Journal of Lie Theory
Volume **30** (2020) 445–459
© 2020 Heldermann Verlag

Fidelity and Metrics on Lorentz Boosts

Sejong Kim

Communicated by G. Ólafsson

Abstract. We see in this article that the extended version of a real counterpart of qubit density matrices introduced by Abraham Ungar, called a Möbius matrix, is indeed a normalized Lorentz boost by finding its spectral decomposition. Using the gyrogroup isomorphism between the set of all Lorentz boosts and the Einstein gyrogroup on the open unit ball of the n-dimensional Euclidean space, we give a gyrogroup structure on the set of Lorentz boosts and compute various metric formulas of Lorentz boosts such as the Riemannian trace metric, Hilbert projective metric, fidelity and Wasserstein distance in terms of Lorentz gamma factors.

Mathematics Subject Classification: 22E43, 20N05, 81P15.

Key Words: Lorentz boost, Einstein gyrogroup, rapidity metric, Hilbert projective metric, fidelity, Wasserstein distance.

1. Introduction

Hendrik Lorentz was seeking the transformation under which Maxwell's equations are invariant when transformed from the ether to a moving frame. In 1905 Henri Poincaré recognized that the set of such transformations form a group named the Lorentz group. Later in the same year Albert Einstein derived the Lorentz transformation under the assumption of the principle of relativity and the constancy of the speed of light in any inertial reference frame. The Lorentz transformation of the relativistically admissible vector is currently an important tool in special relativity, since it enables us to study relativistic mechanics in hyperbolic geometry. It also may include a rotation of space, and especially a rotation-free Lorentz transformation, which is called a *Lorentz boost*:

$$B(\mathbf{u}) = \begin{pmatrix} \gamma_{\mathbf{u}} & \gamma_{\mathbf{u}}\mathbf{u}^T \\ \gamma_{\mathbf{u}}\mathbf{u} & I + \frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}\mathbf{u}\mathbf{u}^T \end{pmatrix},$$

where $\mathbf{u}$ is the column vector in the open unit ball $\mathbf{B}$ of $\mathbb{R}^n$ and $\gamma_{\mathbf{u}} = (1 - \|\mathbf{u}\|^2)^{-1/2}$ is the well-known Lorentz gamma factor. Note that the Lorentz boost is a symmetric, positive definite, and member of the Lorentz group $O(1, n)$. Via a spectral decomposition of Lorentz boost, we mainly study the algebraic structure and metrics on the set of Lorentz boosts in this paper.

In Section 2 we review a non-associative algebra structure (called a gyrogroup) with the typical example on the open unit ball of $\mathbb{R}^n$, the *Einstein gyrogroup* $(\mathbf{B}, \oplus)$,

ISSN 0949–5932 / \$2.50 © Heldermann Verlag

where $\oplus$ denotes the Einstein's relativistic sum (1) of admissible vectors in $\mathbf{B}$. We also review Ungar's gyrometric $\varrho(\mathbf{u},\mathbf{v}) = \|-\mathbf{u}\oplus\mathbf{v}\|$ and a rapidity metric $d_E(\mathbf{u},\mathbf{v}) := \tanh^{-1}\varrho(\mathbf{u},\mathbf{v})$ on the Einstein gyrogroup, which are important notions in this paper. In Section 3 we provide a gyrogroup structure on the set of Lorentz boosts and see a gyrogroup isomorphism between the set of Lorentz boosts and the Einstein gyrogroup $(\mathbf{B},\oplus)$.

In [23, Section 9.5] A. A. Ungar has introduced the real counterpart of qubit density matrices and has suggested its extended version as certain type of higher-level quantum states, called a Möbius matrix, although it is not a natual extension of qubit density matrices. Meanwhile, we find out it as a normalized Lorentz boost via its diagonalization in Section 4. So there is a gyrogroup structure on the set of Lorentz boosts equivalent to the Einstein gyrogroup, which gives us a useful tool to compute easily various distances and fidelity.

In [11] Lawson and Kim have derived the following relationship between the Riemannian trace distance δ of Lorentz boosts and the rapidity metric d_E on the isomorphic Einstein gyrogroup:

$$d_E(\mathbf{u},\mathbf{v}) = \frac{1}{2\sqrt{2}}\delta(B(\mathbf{u})^2, B(\mathbf{v})^2)$$

for $\mathbf{u},\mathbf{v}\in\mathbf{B}$, where $\delta(A,B) := \|\log(A^{-1/2}BA^{-1/2})\|_2$ for positive definite matrices A and B. The Hilbert projective metric has been issued as an important distance measurement for quantum states (see [21]), and so we compute it for Lorentz boosts and Möbius matrices and obtain the relation with the rapidity metric on the Einstein gyrogroup in Section 5. We also calculate the fidelity and Wasserstein distance for Lorentz boosts and give explicit formulas in terms of Lorentz gamma factors in Section 6. Finally we close with some interesting open questions about gyrocentroid and barycenters with respect to the Riemannian trace distance and Wasserstein distance.

2. Gyrogroup and the rapidity metric

We review first the Einstein's relativistic sum of admissible velocities of which magnitude is less than the speed of light $c \risingdotseq 3\times 10^5$ km/sec. In our purpose of this article, we assume the speed of light is normalized by the value 1, so that the admissible vectors are in the open unit ball

$$\mathbf{B} := \{\mathbf{v}\in\mathbb{R}^n : \|\mathbf{v}\| < 1\},$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. Then the relativistic sum of two admissible vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{B}$ is given by

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\mathbf{u}^T\mathbf{v}}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}^T\mathbf{v})\mathbf{u}\right\}, \tag{1}$$

where $\gamma_{\mathbf{u}}$ is the well-known *Lorentz factor*

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1-\|\mathbf{u}\|^2}}. \tag{2}$$

Note that $\mathbf{u}^T\mathbf{v}$ is just the Euclidean inner product of $\mathbf{u}$ and $\mathbf{v}$ written in matrix form.

Definition 2.1. The formula (1) defines a binary operation, called the *Einstein velocity addition*, on the open unit ball $\mathbf{B}$ of $\mathbb{R}^n$.

Remark 2.2. By applying the *Lorentz boost*

$$B(\mathbf{u}) = \begin{pmatrix} \gamma_{\mathbf{u}} & \gamma_{\mathbf{u}}\mathbf{u}^T \\ \gamma_{\mathbf{u}}\mathbf{u} & I + \frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}\mathbf{u}\mathbf{u}^T \end{pmatrix} \tag{3}$$

to the column vector $\begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}}\mathbf{v} \end{pmatrix}$, we obtain that $B(\mathbf{u}) \begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}}\mathbf{v} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{u}\oplus\mathbf{v}} \\ \gamma_{\mathbf{u}\oplus\mathbf{v}}(\mathbf{u}\oplus\mathbf{v}) \end{pmatrix}$, where we use the gamma identity $\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1+\mathbf{u}^T\mathbf{v})$. So we can alternatively get the Einstein addition $\mathbf{u}\oplus\mathbf{v}$ of two admissible vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{B}$.

To abstractly analyze Einstein velocity addition in the theory of special relativity, A. A. Ungar has introduced and studied in several papers and books structures that he has called gyrogroups; see [23] and its bibliography. His algebraic axioms are reminiscent of those for a group, but a gyrogroup operation is neither associative nor commutative in general.

Definition 2.3. A triple $(G, \oplus, 0)$ is a *gyrogroup* if the following axioms are satisfied for all $a, b, c \in G$.

(G1) $0 \oplus a = a \oplus 0 = a$ (existence of identity);

(G2) $a \oplus (-a) = (-a) \oplus a = 0$ (existence of inverses);

(G3) There is an automorphism $\mathrm{gyr}[a,b] : G \to G$ for each $a, b \in G$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a,b]c \text{ (gyroassociativity);}$$

(G4) $\mathrm{gyr}[0,a] = \mathrm{id}_G$;

(G5) $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a,b]$ (loop property).

A gyrogroup $(G, \oplus)$ is *gyrocommutative* if it satisfies

$$a \oplus b = \mathrm{gyr}[a,b](b \oplus a) \text{ (gyrocommutativity).}$$

A gyrogroup is *uniquely* 2*-divisible* if for every $b \in G$, there exists a unique $a \in G$ such that $a \oplus a = b$.

The map $\mathrm{gyr}[a,b]$ is called the *gyroautomorphism* or *Thomas gyration* generated by a and b, which is analogous to the precession map in a loop theory. It has been shown in [22] that gyrocommutative gyrogroups are equivalent to Bruck loops with respect to the same operation. It follows that uniquely 2-divisible gyrocommutative gyrogroups are equivalent to B-loops, uniquely 2-divisible Bruck loops. J. Lawson and Y. Lim have recently introduced dyadic symmetric sets in [13] and showed the equivalence with uniquely 2-divisible gyrocommutative gyrogroups. In our purpose of this article we follow the notion of gyrogroups.

A. A. Ungar has shown in [23, Chapter 3] by computer algebra that Einstein addition on the open unit ball $\mathbf{B}$ is a gyrocommutative gyrogroup operation, and the gyroautomorphisms are orthogonal transformations preserving the Euclidean inner product and the inherited norm. We call $(\mathbf{B}, \oplus)$ the Einstein (gyrocommutative) gyrogroup, where $\oplus$ is defined by the equation (1).

Remark 2.4. We note that the Einstein gyrogroup $(\mathbf{B}, \oplus)$ is uniquely 2-divisible; for any $\mathbf{v} \in \mathbf{B}$ there exists a unique

$$\mathbf{w} = \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} \in \mathbf{B}$$

such that $\mathbf{w} \oplus \mathbf{w} = \mathbf{v}$ (see the equation (6.297) of [23]). We denote it simply by $\mathbf{w} := (1/2) \otimes \mathbf{v}$, or $\mathbf{v} = 2 \otimes \mathbf{w}$.

In [23, Chapter 6], furthermore, he has considered what we call the *Ungar gyrometric* ϱ and the *rapidity metric* d on the Einstein gyrogroup $(\mathbf{B}, \oplus)$ defined by

$$\varrho(\mathbf{u}, \mathbf{v}) = \| - \mathbf{u} \oplus \mathbf{v}\|, \; d(\mathbf{u}, \mathbf{v}) = \tanh^{-1} \varrho(\mathbf{u}, \mathbf{v}).$$

For real numbers s and t, we define

$$s \oplus t = \frac{s+t}{1+st},$$

the restricted Einstein addition analogous to the Einstein sum of parallel vectors. We remark some properties of the Ungar gyrometric and the rapidity metric on $(\mathbf{B}, \oplus)$.

Lemma 2.5. *The following properties hold for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{B}$.

(i) $0 \leq \varrho(\mathbf{u}, \mathbf{v}), d(\mathbf{u}, \mathbf{v})$

(i) $\varrho(\mathbf{u}, \mathbf{v}) = 0 \Leftrightarrow d(\mathbf{u}, \mathbf{v}) = 0 \Leftrightarrow \mathbf{u} = \mathbf{v}$

(iii) $\varrho(\mathbf{u}, \mathbf{v}) = \varrho(\mathbf{v}, \mathbf{u})$, $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

(iv) $\|\mathbf{u} \oplus \mathbf{v}\| \leq \|\mathbf{u}\| \oplus \|\mathbf{v}\| \Leftrightarrow \varrho(\mathbf{u}, \mathbf{w}) \leq \varrho(\mathbf{u}, \mathbf{v}) \oplus \varrho(\mathbf{v}, \mathbf{w})$
$\Leftrightarrow d(\mathbf{u}, \mathbf{w}) \leq d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w})$

(v) $\varrho(\mathbf{u} \oplus \mathbf{v}, \mathbf{u} \oplus \mathbf{w}) = \varrho(\mathbf{v}, \mathbf{w})$ *and* $d(\mathbf{u} \oplus \mathbf{v}, \mathbf{u} \oplus \mathbf{w}) = d(\mathbf{v}, \mathbf{w})$

The next proposition and proof appear in Theorem 3.46 of [23].

Proposition 2.6. *The inequality* $\|\mathbf{u} \oplus \mathbf{v}\| \leq \|\mathbf{u}\| \oplus \|\mathbf{v}\|$ *holds in the Einstein gyrogroup* $(\mathbf{B}, \oplus)$*, and hence the rapidity metric* d *for this gyrogroup is indeed a metric.*

The following shows a relationship of metrics on gyrocommutative gyrogroups under an injective homomorphism that plays an important role in our results.

Lemma 2.7. ([10, Lemma 4.2]) *Let* $(G_1, \oplus_1, 0_1)$ *and* $(G_2, \oplus_2, 0_2)$ *be gyrocommutative gyrogroups equipped with metrics* d_1 *and* d_2 *invariant under left translations, respectively. Let* $f : G_1 \to G_2$ *be an injective gyrogroup homomorphism. Then* $d_2(f(x), 0_2) \leq \kappa d_1(x, 0_1)$ *for each* $x \in G_1$ *and for some* $\kappa > 0$ *if and only if* $d_2(f(x), f(y)) \leq \kappa d_1(x, y)$ *for all* $x, y \in G_1$*. The equalities are preserved each other.*

3. Gyro structure for Lorentz boosts

Let $\mathbb{P}_{n+1}$ be the set of all Lorentz boosts given in the equation (3). A Lorentz boost is a positive definite member of the *Lorentz group* $O(1, n)$, the group (under composition) of all linear transformations preserving the Lorentz form $\mathcal{L}$ defined by

$$\mathcal{L}\langle(s, x_1, \ldots, x_n), (t, y_1, \ldots, y_n)\rangle = -st + \sum_{i=1}^{n} x_i y_i.$$

Indeed, a Lorentz boost is a member of the *restricted Lorentz group* $SO^+(1,n)$, the identity component of the Lorentz group consisting of all proper orthochronous maps. The following is a basic fact; for example, see (10.1) of [9].

Lemma 3.1. *The mapping* $\mathbf{v} \to B(\mathbf{v})$ *is a bijection from the open unit ball* $\mathbf{B}$ *to the set of positive definite elements in* $SO^+(1,n)$.

We provide a diagonalization of Lorentz boost to obtain our results later.

Lemma 3.2. ([11, Theorem 5.6]) *For each* $\mathbf{v} \in \mathbf{B}$ *there exist an orthogonal matrix* $O_\mathbf{v}$ *and a diagonal matrix* $D_\mathbf{v}$

$$O_\mathbf{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2}\|\mathbf{v}\|}\mathbf{v} & \frac{1}{\sqrt{2}\|\mathbf{v}\|}\mathbf{v} & \mathbf{u}_1 & \cdots & \mathbf{u}_{n-1} \end{pmatrix}, \quad D_\mathbf{v} = \begin{pmatrix} \begin{matrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{matrix} & 0 \\ 0 & I_{n-1} \end{pmatrix}$$

such that $B(\mathbf{v}) = O_\mathbf{v} D_\mathbf{v} O_\mathbf{v}^T$, *where* $\{\mathbf{u}_j \;:\; \mathbf{v}^T\mathbf{u}_j = 0$ *for all* $j = 1, 2, \ldots, n-1\}$ *is an orthonormal set obtained by the Gram-Schmidt process, and*

$$\lambda = \sqrt{\frac{1 + \|\mathbf{v}\|}{1 - \|\mathbf{v}\|}} > 1.$$

From the polar decomposition of $B(\mathbf{u})B(\mathbf{v})$ for $\mathbf{u}, \mathbf{v} \in \mathbf{B}$ we have the relation

$$B(\mathbf{u} \oplus \mathbf{v}) = \left(B(\mathbf{u})B(\mathbf{v})^2 B(\mathbf{u})\right)^{1/2}, \tag{4}$$

see [11] for more details. Hence, we obtain

Theorem 3.3. *The Lorentz boost map* B *is an isomorphism from* $(\mathbf{B}, \oplus, \mathbf{0})$ *to* $(\mathbb{P}_{n+1}, \star, I)$, *where* $B(\mathbf{u}) \star B(\mathbf{v}) = (B(\mathbf{u})B(\mathbf{v})^2 B(\mathbf{u}))^{1/2}$.
Furthermore, the powers and roots in $(\mathbb{P}_{n+1}, \star)$ *agree with those of matrix multiplication.*

On the cone Ω_{n+1} of all $(n+1) \times (n+1)$ positive definite Hermitian matrices, the squaring map $D : \Omega_{n+1} \to \Omega_{n+1}$, $D(A) = A^2$ gives us a different algebraic structure on the set $\mathbb{P}_{n+1}$. We note that the squaring map D is a bijection since any positive definite Hermitian matrix has a unique square root in Ω_{n+1}.

Theorem 3.4. *The composition* $D \circ B : (\mathbf{B}, \oplus, \mathbf{0}) \to (\mathbb{P}_{n+1}, *, I)$ *is also an isomorphism, where* $B(\mathbf{u}) * B(\mathbf{v}) = B(\mathbf{u})^{1/2} B(\mathbf{v}) B(\mathbf{u})^{1/2}$.

Remark 3.5. From Theorem 3.3 and Theorem 3.4 we see that both $(\mathbb{P}_{n+1}, \star, I)$ and $(\mathbb{P}_{n+1}, *, I)$ are uniquely 2-divisible gyrocommutative gyrogroups. Moreover, we have $B(2 \otimes \mathbf{v}) = B(\mathbf{v})^2$, $B((1/2) \otimes \mathbf{v}) = B(\mathbf{v})^{1/2}$ for any $\mathbf{v} \in \mathbf{B}$.

4. Möbius matrices and Lorentz boosts

A qubit density matrix is a 2×2 positive semidefinite Hermitian matrix with trace 1. It can be described by a Bloch vector $\mathbf{v}=(v_1,v_2,v_3)\in\mathbb{R}^3$ such as

$$\rho_{\mathbf{v}}=\frac{1}{2}\begin{pmatrix} 1+v_3 & v_1-iv_2 \\ v_1+iv_2 & 1-v_3 \end{pmatrix}=\frac{1}{2}(I_2+v_1\sigma_x+v_2\sigma_y+v_3\sigma_z), \tag{5}$$

where I_2 denotes the 2×2 identity matrix and

$$\sigma_x=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices. It is known that all qubit mixed states, or 2×2 invertible density matrices, are parameterized by the open unit ball $\mathbf{B}$ in $\mathbb{R}^3$.
In general, it is difficult to extend the qubit mixed state $\rho_{\mathbf{v}}$ to a density matrix which is parametrized by an n-dimensional Bloch vector $\mathbf{v}\in\mathbf{B}$ for $n>3$. On the other hand, A. A. Ungar has suggested in [23, Section 9.5] the real counterpart $\mu_{3,\mathbf{v}}$ of $\rho_{\mathbf{v}}$ that shares similar properties with $\rho_{\mathbf{v}}$ and its extended version such as

$$\mu_{n,\mathbf{v}}=\frac{2\gamma_{\mathbf{v}}^2}{(n-3)+4\gamma_{\mathbf{v}}^2}\begin{pmatrix} 1-\frac{1}{2\gamma_{\mathbf{v}}^2} & \mathbf{v}^T \\ \mathbf{v} & \frac{1}{2\gamma_{\mathbf{v}}^2}I_n+\mathbf{v}\mathbf{v}^T \end{pmatrix}$$

$$=\frac{2\gamma_{\mathbf{v}}^2}{(n-3)+4\gamma_{\mathbf{v}}^2}\begin{pmatrix} 1-\frac{1}{2\gamma_{\mathbf{v}}^2} & v_1 & v_2 & v_2 & \cdots & v_n \\ v_1 & \frac{1}{2\gamma_{\mathbf{v}}^2}+v_1^2 & v_1v_2 & v_1v_3 & \cdots & v_1v_n \\ v_2 & v_1v_2 & \frac{1}{2\gamma_{\mathbf{v}}^2}+v_2^2 & v_2v_3 & \cdots & v_2v_n \\ v_3 & v_1v_3 & v_2v_3 & \frac{1}{2\gamma_{\mathbf{v}}^2}+v_3^2 & \cdots & v_3v_n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & v_1v_n & v_2v_n & v_3v_n & \cdots & \frac{1}{2\gamma_{\mathbf{v}}^2}+v_n^2 \end{pmatrix},$$

where I_n is the $n\times n$ identity matrix. One can see that $\mu_{n,\mathbf{v}}$ is an $(n+1)\times(n+1)$ symmetric matrix, called a *Möbius matrix*, parameterized by the vector $\mathbf{v}\in\mathbf{B}$. Although it is not a natural extension of the qubit density matrix, it is meaningful that we explore $\mu_{n,\mathbf{v}}$ as a density matrix in the study of higher-level quantum states. Via a diagonalization of $\mu_{n,\mathbf{v}}$ we confirm that it is a normalized Lorentz boost generated by the vector $2\otimes\mathbf{v}$.

Theorem 4.1. *For each $\mathbf{v}\in\mathbf{B}$ there exist an orthogonal matrix $O_{\mathbf{v}}$ and a diagonal matrix $D_{\mathbf{v}}$*

$$O_{\mathbf{v}}=\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2}\|\mathbf{v}\|}\mathbf{v} & \frac{1}{\sqrt{2}\|\mathbf{v}\|}\mathbf{v} & \mathbf{u}_1 & \cdots & \mathbf{u}_{n-1} \end{pmatrix},$$

$$D_{\mathbf{v}}=\frac{1}{(n-3)+4\gamma_{\mathbf{v}}^2}\begin{pmatrix} \lambda^2 & 0 & \\ 0 & \frac{1}{\lambda^2} & 0 \\ & 0 & I_{n-1} \end{pmatrix}$$

such that $\mu_{n,\mathbf{v}}=O_{\mathbf{v}}^TD_{\mathbf{v}}O_{\mathbf{v}}$, where $\{\mathbf{u}_j\ :\ \mathbf{v}^T\mathbf{u}_j=0$ for all $j=1,2,\ldots,n-1\}$ is an orthonormal set obtained by the Gram-Schmidt process, and

$$\lambda=\sqrt{\frac{1+\|\mathbf{v}\|}{1-\|\mathbf{v}\|}}>1.$$

Proof. Let $A = \frac{(n-3)+4\gamma_{\mathbf{v}}^2}{2\gamma_{\mathbf{v}}^2}\mu_{n,\mathbf{v}} = \begin{pmatrix} 1-\frac{1}{2\gamma_{\mathbf{v}}^2} & \mathbf{v}^T \\ \mathbf{v} & \frac{1}{2\gamma_{\mathbf{v}}^2}I_n + \mathbf{v}\mathbf{v}^T \end{pmatrix}$.

It is enough to show that $A = O_{\mathbf{v}}^T \cdot \frac{1}{2\gamma_{\mathbf{v}}^2}\begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \frac{1}{\lambda^2} & \\ & 0 & I_{n-1} \end{pmatrix} \cdot O_{\mathbf{v}}$.

Indeed,
$$A\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}\|\mathbf{v}\|}\mathbf{v} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(1-\frac{1}{2\gamma_{\mathbf{v}}^2}\right) + \frac{1}{\sqrt{2}\|\mathbf{v}\|}\mathbf{v}^T\mathbf{v} \\ \frac{1}{\sqrt{2}}\mathbf{v} + \left(\frac{1}{2\gamma_{\mathbf{v}}^2}I_n + \mathbf{v}\mathbf{v}^T\right)\frac{1}{\sqrt{2}\|\mathbf{v}\|}\mathbf{v} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2\sqrt{2}}(1+\|\mathbf{v}\|)^2 \\ \frac{1}{2\sqrt{2}\|\mathbf{v}\|}(1+\|\mathbf{v}\|)^2\mathbf{v} \end{pmatrix} = \frac{(1+\|\mathbf{v}\|)^2}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}\|\mathbf{v}\|}\mathbf{v} \end{pmatrix}.$$

Here, $\frac{(1+\|\mathbf{v}\|)^2}{2} = \frac{1}{2\gamma_{\mathbf{v}}^2}\cdot\frac{1+\|\mathbf{v}\|}{1-\|\mathbf{v}\|} = \frac{\lambda^2}{2\gamma_{\mathbf{v}}^2}$. Similarly,
$$A\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}\|\mathbf{v}\|}\mathbf{v} \end{pmatrix} = \frac{(1-\|\mathbf{v}\|)^2}{2}\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}\|\mathbf{v}\|}\mathbf{v} \end{pmatrix} = \frac{1}{2\gamma_{\mathbf{v}}^2\lambda^2}\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}\|\mathbf{v}\|}\mathbf{v} \end{pmatrix}.$$

Finally for each $j = 1, 2, \ldots, n-1$
$$A\begin{pmatrix} 0 \\ \mathbf{u}_j \end{pmatrix} = \begin{pmatrix} \mathbf{v}^T\mathbf{u}_j \\ \left(\frac{1}{2\gamma_{\mathbf{v}}^2}I_n + \mathbf{v}\mathbf{v}^T\right)\mathbf{u}_j \end{pmatrix} = \frac{1}{2\gamma_{\mathbf{v}}^2}\begin{pmatrix} 0 \\ \mathbf{u}_j \end{pmatrix}$$
since $\mathbf{v}^T\mathbf{u}_j = 0$. ■

Remark 4.2. By Theorem 4.1 we have that the matrix $\mu_{n,\mathbf{v}}$ is positive definite,
$$\mathrm{tr}\mu_{n,\mathbf{v}} = \frac{1}{(n-3)+4\gamma_{\mathbf{v}}^2}\left(\lambda^2 + \frac{1}{\lambda^2} + n - 1\right) = 1, \tag{6}$$
and $\det\mu_{n,\mathbf{v}} = \left(\frac{1}{(n-3)+4\gamma_{\mathbf{v}}^2}\right)^{n+1} = \left(\frac{1-\|\mathbf{v}\|^2}{(n+1)-(n-3)\|\mathbf{v}\|^2}\right)^{n+1} > 0.$

We proved equation 6 in [23] and that $\mu_{n,\mathbf{v}}$ is an $(n+1)\times(n+1)$ real mixed state.

From Lemma 3.2 we have seen that the Lorentz boost $B(\mathbf{v})$ generated by the vector $\mathbf{v} \in \mathbf{B}$ can be decomposed as
$$B(\mathbf{v}) = O_{\mathbf{v}}^T\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \frac{1}{\lambda} & \\ & 0 & I_{n-1} \end{pmatrix}O_{\mathbf{v}},$$
where we use the same notations for an orthogonal matrix $O_{\mathbf{v}}$ and an eigenvalue λ as in Theorem 4.1. So we have the following proposition, since
$$\mathrm{tr}B(\mathbf{v})^2 = \lambda^2 + \frac{1}{\lambda^2} + n - 1 = 2\gamma_{\mathbf{v}}^2(1+\|\mathbf{v}\|^2) + n - 1 = (n-3) + 4\gamma_{\mathbf{v}}^2.$$

Proposition 4.3. *For each* $\mathbf{v} \in \mathbf{B}$*:* $\mu_{n,\mathbf{v}} = \frac{1}{trB(\mathbf{v})^2}B(\mathbf{v})^2 = \frac{1}{trB(2\otimes\mathbf{v})}B(2\otimes\mathbf{v}).$

5. The Hilbert projective metric

For Hermitian matrices X and Y, we write that $X \leq Y$ if $Y - X$ is positive semidefinite, and $X < Y$ if $Y - X$ is positive definite. This gives us a partial order on the real vector space $\mathbb{H}_{n+1}$ of all $(n+1) \times (n+1)$ Hermitian matrices, known as the *Loewner order*. It is not a total order on a Banach space $\mathbb{H}_{n+1}$ with the inner product $\langle A, B\rangle := \mathrm{tr}(AB)$ and its associated norm $\|A\| = [\mathrm{tr}(A^2)]^{1/2}$; see [6, Section 7.7] for more information.

For any $A, B \in \Omega_{n+1}$, we define

$$m(A,B) := \sup\{\mu > 0 : \mu B \leq A\} = \lambda_n(B^{-1/2}AB^{-1/2}),$$
$$M(A,B) := \inf\{\lambda > 0 : A \leq \lambda B\} = \lambda_1(B^{-1/2}AB^{-1/2}),$$

where $\lambda_1(X)$ and $\lambda_n(X)$ are the largest and smallest eigenvalues of X, respectively. The *Hilbert projective metric* on Ω_{n+1} is given by

$$p(A,B) := \ln \frac{M(A,B)}{m(A,B)}. \tag{7}$$

Since $m(A,B) = M(B,A)^{-1}$, one may define the Hilbert projective metric such as

$$p(A,B) = \ln M(A,B) + \ln M(B,A).$$

We recall from [12, Proposition 5.1] some properties of the Hilbert projective metric on Ω_{n+1}, and see more from [16, 20].

Lemma 5.1. *The following properties hold for all $A, B, C \in \Omega_{n+1}$.*

(i) $0 \leq p(A,B)$.

(ii) $p(A,B) = 0$ *if and only if there is* $\lambda > 0$ *such that* $B = \lambda A$.

(iii) $p(A,B) = p(B,A)$.

(iv) $p(A,C) \leq p(A,B) + p(B,C)$.

Remark 5.2. Lemma 5.1 says that (Ω_{n+1}, p) is a pseudo-metric space, meanwhile, $(\mathbb{P}_{n+1}, h)$ is a metric space since the determinant of Lorentz boosts is 1.

In the following we denote by $X^\dagger$ the complex conjugate transpose of X.

Lemma 5.3. *The following properties hold for all $A, B \in \Omega_{n+1}$.*

(i) $p(\alpha A, \beta B) = p(A,B)$ *for any* $\alpha, \beta > 0$.

(ii) $p(PAP^\dagger, PBP^\dagger) = p(A,B)$ *for any invertible matrix* P.

(iii) $p(A^{-1}, B^{-1}) = p(A,B)$.

(iv) $p(A^t, I) = |t| p(A, I)$ *for any* $t \in \mathbb{R}$.

(v) $p(A^t, B^t) \leq |t| p(A,B)$ *for any* $|t| \leq 1$.

Proof. Let $A, B \in \Omega_{n+1}$.

(i) Using the definition of p this is proved.

(ii) This is easily seen since $B^{-1/2}AB^{-1/2}$ is similar to $B^{-1}A$ and since $B^{-1}A$ is similar to $(P^\dagger)^{-1}(B^{-1}A)P^\dagger = (PBP^\dagger)^{-1}(PAP^\dagger)$ for any invertible matrix P.

(iii) Since $m(A^{-1}, B^{-1}) = M(A,B)^{-1}$, it is proved.

(iv) From (iii) it is enough to show for the case when $t > 0$. Since $m(A^t, I) = \lambda_n(A)^t$ and $M(A^t, I) = \lambda_1(A)^t$, it is proved.

(v) This follows from [12, Proposition 4.1]. ■

We now see the interesting relation between the Hilbert projective metric for Lorentz boosts and the rapidity metric for Einstein admissible vectors.

Theorem 5.4. *For any* $\mathbf{u}, \mathbf{v} \in (\mathbf{B}, \oplus)$: $p(B(\mathbf{u})^2, B(\mathbf{v})^2) = 4d(\mathbf{u}, \mathbf{v})$.

Proof. We have seen from Theorem 3.4 that the composition $D \circ B : (\mathbf{B}, \oplus) \to (\mathbb{P}_{n+1}, *)$ is an isomorphism of gyrocommutative gyrogroups. By Lemma 2.5 (v) and Lemma 5.3 (ii), we have that the rapidity metric on $\mathbf{B}$ and the Hilbert projective metric on Ω_{n+1}, especially on $\mathbb{P}_{n+1}$, are invariant under left translations. Furthermore, for any $\mathbf{v} \in \mathbf{B}$

$$p(B(\mathbf{v})^2, I) = \ln \frac{\lambda_1(B(\mathbf{v})^2)}{\lambda_n(B(\mathbf{v})^2)} = 2 \ln \frac{1 + \|\mathbf{v}\|}{1 - \|\mathbf{v}\|} = 4 \tanh^{-1} \|\mathbf{v}\| = 4d(\mathbf{v}, \mathbf{0}).$$

The first equality follows from the definition of p, and the second equality follows from Lemma 3.2 and

$$\lambda_1(B(\mathbf{v})) = \sqrt{\frac{1 + \|\mathbf{v}\|}{1 - \|\mathbf{v}\|}} = \frac{1}{\lambda_n(B(\mathbf{v}))}.$$

By Lemma 2.7 the proof is complete. ■

Proposition 5.5. *For any* $\mathbf{u}, \mathbf{v} \in \mathbf{B}$: $p(\mu_{n,\mathbf{u}}, \mu_{n,\mathbf{v}}) = 4d(\mathbf{u}, \mathbf{v})$.

Proof. By Proposition 4.3 and Lemma 5.3 (i) we have

$$p(\mu_{n,\mathbf{u}}, \mu_{n,\mathbf{v}}) = p\left(\frac{1}{\mathrm{tr}B(\mathbf{u})^2} B(\mathbf{u})^2, \frac{1}{\mathrm{tr}B(\mathbf{v})^2} B(\mathbf{v})^2\right) = p(B(\mathbf{u})^2, B(\mathbf{v})^2) = 4d(\mathbf{u}, \mathbf{v}). \quad ■$$

Corollary 5.6. *For any* $\mathbf{u}, \mathbf{v} \in (\mathbf{B}, \oplus)$: $p(B(\mathbf{u}), B(\mathbf{v})) \leq 2d(\mathbf{u}, \mathbf{v})$.

Proof. From Proposition 7.8 of [11] we have $d(1/2 \otimes \mathbf{u}, 1/2 \otimes \mathbf{v}) \leq (1/2)d(\mathbf{u}, \mathbf{v})$.

Thus,

$$\begin{aligned} p(B(\mathbf{u}), B(\mathbf{v})) &= p(B(1/2 \otimes \mathbf{u})^2, B(1/2 \otimes \mathbf{v})^2) \\ &= 4d(1/2 \otimes \mathbf{u}, 1/2 \otimes \mathbf{v}) \leq 2d(\mathbf{u}, \mathbf{v}). \end{aligned}$$ ■

Remark 5.7. By Theorem 7.6 of [11] it has been proved the relation between the rapidity metric and the Riemannian trace metric δ: for $\mathbf{u}, \mathbf{v} \in \mathbf{B}$

$$2\sqrt{2}d(\mathbf{u}, \mathbf{v}) = \delta(B(\mathbf{u})^2, B(\mathbf{v})^2).$$

From Theorem 5.4 and Remark 3.5 we obtain

$$p(B(\mathbf{u}), B(\mathbf{v})) = \sqrt{2}\delta(B(\mathbf{u}), B(\mathbf{v})).$$

6. Fidelity and Wasserstein distance

It has been issued how to measure the distance of quantum states represented by density matrices, i.e., positive semidefinite Hermitian matrices with trace 1. The fidelity is one of the crucial measurements although it is actually not a metric for quantum states. On the other hand, it is a measure of the closedness of two quantum states, that is, the fidelity is 1 if and only if two quantum states are identical. Moreover, it does give rise to a useful metric such as Bures (or Wasserstein) distance, and is able to apply for a variety of research areas in quantum information and computation theory; see [19] and [17, Section 9.2.2].

The *fidelity* for density matrices ρ and σ is defined by

$$F(\rho,\sigma) := \operatorname{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}. \tag{8}$$

We review some basic properties of the fidelity.

Lemma 6.1. *The following are satisfied for any density matrices ρ and σ.*

(i) $0 \le F(\rho,\sigma) \le 1$.

(ii) $F(\rho,\sigma) = 1$ *if and only if* $\rho = \sigma$.

(iii) $F(\rho,\sigma) = F(\sigma,\rho)$.

(iv) $F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho,\sigma)$ *for any unitary matrix* U.

The property (iv) of Lemma 6.1 is called the invariance under unitary congruence transformation, so that the fidelity is basis-independent.

Remark 6.2. The fidelity F can be quite difficult to calculate, but it takes a simple form for the 2-by-2 density matrices ρ and σ: see the equation (8.52) in [3],

$$F(\rho,\sigma)^2 = \operatorname{tr}(\rho\sigma) + 2\sqrt{\det(\rho)\det(\sigma)}. \tag{9}$$

From the equations (9.64) and (9.68) in [23] we have alternative expression of the fidelity for the 2-by-2 density matrices $\rho_{\mathbf{u}}$ and $\rho_{\mathbf{v}}$ given in (5):

$$F(\rho_{\mathbf{u}},\rho_{\mathbf{v}})^2 = \frac{1+\gamma_{\mathbf{u}\oplus\mathbf{v}}}{2\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} = \frac{1}{2}\left\{1 + \mathbf{u}^T\mathbf{v} + \sqrt{1-\|\mathbf{u}\|^2}\sqrt{1-\|\mathbf{v}\|^2}\right\}. \tag{10}$$

One can verify that two equations (9) and (10) are the same.

The original fidelity of density matrices can be naturally generalized to positive semi-definite Hermitian matrices such as

$$F(A,B) = \operatorname{tr}(A^{1/2}BA^{1/2})^{1/2}$$

for any positive semi-definite Hermitian matrices A and B. Recently, a new metric on the cone Ω of positive definite Hermitian matrices have been introduced:

$$d^W(A,B) := \left[\operatorname{tr}\left(\frac{A+B}{2}\right) - \operatorname{tr}(A^{1/2}BA^{1/2})^{1/2}\right]^{1/2}. \tag{11}$$

It is the 2-Wasserstein distance for two Gaussian probabilities with means 0 and covariance matrices $A, B \in \Omega$, so we call d^W the *Wasserstein distance*. As the least squares mean for the Wasserstein distance, the Wasserstein barycenter has been widely studied: see [1, 2, 4, 5, 7, 15]. In the following we investigate the fidelity and Wasserstein distance of Lorentz boosts as elements of $SO^+(1,n)$.

Lemma 6.3. *For any* $\mathbf{u}, \mathbf{v} \in (\mathbf{B}, \oplus)$: $tr[B(\mathbf{u})B(\mathbf{v})^2B(\mathbf{u})]^{1/2} = 2\gamma_{\mathbf{u}\oplus\mathbf{v}} + n - 1.$

Proof. This follows from equation (4) and Lemma 3.2. ■

Theorem 6.4. *For any* $\mathbf{u}, \mathbf{v} \in (\mathbf{B}, \oplus)$ *and* $\mathbf{w} = (1/2) \otimes \mathbf{u} \oplus (1/2) \otimes \mathbf{v}$ *we have*

$$F(B(\mathbf{u}), B(\mathbf{v})) = 2\gamma_{\mathbf{w}} + n - 1, \quad \text{and} \quad d^W(B(\mathbf{u}), B(\mathbf{v})) = \gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} - 2\gamma_{\mathbf{w}}.$$

Proof. Let $\mathbf{u}' := (1/2) \otimes \mathbf{u}$ and $\mathbf{v}' := (1/2) \otimes \mathbf{v}$. Then

$$F(B(\mathbf{u}), B(\mathbf{v})) = \mathrm{tr}[B(\mathbf{u}')B(\mathbf{v}')^2B(\mathbf{u}')]^{1/2} = 2\gamma_{\mathbf{u}'\oplus\mathbf{v}'} + n - 1.$$

The first equality follows from Theorem 3.3, and the second follows from Lemma 6.3. Since $\mathrm{tr}B(\mathbf{v}) = 2\gamma_{\mathbf{v}} + n - 1$ by Lemma 6.3, the formula of Wasserstein distance $d^W(B(\mathbf{u}), B(\mathbf{v}))$ can be obtained. ■

For any $\mathbf{u}, \mathbf{v} \in \mathbf{B}$, we have in general,

$$(1/2) \otimes \mathbf{u} \oplus (1/2) \otimes \mathbf{v} \neq (1/2) \otimes (\mathbf{u} \oplus \mathbf{v}),$$

see [23, Chapter 6] for more details. We give a formula for the Lorentz factor for $\mathbf{w} = (1/2) \otimes \mathbf{u} \oplus (1/2) \otimes \mathbf{v}$, so that the fidelity for Lorentz boosts can be simply calculated.

Lemma 6.5. *For any* $\mathbf{u}, \mathbf{v} \in (\mathbf{B}, \oplus)$ *and* $\mathbf{w} = (1/2) \otimes \mathbf{u} \oplus (1/2) \otimes \mathbf{v}$.

$$\gamma_{\mathbf{w}} = \frac{1}{\sqrt{1+2\gamma_{\mathbf{u}}}} \frac{1}{\sqrt{1+2\gamma_{\mathbf{v}}}} ((1+\gamma_{\mathbf{u}})(1+\gamma_{\mathbf{v}}) + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\mathbf{u}^T\mathbf{v}),$$

Proof. Let $\mathbf{u}' := (1/2) \otimes \mathbf{u}$ and $\mathbf{v}' := (1/2) \otimes \mathbf{v}$. By Remark 2.4

$$\gamma_{\mathbf{v}'} = \frac{1}{\sqrt{1 - \|\mathbf{v}'\|^2}} = \frac{1}{\sqrt{1 - \gamma_{\mathbf{v}}^2/(1+\gamma_{\mathbf{v}})^2}} = \frac{1+\gamma_{\mathbf{v}}}{\sqrt{1+2\gamma_{\mathbf{v}}}}.$$

Applying the gamma identity $\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1 + \mathbf{u}^T\mathbf{v})$ to $\mathbf{u}'$ and $\mathbf{v}'$, it is proved. ■

We now provide the fidelity of Möbius matrices, which is a generalization of the qubit density matrices (10).

Corollary 6.6. *For* $n \geq 3$ *and any* $\mathbf{u}, \mathbf{v} \in (\mathbf{B}, \oplus)$

$$F(\mu_{n,\mathbf{u}}, \mu_{n,\mathbf{v}}) = \frac{2\gamma_{\mathbf{u}\oplus\mathbf{v}} + n - 1}{\sqrt{(4\gamma_{\mathbf{u}}^2 + n - 3)(4\gamma_{\mathbf{v}}^2 + n - 3)}}.$$

Proof. Note from the definition of fidelity that

$$F(\alpha A, \beta B) = \sqrt{\alpha\beta} F(A, B)$$

for any $A, B \in \Omega$ and $\alpha, \beta > 0$. Since

$$\mu_{n,\mathbf{v}} = \frac{1}{\mathrm{tr} B(\mathbf{v})^2} B(\mathbf{v})^2 = \frac{1}{\mathrm{tr} B(2 \otimes \mathbf{v})} B(2 \otimes \mathbf{v})$$

by Proposition 4.3, we have by Lemma 6.3

$$\begin{aligned} F(\mu_{n,\mathbf{u}}, \mu_{n,\mathbf{v}}) &= \frac{1}{\sqrt{\mathrm{tr} B(\mathbf{u})^2 \mathrm{tr} B(\mathbf{v})^2}} F(B(\mathbf{u})^2, B(\mathbf{v})^2) \\ &= \frac{2\gamma_{\mathbf{u}\oplus\mathbf{v}} + n - 1}{\sqrt{(4\gamma_{\mathbf{u}}^2 + n - 3)(4\gamma_{\mathbf{v}}^2 + n - 3)}}. \end{aligned}$$

■

Remark 6.7. Especially, for $n = 3$

$$F(\mu_{3,\mathbf{u}}, \mu_{3,\mathbf{v}}) = \frac{\gamma_{\mathbf{u}\oplus\mathbf{v}} + 1}{2\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} = F(\rho_{\mathbf{u}}, \rho_{\mathbf{v}})^2.$$

The following provides the upper bound for the Wasserstein distance of Möbius matrices.

Theorem 6.8. *For $n \geq 3$ and any $\mathbf{u}, \mathbf{v} \in (\mathbf{B}, \oplus)$*

$$\sqrt{1 - F(\mu_{n,\mathbf{u}}, \mu_{n,\mathbf{v}})^2} \leq \varrho(\mathbf{u}, \mathbf{v}),$$

where $\varrho(\mathbf{u}, \mathbf{v}) = \| -\mathbf{u} \oplus \mathbf{v}\|$ is the Ungar gyrometric. Thus,

$$d^W(\mu_{n,\mathbf{u}}, \mu_{n,\mathbf{v}}) \leq \varrho(\mathbf{u}, \mathbf{v}).$$

Proof. By Proposition 18 in [21] it has been shown that

$$\sqrt{1 - F(\rho, \sigma)^2} \leq \tanh \frac{h(\rho, \sigma)}{4}$$

for any density matrices ρ and σ, where h is the Hilbert projective metric. So for any $\mathbf{u}, \mathbf{v} \in (\mathbf{B}, \oplus)$,

$$\sqrt{1 - F(\mu_{n,\mathbf{u}}, \mu_{n,\mathbf{v}})^2} \leq \tanh \frac{h(\mu_{n,\mathbf{u}}, \mu_{n,\mathbf{v}})}{4}.$$

By Lemma 5.3 (i) and Theorem 5.4 we have

$$\tanh \frac{h(\mu_{n,\mathbf{u}}, \mu_{n,\mathbf{v}})}{4} = \tanh \frac{h(B(\mathbf{u})^2, B(\mathbf{v})^2)}{4} = \tanh d(\mathbf{u}, \mathbf{v}) = \| -\mathbf{u} \oplus \mathbf{v}\|.$$

Therefore, we obtain the desired first inequality. Moreover,

$$d^W(\mu_{n,\mathbf{u}}, \mu_{n,\mathbf{v}}) = \sqrt{1 - F(\mu_{n,\mathbf{u}}, \mu_{n,\mathbf{v}})} \leq \sqrt{1 - F(\mu_{n,\mathbf{u}}, \mu_{n,\mathbf{v}})^2},$$

since $0 \leq F(\rho, \sigma) \leq 1$ for any density matrices ρ and σ by Lemma 6.1 (i). Hence, we get the desired second inequality. ■

7. Final remarks and open questions

In [23] A. A. Ungar has provided a simple algebraic tool, called a gyrogroup, to study analytic hyperbolic geometry such as

(I) the Poincaré ball model regulated by Möbius gyrogroup,

(II) the Beltrami-Klein ball model regulated by Einstein gyrogroup, and

(III) the PV (Proper Velocity) ball model regulated by PV gyrogroup.

We considered in the article mainly the Einstein gyrogroup on the open unit ball $\mathbf{B}$ of the n-dimensional Euclidean space $\mathbb{R}^n$ arisen from the Einstein velocity addition $\oplus$. Furthermore, we constructed the gyrogroup structure on the set $\mathbb{P}$ of Lorentz boosts isomorphic to the Einstein gyrogroup $(\mathbf{B}, \oplus)$. We then introduced the Hilbert projective metric p to distinguish two Lorentz boosts. Based on this isomorphism we computed the metrics p, and obtained the simple connection with the rapidity metric d on the Einstein gyrogroup $(\mathbf{B}, \oplus)$ such as

$$p(B(\mathbf{u})^2, B(\mathbf{v})^2) = 4d(\mathbf{u}, \mathbf{v})$$

for any $\mathbf{u}, \mathbf{v} \in \mathbf{B}$: see Theorem 5.4.

In quantum information, the fidelity F is an important tool to measure two quantum states usually described by density matrices. It can be simply calculated for any 2-by-2 density matrices (see Remark 6.2), or for any pure states $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$ such as

$$F(\rho, \sigma) = |\langle\psi|\phi\rangle|.$$

However, it is quite difficult to calculate the fidelity for higher-dimensional mixed states. For certain quantum states derived by Lorentz boosts

$$\rho^B(\mathbf{v}) = \frac{1}{\mathrm{tr}B(\mathbf{v})} B(\mathbf{v}), \ \mathbf{v} \in \mathbf{B} \subseteq \mathbb{R}^n,$$

which is an $(n+1) \times (n+1)$ mixed state, we can provide an explicit formula of the fidelity for them by using our previous results in Section 6 such as

$$F(\rho^B(\mathbf{u}), \rho^B(\mathbf{v})) = \frac{2\gamma_{\mathbf{w}} + n - 1}{\sqrt{(2\gamma_{\mathbf{u}} + n - 1)(2\gamma_{\mathbf{v}} + n - 1)}}$$

for any $\mathbf{u}, \mathbf{v} \in (\mathbf{B}, \oplus)$, where $\mathbf{w} = (1/2) \otimes \mathbf{u} \oplus (1/2) \otimes \mathbf{v}$. Via a formula of the gamma factor of $\mathbf{w}$ in Lemma 6.5 one might be able to compute the fidelity easier.

The Einstein gyrovector space $(\mathbf{B}, \oplus, \otimes)$ gives an algebraic tool for the Beltrami-Klein ball model of hyperbolic geometry, where the scalar multiplication $\otimes : \mathbb{R} \times \mathbf{B} \to \mathbf{B}$ is given by

$$r \otimes \mathbf{v} := \tanh(r \tanh^{-1} \|\mathbf{v}\|) \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

for $r \in \mathbb{R}$ and $\mathbf{v}(\neq \mathbf{0}) \in \mathbf{B}$, and $r \otimes \mathbf{0} := \mathbf{0}$. Gyrolines as geodesics in the Einstein gyrovector space are Euclidean straight lines, and for given $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{B}$ three gyromedians are concurrent. The point of concurrency, named the *gyrocentroid* $C_{\mathbf{uvw}}$, has the elegant form

$$C_{\mathbf{uvw}} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}}}.$$

Note that the gyrocentroid is invariant under a left translation and is related with the relativistic center of momentum velocity: see Chapters 6 and 11 in [23].

On the other hand, one can consider the barycenters of Lorentz boosts with respect to the Riemannian trace distance δ and Wasserstein distance d^W. That is, for given $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{B}$

$$\begin{aligned} G(B(\mathbf{u}), B(\mathbf{v}), B(\mathbf{w})) &:= \underset{X \in \Omega_{n+1}}{\arg\min}\, \delta(X, B(\mathbf{u}))^2 + \delta(X, B(\mathbf{v}))^2 + \delta(X, B(\mathbf{w}))^2 \\ W(B(\mathbf{u}), B(\mathbf{v}), B(\mathbf{w})) &:= \underset{X \in \Omega_{n+1}}{\arg\min}\, d^W(X, B(\mathbf{u}))^2 + d^W(X, B(\mathbf{v}))^2 + d^W(X, B(\mathbf{w}))^2. \end{aligned}$$

It has been known from [8, 18, 14] and [1, 2], respectively, that such barycenters G and W exist in the open convex cone Ω_{n+1} of positive definite Hermitian matrices. So it is a natural question that $G(B(\mathbf{u}), B(\mathbf{v}), B(\mathbf{w}))$ and $W(B(\mathbf{u}), B(\mathbf{v}), B(\mathbf{w}))$ are again Lorentz boosts, that is, elements of the restricted Lorentz group $SO^+(1,n)$. If yes, that is, $G(B(\mathbf{u}), B(\mathbf{v}), B(\mathbf{w})) = B(\mathbf{m})$ and $W(B(\mathbf{u}), B(\mathbf{v}), B(\mathbf{w})) = B(\mathbf{n})$ for some $\mathbf{m}, \mathbf{n} \in \mathbf{B}$, then one can also ask whether or not their generators $\mathbf{m}$ and $\mathbf{n}$ are related with gyrocentroid $C_{\mathbf{uvw}}$.

Acknowledgement. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (No. NRF-2018 R1C1B 6001 394).

References

[1] M. Agueh, G. Carlier: *Barycenters in the Wasserstein space*, SIAM J. Math. Anal. Appl. 43 (2011) 904–924.

[2] P. C. Alvarez-Esteban, E. del Barrio, J. A. Cuesta-Albertos, C. Matran: *A fixed point approach to barycenters in Wasserstein spaces*, J. Math. Anal. Appl. 441 (2016) 744–762.

[3] S. M. Barnett: *Quantum Information*, Oxford University Press, Oxford (2009).

[4] R. Bhatia, T. Jain, Y. Lim: *Inequalities for the Wasserstein mean of positive definite matrices*, Linear Algebra Appl. 576 (2019) 108–123.

[5] R. Bhatia, T. Jain, Y. Lim: *On the Bures-Wasserstein distance between positive definite matrices*, Expositiones Mathematae 37(2) (2019) 165–191.

[6] R. Horn, C. Johnson: *Matrix Analysis*, Cambridge University Press, Cambridge (2009).

[7] J. Hwang, S. Kim: *Bounds for the Wasserstein mean with applications to the Lie-Trotter mean*, J. Math. Anal. Appl. 475 (2019) 1744–1753.

[8] H. Karcher: *Riemannian center of mass and mollifier smoothing*, Comm. Pure Appl. Math. 30 (1977) 509–541.

[9] H. Kiechle: *Theory of K-Loops*, Lecture Notes in Mathematics 1778, Springer, Berlin (2002).

[10] S. Kim: *Distances of qubit density matrices on Bloch sphere*, J. Math. Phys. 52 (2011) 102303.

[11] S. Kim, J. D. Lawson: *Unit balls, Lorentz boosts, and hyperbolic geometry*, Results Math. 63 (2013) 1225–1242.

[12] K. Koufany: *Application of Hilbert's projective metric on symmetric cones*, Acta Math. Sinica 22(5) (2006) 1467–1472.

[13] J. D. Lawson, Y. Lim: *Symmetric sets with midpoints and algebraically equivalent theories*, Results Math. 46 (2004) 37–56.

[14] J. D. Lawson, Y. Lim: *Karcher means and Karcher equations of positive definite operators*, Trans. Amer. Math. Soc. Series B 1 (2014) 1–22.

[15] H. Lee, S. Kim: *Inequalities of the Wasserstein mean with other matrix means*, Annals of Functional Analysis 11 (2020) 194–207.

[16] B. Lemmens: *Nonexpansive mappings on Hilbert's metric spaces*, Topol. Methods Nonlinear Analysis 38 (2011) 45–58.

[17] Z. Ma, F.-L. Zhang, J.-L. Chen: *Fidelity induced distance measures for quantum states*, Phys. Letters A 373 (2009) 3407–3409.

[18] M. Moakher: *A differential geometric approach to the geometric mean of symmetric positive-definite matrices*, SIAM J. Matrix Anal. Appl. 26(3) (2005) 735–747.

[19] M. Nielsen, I. Chuang: *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge (2010).

[20] R. D. Nussbaum: *Hilbert Projective Metric and Iterated Nonlinear Maps*, Memoirs of the American Mathematical Society 391, American Mathematical Society, Providence (1988).

[21] D. Reeb, M. Kastoryano, M. Wolf: *Hilbert's projective metric in quantum information theory*, J. Math. Physics 52 (2011) 082201.

[22] L. V. Sabinin, L. L. Sabinina, L. V. Sbitneva: *On the notion of a gyrogroup*, Aequationes Math. 56 (1998) 11–17.

[23] A. A. Ungar: *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*, World Scientific, Singapour (2008).

Sejong Kim
Department of Mathematics
Chungbuk National University
Cheongju 28644, Korea
skim@chungbuk.ac.kr

Received August 12, 2019
and in final form September 12, 2019