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Hadamard Semigroups of Off-Diagonal Constant Matrices

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Abstract. The convex cone of positive semidefinite matrices of fixed size forms a commutative topological semigroup under the Hadamard product. In this paper we consider the closed subsemigroup of off-diagonal constant matrices, matrices having the same value in the off-diagonal positions, and its compact and convex subsemigroup of matrices with diagonal entries in the unit interval. Several results on these topological semigroups are presented: the group of units, (Löwner) ordered semigroup structures, one-parameter semigroups. An application of Hadamard powers obtained by FitzGerald and Horn and related open problems on Euclidean Jordan algebras are discussed.

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1. Introduction

The set of $m \times m$ real matrices is denoted by $\mathbb{M}_m(\mathbb{R})$. The symbol I_m stands for the $m \times m$ identity matrix. Let $\mathbb{S}_m$ be the Euclidean space of $m \times m$ real symmetric matrices equipped with the trace inner product $\langle X, Y\rangle = \mathrm{tr}(XY)$, and let $\mathbb{P}_m$ be the closed convex cone of $m \times m$ positive semidefinite matrices. For $X, Y \in \mathbb{S}_m$, $X \leq Y$ means $Y - X \in \mathbb{P}_m$, and $X < Y$ means $Y - X \in \mathbb{P}_m^\circ$, where $\mathbb{P}_m^\circ$ denotes the open convex cone of $m \times m$ positive definite matrices.

The Hadamard product (or the entrywise product, or Schur product) of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the matrix $A \circ B = [a_{ij}b_{ij}]$. Then the space $\mathbb{S}_m$ becomes a commutative semigroup under the Hadamard product, called the *Hadamard semigroup* for simplicity. The matrix J_m with all entries equal to 1 is the identity for the semigroup $(\mathbb{S}_m, \circ)$.

The most interesting theorem about Hadamard products (also known as the Schur product theorem) was proved by Issai Schur [18]. It says that if A and B are positive semidefnite (resp. definite), then so is $A \circ B$. That is, the closed convex cone $\mathbb{P}_m$ is a subsemigroup of the Hadamard semigroup $\mathbb{S}_m$.

In this paper, we consider a Hadamard subsemigroup $\mathcal{S}_m$ consisting of $m \times m$ positive semidefinite matrices of the form

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$$\mathbf{M_a}(x) := \begin{bmatrix} a_1 & x & \cdots & x \\ x & a_2 & \cdots & x \\ \vdots & \vdots & \vdots & \vdots \\ x & x & \cdots & a_m \end{bmatrix} \geq 0$$

where $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$ and x varies over real numbers, respectively. It is indeed closed under the Hadamard product by the Schur product theorem and

$$\mathbf{M_a}(x) \circ \mathbf{M_b}(y) = \mathbf{M_{ab}}(xy),$$

where $\mathbf{ab} = (a_1b_1, \ldots, a_mb_m)$ for $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ in $\mathbb{R}^m$. The matrix $\mathbf{M_a}(x)$, called an off-diagonal constant matrix, appears particularly in Grover diffusion matrices in quantum computation and compound symmetry correlation matrices. We note that for given positive diagonal entries $a_1, \ldots, a_m$, it is non-trivial to determine x for which $\mathbf{M_a}(x)$ is positive semidefinite [8]. Indeed, it is shown that $\mathbf{M_a}(x) \geq 0$ if and only if $x \in [\theta_m^-(\mathbf{a}), \theta_m^+(\mathbf{a})]$, where $\theta_m^{\pm}(\mathbf{a})$ are the smallest positive and the unique negative roots of the polynomial $\det \mathbf{M_a}(x)$ in variable x, respectively. The main concern of this paper is to investigate some topological semigroup structures on $\mathcal{S}_m$ and to find their relations with the theory of matrix analysis. We then obtain some new results on the root maps $\theta_m^{\pm}$ based on these topological semigroup structures.

In what follows $\mathbb{R}_+ = [0, \infty), \mathbb{R}_{++} = (0, \infty)$ and $m \geq 2$.

2. A decomposition of $\mathcal{S}_m$

The set of $\mathbf{M_a}(x)$ varying over $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$ and $x \in \mathbb{R}$ forms a subspace of $\mathbb{S}_m$ and is linear isomorphic to $\mathbb{R}^{m+1}$ via $\mathbf{M_a}(x) \mapsto (a_1, \ldots, a_m, x)$;

$$t\mathbf{M_a}(x) + s\mathbf{M_b}(y) = \mathbf{M}_{t\mathbf{a}+s\mathbf{b}}(tx + sy),$$

where $t\mathbf{a} = (ta_1, \ldots, ta_m)$. The *Hadamard subsemigroup*

$$\mathcal{S}_m = \{\mathbf{M_a}(x) \geq 0 : \mathbf{a} \in \mathbb{R}^m, x \in \mathbb{R}\}$$

is a closed convex subcone of $\mathbb{S}_m$ with the *relative interior*

$$\mathcal{S}_m^\circ := \{\mathbf{M_a}(x) > 0 : \mathbf{a} \in \mathbb{R}^m, x \in \mathbb{R}\},$$

which is a subsemigroup of $\mathcal{S}_m$ by the Schur product theorem.

Remark 2.1. Since every positive semidefinite matrix has nonnegative diagonal entries, $\mathbf{M_a}(x) \geq 0$ (resp. $\mathbf{M_a}(x) > 0$) implies that $\mathbf{a} \in \mathbb{R}_+^m$ (resp. $\mathbf{a} \in \mathbb{R}_{++}^m$). Hence

$$\mathcal{S}_m = \left\{\mathbf{M_a}(x) \geq 0 : \mathbf{a} \in \mathbb{R}_+^m, x \in \mathbb{R}\right\}, \quad \text{and} \quad \mathcal{S}_m^\circ = \left\{\mathbf{M_a}(x) > 0 : \mathbf{a} \in \mathbb{R}_{++}^m, x \in \mathbb{R}\right\}.$$

The Hadamard semigroup $\mathcal{S}_m$ is a commutative semigroup with the identity J_m. Indeed, J_m is positive semidefinite from

$$J_m = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T \geq 0.$$

We denote the *boundary* of $\mathcal{S}_m$ by $\mathcal{S}_m^\partial := \mathcal{S}_m \setminus \mathcal{S}_m^\circ$.

It consists of all positive semidefinite matrices $\mathbf{M}_\mathbf{a}(x)$ with determinant zero. For example, $J_m \in \mathcal{S}_m^\partial$. Let $\mathbb{D}_m \subset \mathbb{S}_m$ be the subspace of all diagonal matrices and let $\mathcal{D}_m := \mathbb{D}_m \cap \mathcal{S}_m$, the set of all diagonal matrices with nonnegative entries. Define $\mathcal{D}_m^\circ = \mathbb{D}_m \cap \mathcal{S}_m^\circ, \mathcal{D}_m^\partial = \mathbb{D}_m \cap \mathcal{S}_m^\partial$, and

$$\mathcal{T}_m^\partial := \mathcal{S}_m^\partial \setminus \mathcal{D}_m^\partial.$$

Then $\mathcal{S}_m = \mathcal{S}_m^\circ \cup \mathcal{T}_m^\partial \cup \mathcal{D}_m^\partial$.

Example 2.2. $(m = 2)$ For two real numbers a and b, the 2×2 real symmetric matrix $\begin{bmatrix} a & x \\ x & b \end{bmatrix}$ is positive semidefinite if and only if a and b are nonnegative and $-\sqrt{ab} \le x \le \sqrt{ab}$. Moreover, it is positive definite if and only if $a, b > 0$ and $-\sqrt{ab} < x < \sqrt{ab}$. Hence

$$\begin{aligned}
\mathcal{S}_2 &= \left\{\mathbf{M}_{(a,b)}(x) : a, b \ge 0, -\sqrt{ab} \le x \le \sqrt{ab}\right\}, \\
\mathcal{S}_2^\circ &= \left\{\mathbf{M}_{(a,b)}(x) : a, b > 0, -\sqrt{ab} < x < \sqrt{ab}\right\}, \\
\mathcal{D}_2^\partial &= \left\{\mathbf{M}_{(a,b)}(0) : a, b \ge 0, ab = 0\right\}, \\
\mathcal{T}_2^\partial &= \left\{\mathbf{M}_{(a,b)}(\pm\sqrt{ab}) : a, b > 0\right\}.
\end{aligned}$$

Note that $\mathcal{T}_2^\partial$ is a subsemigroup of $\mathcal{S}_2$.

3. A description of $\mathcal{S}_m$

In this section we will prove the following.

Theorem 3.1. *There exist continuous maps* $\theta_m^+ : \mathbb{R}_+^m \to [0, \infty)$ *and* $\theta_m^- : \mathbb{R}_+^m \to (-\infty, 0]$ *such that for* $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}_+^m$,

$$\theta_m^\pm(\mathbf{a}) = 0 \iff \prod_{j=1}^m a_j = 0 \tag{1}$$

(*vanishes at the boundary*) *and*

$$\begin{aligned}
\mathcal{S}_m &= \left\{\mathbf{M}_\mathbf{a}(x) : \mathbf{a} \in \mathbb{R}_+^m, \theta_m^-(\mathbf{a}) \le x \le \theta_m^+(\mathbf{a})\right\}, &(2)\\
\mathcal{S}_m^\circ &= \left\{\mathbf{M}_\mathbf{a}(x) : \mathbf{a} \in \mathbb{R}_{++}^m, \theta_m^-(\mathbf{a}) < x < \theta_m^+(\mathbf{a})\right\}, &(3)\\
\mathcal{S}_m^\partial &= \left\{\mathbf{M}_\mathbf{a}(\theta_m^\pm(\mathbf{a})) : \mathbf{a} \in \mathbb{R}_+^m\right\}, &(4)\\
\mathcal{T}_m^\partial &= \left\{\mathbf{M}_\mathbf{a}(\theta_m^\pm(\mathbf{a})) : \mathbf{a} \in \mathbb{R}_{++}^m\right\}, &(5)\\
\mathcal{D}_m^\partial &= \left\{\mathbf{M}_\mathbf{a}(\theta_m^\pm(\mathbf{a})) : \mathbf{a} \in \mathbb{R}_+^m, \prod_{j=1}^m a_j = 0\right\}. &(6)
\end{aligned}$$

To construct $\theta_m^\pm$, we let for $\mathbf{a} \in \mathbb{R}_+^m$,

$$\mathbf{S}_\mathbf{a} := \{x \in \mathbb{R} : \mathbf{M}_\mathbf{a}(x) \ge 0\}, \quad \text{and} \quad \mathbf{S}_\mathbf{a}^\circ := \{x \in \mathbb{R} : \mathbf{M}_\mathbf{a}(x) > 0\}.$$

Then

$$\mathcal{S}_m = \left\{\mathbf{M}_\mathbf{a}(x) : \mathbf{a} \in \mathbb{R}_+^m, x \in \mathbf{S}_\mathbf{a}\right\}, \quad \text{and} \quad \mathcal{S}_m^\circ = \left\{\mathbf{M}_\mathbf{a}(x) : \mathbf{a} \in \mathbb{R}_{++}^m, x \in \mathbf{S}_\mathbf{a}^\circ\right\}.$$

We note that $\mathbf{S_a}$ is a closed subset of $\mathbb{R}$ containing 0, and $\mathbf{S_a^\circ}$ is an open set around 0 if $\mathbf{a} \in \mathbb{R}_{++}$.

Example 3.2 $(m = 2)$**.** For $a, b > 0$,

$$\mathbf{S}_{(a,b)} = \left[-\sqrt{ab}, \sqrt{ab}\right] \quad \text{and} \quad \mathbf{S}^\circ_{(a,b)} = \left(-\sqrt{ab}, \sqrt{ab}\right).$$

We first describe the set $\mathbf{S_a}$ when $\prod_{j=1}^m a_j = 0$. If $a, b \geq 0$ and $ab = 0$, then

$$\begin{bmatrix} a & x \\ x & b \end{bmatrix} \geq 0 \Longleftrightarrow x = 0.$$

Since every principal submatrix of a positive semidefinite matrix is again positive semidefinite [13], we have that for $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m_+$,

$$\mathbf{S_a} = \{0\}, \qquad \text{if } a_j = 0 \text{ for some } j = 1, \ldots, m. \tag{7}$$

The reverse implication holds true. That is, if $a_j > 0$ for all $j = 1, 2, \ldots, m$, then $\mathbf{S_a} \neq \{0\}$, that follows from Theorem 3.4 below.

Proposition 3.3. *For* $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m_+$,

$$\mathbf{S_a} = \{0\} \Longleftrightarrow \prod_{j=1}^m a_j = 0. \tag{8}$$

For $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$, let

$$D_\mathbf{a}(x) := \det \begin{bmatrix} a_1 & x & \cdots & x \\ x & a_2 & \cdots & x \\ \vdots & \vdots & \vdots & \vdots \\ x & x & \cdots & a_m \end{bmatrix}.$$

It is shown in [8] that

$$D_\mathbf{a}(x) = \sum_{k=0}^m (-1)^{m-k-1} \, (m-k-1) \, S_k(\mathbf{a}) x^{m-k}, \tag{9}$$

where $S_k(\mathbf{a})$ denotes the k*th elementary symmetric sum* of $\mathbf{a} = (a_1, \ldots, a_m)$:

$$S_0(\mathbf{a}) = 1, \qquad S_k(\mathbf{a}) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq m} a_{j_1} a_{j_2} \cdots a_{j_k}, \qquad 1 \leq k \leq m.$$

If $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m_{++}$, then the polynomial $D_\mathbf{a}(x)$ in variable x has only real roots and has a unique negative root, say $\theta^-_m(\mathbf{a})$. Moreover,

Theorem 3.4. ([8]) *For every* $\mathbf{a} \in \mathbb{R}^m_{++}$,

$$\mathbf{S_a} \;=\; [\theta^-_m(\mathbf{a}), \theta^+_m(\mathbf{a})], \qquad \mathbf{S^\circ_a} = (\theta^-_m(\mathbf{a}), \theta^+_m(\mathbf{a})), \tag{10}$$

where $\theta^+_m(\mathbf{a})$ *is the smallest positive root of the polynomial* $D_\mathbf{a}(x)$.

For $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m_{++}$, let $\xi_1 = \theta^-_m(\mathbf{a}), \xi_2 = \theta^+_m(\mathbf{a}), \ldots, \xi_m$ be the real roots of $D_\mathbf{a}(x)$ in nondecreasing order $\xi_1 < 0 < \xi_2 \leq \xi_2 \leq \cdots \leq \xi_m$, and let σ be a permutation on m-letters such that $a_{\sigma(1)} \leq a_{\sigma(2)} \leq \cdots \leq a_{\sigma(m)}$.

It is further shown in [8] that for $j = 2, \dots, m$,

(S1) $\xi_j \in [a_{\sigma(j-1)}, a_{\sigma(j)}]$, and if $a_{\sigma(j)} < a_{\sigma(j+1)}$, then $\xi_j \in (a_{\sigma(j-1)}, a_{\sigma(j)})$;

(S2) if $a_{\sigma(j-1)} = a_{\sigma(j)}$, then $\xi_j = a_{\sigma(j)}$ and the multiplicity of ξ_j is one less than that of $a_{\sigma(j-1)}$ in the m-tuple $(a_1, \dots, a_m)$.

Next, suppose that $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}^m_+$. Letting

$$\mathbf{a}_\varepsilon = (a_1 + \varepsilon, \dots, a_m + \varepsilon) \in \mathbb{R}^m_{++}, \qquad \varepsilon > 0$$

we have real roots of $D_{\mathbf{a}_\varepsilon}(x) = 0$;

$$\theta^-_m(\mathbf{a}_\varepsilon) = \xi_1(\varepsilon) < 0 < \theta^+_m(\mathbf{a}_\varepsilon) = \xi_2(\varepsilon) \le \xi_2(\varepsilon) \le \cdots \le \xi_m(\varepsilon).$$

From the continuity of the roots of polynomials, the equation $D_{\mathbf{a}}(x) = 0$ has m real roots $\xi_1, \dots, \xi_m$ in nondecreasing order;

$$\xi_1 := \lim_{\varepsilon \to 0^+} \xi_1(\varepsilon) \le 0 \ \text{ and } \ \xi_j := \lim_{\varepsilon \to 0^+} \xi_j(\varepsilon) \ge 0 \quad \text{for} \quad j = 2, \dots, m.$$

Setting
$$\theta^-_m(\mathbf{a}) := \xi_1 = \lim_{\varepsilon \to 0^+} \xi_1(\varepsilon) = \lim_{\varepsilon \to 0^+} \theta^-_m(\mathbf{a}_\varepsilon) \tag{11}$$
and
$$\theta^+_m(\mathbf{a}) := \xi_2 = \lim_{\varepsilon \to 0^+} \xi_2(\varepsilon) = \lim_{\varepsilon \to 0^+} \theta^+_m(\mathbf{a}_\varepsilon), \tag{12}$$

the maps $\theta^\pm_m$ are defined on $\mathbb{R}^m_+$;

$$\theta^+_m : \mathbb{R}^m_+ \to [0, \infty), \qquad \theta^-_m : \mathbb{R}^m_+ \to (-\infty, 0]. \tag{13}$$

We will show that for $\mathbf{a} \in \mathbb{R}^m_+$,

$$\theta^\pm_m(\mathbf{a}) = 0 \iff \prod_{j=1}^m a_j = 0. \tag{14}$$

Since $\mathbf{M}_{\mathbf{a}_\varepsilon}(\theta^\pm_m(\mathbf{a}_\varepsilon)) \ge 0$ for all $\varepsilon > 0$, their limits are positive semidefinite as $\varepsilon \to 0$:

$$\mathbf{M}_{\mathbf{a}}(\theta^\pm_m(\mathbf{a})) \ge 0,$$

that is, $\theta^\pm_m(\mathbf{a}) \in \mathbf{S}_{\mathbf{a}}$. The assertion then follows from (8) and Theorem 3.4.

It is shown [8] that the maps $\theta^\pm_m$ on $\mathbb{R}^m_{++}$ are continuous. Then by the continuity of roots of polynomials or by (11) and (12), their extensions $\theta^\pm_m$ on $\mathbb{R}^m_+$ are continuous. The properties (2)-(6) hold true by Theorem 3.4 and (8). This completes the proof of Theorem 3.1.

To provide some basic properties of $\theta^\pm_m$, we need the following further notions.

Definition 3.5. For $\mathbf{a} = (a_1, \dots, a_m), \mathbf{b} = (b_1, \dots, b_m) \in \mathbb{R}^m$ and a permutation σ on m-letters,

$$\mathbf{a} \le \mathbf{b} \iff a_j \le b_j, \forall j = 1, \dots, m,$$

$$\mathbf{ab} = (a_1 b_1, \dots, a_m b_m), \qquad \alpha \mathbf{a} = (\alpha a_1, \dots, \alpha a_m),$$

$$\mathbf{a}^{-1} = (a_1^{-1}, \dots, a_m^{-1}), \qquad \mathbf{a}_{\neq m} = (a_1, \dots, a_{m-1}), \qquad \mathbf{a}_\sigma = (a_{\sigma(1)}, \dots, a_{\sigma(m)}).$$

The following properties of $\theta_m^\pm$ on $\mathbb{R}^m_{++}$ appear in [8]. We have similar properties on $\mathbb{R}^m_+$ by continuity. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m_{++}, \alpha > 0$, and σ be a permutation on m-letters,

(P1) $\theta_m^\pm(\alpha\mathbf{a}) = \alpha\theta_m^\pm(\mathbf{a})$.

(P2) $\theta_m^\pm(a_1, \ldots, a_m) = \theta_m^\pm(a_{\sigma(1)}, \ldots, a_{\sigma(m)})$.

(P3) $-\theta_m^- \le \theta_m^+$ and $\pm\theta_m^\pm$ are concave and monotonic.

(P4) $\theta_m^+(\mathbf{a})\theta_m^+(\mathbf{b}) \le \theta_m^+(\mathbf{ab})$ and $\theta_m^-(\mathbf{a})\theta_m^-(\mathbf{b}) \le -\theta_m^-(\mathbf{ab})$. In particular,

$$\theta_m^+(\mathbf{a}) \le \theta_m^+(\mathbf{a}^{-1})^{-1}, \quad \theta_m^-(\mathbf{a}^{-1})^{-1} \le \theta_m^-(\mathbf{a}).$$

(P5) For $m \ge 3$, $\theta_m^+(\mathbf{a}) \le (a_1 \cdots a_m)^{\frac{1}{m}}$. Equality holds if and only if $a_1 = \cdots = a_m$.

(P6) $\theta_m^+(a, a, a_1, \ldots, a_{m-2}) = a, \qquad$ if $a \le a_j, j = 1, \ldots, m-2$.

(P7) $\theta_m^\pm(a, b, a_1, \ldots, a_{m-2}) \to \pm\sqrt{ab}$ as $a_j \to \infty$ for $j = 1, \ldots, m-2$.

(P8) $\theta_m^-(a, \ldots, a) = -\frac{a}{m-1}$.

(P9) Let σ be a permutation on m-letters such that $a_{\sigma(1)} \le a_{\sigma(2)} \le \cdots \le a_{\sigma(m)}$. Then

(i) if $a_{\sigma(1)} < a_{\sigma(2)}$, then $\theta_{m-1}^-(\mathbf{a}_{\neq m}) < \theta_m^-(\mathbf{a}) < 0 < \theta_m^+(\mathbf{a}) < \theta_{m-1}^+(\mathbf{a}_{\neq m})$.

(ii) if $a_{\sigma(1)} = a_{\sigma(2)}$, then $\theta_{m-1}^-(\mathbf{a}_{\neq m}) < \theta_m^-(\mathbf{a}) < 0 < \theta_m^+(\mathbf{a}) = \theta_{m-1}^+(\mathbf{a}_{\neq m}) = a_{\sigma(1)}$.

(P10) For $0 < a \le b \le c$,

$$\theta_3^-(a, a, c) = \frac{c - \sqrt{c^2 + 8ac}}{4}, \qquad \theta_3^-(a, b, b) = \frac{a - \sqrt{a^2 + 8ab}}{4},$$

and $\theta_3^+(a, b, b) = \frac{a + \sqrt{a^2 + 8ab}}{4}$.

By Theorem 3.1, (14), and (P6),

Corollary 3.6. *The set $E(\mathcal{S}_m)$ of all idempotents of the Hadamard semigroup $\mathcal{S}_m$ is*

$$E(\mathcal{S}_m) = \{\mathbf{M}_\mathbf{a}(0) : \mathbf{a} \in \{0, 1\}^m\} \cup \{J_m\}. \tag{15}$$

Furthermore, $\mathcal{T}_m^\partial$ consists of all singular matrices in $\mathcal{S}_m$ with positive diagonal entries and forms a cone (closed under positive scalar multiplication).

We observe that $E(\mathcal{S}_m)$ is a subsemigroup of $\mathcal{S}_m$, and that every idempotent has an open neighborhood containing no other idempotents. In particular, the identity matrix I_m is the unique idempotent in $\mathcal{S}_m^\circ$ and J_m is the only idempotent in $\mathcal{T}_m^\partial$.

Remark 3.7. For $\mathbf{a} \in \mathbb{R}^m_{++}$, the curve $\mathbb{R} \to \mathbb{S}_m, x \mapsto \mathbf{M}_\mathbf{a}(x)$, lies in $\mathcal{S}_m^\circ \subset \mathbb{P}_m^\circ$ for $\theta_m^-(\mathbf{a}) < x < \theta_m^+(\mathbf{a})$, passes through its boundary $\mathcal{S}_m^\partial$ at $x = \theta_m^\pm(\mathbf{a})$, and leaves off for other x. If $a_j = 0$ for some j, then it intersects $\mathcal{S}_m$ only at $x = 0$.

Remark 3.8. (Barrier functions) The map $\mathbf{a} \mapsto \theta_m^+(\mathbf{a})^{-1}$ is a convex barrier function on the positive octant. This follows from (14) and (P3). Here a barrier function is a continuous function whose value on a point increases to infinity as the point approaches the boundary. A well-known barrier function is $\mathbf{a} \mapsto -\log\prod_{j=1}^m a_j$, which is strictly convex. Our barrier function is positive valued.

4. Ordered and compact subsemigroups

The Löwner order $A \leq B$ on the space $\mathbb{S}_m$ is a closed partial order and hence on the Hadamard semigroup $\mathcal{S}_m$. From the results obtained in Section 3, we have an explicit formula for the Löwner order between off-diagonal constant matrices.

Corollary 4.1. (Löwner ordering) *For* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$,

$$\mathbf{M}_{\mathbf{a}}(x) \leq \mathbf{M}_{\mathbf{b}}(y) \Longleftrightarrow \mathbf{a} \leq \mathbf{b} \text{ and } y - x \in [\theta_m^-(\mathbf{b}-\mathbf{a}), \theta_m^+(\mathbf{b}-\mathbf{a})]. \tag{16}$$

If $a_j = b_j$ *for some* j, *then* $\mathbf{M}_{\mathbf{a}}(x) \leq \mathbf{M}_{\mathbf{b}}(y) \Longleftrightarrow \mathbf{a} \leq \mathbf{b}$ and $x = y$.

Proof. By (2), $\mathbf{M}_{\mathbf{a}}(x) \leq \mathbf{M}_{\mathbf{b}}(y)$ if and only if $\mathbf{M}_{\mathbf{b}-\mathbf{a}}(y-x) = \mathbf{M}_{\mathbf{b}}(y) - \mathbf{M}_{\mathbf{a}}(x) \geq 0$ if and only if $\mathbf{b} - \mathbf{a} \geq 0$ and $y - x \in [\theta_m^-(\mathbf{b}-\mathbf{a}), \theta_m^+(\mathbf{b}-\mathbf{a})]$. The remaining part of proof follows from (14). ■

An *ordered* semigroup is a semigroup S together with a partial order $\leq$ (it is also a closed subset of $S \times S$ in the case of topological semigroup [7]) that is compatible with the semigroup operation, meaning that $x \leq y$ implies $zx \leq zy$ and $xz \leq yz$ for all x, y, z in S. By the Schur product theorem, the Hadamard semigroup $\mathbb{P}_m$ equipped with the Löwner order $A \leq B$ is an ordered topological semigroup. Indeed, if $A \leq B$ and $C \geq 0$, then

$$0 \leq (B - A) \circ C = B \circ C - A \circ C. \tag{17}$$

In particular,

Proposition 4.2. $(\mathcal{S}_m, \leq)$ *is an ordered topological semigroup.*

Let $A = \mathbf{M}_{\mathbf{a}}(x), B = \mathbf{M}_{\mathbf{b}}(y), C = \mathbf{M}_{\mathbf{c}}(z)$ be in $\mathcal{S}_m$. Suppose that $A \leq B$. By (16),

$$\mathbf{a} \leq \mathbf{b} \qquad \text{and} \qquad y - x \in [\theta_m^-(\mathbf{b}-\mathbf{a}), \theta_m^+(\mathbf{b}-\mathbf{a})].$$

By (17), $A \circ C \leq B \circ C$, which is equivalent to $\mathbf{M}_{\mathbf{ac}}(xz) \leq \mathbf{M}_{\mathbf{bc}}(yz)$.
By (16), this is equivalent to

$$\mathbf{ac} \leq \mathbf{bc} \qquad \text{and} \qquad z(y - x) \in [\theta_m^-(\mathbf{c}(\mathbf{b}-\mathbf{a})), \theta_m^+(\mathbf{c}(\mathbf{b}-\mathbf{a}))].$$

Varying over $z \in [\theta_m^-(\mathbf{c}), \theta_m^+(\mathbf{c})]$ leads to

$$[\theta_m^-(\mathbf{c}), \theta_m^+(\mathbf{c})](y - x) \subset [\theta_m^-(\mathbf{c}(\mathbf{b}-\mathbf{a})), \theta_m^+(\mathbf{c}(\mathbf{b}-\mathbf{a}))]. \tag{18}$$

Corollary 4.3. *For* $\mathbf{a}, \mathbf{b}, \mathbf{c} \geq 0$ *with* $\mathbf{a} \leq \mathbf{b}$,

$$\theta_m^-(\mathbf{c}(\mathbf{b}-\mathbf{a})) \leq \theta_m^-(\mathbf{c})\theta_m^+(\mathbf{b}-\mathbf{a}). \tag{19}$$

In particular, for every $\mathbf{a}, \mathbf{b} \geq 0$, *we have* $\theta_m^-(\mathbf{ab}) \leq \theta_m^-(\mathbf{a})\theta_m^+(\mathbf{b})$.

Proof. Let $x = 0$ and $y = \theta_m^+(\mathbf{b}-\mathbf{a})$. Then $A := \mathbf{M}_{\mathbf{a}}(x) \geq 0$ and by monotonicity of θ_m^+, $0 \leq y \leq \theta_m^+(\mathbf{b})$ and hence $B := \mathbf{M}_{\mathbf{b}}(y) \geq 0$. Since $y = \theta_m^+(\mathbf{b}-\mathbf{a}) \geq 0$, $A \leq B$ by (16) and hence (19) follows from (18). ■

In the following we focus on members of $\mathcal{S}_m$ with nonnegative entries.

Definition 4.4. Define

$$\mathcal{S}_m^+ = \{\mathbf{M_a}(x) \geq 0 : x \geq 0\}, \quad \text{and} \quad \mathcal{S}_m^- = \{\mathbf{M_a}(x) \geq 0 : x \leq 0\}.$$

Note that $\mathcal{S}_m = \mathcal{S}_m^+ \cup \mathcal{S}_m^-$ and $\mathcal{S}_m^+ \cap \mathcal{S}_m^- = \mathcal{D}_m \cap \mathbb{P}_m = \left\{\mathrm{diag}(\mathbf{a}) : \mathbf{a} \in \mathbb{R}_+^m\right\}$. It is direct to see that $\mathcal{S}_m^+$ is a closed subsemigroup of $\mathcal{S}_m$, and $\mathcal{S}_m^+ \cap \mathcal{S}_m^-$ is a closed ideal of $\mathcal{S}_m$. Moreover, $\mathcal{S}_m^+ \circ \mathcal{S}_m^- \subset \mathcal{S}_m^-$ and $\mathcal{S}_m^- \circ \mathcal{S}_m^- \subset \mathcal{S}_m^+$.

Next, we introduce some compact subsemigroups of $\mathcal{S}_m$.

Definition 4.5. Define

$$\mathcal{C}_m = \{\mathbf{M_a}(x) \geq 0 : \mathbf{a} \in [0,1]^m\}, \quad \text{and} \quad \mathcal{C}_m^+ = \{\mathbf{M_a}(x) \geq 0 : \mathbf{a} \in [0,1]^m, x \geq 0\}.$$

By the Schur product theorem, $\mathcal{C}_m$ and $\mathcal{C}_m^+$ are subsemigroups of $\mathcal{S}_m$. From

$$(1-t)\mathbf{M_a}(x) + t\mathbf{M_b}(y) = \mathbf{M}_{(1-t)\mathbf{a}+t\mathbf{b}}((1-t)x + ty)$$

for every $t \in [0,1]$, $\mathcal{C}_m$ and $\mathcal{C}_m^+$ are convex subsets of $\mathbb{S}_m$. We shall show that they are closed and hence compact. Note from (P3), (P6) and P(8) that

$$\begin{aligned} \mathcal{C}_m &= \left\{\mathbf{M_a}(x) : \mathbf{a} \in [0,1]^m, x \in [\theta_m^-(\mathbf{a}), \theta_m^+(\mathbf{a})]\right\} \\ &\subset \{\mathbf{M_a}(x) : \mathbf{a} \in [0,1]^m, x \in [-1,1]\} \equiv [0,1]^m \times [-1,1]. \end{aligned}$$

Suppose that $\mathbf{M}_{\mathbf{a}_k}(x_k) \in \mathcal{C}_m$ converges to $A = [a_{ij}] \in \mathbb{S}_m$. Then $\lim_{k\to\infty}(\mathbf{a}_k)_j = a_{jj} \in [0,1]$ for all j and

$$x_k \in [\theta_m^-(\mathbf{a}_k), \theta_m^+(\mathbf{a}_k)], \qquad \forall k \in \mathbb{N}.$$

By the continuity of $\theta_m^\pm$,

$$a_{ij} = \lim_{k\to\infty} x_k \in [\theta_m^-(a_{11}, \ldots, a_{mm}), \theta_m^+(a_{11}, \ldots, a_{mm})], \qquad 1 \leq i \neq j \leq m.$$

Therefore $A \in \mathcal{C}_m$. This shows that $\mathcal{C}_m$ is a closed and hence a compact subset of $\mathbb{S}_m$. Similarly $\mathcal{C}_m^+$ is a compact subset of $[0,1]^m \times [0,1]$.

By the homogeneity of θ_m^+,

$$\theta_m^+(\mathbf{a}) = \mathbf{a}_{\max}\theta_m^\pm\left(\frac{1}{\mathbf{a}_{\max}}\mathbf{a}\right),$$

where $\mathbf{a} \in \mathbb{R}_{++}^m$ and $\mathbf{a}_{\max} := \max\{a_j\}$. From $\frac{1}{\mathbf{a}_{\max}}\mathbf{a} \in [0,1]^m$ and

$$\mathbf{M_a}(x) = \mathbf{a}_{\max}\mathbf{M}_{\frac{1}{\mathbf{a}_{\max}}\mathbf{a}}\left(\frac{1}{\mathbf{a}_{\max}}x\right).$$

Proposition 4.6. *We have that $\mathcal{C}_m$ and $\mathcal{C}_m^+$ are compact convex subsemigroups of $\mathcal{S}_m$. Moreover, $\mathcal{S}_m = \mathbb{R}_{++}\mathcal{C}_m$ and $\mathcal{S}_m^+ = \mathbb{R}_{++}\mathcal{C}_m^+$.*

Remark 4.7. By Proposition 4.2, $(\mathcal{C}_m^+, \leq)$ is an ordered compact commutative semigroup containing all idempotents of $\mathcal{S}_m$.

For $r \in \mathbb{R}$ and $A = [a_{ij}] \in \mathbb{S}_m$, we denote the rth Hadamard power of A by, whenever it exists,

$$A^{(r)} := [a_{ij}^r].$$

For $A \in \mathcal{C}_m$, the closure of the semigroup generated by A

$$\Gamma(A) = \overline{\{A^{(k)} : k = 1, 2, \ldots, .\}}$$

is a compact subsemigroup of $\mathcal{C}_m$. One can see that the sequence $\{A^{(k)}\}$ converges to an idempotent for every $A \in \mathcal{C}_m$. Indeed, for $\mathbf{a} \in [0,1]^m$ with $a_j \neq 1$ for some j, $\theta_m^+(\mathbf{a}) < 1$ by the geometric mean inequality (P5) and hence $\theta_m^-(\mathbf{a}) > -1$ from (P3).

Remark 4.8. Let I be a subsemigroup of the additive semigroup $\mathbb{R}_+^m$ and let

$$\mathcal{S}_m(I) := \{\mathbf{M_a}(x) \geq 0 : \mathbf{a} \in I\}.$$

Then it is a subsemigroup of $\mathcal{S}_m$ and is an ideal if I is. One can see that if I is compact, then $\mathcal{S}_m(I)$ is also. For example, let

$$I = \{\mathbf{a} : 0 \leq a_1 = a_2 \leq a_3 \leq \cdots \leq a_m \leq 1\}.$$

By (P6), $\mathcal{S}_m(I) = \{\mathbf{M_a}(x) : \mathbf{a} = (a_1, \ldots, a_m) \in I, -a_1 \leq x \leq a_1\}$ and is a compact subsemigroup.

Remark 4.9. (Radially convex metric [6]) There exists an equivalent metric d on $\mathcal{C}_m^+$ (on any compact subsemigroup of $\mathcal{S}_m$) such that for $A \leq B \leq C$ with $B \neq C$, $d(A,B) < d(A,C)$. Note that the Thompson metric $d_T(A,B) = ||\log A^{-1}B||$ on $\mathbb{P}_m^\circ$ or $\mathcal{S}_m^\circ$ satisfies $d_T(B,C) \leq d_T(A,D)$ for $0 < A \leq B, A \leq C \leq D$ (see Lemma 5.8 of [14]).

5. The group of units

Let $E(S)$ be the set of all idempotents of a semigroup S. Each idempotent e lies in the maximal subgroup $H(e)$ with the identity e:

$$H(e) := \{x \in S : xe = ex = x, xy = yx = e, \text{for some } y \in S\}.$$

The union of all maximal subgroups of S is denoted by

$$H(S) = \bigcup_{e \in E(S)} H(e).$$

One can see directly that for $\mathbf{e} = (e_1, \ldots, e_m) \in \{0,1\}^m$,

$$H(\mathbf{M_e}(0)) = \{\mathrm{diag}(a_1, \ldots, a_m) \geq 0 : e_j = 0 \Longleftrightarrow a_j = 0, j = 1, \ldots, m\}.$$

In particular, $H(I_m) = \mathcal{D}_m^\circ$ and $\mathcal{D}_m = \bigcup_{\mathbf{e} \in \{0,1\}^m} H(\mathbf{M_e}(0))$.

However, it is non-trivial to describe $H(J_m)$, the group of units of $\mathcal{S}_m$:

$$\begin{aligned} H(J_m) &= \{A \in \mathcal{S}_m : A \circ B = J_m \text{ for some } B \in \mathcal{S}_m\} \\ &= \{A \in \mathcal{S}_m : A^{(-1)} \in \mathcal{S}_m\}. \end{aligned}$$

Note that if $\mathbf{M_a}(x) \in H(J_m)$, then $\mathbf{a} > 0$ and $x \neq 0$.

Proposition 5.1. *For every* $\mathbf{a} \in \mathbb{R}^m_{++}$ *with* $m \geq 3$,

$$\theta^+_m(\mathbf{a}) = \theta^+_m(\mathbf{a}^{-1})^{-1} \Longleftrightarrow a_1 = \cdots = a_m.$$

Proof. Suppose that $\theta^+_m(\mathbf{a}) = \theta^+_m(\mathbf{a}^{-1})^{-1}$. The geometric mean inequality (P5),

$$\theta^+_m(\mathbf{a}^{-1})^{-1} \geq (a_1 \cdots a_m)^{\frac{1}{m}} \geq \theta^+_m(\mathbf{a})$$

and hence $a_1 = \cdots = a_m$. ∎

Remark 5.2. Recall from Corollary 3.6 that $\mathcal{T}^\partial_m$ consists of all singular matrices in $\mathcal{S}_m$ with positive diagonal entries. The result in above shows that the set $\mathcal{T}^\partial_m$ for $m \geq 3$ is not closed under the Hadamard product. Indeed let $\mathbf{a} = (a_1, \ldots, a_m) > 0$ with $a_i \neq a_j$ for some $i \neq j$ and consider

$$\mathbf{M}_\mathbf{a}(\theta^+_m(\mathbf{a})), \mathbf{M}_{\mathbf{a}^{-1}}(\theta^+_m(\mathbf{a}^{-1})) \in \mathcal{T}^\partial_m.$$

Suppose that their Hadamard product is in $\mathcal{T}^\partial_m$, that is,

$$\mathbf{M}_{(1,\ldots,1)}(\theta^+_m(\mathbf{a})\theta^+_m(\mathbf{a}^{-1})) = \mathbf{M}_\mathbf{a}(\theta^+_m(\mathbf{a})) \circ \mathbf{M}_{\mathbf{a}^{-1}}(\theta^+_m(\mathbf{a}^{-1})) \in \mathcal{T}^\partial_m.$$

By (5), $\theta^+_m(\mathbf{a})\theta^+_m(\mathbf{a}^{-1}) = 1$ and hence $a_1 = \cdots = a_m$, by the previous proposition.

Theorem 5.3. *For* $m \geq 3$, $H(J_m) = \mathbb{R}_{++}J_m$. *In particular,* $H(\mathcal{S}_m) = \mathcal{D}_m \cup \mathbb{R}_{++}J_m$.

Proof. Let $m \geq 3$ and let $\mathbf{a} \in \mathbb{R}^m_{++}$. It suffices to show that for $x \in [\theta^-_m(\mathbf{a}), \theta^+_m(\mathbf{a})]$ with $x \neq 0$,

$$x^{-1} \in [\theta^-_m(\mathbf{a}^{-1}), \theta^+_m(\mathbf{a}^{-1})] \Longleftrightarrow x = a_1 = \cdots = a_m.$$

Let $x \in [\theta^-_m(\mathbf{a}), \theta^+_m(\mathbf{a})]$ and $x \neq 0$. Suppose that $M_{\mathbf{a}^{-1}}(x^{-1}) \geq 0$.

Case 1: $x < 0$. Then $\theta^-_m(\mathbf{a}^{-1}) \leq x^{-1}$, that is, $x \leq \theta^-_m(\mathbf{a}^{-1})^{-1}$. Since $\theta^-_m(\mathbf{a}) \leq x$, we have by (P4), $\theta^-_m(\mathbf{a}) \leq x \leq \theta^-_m(\mathbf{a}^{-1})^{-1} \leq \theta^-_m(\mathbf{a})$. This implies that

$$\theta^-_m(\mathbf{a}) = x = \theta^-_m(\mathbf{a}^{-1})^{-1}. \tag{20}$$

By (P3), $\theta^+_m(\mathbf{a}^{-1})^{-1} \leq -\theta^-_m(\mathbf{a}^{-1})^{-1} = -x = -\theta^-_m(\mathbf{a}) \leq \theta^+_m(\mathbf{a})$.

Again by (P4), $\theta^+_m(\mathbf{a}^{-1})^{-1} = \theta^+_m(\mathbf{a})$. By the preceding proposition, $a := a_1 = \cdots = a_m$. This together with (20) and (P8) leads to $\frac{a}{m-1} = (m-1)a$. This is impossible because $m \geq 3$.

Case 2: $x > 0$. Since $x^{-1} \leq \theta^+_m(\mathbf{a}^{-1})$ and $x \leq \theta^+_m(\mathbf{a})$, we have from (P4) that

$$\theta^+_m(\mathbf{a}) \leq \theta^+_m(\mathbf{a}^{-1})^{-1} \leq x \leq \theta^+_m(\mathbf{a}).$$

That is, $\theta^+_m(\mathbf{a}) = x = \theta^+_m(\mathbf{a}^{-1})^{-1}$. By Proposition 5.1, $a_1 = \cdots = a_m = \theta^+_m(\mathbf{a}) = x$. This completes the proof. ∎

Remark 5.4. (Units) Let $m \geq 3$ and $\mathbf{a} > 0$ with $a_i \neq a_j$ for some $i \neq j$. Then for $x \neq 0$,

$$\mathbf{M}_\mathbf{a}(x) \geq 0 \Longrightarrow \mathbf{M}_\mathbf{a}(x)^{(-1)} = \mathbf{M}_{\mathbf{a}^{-1}}(x^{-1}) \ngeq 0.$$

In particular, the compact semigroup $\mathcal{C}^+_m$ has no units (invertible elements) except J_m. We know that for a compact topological monoid, the group of units is compact [7].

Remark 5.5. ([11, 17]) Every compact semigroup S has a unique minimal ideal, called the kernel of S and denoted by $M(S)$. It is closed and homeomorphic to a paragroup (a Rees product of groups). An interesting problem is to determine the minimal ideal of $\mathcal{C}_m$.

6. One-parameter semigroups and infinite divisibility

Definition 6.1. A one-parameter semigroup on the Hadamard semigroup $\mathcal{S}_m$ based at J_m is a continuous homomorphism $\gamma\colon [0,\infty) \to \mathcal{S}_m$ with $\gamma(0) = J_m$.

Suppose that $A = [a_{ij}]$ is positive semidefinite and that $a_{ij} \geq 0$ for all i and j. We say that A is *infinitely divisible* if the matrix $A^{(r)}$ is positive semidefinite for every nonnegative r. We note that every 2×2 positive semidefinite matrix with nonnegative entries is infinitely divisible [3, 4]. In terms of semigroups, A is infinitely divisible if and only if it generates a one-parameter semigroup $t \mapsto A^{(t)}$ starting at J_m.

By the Schur product theorem and continuity, a positive semidefinite matrix A with nonnegative entries is infinitely divisible if and only if $A^{(1/k)}$ is positive semidefinite for all $k \in \mathbb{N}$.

Example 6.2. We note that for $a, x \geq 0$,

$$\gamma(t) = \begin{bmatrix} a^t & x^t & x^t \\ x^t & a^t & x^t \\ x^t & x^t & a^t \end{bmatrix} \geq 0 \Longleftrightarrow x^t \leq a^t = \theta_m^+(a^t, a^t, \ldots, a^t),$$

which is equivalent to $x \leq a$. Thus, γ is a one-parameter semigroup if and only if $x \leq a$.

By super-multiplicativity (P4) of θ_m^+, we have that for every $\mathbf{a} = (a_1, \ldots, a_m) \geq 0$,

$$\theta_m^+(a_1, \ldots, a_m)^k \leq \theta_m^+(a_1^k, \ldots, a_m^k), \qquad \forall k \in \mathbb{N},$$

and hence

$$\theta_m^+(a_1^{\frac{1}{k}}, \ldots, a_m^{\frac{1}{k}})^k \leq \theta_m^+(a_1, \ldots, a_m) \leq \theta_m^+(a_1^k, \ldots, a_m^k)^{\frac{1}{k}}, \qquad \forall k \in \mathbb{N}. \tag{21}$$

Proposition 6.3. *Let* $\mathbf{a} = (a_1, \ldots, a_m) \geq 0$. *Then* $\mathbf{M_a}(x)$ *is infinitely divisible if and only if* $0 \leq x \leq \theta_m^+(a_1^{\frac{1}{k}}, \ldots, a_m^{\frac{1}{k}})^k$ *for all* $k \in \mathbb{N}$. *In particular,* $\mathbf{M_a}(\theta_m^+(\mathbf{a}))$ *is infinitely divisible if and only if*

$$\theta_m^+(a_1, \ldots, a_m) = \theta_m^+(a_1^{\frac{1}{k}}, \ldots, a_m^{\frac{1}{k}})^k, \qquad \forall k \in \mathbb{N}.$$

Proof. We may assume that $\mathbf{a} = (a_1, \ldots, a_m) > 0$ and $x > 0$. Then $A := \mathbf{M_a}(x)$ is infinitely divisible if and only if $A^{(1/k)} = \mathbf{M}_{\mathbf{a}^{1/k}}(x^{1/k})$ is positive semidefinite for all k if and only if $x^{1/k} \leq \theta_m^+(a_1^{1/k}, \ldots, a_m^{1/k})$ for all k if and only if $x \leq \theta_m^+(a_1^{1/k}, \ldots, a_m^{1/k})^k$ for all $k \in \mathbb{N}$.

The remaining part of proof follows by (21). ■

For $\mathbf{a} = (a_1, \ldots, a_m) \geq 0$, let $\mathbf{a}_{\min} := \min\{a_j : j = 1, \ldots, m\}$. Since $\mathbf{a} \geq \mathbf{a}_{\min}(1, \ldots, 1)$, we have by monotonicity of θ_m^+ that $\mathbf{a}_{\min} \leq \theta_m^+(\mathbf{a}^{\frac{1}{k}})^k$ for all $k \in \mathbb{N}$.

Corollary 6.4. *Let* $\mathbf{a} \geq 0$. *Then* $\mathbf{M}_{\mathbf{a}}(x)$ *is infinitely divisible for every* $0 \leq x \leq \inf \theta_m^+(\mathbf{a}^{\frac{1}{k}})^k$. *In particular the map* $\gamma : [0, \infty) \to \mathcal{S}_m^+$ *defined by* $\gamma(t) = \mathbf{M}_{\mathbf{a}}(x)^{(t)} = \mathbf{M}_{\mathbf{a}^t}(x^t)$ *is a one-parameter semigroup for every* $0 \leq x \leq \inf \theta_m^+(\mathbf{a}^{\frac{1}{k}})^k$.

Problem 1. Prove that the sequence $\theta_m^+(\mathbf{a}^{\frac{1}{k}})^k$ is nonincreasing and hence it converges. Numerical computations shows that this holds true. This is equivalent to

$$\mathbf{M}_{\mathbf{a}^{\frac{1}{k}}}\left(\theta_m^+\left(\mathbf{a}^{\frac{1}{k+1}}\right)^{\frac{k+1}{k}}\right) \geq 0, \qquad \forall k \in \mathbb{N}.$$

Example 6.5. Let $\mathbf{a} = (a_1, \ldots, a_m) \in [0,1]^m$ with $0 < a_1 \leq a_2 \leq \cdots \leq a_m$, and let $x \in (0, a_1]$. Then $\gamma(t) = \mathbf{M}_{\mathbf{a}}(x)^{(t)}$ is a one-parameter semigroup of $\mathcal{C}_m^+$. If further $a_m < 1$, then $\gamma(t) \to \mathbf{0}_m$, the zero element, as $t \to \infty$. Otherwise it tends to an idempotent $\mathrm{diag}(0, \ldots, 0, 1, \ldots, 1)$ where the number of 1 is that of 1 in a_j.

Problem 2. Find all one-parameter semigroups of $\mathcal{C}_m^+$ based at J_m.

7. A theorem of FitzGerald and Horn

In [10], FitzGerald and Horn showed that for $m \times m$ matrices A and B with nonnegative real entries,

$$0 \leq A \leq B \Longrightarrow 0 \leq A^{(t)} \leq B^{(t)}, \qquad \forall t \geq m-2. \tag{22}$$

It turns out that the bound is sharp. This includes the following elegant result; for $m \times m$ matrix A with nonnegative entries,

$$A \geq 0 \Longrightarrow A^{(t)} \geq 0, \qquad t \geq m-2. \tag{23}$$

Application to $A \in \mathcal{S}_m^+$ yields

$$\mathbf{M}_{\mathbf{a}}(x) \in \mathcal{S}_m^+ \Longrightarrow \mathbf{M}_{\mathbf{a}}(x)^{(t)} = \mathbf{M}_{\mathbf{a}^t}(x^t) \in \mathcal{S}_m^+, \qquad t \geq m-2.$$

In other words,

Corollary 7.1. *For all* $\mathbf{a} \in \mathbb{R}_+^m$ *and* $t \geq m-2$: $0 \leq x \leq \theta_m^+(\mathbf{a}) \Longrightarrow x \leq \theta_m^+(\mathbf{a}^t)^{\frac{1}{t}}$.

In particular,

$$\theta_m^+(\mathbf{a}) \leq \theta_m^+(\mathbf{a}^t)^{\frac{1}{t}}, \qquad \forall t \geq m-2. \tag{24}$$

We note that (24) is equivalent to

$$\theta_m^+(\mathbf{a}^{\frac{1}{t}})^t \leq \theta_m^+(\mathbf{a}), \qquad \forall t \geq m-2 \tag{25}$$

or (for $m \geq 3$)

$$\theta_m^+(\mathbf{a}^t)^{\frac{1}{t}} \leq \theta_m^+(\mathbf{a}), \qquad \forall 0 < t \leq \frac{1}{m-2}. \tag{26}$$

By super-multiplicativity of θ_m^+, we have $\theta_m^+(\mathbf{a}^{\frac{1}{k}})^k \leq \theta_m^+(\mathbf{a})$ for all $k \in \mathbb{N}$. Equivalently, $\theta_m^+(\mathbf{a}) \leq \theta_m^+(\mathbf{a}^k)^{\frac{1}{k}}$ for all $k \in \mathbb{N}$.

Example 7.2. $(m=2)$ By (24) and (25),

$$\theta_2^+(\mathbf{a}^{\frac{1}{t}})^t = \theta_2^+(\mathbf{a}) \tag{27}$$

for all $\mathbf{a} \in \mathbb{R}_+$ and $t > 0$. Indeed, it follows also from $\theta_2^+(a,b) = \sqrt{ab}$.

We have shown the following

Proposition 7.3. *Let $m \geq 3$ and $\mathbf{a} \in \mathbb{R}_+^m$. Then*

$$\theta_m^+(\mathbf{a}^s)^{\frac{1}{s}} \leq \theta_m^+(\mathbf{a}) \leq \theta_m^+(\mathbf{a}^t)^{\frac{1}{t}} \tag{28}$$

for all $s \in (0, \frac{1}{m-2}] \cup \{1/k\}_{k=1}^{\infty}$ and $t \in [m-2, \infty) \cup \mathbb{N}$.

Example 7.4. $(m=3)$

$$\theta_3^+(\mathbf{a}^s)^{\frac{1}{s}} \leq \theta_3^+(\mathbf{a}) \leq \theta_3^+(\mathbf{a}^t)^{\frac{1}{t}}, \qquad 0 < s \leq 1 \leq t. \tag{29}$$

Problem 3. Does (29) hold for every $m \geq 3$?

Let $\mathbf{M}_{\mathbf{a}}(x), \mathbf{M}_{\mathbf{b}}(y) \in \mathcal{S}_m^+$. By (22)

$$\mathbf{M}_{\mathbf{a}}(x) \leq \mathbf{M}_{\mathbf{b}}(y) \implies 0 \leq \mathbf{M}_{\mathbf{a}}(x)^{(t)} \leq \mathbf{M}_{\mathbf{b}}(y)^{(t)}, \qquad \forall t \geq m-2.$$

By Corollary 4.1, the latter is equivalent to

(i) $\mathbf{a} \leq \mathbf{b}$;

(ii) $x^t \leq \theta_m^+(\mathbf{a}^t), y^t \leq \theta_m^+(\mathbf{b}^t)$; and

(iii) $y^t - x^t \in [\theta_m^-(\mathbf{b}^t - \mathbf{a}^t), \theta_m^+(\mathbf{b}^t - \mathbf{a}^t)]$

for all $t \geq m-2$.

Corollary 7.5. *Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+^m$ with $\mathbf{a} \leq \mathbf{b}$.*
Then for $0 \leq x \leq \theta_m^+(\mathbf{a})$ and $0 \leq y \leq \theta_m^+(\mathbf{b})$,

$$y - x \in [\theta_m^-(\mathbf{b}-\mathbf{a}), \theta_m^+(\mathbf{b}-\mathbf{a})] \Longrightarrow y^t - x^t \in [\theta_m^-(\mathbf{b}^t - \mathbf{a}^t), \theta_m^+(\mathbf{b}^t - \mathbf{a}^t)] \tag{30}$$

for all $t \geq m-2$.

If $y \geq x$, then (30) is equivalent to

$$y - x \leq \theta_m^+(\mathbf{b}-\mathbf{a}) \Longrightarrow y^t - x^t \leq \theta_m^+(\mathbf{b}^t - \mathbf{a}^t) \tag{31}$$

for all $t \geq m-2$. Application with $x = \theta_m^+(\mathbf{a}) \leq y = \theta_m^+(\mathbf{b})$, yields

Corollary 7.6. *Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+^m$ with $\mathbf{a} \leq \mathbf{b}$. Then*

$$\theta_m^+(\mathbf{b}) - \theta_m^+(\mathbf{a}) \leq \theta_m^+(\mathbf{b}-\mathbf{a}) \Longrightarrow \theta_m^+(\mathbf{b})^t - \theta_m^+(\mathbf{a})^t \leq \theta_m^+(\mathbf{b}^t - \mathbf{a}^t)$$

for all $t \geq m-2$. Moreover, for all $t \geq m-2$

$$\theta_m^+(\mathbf{b}) = \theta_m^+(\mathbf{b}-\mathbf{a}) + \theta_m^+(\mathbf{a}) \Longrightarrow \theta_m^+(\mathbf{b})^t = \theta_m^+(\mathbf{b}^t - \mathbf{a}^t) + \theta_m^+(\mathbf{a})^t.$$

Remark 7.7. We have seen that for every $A \in \mathcal{S}_m^+$, the map

$$\gamma\colon [m-2,\infty) \to \mathcal{S}_m^+, \qquad t \mapsto A^{(t)}$$

is well-defined and satisfies

$$\gamma(t+s) = \gamma(t) \circ \gamma(s), \qquad \forall t, s \geq m-2.$$

If $A^{(2-m)} \geq 0$, then the map $\beta(t) = A^{(t)} = A^{(2-m)} \circ A^{(m-2+t)}$ is a one-parameter subsemigroup on $\mathcal{S}_m^+$. However, this happens only for diagonal matrices from Remark 5.4.

8. Final remarks

(1) The super-multiplicativity (P4) of $\theta_m^{\pm}$ that follows from the Schur product theorem is crucial for the semigroup structures of $\mathcal{S}_m$. Another such a mean on positive real numbers is the geometric mean $G(a_1,\ldots,a_m) = (a_1\cdots a_m)^{\frac{1}{m}}$. By (P3), we have a Hadamard semigroup containing $\mathcal{S}_m$ defined by $\left\{(\mathbf{a},x) \in \mathbb{R}_{++}^m \times \mathbb{R} : |x| \leq G(\mathbf{a})\right\}$. Although the theory of positive semidefinite matrices can not be applied in this Hadamard semigroup, it provides a notion of *semigroups with means*.

Let S be an ordered topological semigroup with a continuous super-multiplicative mean $G : S^m \to S$. Then $\{(\mathbf{a},x) \in S^{m+1} : x \leq G(\mathbf{a})\}$ is a closed subsemigroup of the product semigroup S^{m+1}. For example, consider the Hadamard semigroup $\mathbb{P}$ of all $k \times k$ positive semidefinite matrices and a multivariate geometric mean: e.g., the Karcher mean [15, 16], ALM and BMP means [2, 5] which are super-multiplicative with respect to the Hadamard product (see Theorem 13 of [1] for $m = 2$.) It then gives rise to a topological semigroup $\{(A_1,\ldots,A_m,X) \in \mathbb{P}^{m+1} : X \leq G(A_1,\ldots,A_m)\}$. The following are compact subsemigroups

$$\{(A_1,\ldots,A_m,X) \in \mathbb{P}^{m+1} : 0 \leq A_j \leq I_k, j = 1,\ldots,m, X \leq G(A_1,\ldots,A_m)\},$$

$$\{(A_1,\ldots,A_m,X) \in \mathbb{P}^{m+1} : 0 \leq A_j \leq J_k, j = 1,\ldots,m, X \leq G(A_1,\ldots,A_m)\}.$$

The later is a monoid; it contains the Hadamard identity $(J_k,\ldots,J_k)$. A study on these semigroups is of interest for further work. We expect that the theory of topological semigroup plays a role in the field of matrix means and matrix analysis.

(2) Not much known about Hadamard (sub)semigroups in context of topological semigroups [7]; homomorphisms, one-parameter semigroups, congruences, quotient semigroups, compact semigroups, projective systems of semigroups, ordered semigroups. Although we have restricted our attention to real x of the symmetric matrix $\mathbf{M}_{\mathbf{a}}(x)$, similar problems arise for the Hermitian matrices over complex numbers, thanks to the Schur product theorem for Hermitian matrices. However, we do not have a notion of Hadamard product (and hence Schur's product theorem) and of off-diagonal constant matrices on a general Euclidean Jordan algebra. For details on Euclidean Jordan algebras and symmetric cones, see [9]. We close this paper with an affirmative answer on spin factors.

Let $V = \mathbb{R}^d \times \mathbb{R}, d \geq 2$, with the Jordan product $(\boldsymbol{u},t)*(\boldsymbol{v},s) := (s\boldsymbol{u} + t\boldsymbol{v}, ts + \langle \boldsymbol{u},\boldsymbol{v}\rangle)$. The element $e = (\mathbf{0},1)$ is the Jordan identity and the symmetric cone of invertible squares is the forward light cone $\Omega := \left\{(\boldsymbol{u},t) \in \mathbb{R}^d \times \mathbb{R} : ||\boldsymbol{u}|| < t\right\}$. For $\mathbf{x},\mathbf{y} \in V$, we define $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{y} - \mathbf{x} \in \overline{\Omega}$. The Pierce decomposition for the Jordan frame $\{c_1, c_2\}$ is

$$V = \boldsymbol{e}_d^{\perp} \times \{0\} \oplus \mathbb{R}c_1 \oplus \mathbb{R}c_2,$$

where $c_1 = \frac{1}{2}(\boldsymbol{e}_d, 1)$, $c_2 = \frac{1}{2}(-\boldsymbol{e}_d, 1)$, and $\boldsymbol{e}_d := (0, \ldots, 0, 1) \in \mathbb{R}^d$.

Definition 8.1. For $a, b \in \mathbb{R}$ and $\boldsymbol{u} = (u_1, \ldots, u_{d-1}) \in \mathbb{R}^{d-1}$, define

$$\mathbf{M}_{(a,b)}(\boldsymbol{u}) = \left(u_1, u_2, \ldots, u_{d-1}, \frac{a-b}{2}, \frac{a+b}{2}\right) \in V.$$

An element $\mathbf{x} \in V$ is called *off-diagonal constant* if

$$\mathbf{x} = \left(x, x, \ldots, x, \frac{a-b}{2}, \frac{a+b}{2}\right) = \mathbf{M}_{(a,b)}(x, \ldots, x)$$

for some $x, a, b \in \mathbb{R}$. The Hadamard product on V is defined by

$$\mathbf{M}_{(a,b)}(\boldsymbol{u}) \circ \mathbf{M}_{(c,d)}(\boldsymbol{v}) = \mathbf{M}_{(ac,bd)}(\boldsymbol{uv}).$$

Then $(V, \circ)$ is a commutative semigroup with the identity $e = (\mathbf{0}, 1)$. By a direct computation, we have

Theorem 8.2. (i) $\mathbf{M}_{(a,b)}(\boldsymbol{u}) \geq 0$ *if and only if* $a, b \geq 0$ *and* $\sum_{j=1}^{d-1} u_j^2 \leq ab$.

(ii) $\mathbf{M}_{(a,b)}(\boldsymbol{u}) \circ \mathbf{M}_{(c,d)}(\boldsymbol{v}) \geq 0$, *if* $\mathbf{M}_{(a,b)}(\boldsymbol{u}), \mathbf{M}_{(c,d)}(\boldsymbol{v}) \geq 0$.

(iii) $\mathbf{M}_{(a,b)}(x, x, \ldots, x) \geq 0$ *if and only if* $a, b \geq 0$ *and* $|x| \leq \sqrt{\frac{ab}{d-1}}$.

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