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A Banach Algebra Approach to Loos Symmetric Cones

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Abstract. We consider Loos symmetric spaces on an open cone Ω in the Banach space setting and show how such Loos symmetric spaces may be realized from the set of elements inverted by an involution on a Banach-Lie group. The group is a subgroup of the group of invertible elements of the Banach algebra of all bounded linear transformations on the Banach space $V = \Omega - \Omega$. This construction connects the theory of Loos symmetric cones to that of involutive Lie groups.

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Key Words: Loos symmetric cone, normal cone, symmetric space, Banach-Lie group, involutive group.

1. Introduction

Loos symmetric cones were introduced in [10] as a generalization of the positive cone $\mathbb{A}^+$ of a unital C^*-algebra $\mathbb{A}$. Earlier E. Andruchow, G. Corach, H. Porta, L. Recht, and D. Stojanoff studied the geometry of the cones $\mathbb{A}^+$, derived interesting connections between the geometry and various operator inequalities, and gave geometric proofs of inequalities in [1], [2], and [3]. In [10] metric Loos symmetric cones were introduced, basic algebraic, order, and metric properties were derived, and several basic inequalities were derived in this general setting. The study was extended to smooth Loos symmetric cones in [7]. The smooth structure gives rise to an exponential function, which provides a significant enrichment of the theory. Additional inequalities were derived, which one might think of in some sense as "universal" operator inequalities.

In the study of finite-dimensional symmetric spaces the theory is closely connected with that of involutive Lie groups, Lie groups G equipped with an involution. The set of fixed points is a closed subgroup K and G/K yields a symmetric space. One can alternatively consider the set of points inverted by the involution as a symmetric space. In these cases there is a Lie triple system embedded in the Lie algebra of G that may be viewed as the tangent space to some distinguished point of the symmetric space.

In this paper we extend this approach to the setting of (infinite-dimensional) Loos symmetric cones. For a Loos symmetric cone Ω embedded in a Banach space $V = \Omega - \Omega$, we find an involutive Lie subgroup of the group $\mathcal{L}^\times(V)$ of invertible elements of the Banach algebra $\mathcal{L}(V)$ of bounded linear operators on V which gives

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rise (up to isomorphism) to the given Loos symmetric cone in the fashion just referred to as well as to the corresponding Lie triple system.

2. Loos systems

We recall from [8] the underlying algebraic structure with which we work and basic properties thereof. A *Loos system* consists of a binary system $(X, \bullet)$, with left translation $S_x y := x \bullet y$ representing the point symmetry (reflection) through x, satisfying for all $a, b, c \in X$:

(S1) $a \bullet a = a \quad (S_a a = a)$;

(S2) $a \bullet (a \bullet b) = b \quad (S_a S_a = \mathrm{id}_X)$;

(S3) $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c) \quad (S_a S_b = S_{S_a b} S_a)$;

(S4) the equation $x \bullet a = b$ $(S_x a = b)$ has a unique solution $x \in X$, called the *midpoint* or *geometric mean* of a and b, and denoted by $a \# b$.

The axioms bear close resemblance to the Loos axioms for a symmetric space [11]. Systems satisfying only Axioms (1)-(3) are called *symmetric sets.* Axiom (S4) asserts the unique existence of a "midpoint of symmetry" $a\#b$ that reflects a to b and by (S2) also reflects b to a.

A *pointed Loos system* is a triple $(X, \bullet, \varepsilon)$, where $(X, \bullet)$ is a Loos system and $\varepsilon \in X$ is some distinguished point, called the base point. In this setting we define

$$x^{-1} := S_\varepsilon x, \quad x^2 := S_x \varepsilon, \quad x^{1/2} := \varepsilon \# x \tag{1}$$

and inductively from these definitions all dyadic powers are defined ([9, Section 2]) so that the following rules are satisfied:

$$(x^r)^s = x^{rs}, \quad x^r \bullet x^s = x^{2r-s}, \quad x^r \# x^s = x^{\frac{(r+s)}{2}}. \tag{2}$$

If we consider the dyadic rationals $\mathbb{D}$ endowed with the structure $a \bullet b = 2a - b$ (the reflection of b through a), then $a\#b = (a+b)/2$, the usual midpoint, and the map $t \mapsto x^t : \mathbb{D} \to X$ is both a $\bullet$-homomorphism and $\#$-homomorphism. From this fact the preceding rules (and others) easily follow.

A *twisted subgroup* P of a group G is a subset containing the identity e that is closed with respect to the operation $x \bullet y = xy^{-1}x$. The system $(P, \bullet)$ satisfies (S1), (S2), and (S3), and further satisfies (S4) if and only if it is uniquely 2-divisible (every element of P has a unique square root in P); see [8, Proposition 4.2]. It is natural to choose e for the distinguished point of P. As a special case, consider the group $(\mathbb{R}, +)$ of additive real numbers, which is a uniquely 2-divisible twisted subgroup of itself. Then $(\mathbb{R}, \bullet, 0)$ is a pointed Loos system, where $s \bullet t = 2s - t$.

The displacement group $G(X)$ (also called the transvection group) of a Loos system X is the group generated under the composition by all transformations of the form $S_x S_y$, $x, y \in X$. It follows from Axioms (S2) and (S3) that these are automorphisms and thus there is a group action $(g, x) \mapsto g.x : G(X) \times X \to X$ with $G(X)$ acting as automorphisms. If X is pointed with base point ε, then $G(X)$ is generated by all displacements $S_x S_\varepsilon$ and X embeds into $G(X)$ as a twisted subgroup (closed under $g \bullet h = gh^{-1}g$) via the *quadratic representation* $Q : X \to G(X)$ defined by $Q(x) = S_x S_\varepsilon$. The image $Q(X)$ is a Loos system under the preceding $\bullet$-operation and the quadratic representation is an isomorphism between X and $Q(X)$. In particular, $Q(x \# y) = Q(x) \# Q(y)$ and $Q(x^{1/2}) = Q(x)^{1/2}$, indeed $Q(x^r) = Q(x)^r$ for all dyadic rationals r. For $x, y \in X$, we write interchangeably as convenient $Q(x)y$ or $Q(x)(y)$.

Remark 2.1. The following useful calculation rules are derived in [8] or can easily be derived by the methods there:

(1) $Q(Q(x)y) = Q(x)Q(y)Q(x)$.
(2) $(Q(x))^{-1} = Q(x^{-1})$.
(3) $(Q(x)y)^{-1} = Q(x^{-1})y^{-1}$.
(4) $Q(x^r)x^s = x^{2r+s}$,

3. Generalities on cones

Let us recall some elementary standard constructions about cones and vector spaces, where a cone is defined to be a nonempty subset of a vector space that is closed under addition and multiplication by positive scalars. Let V be a topological vector space over the real numbers and C be an open cone contained in V. Then it is straightforward to check that $C - C$ is an open subspace of V and hence all of V, since it is a neighborhood of 0. The subtraction map $S : C \oplus C \to V$ defined by $S(x, y) = x - y$ is an open surjective map. The relation $\sim$ induced on $C \oplus C$ by S is given by

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow S(x_1, y_1) = S(x_2, y_2) \Leftrightarrow x_1 + y_2 = y_1 + x_2,$$

and is easily seen to be a closed equivalence relation, indeed closed congruence relation. Set $\widetilde{V} = \{[x, y] : x, y \in C\}$, where $[x, y]$ denotes the equivalence class of (x, y) under $\sim$. Defining $[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$ and $t[x, y] = [tx, ty]$ yields an abelian group addition and a scalar multiplication for positive scalars on $\widetilde{V}$, and the scalar multiplication uniquely extends to all real scalars in a unique way to yield a real vector space. If $\widetilde{V}$ is given the quotient topology induced by the map S, then C embeds topologically and algebraically into $\widetilde{V}$ by $\eta(x) = [2x, x]$, and the map $[x, y] \mapsto x - y$ is a topological isomorphism from $\widetilde{V}$ to V. Thus up to topological isomorphism V is uniquely determined by C.

Remark 3.1. As an extension of the previous comments, we note that η is universal in the sense that if $f : C \to W$ is a continuous linear map (with respect to positive scalars) into a topological vector space W, then f extends uniquely to a continuous linear map $\widetilde{F} : \widetilde{V} \to W$ (defined by $\widetilde{F}([x, y]) = f(x) - f(y)$) such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\eta} & \widetilde{V} \\ & \searrow^{f} & \downarrow{\scriptstyle \widetilde{F}} \\ & & W \end{array}$$

Let V be a Banach space and let Ω be an open convex cone of V. We define $x \leq y$ if $y - x \in \overline{\Omega}$, the closure of Ω. We assume that Ω is a normal cone: that is, there exists a constant K with $||x|| \leq K||y||$ for all $x, y \in \Omega$ with $x \leq y$.

For a normal cone Ω, the relation

$$x \le y \ \text{ if and only if } y - x \in \overline{\Omega}$$

is a partial order. We write $x < y$ if $y - x \in \Omega$.

Any member ε of Ω is an order unit for the ordered space $(V, \le)$, and the cone is normal if and only if the order unit norm determined by ε is compatible, i.e., determines the topology of V. In this case $x \le y$ implies $\|x\| \le \|y\|$ with respect to the order unit norm, that is, we may assume without loss of generality that $K = 1$. We henceforth make this assumption.

A. C. Thompson [16] (cf. [14], [15]) has proved that Ω is a complete metric space with respect to the Thompson part metric defined by

$$d(x, y) = \max\{\log M(x/y), \log M(y/x)\}$$

where $M(x/y) := \inf\{\lambda > 0 : x \le \lambda y\}$. He also showed that the metric topology is equal to the relative norm topology.

4. Loos symmetric cones

We recall from [7] the notion of a smooth symmetric cone, where "smooth" means C^∞. We note that item (4) below connects the symmetric structure to the linear structure and item (5) to the order structure.

Definition 4.1. A *pointed Loos symmetric cone* is a triple $(\Omega, \bullet, \varepsilon)$ satisfying

(1) Ω is an open cone in a Banach space V for which $\overline{\Omega}$ is a normal cone and $\varepsilon \in \Omega$;
(2) $(\Omega, \bullet)$ is a Loos system;
(3) the map $(t, x, y) \mapsto x\#_t y : \mathbb{R} \times \Omega \times \Omega \to \Omega$ is smooth, where $t \mapsto x\#_t y : \mathbb{R} \to \Omega$ is the unique continuous homomorphism from $(\mathbb{R}, \bullet)$ into $(\Omega, \bullet)$ mapping 0 to x and 1 to y;
(4) every basic displacement $Q(x) = S_x S_\varepsilon$ is additive and positively homogeneous on Ω;
(5) $x^{1/2} \le (\varepsilon + x)/2$ for all $x \in \Omega$.

As worked out in the general setting of Loos symmetric spaces [12] and applied to Loos symmetric cones in [7], the Loos structure induces a spray on Ω, which in turn induces an exponential function on the tangent bundle $\exp : T\Omega \to \Omega$. The following appears as Remark 6.2 of [7].

Remark 4.2. Each of the following equivalent conditions is equivalent to condition (3) of the previous definition:

(3a) For any $x \in \Omega$ there exists a continuous $\bullet$-homomorphism from $\mathbb{R}$ to Ω with image containing both x and ε, and $x \mapsto \varepsilon\# x$ is smooth on Ω.
(3b) The exponential $\exp_x : T_x\Omega \to \Omega$ is a diffeomorphism for all $x \in \Omega$.

Let $(\Omega, \bullet, \varepsilon)$ be a pointed Loos symmetric cone, $V = \Omega - \Omega$. Let $(\mathcal{L}(V), \|\cdot\|)$ be the unital real Banach algebra of bounded linear operators on V equipped with the operator norm. It is a standard result that the map $(A, x) \mapsto A(x) : \mathcal{L}(V) \times V \to V$ is continuous, indeed a smooth map.

Let $\mathcal{L}^\times(V)$ denote the group (under composition) of all invertible elements of $\mathcal{L}(V)$. The group $\mathcal{L}^\times(V)$ is open in $\mathcal{L}(V)$, $\mathcal{L}(V)$ equipped with the commutator product $[A,B] = AB - BA$ is a Banach-Lie algebra, and the standard exponential map $\exp : \mathcal{L}(V) \to \mathcal{L}^\times(V)$ given by the standard exponential power series yields a Banach Lie group; see, for example, [17, Sections 2 and 6].

Since by definition each basic displacement $Q(x)$ is additive and positively homogeneous, we know as a consequence of Remark 3.1 (or see [10, Theorem 4.2, Step 1]) that $Q(x)$ uniquely extends to a continuous (hence bounded) linear transformation on V, hence to a member of $\mathcal{L}(V)$. We record an elementary lemma.

Lemma 4.3. *Each member of the displacement group $G(\Omega)$ extends uniquely to a member of $\mathcal{L}(V)$. Mapping a member of $G(\Omega)$ to its bounded linear extension yields an isomorphic embedding of $G(\Omega)$ into $\mathcal{L}^\times(V)$.*

Proof. Any member of $g \in G(\Omega)$ is a composition of basic displacements, each of which extends continuously and linearly to V. The composition of the extensions gives a continuous linear extension of g. It is immediate that this map is a homomorphism from $G(\Omega)$ into $\mathcal{L}^\times(V)$. The mapping from $G(\Omega)$ to $\mathcal{L}(V)$ is injective since two continuous linear maps agree on V if and only if their restrictions to Ω agree, since $V = \Omega - \Omega$. ■

Remark 4.4. We write the isomorphic image of the displacement group $G(\Omega)$ as $\widehat{G}(\Omega)$, which by the preceding lemma is isomorphic to $G(\Omega)$. We equip $\widehat{G}(\Omega)$ with the relative topology from the operator norm topology of $\mathcal{L}(V)$. The inverse isomorphism from $\widehat{G}(\Omega)$ to $G(\Omega)$ is, of course, restriction to Ω.

Lemma 4.5. *The mapping $\gamma : \widehat{G}(\Omega) \to \Omega$ given by $\gamma(g) = (g(\varepsilon))^{1/2}$ is smooth. Its restriction to extended basic displacements is given by $\gamma(Q(a)) = a$.*

Proof. The smoothness follows immediately from the smoothness of the evaluation map on $\mathcal{L}(V) \times V$ and the smoothness of the square root map from Ω to Ω (Definition 4.1(3)). For the last assertion, note that

$$\gamma(Q(a)) = (Q(a)(\varepsilon))^{1/2} = (S_a S_\varepsilon(\varepsilon))^{1/2} = (a^2)^{1/2} = a. \qquad ■$$

Lemma 4.6. *The map $Q : \Omega \to \mathcal{L}^\times(V)$ that sends a to $Q(a)$ is smooth, in particular continuous.*

Proof. Condition (3) of Definition 4.1 implies that the map $F(x,y) = x \bullet y = x\#_{-1}y = Q(x)y$ is smooth. If we consider the map $y \mapsto x \bullet y$ on Ω, then it is the restriction to the open set Ω of the map $y \mapsto Q(x)y$ on V, which has constant derivative $Q(x)$ on V, since $Q(x)$ is a bounded linear map. Thus D_2F, the partial derivative with respect to the second coordinate of the map F, is given by $D_2F(x,y) = Q(x) \in \mathcal{L}^\times(V)$ and is smooth by [6, Proposition I-3.5], and hence the map Q of the lemma, which is equal to $x \mapsto D_2F(x,y)$ for some fixed $y \in \Omega$ is also smooth. ■

We recall some basic facts from [13] and [4] about subgroups of Banach Lie groups, particularly closed subgroups.

Definition 4.7. An injective morphism $\iota : H \to G$ of Lie groups is called *initial* if for each smooth map $f : M \to G$ from a smooth manifold M to G with $\operatorname{im}(f) \subseteq \iota(H)$, the corresponding map $\iota^{-1} \circ f : M \to H$ is smooth. We then call H an *initial Lie subgroup.*

Remark 4.8. In the following, we shall mostly identify H with the subgroup $\iota(H)$ of G.

Definition 4.9. Let G be a Lie group. An *integral subgroup* is an injective morphism $\iota : H \to G$ of Lie groups for which H is connected and the differential $\mathbf{L}(\iota) : \mathbf{L}(H) \to \mathbf{L}(G)$ is injective.

Theorem 4.10. *Let G be a Banach Lie group and $H \subseteq G$ a closed subgroup. Then H carries the structure of an initial Lie subgroup, whose C^1-arc component is an integral subgroup for the closed Lie subalgebra*

$$\mathbf{L}(H) = \{X \in \mathbf{L}(G) : \exp(tX) \in H \textit{ for all } t \in \mathbb{R}\}.$$

Proof. This is essentially Theorem IV.4.16 of [13], or Theorem 8.3.25 of [4]. (The C^1-arc component of H consists of all points in H that can be connected to the identity element with a C^1-path from $[0,1]$ into H.) ■

We return to the setting of a pointed Loos symmetric cone $(\Omega, \bullet, \varepsilon)$. By Lemma 4.3 we have an isomorphic embedding $j : G(\Omega) \to \mathcal{L}^\times(V)$.

Definition 4.11. We define $\mathcal{L}^\times(\Omega)$ to consist of all $g \in \mathcal{L}^\times(V)$ satisfying

(i) $g(\Omega) = \Omega$,

(ii) $g(a \bullet b) = g(a) \bullet g(b)$ for all $a, b \in \Omega$,

(iii) $S_\varepsilon g S_\varepsilon$ preserves addition and scalar multiplication on Ω, and hence extends to a member of $\mathcal{L}^\times(\Omega)$.

Remark 4.12. (a) Each member of $\mathcal{L}^\times(\Omega)$ restricted to Ω is both a linear isomorphism and an isomorphism with regard to the symmetric cone structure. If follows that $g^{-1} \in \mathcal{L}^\times(\Omega)$ whenever g is and that the composition of two members of $\mathcal{L}^\times(\Omega)$ is again in $\mathcal{L}^\times(\Omega)$, and hence $\mathcal{L}^\times(\Omega)$ is a group, a subgroup of $\mathcal{L}^\times(V)$.

(b) $\mathcal{L}^\times(\Omega)$ is closed in $\mathcal{L}^\times(V)$.

Proof. The proof of (a) is straightforward and left to the reader. Suppose that $\{g_n\}$ is a sequence in $\mathcal{L}^\times(\Omega)$ converging to $g \in \mathcal{L}^\times(V)$. Since $\mathcal{L}^\times(V)$ is a Lie group, in particular a topological group, $g_n^{-1} \to g^{-1}$. By pointwise limit arguments both g and g^{-1} carry C into $\overline{C}$, hence by continuity $\overline{C}$ into $\overline{C}$, and thus carry the interior C of $\overline{C}$ into itself. Since g and g^{-1} are inverses they must each carry C to C. Conditions (ii) and (iii) follow from straightforward pointwise limit arguments. Thus (b) holds. ■

Theorem 4.13. *The group $\mathcal{L}^\times(\Omega)$ of $\mathcal{L}^\times(V)$ carries the structure of an initial Lie subgroup. Since the map $a \mapsto Q(a)$ from Ω into $\mathcal{L}^\times(V)$ is continuous, the C^1-arccomponent $\mathcal{L}_0^\times(\Omega)$ of the identity of $\mathcal{L}^\times(\Omega)$ is an integral subgroup for the closed Lie subalgebra*

$$\mathbf{L}(\mathcal{L}^\times(\Omega)) = \{X \in \mathcal{L}(V) : \exp(tX) \in \mathcal{L}^\times(\Omega) \text{ for all } t \in \mathbb{R}\},$$

a subalgebra of $\mathcal{L}(V)$ equipped with the commutator product. In particular, $\mathcal{L}_0^\times(\Omega)$ is a Banach-Lie group.

Proof. By Remark 4.12 $\mathcal{L}^\times(\Omega)$ is a closed subgroup of the Banach-Lie group $\mathcal{L}^\times(V)$. By Theorem 4.10 $\mathcal{L}^\times(\Omega)$ (with injective map inclusion) carries the structure of an initial Lie subgroup and its C^1-arc component $\mathcal{L}_0^\times(\Omega)$ is an integral subgroup for the closed Lie subalgebra

$$\mathbf{L}(\mathcal{L}^\times(\Omega)) = \{X \in \mathcal{L}(V) : \exp(tX) \in \mathcal{L}^\times(\Omega) \text{ for all } t \in \mathbb{R}\}.$$

By Lemma 4.6 $Q : \Omega \to Q(\Omega) \subseteq \mathcal{L}^\times(V)$ is smooth. Since $Q(\Omega) \subseteq \mathcal{L}^\times(\Omega)$, an initial Lie group, we conclude that $Q : \Omega \to \mathcal{L}^\times(\Omega)$ is also smooth. It follows that $Q(\Omega)$ lies in the C^1-arc component of the identity since each $x \in \Omega$ is connected to ε by the smooth path x^t, $0 \leq t \leq 1$. Since this C^1-arc component $\mathcal{L}_0^\times(\Omega)$ is a subgroup, we see that the displacement group $\widehat{G}(\Omega)$ is contained in $\mathcal{L}_0^\times(\Omega)$. By Theorem 4.10 $\mathcal{L}_0^\times(\Omega)$ is an integral subgroup for the Lie subalgebra

$$\mathbf{L}(\mathcal{L}^\times(\Omega)) = \{X \in \mathcal{L}(V) : \exp(tX) \in \mathcal{L}^\times(\Omega) \text{ for all } t \in \mathbb{R}\}.$$

In the setting of Banach-Lie groups, an integral subgroup for a closed Lie subalgebra is again a Banach-Lie group. ■

We next consider the map J on $\mathcal{L}_0^\times(\Omega)$ that sends g to the extension of $S_\varepsilon g S_\varepsilon$ to V. Since $S_\varepsilon \circ S_\varepsilon$ is the identity on Ω, this map is an involution. We note also that J is an isomorphism since

$$J(g_1g_2) = S_\varepsilon g_1 g_2 S_\varepsilon = S_\varepsilon g_1 S_\varepsilon S_\varepsilon g_2 S_\varepsilon = J(g_1)J(g_2).$$

We define
$$K(\Omega) = \{g \in \mathcal{L}_0^\times(\Omega) : J(g) = g\}$$

and note that $K(\Omega)$ is a closed subgroup of $\mathcal{L}_0^\times(\Omega)$.

Lemma 4.14. *For $g \in \mathcal{L}_0^\times(\Omega)$, $g \in K(\Omega)$ if and only if $g(\varepsilon) = \varepsilon$. Furthermore, $J(Q(a)) = S_\varepsilon Q(a) S_\varepsilon = Q(a^{-1}) = (Q(a))^{-1}$ for $a \in \Omega$.*

Proof. If $g \in \mathcal{L}_0^\times(\Omega)$, then $S_\varepsilon g S_\varepsilon(\varepsilon) = g(\varepsilon)^{-1}$; we conclude that g is not fixed by conjugation by S_ε if $g(\varepsilon) \neq \varepsilon$. Conversely if $g(\varepsilon) = \varepsilon$, then

$$S_\varepsilon g S_\varepsilon(x) = S_\varepsilon(g(\varepsilon) \bullet g(x)) = S_\varepsilon(g(x)^{-1}) = g(x),$$

so g is fixed by conjugation by S_ε.
We note further that $S_\varepsilon Q(a) S_\varepsilon = S_\varepsilon S_a S_\varepsilon S_\varepsilon = (S_a S_\varepsilon)^{-1} = Q(a)^{-1}$. ■

Lemma 4.15. *Every $g \in \mathcal{L}_0^\times(\Omega)$, $g = Q(a^{1/2})(Q(a^{-1/2})g)$, where $a = g(\varepsilon)$, is unique representation of g of the form $g = Q(x)k$ for some $x \in \Omega$, $k \in K(\Omega)$.*

Proof. Let $g(\varepsilon) = a$. Then $Q(a^{1/2})(\varepsilon) = a$, so $k := Q(a^{-1/2})g$ fixes ε and is hence in $K(\Omega)$. Thus we have the factorization $g = Q(a^{1/2})k \in Q(\Omega)K(\Omega)$. If $Q(x)k = Q(y)k'$, then applying both sides to ε yields $x^2 = Q(x)\varepsilon = Q(y)\varepsilon = y^2$, and thus $x = x^2\#\varepsilon = y^2\#\varepsilon = y$. It follows that $k = k'$. ∎

Lemma 4.16. *The map $J : \mathcal{L}_0^\times(\Omega) \to \mathcal{L}_0^\times(\Omega)$ is a unique smooth involution that fixes the points of $K(\Omega)$ and inverts the points of $Q(\Omega)$.*

Proof. We note first for $g = Q(a^{1/2})(Q(a^{-1/2})g)$,

$$J(g) = J(Q(a^{1/2})Q(a^{-1/2})g) = J(Q(a^{1/2}))J(Q(a^{-1/2})g) = Q(a^{-1/2})Q(a^{-1/2})g.$$

We consider the composition

$$\begin{aligned} g &\mapsto \big(Q(g(\varepsilon)^{1/2}), Q(g(\varepsilon)^{-1/2})g\big) \mapsto \big(Q(g(\varepsilon)^{-1/2}), Q(g(\varepsilon)^{-1/2})g\big) \\ &\mapsto Q(g(\varepsilon)^{-1/2})Q(g(\varepsilon)^{-1/2})g = J(g). \end{aligned}$$

In this way we can write J as a composition of smooth maps, the smoothness having been established in the preceding results, and hence obtain that J is smooth. ∎

The smooth involution J induces an involution $\mathbf{L}J$ on the Lie algebra $\mathfrak{g} = \mathbf{L}(\mathcal{L}_0^\times(\Omega))$ that is a Lie algebra isomorphism. By standard basic theory of Lie algebras, we can write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$, where $\mathfrak{k}$ is the $+1$-eigenspace of J and $\mathfrak{q}$ is the -1-eigenspace, both closed subspaces of $\mathfrak{g}$. It also follows that

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{q}] \subseteq \mathfrak{q}, \qquad [\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{k}.$$

From the definition of $K(\Omega)$ and elementary Lie theory it follows that $\mathfrak{k} = \mathbf{L}(K(\Omega))$, the Lie algebra of the closed subgroup $K(\Omega)$.

It is shown in [7] that a pointed Loos symmetric cone $(\Omega, \bullet, \varepsilon)$ considered as a smooth manifold has induced from its symmetric structure an exponential map $\exp_x : T_x\Omega \to \Omega$ that is a C^∞-diffeomorphism for all $x \in \Omega$. We focus on $\exp_\varepsilon : T_\varepsilon\Omega \to \Omega$. For the Banach-Lie group $\mathcal{L}^\times(V)$, we identify the tangent space at the identity e as $\mathcal{L}(V)$ and have the exponential mapping given by the usual exponential power series. The map $Q : \Omega \to \mathcal{L}_0^\times(\Omega)$ is a smooth monomorphism of Loos symmetric spaces, and one can easily see from the appropriate commutative diagram that the exponential function $\exp : \mathcal{L}(V) \to \mathcal{L}^\times(V)$ carries $\mathbf{L}Q(T_\varepsilon\Omega)$ diffeomorphically to $Q(\Omega)$. Since $J(Q(x)) = Q(x)^{-1}$ for $x \in \Omega$, it follows that $\mathbf{L}J : \mathbf{L}(\mathcal{L}_0^\times(\Omega)) \to \mathbf{L}(\mathcal{L}_0^\times(\Omega))$ restricted to $\mathbf{L}Q(T_\varepsilon\Omega)$ is multiplication by -1, and thus $\mathbf{L}Q(T_\varepsilon\Omega) \subseteq \mathfrak{q}$.

Since $\mathcal{L}_0^\times(\Omega)$ is an integral Lie subgroup for $\mathfrak{g}$, the exponential map $\exp : \mathfrak{g} \to \mathcal{L}_0^\times(\Omega)$ is locally a diffeomorphism from some neighborhood U of $0 \in \mathfrak{g}$ to a neighborhood V of e in $\mathcal{L}_0^\times(\Omega)$ with derivative $d\exp(0)$ being the identity map. From the Baker-Campbell-Hausdorff formula, it follows that the map from $\mathfrak{g} \times \mathfrak{g} \to \mathcal{L}_0^\times(\Omega)$ sending (u, v) to $\exp(u)\exp(v)$ has derivative $(u, v) \mapsto u + v$ at $(0, 0)$. We consider the composition Λ from $\mathfrak{g}$ to $\mathcal{L}_0^\times(\Omega)$

$$w \mapsto (u = \pi_{\mathfrak{q}}(w), v = \pi_{\mathfrak{k}}(w)) \mapsto (\exp u)(\exp v).$$

From the Chain Rule it follows that the derivative of Λ at 0 is given by $d\Lambda(0)(w) = \pi_{\mathfrak{q}}(w) + \pi_{\mathfrak{k}}(w) = w$, and thus by the Inverse Function Theorem the map Λ is a diffeomorphism on some small neighborhood of 0. If there exists v such that $v \in \mathfrak{q}$ but $v \notin \mathbf{L}Q(T_\varepsilon\Omega)$, then for v small enough we may write $\exp v$ both as $\Lambda(v)$ and from Lemma 4.15 as

$$\exp v = Q(\exp(v)^{1/2})(Q(\exp(v)^{-1/2})\exp v).$$

We note $Q(\exp(v)^{1/2}) \in Q(\exp_\varepsilon(T_\varepsilon\Omega)) = \exp(\mathbf{L}Q(T_\varepsilon(\Omega))) \subseteq \exp(\mathfrak{q})$. Also for small v, $\log\big(Q(\exp(v)^{-1/2})\exp v\big)$ exists and is in $\mathfrak{k}$. Thus

$$w = \log\big(Q(\exp(v)^{1/2})\big) + \log\big(Q(\exp(v)^{-1/2})\exp v)\big)$$

is another element with $\Lambda(w) = \Lambda(v)$, which contradicts the fact Λ is a diffeomorphism on the neighborhood under consideration. Putting this information all together, we have the following result.

Theorem 4.17. *Let $(\Omega, \bullet, \varepsilon)$ be a pointed Loos symmetric cone, and let $\mathcal{L}^\times(V)$ be the Banach-Lie group of invertible operators in the Banach algebra $\mathcal{L}(V)$ of bounded linear operators on $V = \Omega - \Omega$. Then there exists an integral subgroup $\mathcal{L}_0^\times(\Omega)$ (see Theorem 4.13) and a smooth involution J on $\mathcal{L}_0^\times(\Omega)$ inducing a direct sum decomposition $\mathfrak{g} = \mathfrak{q} + \mathfrak{k}$ on the Lie algebra $\mathfrak{g}$ of $\mathcal{L}_0^\times(\Omega)$ such that $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{q}] \subseteq \mathfrak{q}$, and $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{k}$. The following conclusions also hold:*

(1) *The exponential function* exp *restricted to $\mathfrak{q}$ is a diffeomorphism onto $Q(\Omega)$.*

(2) *The group $K(\Omega) = \{g \in \mathcal{L}_0^\times(\Omega) : g(\varepsilon) = \varepsilon\}$ is the fixed point set of J and has Lie algebra $\mathfrak{k}$.*

(3) *Every $g \in \mathcal{L}_0^\times(\Omega)$ has a unique factorization $g = qh \in Q(\Omega)K(\Omega)$ with $q = Q(g(\varepsilon)^{1/2})$.*

(4) *The displacement group $\widehat{G}(\Omega)$ is contained in $\mathcal{L}_0^\times(\Omega)$.*

Remark 4.18. As a special case of the discussion immediately preceding Remark 2.1, we have that the quadratic representation $Q : \Omega \to \mathcal{L}_0^\times(\Omega)$ corestricts to an isomorphism from $(\Omega, \bullet, \varepsilon)$ to $(Q(\Omega), \bullet, e)$, where where the $\bullet$-operation on the twisted subgroup $Q(\Omega)$ is $g \bullet h = gh^{-1}g$ and e is the identity of the group $\mathcal{L}_0^\times(\Omega)$. By Lemma 4.6 Q is smooth.

5. Examples

The basic example of a Loos symmetric cone arises in the context of JB-algebras. Recall that a Jordan algebra is a vector space with a commutative multiplication xy such that $x(x^2y) = x^2(xy)$ holds for x, y. A JB-algebra V is a real Jordan algebra with unit e endowed with a complete norm $||\cdot||$ such that

$$||zw|| \;\le\; ||z||\,||w||, \;\; ||z^2|| = ||z||^2, \;\; ||z||^2 \le ||z^2 + w^2||.$$

For $x \in V$ we write $L(x)(y) = xy$, the multiplication operator. We consider the set

$$\Omega := \{x \in V : \mathrm{Spec}(L(x)) \subset (0, \infty)\}.$$

Then Ω is an open convex cone of V (see Section 21 of [17], particularly Proposition 21.19 and Corollary 21.22, also Section 3.3 of [5]) and is realized as

$$\Omega = \exp(V) := \{\exp(x) : x \in V\},$$

and the map $\exp : V \to \Omega$ is a diffeormorphism. Thus C has the alternative characterization of being the exponential image of V.

The following appears as Corollary 9.1 in [7].

Proposition 5.1. *Let V be a JB-algebra with identity and let Ω be the associated symmetric cone. Then Ω is a Loos symmetric cone with*

(1) $Q(x) = P(x)$, *where* $P(x) = 2L(x)^2 - L(x^2)$ *for* $L(x)$ *left translation.*

(2) $x \bullet y = P(x)y^{-1}$, *where* y^{-1} *is the multiplicative inverse of* y *in* V.

(3) $x^{-1} = e \bullet x = \exp(-a)$, *where* $a = \log x$.

(4) $x\#y = P(x^{1/2})(P(x^{-1/2})y)^{1/2}$.

The case of unital C^*-algebras may be viewed as a special case of the preceding by passing to the associated Jordan algebra $a \cdot b = (1/2)(ab + ba)$.

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