfrak{p}}], \frac{dL_{\mathfrak{p}}}{dt} = [dH_{\mathfrak{k}}, L_{\mathfrak{p}}] + s[dH_{\mathfrak{k}}, L_{\mathfrak{p}}], s = 0, 1. \tag{16}$$

One can show that

$$\mathcal{O}(L_0) = \{P = Ad_h(P_0), Q = [Ad_h(P_0), X] + Ad_h(Q_0), (X, h) \in G_0\}. \tag{17}$$

is the coadjoint orbit through $P_0 \in \mathfrak{p}, Q_0 \in \mathfrak{k}$ under the action of $G_0 = \mathfrak{p} \rtimes K$, when $\ell_0 \in \mathfrak{g}_s^*$ is identified with $L_0 = P_0 + Q_0$ in $\mathfrak{g}_0$, and when $\ell = \mathrm{Ad}^*_{(X,h)}(\ell_0)$ is identified with $L = P + Q$ ([11]).

The adjoint orbits of a non-compact semi-simple Lie group G are often symplectomorphic with the cotangent bundles of manifolds ([6]). It appears that the same is true for coadjoint orbits under the action of semi-direct products. We will now single out two such situations which are relevant for our results.

Space forms. Return now to the isometry groups $G = SO_\epsilon(n + 1)$, $\epsilon = \pm 1$, introduced earlier, their Lie algebras $\mathfrak{so}_\epsilon(n + 1)$ with the canonical trace form $\langle A, B\rangle = -\frac{1}{2}Tr(AB)$, and $K = \{1\} \times SO(n)$ the isotropy group of e_0. Relative to $SO_\epsilon(n + 1)$ we define its invariant bilinear form $(x, y)_\epsilon = x_0y_0 + \epsilon \sum_{i=1}^n x_iy_i$ in the ambient space $\mathbb{R}^{n+1}$.

For any $a,b\in\mathbb{R}^{n+1}$, $a\otimes_\epsilon b$ denotes the matrix defined by $(a\otimes_\epsilon b)x = (a,x)_\epsilon b,\ x\in\mathbb{R}^{n+1}$, and then $a\wedge_\epsilon b$ is the matrix $a\otimes_\epsilon b - b\otimes_\epsilon a$. Since $((a\wedge_\epsilon b)x,y)_\epsilon + (x,(a\wedge_\epsilon b)y)_\epsilon = 0$, $a\wedge_\epsilon b$ belongs to $\mathfrak{so}_\epsilon(n+1)$ for any $a,b\in\mathbb{R}^{n+1}$.

It is easy to show that the Lie algebra $\mathfrak{k}$ and its orthogonal complement $\mathfrak{p}_\epsilon$ in $\mathfrak{so}_\epsilon(n+1)$ are given by the following expressions:

$$\mathfrak{p}_\epsilon = \{A = u\wedge_\epsilon e_0 : u\in\mathbb{R}^{n+1}, (u,e_0)_\epsilon = 0\}, \tag{18}$$

$$\mathfrak{k} = \{B = v\wedge_\epsilon w : v\in\mathbb{R}^{n+1}, w\in\mathbb{R}^{n+1}, (v,e_0)_\epsilon = (w,e_0)_\epsilon = 0\}, \tag{19}$$

The preceding matrices can be written also as

$$A = \begin{pmatrix} 0 & -\epsilon u^* \\ u & 0\end{pmatrix},\quad B = \begin{pmatrix} 0 & 0 \\ 0 & v\wedge w\end{pmatrix},\ u,v,w\in\mathbb{R}^n.$$

Proposition 2.3. *The coadjoint orbit $\mathcal{O}(P_0)$ through $P_0 = p_0\wedge_\epsilon e_0$, $(p_0,e_0)_\epsilon = 0$, $Q_0 = 0$ under the action of the semi-direct product $\mathfrak{p}_\epsilon \rtimes K$ is diffeomorphic to the tangent bundle of the connected "sphere" $S^n_\epsilon = \{p\in\mathbb{R}^{n+1} : (p,p)_\epsilon = (p_0,p_0)_\epsilon\}$ that contains p_0.*

Proof. Let $h\in K$, and $X = x\wedge e_o$, $(x,e_0)_\epsilon = 0$. Then

$$P = Ad_h(P_0) = h(p_0)\wedge_\epsilon h(e_0) = p\wedge_\epsilon e_0, p = h(p_0)$$

$$Q = [Ad_h(P_0), X] = [p\wedge_\epsilon e_0, x\wedge_\epsilon e_0] = p\wedge_\epsilon x = p\wedge_\epsilon x_p^\perp,$$

where $x_p^\perp$ is the projection of x on the orthogonal complement of p in $\mathbb{R}^{n+1}$.

Therefore,
$$(p, x_p^\perp) \Rightarrow p\wedge_\epsilon e_0 + p\wedge x_p^\perp$$

is the desired diffeomorphism from the tangent bundle of the connected sphere S^n_ϵ onto the coadjoint orbit $\{Ad_h(P_0) + [Ad_h(P_0), X], (X,h)\in\mathfrak{p}_\epsilon\rtimes K\}$. ■

The above diffeomorphism is actually a symplectomorphism from the cotangent bundle of either the Euclidean sphere S^n when $\epsilon = 1$, or the hyperboloid of one sheet when $\epsilon = -1$, to the appropriate coadjoint orbit, but we will not go into these details

Coadjoint orbits and flag manifolds. We will now turn our attention to the reductive pair $G = SL(n), K = SO(n)$. Then $\mathfrak{k} = \mathfrak{so}(n)$ is the Lie algebra of K, and $\mathfrak{p}$, the space of symmetric $n\times n$ matrices of trace zero, is the orthogonal complement relative to the trace form. Every symmetric matrix S can be written as $S = S_0 + \frac{Tr(S)}{n}I$, $S_0\in\mathfrak{p}$. An easy inspection of (17) shows that the orbit through S differs by a constant factor $\frac{Tr(S)}{n}I$ from the orbit through S_0. So the zero-trace requirement is inessential for the structure of coadjoint orbits.

Consider next the coadjoint orbit through a symmetric matrix P_0 with distinctive non-zero eigenvalues $\alpha_1,\dots,\alpha_k$ under the action of $G_o = \mathfrak{p}\rtimes SO(n)$.

Proposition 2.4. *The coadjoint orbit through P_0 given by*

$$P = Ad_h(P_0),\quad Q = [Ad_h(P_0), X],\quad (X,h)\in\mathfrak{p}\rtimes SO(n)$$

is diffeomorphic to the tangent bundle of the flag manifold $\mathbb{F}(1,2,\dots,k)$ consisting of subspaces $V_1\subset V_2\cdots\subset V_k$ with $dimV_i = i$.

Sketch of the proof: Let P_0 denote a symmetric matrix with distinct non-zero eigenvalues $\alpha_1 < \alpha_2 < \cdots < \alpha_k$. Then the matrix P_0 can be identified with a point $(V_1 \subset V_2 \subset \cdots \subset V_k)$ in $\mathbb{F}(1,\dots,k)$ where each subspace V_i is equal to the linear span $< a_1,\dots,a_i >$ of unit eigenvectors of P_0, $P_0(a_i) = \alpha_i a_i$, $||a_i|| = 1$, $i = 1,\dots,k$. If P_0 is represented by the matrix $\sum_{i=1}^k \alpha_i(a_i \otimes a_i)$, then $Ad_h(P_0)$ is represented by the matrix $\sum_{i=1}^k \alpha_i(h(a_i) \otimes h(a_i))$ that corresponds to the point

$$F_h = (hV_1 \subset hV_2 \subset \cdots \subset hV_k)$$

in $\mathbb{F}(1,\dots,k)$. The correspondence $Ad_h(P_0) \to F_h$ is a diffeomorphism from the orbit $\{Ad_h(P_0), h \in SO(n)\}$ onto $\mathbb{F}(1,\dots,k)$.

Let now St_k^n denote the Stiefel manifold of k-orthonormal frames $[a_1,\dots,a_k]$ in $\mathbb{R}^n$. Points of St_k^n can be represented by $n \times k$ matrices M with columns $a_1,\dots,a_k$ that satisfy $M^T M = I_k$, where M^T denotes the matrix transpose of M, and where I_k is the k-dimensional identity matrix. Let $\phi : St_k^n \to \mathbb{F}(1,\dots,k)$ be the embedding

$$M = [a_1,\dots,a_k] \to F_M = (V_1 \subset V_2 \cdots \subset V_k), \;\; V_i =< a_1,\dots,a_i > .$$

Then $\phi^{-1}(F_M) = MD$, where D is a diagonal $k \times k$ matrix with its diagonal entries equal to ± 1. Therefore, $\mathbb{F}(1,\dots,k)$ is a covering space for St_k^n, and hence $\mathbb{F}(1,\dots,k)$ and St_k^n are locally diffeomorphic, that is, every point $M \in St_k^n$ admits an open neighbourhood U such that the restriction of ϕ to U is a diffeomorphism onto $\phi(U)$. It follows that tangent vectors at a point M can be identified with $n \times k$ matrices $\dot{M}$ that satisfy $\dot{M}^T M + M^T \dot{M} = 0$.

Let now U be an open set in St_k^n such that ϕ restricted to U is a diffeomorphism onto $\phi(U)$. For every $F_h \in \phi(U)$, $Ad_h(P_0)$ is identified with $M = [m_1,\dots,m_k]$, $m_i = h(a_i)$, $i = 1,\dots,k$. Then

$$Q = [\mathrm{Ad}_h(P), X] = \sum_{i=1}^k [\alpha_i(m_i \otimes m_i), X] = \sum_{i=1}^k y_i \wedge m_i,$$

with $y_i = \alpha_i X(m_i)$. Since X is symmetric, $\alpha_j(y_i, m_j) = \alpha_i(m_i, y_j)$. Moreover, y_i could be replaced by its orthogonal projection on $m_i^\perp$ without altering the value of Q. So we may assume that $(y_i, m_i) = 0, i = 1,\dots,k$.

It follows that $M^T QM + M^T Q^T M = 0$, hence $\dot{M} = QM$ is a tangent vector at M. The pairs (M, QM) are parametrized by the entries of M and the entries of the matrix Y. The columns $y_i = \alpha_i X m_i$ of Y satisfy $k(k+1)$ constraints $\alpha_j(y_i, m_j) = \alpha_i(y_j, m_i), i \neq j$, and $(y_i, m_i) = 0$. This implies that the manifold of pairs of $n \times k$ matrices (M, Y) subject to the constraints

$$(m_i, m_j) = \delta_{ij}, \alpha_j(y_i, m_j) = \alpha_i(y_j, m_i), i \neq j, (y_i, m_i) = 0, \tag{20}$$

is of the same dimension as the tangent bundle of St_k^n. Therefore, the correspondence $\sum_{i=1}^k \alpha_i(m_i \otimes m_i), \sum_{i=1}^k y_i \wedge m_i \to (M, QM)$ is one to one, and onto the sub-bundle TU over U.

Remark 2.5. I am grateful to the referee for pointing out that the coadjoint orbit through a symmetric matrix P_0 is the flag manifold $\mathbb{F}(1,\dots,k)$ rather than the Stiefel manifold St_k^n, as I had erroneously stated in a previous version of this paper.

Proposition 2.6. *If P_0 is the orthogonal projection on a k-dimensional vector space, i.e., if $P_0 = \sum_{i=1}^{k} a_i \otimes a_i$, for some othronormal vectors $a_1, \dots, a_k$, then the coadjoint orbit through P_0 under the action of the semi-direct product $\mathfrak{p} \rtimes SO_n$ is diffeomorphic to the tangent bundle of the oriented Grassmannian Gr_k^n.*

Here P_0 is identified with the flag consisting of a single k-dimensional vector space V_k spanned by $a_1, \dots, a_k$. Then $\{(hV_k), h \in SO(n)\}$ is diffeomorphic to the oriented Grassmannians Gr_k^n.

3. Affine Hamiltonians and mechanical tops

We will now return to the affine Hamiltonian $H = \frac{1}{2}\langle \mathcal{P}^{-1}(L_{\mathfrak{k}}), L_{\mathfrak{k}} \rangle + \langle A, L_{\mathfrak{p}} \rangle$ on the tangent bundle of a semi-simple reductive group G. As we remarked earlier, the above Hamiltonian can be also viewed as a Hamiltonian on the tangent bundle of the semi-direct product $G_0 = \mathfrak{p} \rtimes K$. According to (16) the Hamiltonian equations are given by

$$\frac{dg}{dt} = g(t)(A + \mathcal{P}^{-1}(L_{\mathfrak{k}}(t))), \quad \frac{dL_{\mathfrak{k}}}{dt} = [\mathcal{P}^{-1}(L_{\mathfrak{k}}(t)), L_{\mathfrak{k}}(t)] + [A, L_{\mathfrak{p}}(t)], \tag{21}$$
$$\frac{dL_{\mathfrak{p}}}{dt} = [\mathcal{P}^{-1}(L_{\mathfrak{k}}(t)), L_{\mathfrak{p}}(t)] + s[A, L_{\mathfrak{k}}(t)], \quad s = 0, 1, \ g \in G_s,$$

$G_1 = G$ and $G_0 = \mathfrak{p} \rtimes K$. Our principal aim is to draw comparisons between the semi-direct Poisson equations

$$\frac{dL_{\mathfrak{k}}}{dt} = [\mathcal{P}^{-1}(L_{\mathfrak{k}}(t)), L_{\mathfrak{k}}(t)] + [A, L_{\mathfrak{p}}(t)], \quad \frac{dL_{\mathfrak{p}}}{dt} = [\mathcal{P}^{-1}(L_{\mathfrak{k}}(t)), L_{\mathfrak{p}}(t)] \tag{22}$$

and the “top-like” equations (12):

$$\frac{dR}{dt} = R(t)(\mathcal{P}^{-1}(M(t))), \quad \frac{dM}{dt} = [\mathcal{P}^{-1}(M(t)), M(t)] + \sum_{i=1}^{n} \alpha_i(t) \wedge \frac{\partial V}{\partial \alpha_i}, \tag{23}$$

associated with the energy Hamiltonian $H = \frac{1}{2}\langle \mathcal{P}^{-1}(M), M \rangle + V(\alpha_1, \dots, \alpha_n)$.

As we mentioned earlier, in “realistic” mechanical tops there is a “physical” constraint $\mathcal{P}(U) = SU + US$ for some positive definite matrix S, thus not every inertia tensor $\langle \mathcal{P}(U), U \rangle$ is allowed ([15]). In three-dimensional tops this constraint implies that the sum of any two eigenvalues of $\mathcal{P}$ cannot be bigger than the third eigenvalue.

Linear potentials. Equations (23) will be referred to as *heavy top-like equations* when there are no constraints on $\mathcal{P}$, and when the potential energy V is generated by a linear Newtonian field, that is, when $V = -\sum_{i=1}^{n} c_i(\alpha_i, a)$, where a is a vector in $\mathbb{R}^n$, and $c_1, \dots, c_n$ are constants. When $a = 0$, (Euler's top), the external torque $\sum_{i=1}^{n} \alpha_i(t) \wedge \frac{\partial V}{\partial \alpha_i}$ is equal to zero, and (23) reduces to the Hamiltonian equation associated with a left-invariant Riemannian metric induced by the operator $\mathcal{P}$.

Heavy top-like equations can be written more compactly as

$$\frac{dR}{dt} = R(t)\Omega(t), \frac{dM}{dt} = [\Omega(t), M(t)] + a \wedge p(t), \tag{24}$$

where $\Omega(t) = \mathcal{P}^{-1}M(t)$, and $p(t) = \sum_{i=1}^{n} c_i\alpha_i(t)$. Since $\alpha_i(t) = R(t)^T e_i$, $p(t)$ is a solution of $\frac{dp}{dt} = -\Omega(t)p(t)$.

Our theorem below relates equations (24) to the Poisson equations (22) on the reductive group $SO_\epsilon(n+1), \epsilon=\pm 1, K=\{1\}\times SO(n)$.

Theorem 3.1. *Heavy top-like equations* (24) *are isomorphic to the Poisson equations*

$$\frac{dL_\mathfrak{k}}{dt}=[\mathcal{P}^{-1}(L_\mathfrak{k}(t)),L_\mathfrak{k}(t)]+[A,L_\mathfrak{p}(t)],\quad \frac{dL_\mathfrak{p}}{dt}=[\mathcal{P}^{-1}(L_\mathfrak{k}(t)),L_\mathfrak{p}(t)]$$

generated by the Hamiltonian $H=\frac{1}{2}\langle\mathcal{P}^{-1}(L_\mathfrak{k}),L_\mathfrak{k}\rangle+\langle A,L_\mathfrak{p}\rangle$ *on the coadjoint orbit through* $P_0=p(0)\wedge_\epsilon e_0$, $Q_0=\begin{pmatrix}0&0\\0&M(0)\end{pmatrix}$ *under the coadjoint action of* $\mathfrak{p}_\epsilon\rtimes SO(n)$. *The passage to the affine Hamiltonian is via the following correspondences*

$$A=\epsilon a\wedge_\epsilon e_0,\ L_\mathfrak{p}=p\wedge_\epsilon e_0,\ L_\mathfrak{k}=\begin{pmatrix}0&0\\0&M\end{pmatrix},\ \mathcal{P}^{-1}(L_\mathfrak{k})=\begin{pmatrix}0&0\\0&\mathcal{P}^{-1}(M)\end{pmatrix}. \tag{25}$$

Proof. Let $(R(t),M(t))$ denote an arbitrary solution of (24). The curve $R(t)$ generates a curve $(X(t),h(t))$ in $G_\epsilon=\mathfrak{p}_\epsilon\rtimes K$ defined by

$$h(t)=\{1\}\times R^T(t), X(t)=\mathrm{Ad}_{h(t)}Y(t), Y(t)=-\int_0^t \mathrm{Ad}_{h^{-1}(s)}A\,ds, A=\epsilon a\wedge_\epsilon e_0,$$

which, in turn, induces a curve $(P(t),Q(t))$ on the coadjoint orbit through P_0,Q_0 given by

$$P(t)=\mathrm{Ad}_{h(t)}(P_0), Q(t)=[\mathrm{Ad}_{h(t)}(P_0),X(t)]+\mathrm{Ad}_{h(t)}Q_0,$$

according to (17). Since

$$h(t)=\{1\}\times R^T(t)\cong\begin{pmatrix}1&0\\0&R^T(t)\end{pmatrix},\quad \text{we obtain}\quad \frac{dh}{dt}=-\tilde{\Omega}(t)h(t)$$

where $\tilde{\Omega}(t)=\begin{pmatrix}0&0\\0&\mathcal{P}^{-1}(M(t))\end{pmatrix}$. Additionally,

$$\frac{dX}{dt}=[\tilde{\Omega}(t),X(t)]+\mathrm{Ad}_{h(t)}\frac{dY}{dt}=[\tilde{\Omega}(t),X(t)]-\mathrm{Ad}_{h(t)}\mathrm{Ad}_{h^{-1}(t)}A=[\tilde{\Omega}(t),X(t)]-A.$$

Let now $L_\mathfrak{p}=p(t)\wedge_\epsilon e_0$. Then,

$$\frac{dL_\mathfrak{p}}{dt}=-\Omega(t)p(t)\wedge_\epsilon e_0=[\tilde{\Omega}(t),L_\mathfrak{p}(t)].$$

Since $P(t)$ and $L_\mathfrak{p}(t)$ satisfy the same equation, they must be equal to each other provided that $P_0=p(0)\wedge_\epsilon e_0$.
In the meantime $Q(t)$ conforms to the following equation

$$\begin{aligned}\frac{dQ}{dt}&=[\frac{dP}{dt},X]+[P,\frac{dX}{dt}]+[\tilde{\Omega}(t),\mathrm{Ad}_{h(t)}Q_0]\\&=[[\tilde{\Omega}(t),P(t)],X(t)]+[P(t),\frac{dX}{dt}]+[\tilde{\Omega}(t),\mathrm{Ad}_{h(t)}Q_0]\\&=-([[X(t),\tilde{\Omega}(t)],P(t)]-[[P(t),X(t)],\tilde{\Omega}(t)])+[P(t),\frac{dX}{dt}]+[\tilde{\Omega}(t),\mathrm{Ad}_{h(t)}Q_0]\\&=[\tilde{\Omega}(t),Q(t)]+[[\tilde{\Omega}(t),X(t)],P(t)]+[P(t),\frac{dX}{dt}]=[\tilde{\Omega}(t),Q(t)]+[A,P(t)].\end{aligned}$$

Thus $$\frac{dQ}{dt} = [\tilde{\Omega}(t), Q(t)] + [A, L_{\mathfrak{p}}(t)].$$

Since $L_{\mathfrak{k}}(t) = \begin{pmatrix} 0 & 0 \\ 0 & M(t) \end{pmatrix}$ satisfies the same equation as $Q(t)$, $L_{\mathfrak{k}}(t) = Q(t)$ provided that $L_{\mathfrak{k}}(0) = Q_0$. We now we have the Poisson equations

$$\frac{dL_{\mathfrak{k}}}{dt} = [\mathcal{P}^{-1}(L_{\mathfrak{k}})(t)), L_{\mathfrak{k}}] + [A, L_{\mathfrak{p}}(t)], \quad \frac{dL_{\mathfrak{p}}}{dt} = [\mathcal{P}^{-1}L_{\mathfrak{k}}(t)), L_{\mathfrak{p}}(t)],$$

associated with $H = \frac{1}{2}\langle \mathcal{P}^{-1}(L_{\mathfrak{k}}), L_{\mathfrak{k}} \rangle + \langle A, L_{\mathfrak{p}} \rangle$. The curve $g(t) = (X(t), h(t))$ in G_ϵ satisfies

$$\frac{d}{dt}(X(t), h(t)) = R_{(X(t),h(t))_*}(A, -\Omega(t)) = (A, -\Omega(t))(X(t), h(t)). \qquad \blacksquare$$

The preceding theorem clarifies the presence of heavy tops in the Hamiltonian equations of elastic rods noted in [9] and [10]. It also proves that the classification of completely integrable elastic rods in [9] and [11] carries over to the heavy top.

Quadratic potentials. We will now show that the tops with quadratic potential V are also present in the equations of affine Hamiltonians, but this time on the tangent bundle of $SL(n)$, or more precisely on the tangent bundle of the semi-direct product $Sym^0(n) \rtimes SO(n)$ where $Sym^0(n)$ denotes the space of symmetric $n \times n$ matrices with zero trace. For that purpose let

$$H(R, M) = \frac{1}{2}\langle \mathcal{P}^{-1}(M), M \rangle + \frac{1}{2}\sum_{i=1}^{n} a_i \langle S\alpha_i, \alpha_i \rangle,$$

with $R \in SO(n), M \in so(n)$, $R^T e_i = \alpha_i$, and S a symmetric $n \times n$ matrix. Again, this Hamiltonian is is slightly more general than that of a mechanical top since $\mathcal{P}$ is unconstrained. In accordance to (23) the Hamiltonian equations of $\vec{H}$ are given by

$$\frac{dR}{dt} = R(t)\Omega(t), \quad \frac{dM}{dt} = [\Omega(t), M(t)] + \sum_{i=1}^{n} a_i \alpha_i(t) \wedge S\alpha_i(t), \tag{26}$$

where $\Omega(t) = \mathcal{P}^{-1}(M(t))$.

Theorem 3.2. *Equations of the top* (26) *are isomorphic with the Poisson equations generated by the affine Hamiltonian* $H = \frac{1}{2}\langle \mathcal{P}^{-1}(L_{\mathfrak{k}}), L_{\mathfrak{k}} \rangle + \langle L_{\mathfrak{p}}, S \rangle$ *on the coadjoint orbit through* $P_0 = \sum_{i=1}^n a_i(\alpha_i(0) \otimes \alpha_i(0)) - (\frac{1}{n}\sum_{i=1}^n a_i)I$, $Q_0 = M(0)$ *in* $Sym^0(n) \rtimes \mathfrak{so}(n)$.

Proof. Every solution $(M(t), R(t))$ of (26) generates symmetric matrices $L_{\mathfrak{p}}(t)$ and $X(t)$ given by

$$L_{\mathfrak{p}}(t) = \sum_{i=1}^{n} a_i(\alpha_i(t) \otimes \alpha_i(t)) - \frac{1}{n}\sum_{i=1}^{n} a_i I,$$
$$X(t) = \mathrm{Ad}_{h(t)} Y(t), Y(t) = -\int_0^t \mathrm{Ad}_{h^{-1}(s)} S\, ds,$$

with $h(t) = R^T(t)$.

Then,
$$\frac{dL_{\mathfrak{p}}}{dt} = -\sum_{i=1}^{n} a_i(\Omega\alpha_i \otimes \alpha_i + \alpha_i \otimes \Omega\alpha_i) = [\Omega, L_{\mathfrak{p}}],$$
$$\frac{dX}{dt} = [\Omega(t), X(t)] + \mathrm{Ad}_{h(t)}\dot{Y} = [\Omega(t), X(t)] - S.$$

Additionally,
$$[S, L_{\mathfrak{p}}(t)] = \sum_{i=1}^{n}(a_i(\alpha_i \otimes \alpha_i)S - Sa_i(\alpha_i \otimes \alpha_i))$$
$$= \sum_{i=1}^{n} a_i\alpha_i \otimes S\alpha_i - a_iS\alpha_i \otimes \alpha_i = \sum_{i=1}^{n} a_i(\alpha_i \wedge S\alpha_i),$$

which in turn implies that (23) can be written as

$$\frac{dR}{dt} = R(t)\Omega(t), \quad \frac{dM}{dt} = [\Omega(t), M(t)] + [S, L_{\mathfrak{p}}(t)].$$

Let now $Q(t) = [\mathrm{Ad}_{h(t)}(P_0), X(t)] + \mathrm{Ad}_{h(t)}Q_0$. Note first that

$$[[\Omega, P], X] = -[[X, \Omega], P] - [[P, X], \Omega] = -[[X, \Omega], P] + [\Omega, Q] - [\Omega, \mathrm{Ad}_h Q_0].$$

Then,

$$\frac{dQ}{dt} = [[\Omega(t), P(t)], X(t)] + [P(t), \frac{dX}{dt}(t)] + [\Omega(t), \mathrm{Ad}_{h(t)}(Q_0]$$
$$= [\Omega(t), Q(t)] + [L_{\mathfrak{p}}, [X, \Omega(t)]] + [L_{\mathfrak{p}}, \frac{dX}{dt}] = [\Omega(t), Q(t)] + [S, L_{\mathfrak{p}}].$$

Therefore $Q(t)$ and $M(t)$ satisfy the same differential equation. Hence $Q(t) = M(t)$ whenever $Q_0 = M(0)$. If we now rename $Q(t)$ as $L_{\mathfrak{k}}(t)$ we get the Poisson equations for the shadow Hamiltonian $H = \frac{1}{2}\langle \mathcal{P}^{-1}(L_{\mathfrak{k}}), L_{\mathfrak{k}}\rangle + \langle S, L_{\mathfrak{p}}\rangle$. ■

Let us now address the relevance of this material for the theory of integrable systems.

4. Affine-quadratic problem and integrable systems

Recall first some classical terminology. A function ϕ is *an integral of motion* for a Hamiltonian H if $\{H, \phi\} = 0$. A Hamiltonian H on a $2n$-dimensional symplectic manifold M is said to be *Liouville integrable*, or *completely integrable*, if there are $n-1$ functionally independent integrals of motion $\phi_1, \ldots, \phi_{n-1}$ for H, also functionally independent from H, that satisfy $\{\phi_i, \phi_j\} = 0$ for all i, j. Any set of functions $\mathcal{F}$ on M whose elements satisfy $\{f, h\} = 0$ for any f, h in $\mathcal{F}$, is said to in *involution*.

Every left invariant Hamiltonian H on a semi-simple Lie group admits extra integrals of motion for the following reasons: the Hamiltonian equation generated by H is of the form $\frac{dL}{dt} = [M(t), L(t)]$, which implies that the eigenvalues of $L(t)$ are integrals of motion, because if $g(t)$ in $GL(n)$ is the solution of $\frac{dg}{dt} = g(t)M(t), g(0) = I$, then

$$\frac{d}{dt}g(t)L(t)g^{-1}(t) = g(t)[L(t), M(t)]g^{-1}(t) + g(t)\frac{dL}{dt}g^{-1}(t) = 0,$$

and hence, $g(t)L(t)g^{-1}(t) = L(0)$. Therefore, the spectrum of $L(t)$ is the same as the spectrum of $L(0)$. The spectral invariants of L are usually expressed through the functions $\phi^{(k)}(L) = trace(L^k), k = 1, \ldots, n-1$. Since they are constant along the integral curves of $\vec{H}$, they Poisson commute with H. Functions that Poisson commute with every function on $\mathfrak{g}$ are called *Casimirs*.

Since left-invariant vector fields commute with right-invariant vector fields, the Hamiltonian lifts of right-invariant vector fields commute with left-invariant Hamiltonians. So every function H on $\mathfrak{g}$, when regarded as a left-invariant Hamiltonian, is automatically embedded in an involutive family of integrals: the Casimirs, and the Hamiltonians generated by the right invariant vector fields with values in a Cartan algebra in $\mathfrak{g}$ (a maximal abelian subalgebra of $\mathfrak{g}$).

This observation implies that every left-invariant Riemannian metric on a three-dimensional Lie algebra results in a completely integrable Hamiltonian system (like Euler's top, for instance). The above carries important implications for the equations of motion of three-dimensional heavy tops as well.

Equations of three-dimensional tops are subordinate to the equations of elastic rods on coadjoint orbits in $\mathfrak{so}_\epsilon(4)$ (Theorem 3.1). This means that the top is completely integrable whenever the elastic rod is completely integrable.

Generic coadjoint orbits in $\mathfrak{so}_\epsilon(4)$ are four dimensional (there are two Casimirs $C_1 = ||L_\mathfrak{p}||^2 + \epsilon ||L_\mathfrak{k}||^2$ and $C_2 = (L_\mathfrak{p}, L_\mathfrak{k})$). So any left invariant Hamiltonian H on $so_\epsilon(4)$, and, hence, any three-dimensional heavy top, is completely integrable whenever there is another integral of motion for H. Therefore, the classification of integrable cases for the elastic Hamiltonian done in ([10] and [11]) is equally valid for the heavy top.

4.1. Integrable systems

We will now single out a remarkable class of affine-quadratic Hamiltonians that plays a prominent role in the theory of integrable systems. It consists of Hamiltonians $H = \frac{1}{2}\langle \mathcal{P}^{-1} L_\mathfrak{k}, L_\mathfrak{k}\rangle + \langle L_\mathfrak{p}, A\rangle$ that admit a spectral representation of the form

$$\frac{dL_\lambda}{dt} = [M_\lambda, L_\lambda], \tag{27}$$

with $M_\lambda = \mathcal{P}^{-1}(L_\mathfrak{k}) - \lambda A$, and $L_\lambda(s) = -L_\mathfrak{p} + \lambda L_\mathfrak{k} + (\lambda^2 - s)B$ for some matrix B that commutes with A, where $L_\mathfrak{p}$ and $L_\mathfrak{k}$ are the solutions of the Poisson equations

$$\frac{dL_\mathfrak{k}}{dt} = [\mathcal{P}^{-1}(L_\mathfrak{k}), L_\mathfrak{k}] + [A, L_\mathfrak{p}], \quad \frac{dL_\mathfrak{p}}{dt} = [\mathcal{P}^{-1}(L_\mathfrak{k}), L_\mathfrak{p}] + s[A, L_\mathfrak{k}], s = 0, 1,$$

associated with H. Such a class is called *isospectral*. This terminology has origins in J. Zimmerman's PhD thesis in 1992, in which he showed that the canonical case $(\mathcal{P} = I)$

$$\frac{dL_\mathfrak{k}}{dt} = [A, L_\mathfrak{p}], \frac{dL_\mathfrak{p}}{dt} = [L_\mathfrak{k}, L_\mathfrak{p}] + s[A, L_\mathfrak{k}],$$

is isospectral with $B = A$ ([23]).

For Hamiltonian systems that admit an isospectral representation, the discrete spectral invariants of L are replaced by the functional invariants $\phi^{(k)}_{\lambda,s}(L) = Trace(L^k_\lambda(s))$. Remarkably, the functional invariants $\phi^k_{\lambda,s}$ are in involution with each other, both with respect to the semi-simple and the semi-direct product Lie bracket, and in some instances generate a sufficient number of integrals of motion to ensure complete integrability. For instance, it is known that the family of functions

$$\mathcal{F}_0 = \{\phi^k_{\lambda,0}, k \geq 1, \lambda \in \mathbb{R}\} \cup \{h_X : [X, B] = 0, X \in \mathfrak{k}\}$$

is completely integrable on each coadjoint orbit in $\mathfrak{p} \rtimes K$ (See [2]). This means that H is completely integrable on each coadjoint orbit in $\mathfrak{p} \rtimes \mathfrak{k}$ whenever H is in involution with the Hamiltonian lifts $h_X(L) = \langle L, X\rangle, X \in \mathfrak{k}, [X, B] = 0$. This implies that the canonical affine Hamiltonian is completely integrable on coadjoint orbits since each left-invariant vector field with values in the isotropy group of A is a symmetry for the canonical system.

It is reasonable to expect that the analogous family of functions in the semi-simple case is also completely integrable on coadjoint orbits of G, but to the best of my knowledge, the proofs have not yet appeared in the literature.

The focus on the affine-quadratic problem and the associated Hamiltonians allows for the following characterization of isospectral Hamiltonians ([11]).

Theorem 4.1. *An affine Hamiltonian $H = \frac{1}{2}\langle \mathcal{P}^{-1} L_\mathfrak{k}, L_\mathfrak{k}\rangle + \langle L_\mathfrak{p}, A\rangle$ is isospectral if and only $[\mathcal{P}^{-1}(L_\mathfrak{k}), B] = [L_\mathfrak{k}, A]$ for some matrix $B \in \mathfrak{p}$ that commutes with A. In the isospectral case, $L_\mathfrak{p} = sB$ is an invariant set for the solutions of* (21)*. The restriction of* (21) *to $L_\mathfrak{p} = sB$ is given by*

$$\frac{dL_\mathfrak{k}}{dt} = [\mathcal{P}^{-1}(L_\mathfrak{k}), L_\mathfrak{k}], \tag{28}$$

with the reduced spectral representation

$$\frac{d}{dt}(L_\mathfrak{k} - \lambda B) = [\mathcal{P}^{-1}(L_\mathfrak{k}) - \lambda A, L_\mathfrak{k} - \lambda B]. \tag{29}$$

This theorem shows that the fundamental results A. T. Fomenko, A. S. Mischenko, and V. V. Trofimov on integrable left-invariant Riemannian metrics on compact Lie groups ([4] and [5]) based on Manakov's seminal work on the n-dimensional Euler's top ([15]) are subordinate to the isospectral properties of the affine Hamiltonian system, in the sense that the spectral invariants of $L_\mathfrak{k} - \lambda B$ on $\mathfrak{k}$ are always in involution with a larger family of functions generated by the spectral invariants of $L_\lambda = -L_\mathfrak{p} + \lambda L_\mathfrak{k} + (\lambda^2 - s)B$ on $\mathfrak{g}_s$ associated with an affine Hamiltonian H.

The identification of cotangent bundles with coadjoint orbits links affine Hamiltonians with geodesic and mechanical systems, and in many instances reveals the "hidden" symmetries behind the integrals of motion. For instance, the Hamiltonian

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{k} ||x_i||^2||y_i||^2 + \frac{1}{2}\sum_{i=1}^{k} \alpha_i(Ax_i, x_i) \tag{30}$$

on the tangent bundle of the Stiefel St^n_k or the Grassmannian Gr^n_k coincides with the restriction of the canonical Hamiltonian $H = \frac{1}{2}\langle L_\mathfrak{k}, L_\mathfrak{k}\rangle - \langle A, L_\mathfrak{p}\rangle$ to the coadjoint orbit through $P_0 = \sum_{i=1}^k \alpha_i a_i \otimes a_i$, $||a_i|| = 1$, $i = 1, \ldots, k$ (Proposition 2.4). It can be readily shown that $\hat{H}$ is the Hamiltonian associated with the motion of a particle on St^n_k or Gr^n_k under the influence of a quadratic potential energy $V = \frac{1}{2}\sum_{i=1}^k (x_i, Ax_i)$. Every such Hamiltonian $\hat{H}$ is completely integrable since H is completely integrable on coadjoint orbits. When A is non-singular, then $[X, A] = 0$, for $X \in \mathfrak{so}(n)$ can hold only for $X = 0$. Therefore, the isospectral integrals of motion generated by

$$L_\lambda = -\sum_{i=1}^k \alpha_i(x_i \otimes x_i) + \lambda \sum_{i=1}^k y_i \wedge x_i - \lambda^2 A. \tag{31}$$

induce a completely integrable family of integrals of motion for $\hat{H}$. In particular, when $k = 1$, $\hat{H}$ is the Hamiltonian associated with C. Neumann's problem on the sphere ([17]). In such a case, the spectral invariants of L_λ yield the integrals of motion found by Neumann

$$F_k = x_k^2 + \sum_{j=1, j\neq k}^{n+1} \frac{(x_j y_k - x_k y_j)^2}{(a_k - a_j)}, \quad k = 1, \dots, n+1, \tag{32}$$

([16], [20]). Analogous calculations for $k > 1$ lead to explicit polynomial integrals of motion of sufficient generality to reconfirm the above claim that C. Neumann's problem is integrable on all Stiefel and Grassmannian manifolds ([3]).

Isospectral systems also provide new insights into the origins of the integrals of motion found in Jacobi's geodesic problem on the ellipsoid

$$E_n = \{x \in \mathbb{R}^{n+1} : (A^{-1}x, x) = 1\}.$$

The link is supplied by the elliptic Hamiltonian

$$H = \frac{1}{2}\langle A^{-1} L_{\mathfrak{k}} A^{-1}, L_{\mathfrak{k}}\rangle + \langle A^{-1}, L_{\mathfrak{p}}\rangle, \tag{33}$$

on $sym(n) \rtimes so(n)$, with A a non-singular and symmetric matrix. Since we have $[A^{-1}L_{\mathfrak{k}}A^{-1}, A] = [L_{\mathfrak{k}}, A^{-1}]$, H is isospectral with $B = A$. Therefore the canonical and the elliptic Hamiltonian share the same spectral matrix $L_\lambda = L_{\mathfrak{p}} - \lambda L_{\mathfrak{k}} + (\lambda^2 - s)A$, and hence share the same isospectral integrals of motion.

On the energy level $H = 0$, the restriction of the elliptic system to the rank-one symmetric matrices coincides with the geodesic problem on the sphere with the elliptic metric $(\dot{x}, A\dot{x})$ ([11]). It then follows that the elliptic geodesic problem on the sphere is completely integrable having the same integrals of motion as the Neumann's mechanical problem on the sphere ((32)). Since Jacobi's problem on the ellipsoid and the elliptic geodesic problem on the sphere are symplectomorphic, both systems are completely integrable. In fact, the mapping $q = A^{\frac{1}{2}}x$, $p = A^{-\frac{1}{2}}(y - \frac{(A^{-1}x,y)}{(A^{-1}x,x)}x)$ takes $\{(x, y) : ||x|| = 1,\ (x, y) = 0\}$ onto $\{(q, p) : (q, A^{-1}q) = 1,\ (A^{-1}q, p) = 0\}$, and transforms the integrals of motion (32) onto the integrals of motion for Jacobi's problem

$$G_k = p_k^2 + \sum_{j=1, j\neq k}^{n+1} \frac{(q_j p_k - q_k p_j)^2}{(a_k - a_j)}, k = 1, \dots, n+1. \tag{34}$$

In the literature on integrable systems, the passage from Neumann's problem to the Jacobi's problem is through a transformation discovered by H. Knörer ([13], [16]).

We conclude this exposition with another beautiful connection between isospectral Hamiltonians and the equations of the top. We will show that O. Bogoyavlensky's famous result that a three-dimensional mechanical top in the presence of a quadratic potential is completely integrable ([1]) is true in any dimension. This conjecture is crucially dependent on Manakov's observation that the inertia tensor $\langle\mathcal{P}(U),U\rangle$ for a rigid body is confined to the transformations $\mathcal{P}(U)=SU+US$, $U\in\mathfrak{so}(n)$, for some positive definite matrix S. For then, $[\mathcal{P}^{-1}(M),S^2]=[M,S]$. Indeed, in this situation $\mathcal{P}(U)=M$, and

$$[\mathcal{P}^{-1}M,S^2]=[U,S^2]=[SU+US,S]=[M,S].$$

This means that the induced affine Hamiltonian

$$\hat{H}=\frac{1}{2}\langle\mathcal{P}^{-1}L_{\mathfrak{k}},L_{\mathfrak{k}}\rangle+\langle S,L_{\mathfrak{p}}\rangle,\quad M=L_{\mathfrak{k}}, \tag{35}$$

on sl_n is isospectral. Since the equations of the Hamiltonian

$$H=\frac{1}{2}\langle\mathcal{P}^{-1}M,M\rangle+\sum_{i=1}^{n}a_i(\alpha_i,S\alpha_i)$$

corresponding to the top with quadratic potential $V=\frac{1}{2}\sum_{i=1}^{n}a_i(S\alpha_i,\alpha_i)$ can be identified with the Poisson equations of $\hat{H}$ on the coadjoint orbit through $L_{\mathfrak{p}}=P_0$, $L_{\mathfrak{k}}=M(0)$, the isospectral invariants of

$$L_\lambda=-\sum_{i=1}^{n}a_i\alpha_i+\lambda M+\lambda^2 S \tag{36}$$

are integrals of motion for the top (Theorem 3.2).
On $\mathfrak{sl}(n)$, $\{X\in\mathfrak{so}(n):[X,S]=0\}=0$ for each non-singular symmetric matrix S. But then the isospectral invariants form a completely integrable family of functions on each coadjoint orbit in $\mathfrak{sl}(n)$, which implies that the top with quadratic potential is completely integrable in all dimensions.

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