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A Survey on Invariant Cones in Infinite Dimensional Lie Algebras

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Abstract. For the Lie algebra $\mathfrak{g}$ of a connected infinite-dimensional Lie group G, there is a natural duality between so-called semi-equicontinuous weak-$*$-closed convex $\mathrm{Ad}^*(G)$-invariant subsets of the dual space $\mathfrak{g}'$ and $\mathrm{Ad}(G)$-invariant lower semicontinuous positively homogeneous convex functions on open convex cones in $\mathfrak{g}$. In this survey, we discuss various aspects of this duality and some of its applications to a more systematic understanding of open invariant cones and convexity properties of coadjoint orbits. In particular, we show that root decompositions with respect to elliptic Cartan subalgebras provide powerful tools for important classes of infinite Lie algebras, such as completions of locally finite Lie algebras, Kac-Moody algebras and twisted loop algebras with infinite-dimensional range spaces. We also formulate various open problems.

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Contents

1. Introduction

Let G be a connected (possibly infinite-dimensional) Lie group with Lie algebra $\mathfrak{g}$ (see [48] for a survey on infinite-dimensional Lie groups). The conjugation action of G on itself induces on $\mathfrak{g}$ the *adjoint action* $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ and by dualization we obtain the *coadjoint action*

$$\mathrm{Ad}^*\colon G \to \mathrm{GL}(\mathfrak{g}') \quad \text{with} \quad \mathrm{Ad}^*_g \lambda := \lambda \circ \mathrm{Ad}_{g^{-1}}.$$

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Here $\mathfrak{g}'$ is the space of continuous linear functionals on $\mathfrak{g}$. We call a subset of $\mathfrak{g}$, resp., $\mathfrak{g}'$ *invariant* if it is invariant under $\mathrm{Ad}(G)$, resp., $\mathrm{Ad}^*(G)$. This survey is primarily concerned with open convex cones $\Omega \subseteq \mathfrak{g}$ which are invariant under the adjoint action of G on $\mathfrak{g}$.

In the context of finite-dimensional groups, invariant cones play a central role in the theory of Lie semigroups [18], the theory of causal symmetric spaces [20], the complex geometry of semigroup complexifications of Lie groups of the type $S = G\exp(i\Omega)$ (Olshanski semigroups) [42], and, last but not least, in the unitary representation theory of Lie groups, where one studies holomorphic extensions to such semigroups ([43]).

The theory of invariant cones in finite-dimensional Lie algebras was initiated by Kostant, I. E. Segal and Vinberg ([77], [80]). A classification of simple Lie algebras with invariant cones was obtained independently by Olshanski, Paneitz, and Kumaresan-Ranjan (cf. [68], [70], [33]). A structure theory of invariant convex cones in general finite-dimensional Lie algebras was developed by Hilgert and Hofmann in [17] and [18]. The characterization of finite-dimensional Lie algebras containing pointed open invariant cones was obtained in [39] in terms of symplectic modules of convex type, and these have been classified in [63]; see [43] for a rather self-contained exposition of this theory. Typical Lie algebras containing open invariant cones are compact ones with non-trivial center (such as $\mathfrak{u}_n(\mathbb{C})$), hermitian Lie algebras corresponding to automorphism groups of bounded symmetric domains (such as $\mathfrak{sp}_{2n}(\mathbb{R})$), and semidirect products, such as the Jacobi Lie algebra $\mathfrak{heis}(\mathbb{R}^{2n}) \rtimes \mathfrak{sp}_{2n}(\mathbb{R})$ (polynomials of degree ≤ 2 on $\mathbb{R}^{2n}$ with the Poisson bracket).

In this survey we review some results extending the theory of invariant cones to infinite-dimensional Lie algebras. Due to the lack of a general structure theory, any effective theory has to build on a well-developed toolbox that applies to important specific classes of Lie algebras. So one of our goals is to contribute to this toolbox by showing how general results on convex sets, invariance under group actions, fixed point theorems and Lie theory can be combined to derive information on invariant cones.

Some types of invariant cones in infinite-dimensional Lie algebras have been discussed in [51], including a classification of invariant cones in the Virasoro algebra and typical relations with representation theory. Invariant cones also arise naturally in differential geometry: In the Poisson-Lie algebra $(C^\infty(M), \{\cdot,\cdot\})$ of a Poisson manifold M, we have the cone of positive functions, and in the Lie algebra $\mathfrak{cont}(M,\eta)$ of contact vector fields on the contact manifold (M,η), the cone of those vector fields X for which $\eta(X) \geq 0$ ([12]).

Any theory of invariant open cones implicitly contains information on invariant convex functions: If $\mathcal{D} \subseteq \mathfrak{g}$ is an open domain and $f \colon \mathcal{D} \to \mathbb{R}$ is a lower-semicontinuous $\mathrm{Ad}(G)$-invariant convex function, then its open epigraph

$$\Omega_f := \{(x,t) \in \mathfrak{g} \times \mathbb{R} \colon f(x) < t\}$$

is an open invariant cone in the direct Lie algebra sum $\mathfrak{g} \oplus \mathbb{R}$.

Invariant convex functions arise naturally in the unitary representation theory of Lie groups, and this connection is an important motivation for the interest in invariant cones. Let $(\pi, \mathcal{H})$ be a unitary representation of G and $x \in \mathfrak{g}$. Then we write

$\partial\pi(x) := \frac{d}{dt}\big|_{t=0}\pi(\exp tx)$ for the infinitesimal generator of the unitary one-parameter group $(\pi(\exp tx))_{t\in\mathbb{R}}$ (Stone's Theorem). If the representation is *smooth* in the sense that the subspace $\mathcal{H}^\infty$ of elements with smooth orbit maps is dense in $\mathcal{H}$, then the *support functional*

$$s_\pi \colon \mathfrak{g} \to \mathbb{R} \cup \{\infty\}, \quad s_\pi(x) := \sup\big(\mathrm{Spec}(-i\partial\pi(x))\big) \tag{1}$$

is a lower semicontinous positively homogeneous convex function. It encodes the spectral bounds of the selfadjoint operators $i\partial\pi(x)$. The relation

$$\partial\pi(\mathrm{Ad}(g)x) = \pi(g)\partial\pi(x)\pi(g)^{-1}$$

implies that it is $\mathrm{Ad}(G)$-invariant. The representation π is said to be *semibounded* if s_π is bounded on some non-empty open subset of $\mathfrak{g}$. We thus obtain a natural connection between unitary representations and invariant convexity in the Lie algebra $\mathfrak{g}$. For a survey on recent progress concerning semibounded representations, we refer to [56]. Particularly striking results concerning so-called smoothing operators and decomposition theory can be found in [58].

A systematic understanding of open invariant cones is required, in particular, to verify that unitary representations are semibounded and to estimate their support functional s_π. It can be used to show that, for certain Lie groups, all semibounded representations are trivial or that all semibounded representations are bounded, which is the case if all non-empty open invariant cones coincide with $\mathfrak{g}$. A particular class of Lie algebras for which this happens are compact semisimple Lie algebras (Remark 4.3), the unitary Schatten Lie algebras $\mathfrak{u}_p(\mathcal{H})$ for $p > 1$ (Proposition 5.5), and the projective unitary Lie algebra $\mathfrak{pu}(\mathcal{H}) = \mathfrak{u}(\mathcal{H})/\mathbb{R}i\mathbf{1}$ of a Hilbert space $\mathcal{H}$ ([53, Thm. 5.6]). This property is also related to interesting fixed point properties of the corresponding groups, and this kind of information is essential for the method of holomorphic induction (see [53, 54, 55]), where one requires that the representation one induces from is bounded.

We start in Section 2 with a discussion of convex subsets of a locally convex space E and its topological dual E'. For our purposes, the most important class of subsets $C \subseteq E'$ are those which are semi-equicontinuous in the sense that their support functional $s_C(v) = \sup\langle C, v\rangle$ is bounded on some non-empty open subset of E. This leads us to the Duality Theorem 2.4 between weak-$*$-closed convex subsets and positively homogeneous lower semicontinuous convex functions on open convex cones. As we want to apply this duality to invariant subsets of Lie algebras and their duals, we discuss in Subsection 2.2 various fixed point theorems, based on compactness, equicontinuity and the Bruhat-Tits Theorem. In Subsection 2.3 we provide a description of closed convex hulls of orbits of Coxeter groups that builds on [22] and makes some results more easily applicable in the Lie algebra context.

In Section 3 we then turn to invariant convexity in Lie algebras, formulate the specific problems (P1-6) one would like to answer for infinite-dimensional Lie algebras and explain how these relate to unitary representations and spectral conditions for corresponding selfadjoint operators (Subsection 3.1). Generalizing root decompositions of finite-dimensional Lie algebras with respect to a compactly embedded Cartan algebra, we introduce in Subsection 3.2 the notion of an elliptic Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$. Here an important point is that it suffices that the sum of the root spaces is dense in $\mathfrak{g}_\mathbb{C}$, which is more natural in the context of Banach and Fréchet-Lie algebras. In this rather general context, the Reduction Theorem 3.10 tells us that the projection $p_{\mathfrak{t}} \colon \mathfrak{g} \to \mathfrak{t}$, resp., the restriction map $p_{\mathfrak{t}'} \colon \mathfrak{g}' \to \mathfrak{t}'$ leave open, resp., semi-equicontinuous weak-$*$-closed invariant convex subsets invariant. So these sets can be studied to a large extent in terms of their intersections with $\mathfrak{t}$ and $\mathfrak{t}'$ respectively. This permits us to generalize various results from the finite-dimensional case. We conclude Section 3 with an explanation of how the algebraic theory of unitary highest weight representations can be used to obtain convexity theorems for coadjoint orbits.

In the brief Section 4 we recall the convexity theorems for adjoint and coadjoint orbits for finite-dimensional Lie algebras and refer to [43] for details. Several specific classes of infinite-dimensional Lie algebras are discussed in Section 5. For nilpotent and 2-step solvable algebras everything reduces to the abelian case (Theorem 5.1), but the class of 3-step solvable algebras contains the oscillator algebras which display various types of non-trivial behavior (Subsection 5.2). For Kac-Moody Lie algebras we formulate the Kac-Peterson Convexity Theorem and we briefly mention some results on Lie algebras of vector fields. Since they display the differences between the finite and infinite-dimensional theory so nicely, we take a closer look at infinite-dimensional versions of unitary groups in Subsection 5.5. We conclude Section 5 with the observation that projective limits of Lie algebras do not lead to new phenomena.

In Section 6 we turn to Lie algebras which contain a dense directed union of finite-dimensional Lie algebras. Typical examples are the Lie algebras $\mathfrak{g} = \mathfrak{u}(J, \mathbb{K})$ of skew-hermitian $J \times J$-matrices with finitely many non-zero entries in $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and their Banach completions. The Lie algebras $\mathfrak{u}(J, \mathbb{K})$ have the interesting property that all semi-equicontinuous orbits in $\mathfrak{g}'$ and all invariant norms extend to the minimal Banach completion, the Lie algebea $\mathfrak{u}_1(\ell^2(J, \mathbb{K}))$ of skew-hermitian trace class operators. We also discuss the existence of invariant scalar products on general direct limits of compact Lie algebras and show that open invariant cones in Hilbert-Lie algebras intersect the center.

Motivated by direct limits of affine Kac-Moody algebras and double extensions of twisted loop algebras, we turn in Section 7 to various aspects of double extensions. The most well-behaved class are double extensions of Lie algebras with an invariant scalar product (euclidean Lie algebras) because they carry an invariant Lorentzian form and this immediately leads to open invariant cones and many semi-equicontinuous orbits. However, a finer analysis of the convexity properties of double extensions turns out to be quite delicate because of the huge variety of double extensions and because many natural representation theoretic constructions lead to non-Lorentzian double extensions. Here we briefly explain the source of these difficulties.

We collect some auxiliary results in appendices on constructing open cones, Lorentzian geometry and the Bruaht-Tits Theorem.

Notation: For a set J, we write S_J for the group of all bijections of J, and $S_{(J)}$ for the subgroup of those elements moving only finitely many elements. These groups act on the space $\mathbb{R}^J$ of functions $J \to \mathbb{R}$ and preserve the subspace $\mathbb{R}^{(J)}$ of finitely supported functions.

For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, we write $\mathfrak{gl}(J, \mathbb{K}) \subseteq \operatorname{End}(\mathbb{K}^{(J)})$ for the Lie algebras of all $J \times J$-matrices with finitely many non-zero entries, $\mathfrak{sl}(J, \mathbb{C}) := \{x \in \mathfrak{gl}(J, \mathbb{C}) \colon \operatorname{tr} x = 0\}$, and $\mathfrak{u}(J,\mathbb{K}) := \{x \in \mathfrak{gl}(J,\mathbb{K})\colon x^* = -x\}$. For topological vector spaces E, F, we write $\mathcal{L}(E,F)$ for the space of continuous linear maps $E \to F$ and $\mathcal{L}(E) = \mathcal{L}(E,E)$. For a subset C of a real linear space E we write $\mathrm{conv}(C)$ for its convex hull and $\mathrm{cone}(C)$ for the convex cone generated by C.

2. Open cones and semi-equicontinuous subsets

This section is independent of Lie algebraic structures. We first recall some basic facts on convex subsets of locally convex spaces and the duality between positively homogeneous lower semicontinuous convex functions on open cones and semi-equicontinuous weak-$*$-closed convex subsets of the topological dual space (Subsection 2.1). We discuss some fixed point results for group actions on convex sets in Subsection 2.2, and in Subsection 2.3 we provide some specific new results on convex hulls of orbits of Coxeter group which can be used to study adjoint and coadjoint orbits in Lie theory.

2.1. Basic facts and concepts

Let E be a real locally convex space and E' be its topological dual, i.e., the space of continuous linear functionals on E. The space of continuous linear endomorphisms of E is denoted $\mathcal{L}(E)$. We write $\langle \alpha, v\rangle = \alpha(v)$ for the natural pairing $E' \times E \to \mathbb{R}$ and endow E' with the weak-$*$-topology, i.e., the coarsest topology for which all evaluation maps $\eta_v\colon E' \to \mathbb{R}, \eta_v(\alpha) := \alpha(v)$ are continuous.

For a subset $C \subseteq E'$, the sets

$$C^\star := \{v \in E\colon (\forall \alpha \in C)\ \alpha(v) \geq 0\} \quad \text{and} \quad B(C) := \{v \in E\colon \inf\langle C, v\rangle > -\infty\}$$

are convex cones. A function $f\colon E \to \mathbb{R} \cup \{\infty\}$ is said to be *convex* (*lower semicontinuous*) if its epigraph $\mathrm{epi}(f) = \{(x,t) \in E \times \mathbb{R}\colon f(x) \leq t\}$ is convex (closed). The *support functional of a subset* $C \subseteq E'$

$$s_C\colon E \to \mathbb{R} \cup \{\infty\}, \quad s_C(v) := \sup\langle C, v\rangle,$$

takes finite values on $-B(C) = B(-C)$. As a sup of a family of continuous linear functionals, s_C is convex, lower semicontinuous and positively homogeneous. Note that s_C does not change if we replace C by its weak-$*$-closed convex hull. It is easy to see that

$$\mathrm{epi}(s_C) = ((-C) \times \{1\})^\star \tag{2}$$

Definition 2.1. We call a subset $C \subseteq E'$ *semi-equicontinuous* if s_C is bounded on some non-empty open subset of E (cf. [50]). This implies that the cone $B(C)$ has interior points and that s_C is continuous on $B(C)^0$ ([49, Prop. 6.8]). ■

Remark 2.2. (a) If the space E is barrelled, which includes in particular Banach and Fréchet spaces and their locally convex direct limits ([13]), then $C \subseteq E'$ is semi-equicontinuous if and only if $B(C)$ has interior points (apply [49, Thm. 6.10] to the lower semicontinuous function s_C).

(b) If $C \subseteq E'$ is semi-equicontinuous, then the lower semicontinuity of s_C implies that, for each $c > 0$, the subset $\{x \in E\colon s_C(x) \leq c\}$ is closed with dense interior. ■

Example 2.3. (a) A subset $C \subseteq E'$ is equicontinuous if $\{v \in E \colon s_C(v) \leq 1\}$ is a 0-neighborhood. This is equivalent to $B(C) = E$ and s_C being continuous. This clearly implies that C is semi-equicontinuous.

(b) If $\Omega \subseteq E$ is an open convex cone, then its dual cone $\Omega^\star = \{\alpha \in E' \colon \alpha|_\Omega \geq 0\}$ is semi-equicontinuous because its support functional $s_{\Omega^\star}$ is constant 0 on the open subset $-\Omega$. If, conversely, $C \subseteq E'$ is a convex cone, then $B(C) = C^\star$, so that C is semi-equicontinuous if and only if $C^\star$ has interior points. ■

Theorem 2.4. (Duality Theorem) *The assignment $C \mapsto s_C|_{B(-C)^0}$ defines a bijection from semi-equicontinuous weak-$*$-closed convex subsets $C \subseteq E'$ to continuous positively homogeneous convex functions defined on non-empty open convex cones.*

Proof. Since C can be reconstructed from $s_C|_{B(-C)^0}$ by

$$C = \{\alpha \in E' \colon (\forall v \in B(-C)^0)\, \alpha(v) \leq s_C(v)\} \tag{3}$$

([49, Prop. 6.4]), the assignment is injective. It remains to show that it is surjective. So let $\emptyset \neq \Omega \subseteq E$ be an open convex cone and let $f \colon \Omega \to \mathbb{R}$ be a continuous, positively homogeneous convex function. We claim that the weak-$*$-closed convex subset

$$C_f := \{\alpha \in E' \colon \alpha|_\Omega \leq f\} \quad \text{is equicontinuous with} \quad s_{C_f}|_\Omega = f. \tag{4}$$

Here $s_{C_f} \leq f$ follows from the definition and since Ω is open, C_f is semi-equicontinuous. To verify (4), let $x \in \Omega$. Then the pair $(x, f(x)) \in E \times \mathbb{R}$ lies in the boundary of the open convex cone $\mathrm{epi}(f)^0 = \{(x,t) \in \Omega \times \mathbb{R} \colon f(x) < t\}$. By the Hahn-Banach Separation Theorem, there exists a non-zero continuous linear functional $(-\alpha, c) \in \mathrm{epi}(f)^\star$ with $-\alpha(x) + cf(x) = 0$. Then $c \geq 0$. To see that we actually have $c > 0$, assume that $c = 0$. Then $\alpha(x) = 0$ and $\alpha \neq 0$, so that $\alpha(\Omega)$ is a neighborhood of 0, contradicting $-\alpha|_\Omega \geq 0$. We conclude that $c > 0$, and $c^{-1}\alpha \in C_f$ satisfies $c^{-1}\alpha(x) = f(x)$. This proves (4).

The dual cone $\mathrm{epi}(f)^\star$ is contained in the closed half space $E' \times [0, \infty)$. Since it contains all elements of the form $(-\alpha, 1)$, $\alpha \in C_f$, the intersection of $\mathrm{epi}(f)^\star$ with the open half space $E' \times (0, \infty)$ is weak-$*$-dense (because open line segments are dense in their closure). Hence $\mathrm{epi}(f)^\star$ is generated by $(-C_f) \times \{1\}$ as a weak-$*$-closed convex cone. We thus obtain with Proposition A.3 and (2)

$$\overline{\mathrm{epi}(f)} = (\mathrm{epi}(f)^\star)^\star = ((-C_f) \times \{1\})^\star = \mathrm{epi}(s_{C_f}). \tag{5}$$

As a consequence, $B(-C_f) \subseteq \overline{\Omega}$, so that $B(-C_f)^0 \subseteq \Omega$ by Lemma A.2(ii). Since $\Omega \subseteq B(-C_f)^0$ by definition, we have equality. This proves that $f = s_{C_f}|_{B(-C_f)^0}$. ■

Remark 2.5. Passing to the larger space $E \times \mathbb{R}$, any continuous, positively homogeneous convex function $f \colon \Omega \to \mathbb{R}$ is determined by its open epigraph $\mathrm{epi}(f)^0 = \{(x,t) \colon t > f(x)\}$ which is an open convex cone. This reduces the analysis of such functions largely to the analysis of open convex cones. As follows from (5), the open cone $\mathrm{epi}(f)^0$ is the interior of the dual cone of the semi-equicontinuous subset $(-C_f) \times \{1\} \subseteq E' \times \mathbb{R}$. ■

Remark 2.6. If $\iota\colon F \to E$ is an injective, continuous linear map with dense range, then its adjoint defines an injection $\iota'\colon E' \hookrightarrow F'$. Then any open convex subset $\Omega \subseteq E$ is determined by its intersection $\Omega \cap F = \iota^{-1}(\Omega)$, which is an open subset of F. Accordingly, any semi-equicontinuous subset $C \subseteq E'$ defines a semi-equicontinuous subset $\iota'(C) \subseteq F'$. This observation can often be used to study open convex subsets of E through F, which may be more accessible. ■

Lemma 2.7. *For a weak-$*$-closed convex subset $C \subseteq E'$, we have:*

(a) *If $A \in \mathcal{L}(E)$, then the adjoint $A'\colon E' \to E', \lambda \mapsto \lambda \circ A$ satisfies $A'C \subseteq C$ if and only if $s_C(Av) \leq s_C(v)$ for every $v \in E$.*

(b) *$\mathcal{L}(E)_C := \{A \in \mathcal{L}(E)\colon A'C \subseteq C\}$ is closed in the topology of pointwise convergence.*

Proof. (a) By the Hahn-Banach Separation Theorem,

$$C = \{\alpha \in E'\colon (\forall v \in E)\, \alpha(v) \leq s_C(v)\}.$$

Therefore $A'C \subseteq C$ is equivalent to $s_C \circ A = s_{A'C} \leq s_C$.
(b) Since s_C is lower semicontinuous, the subset $\{w \in E\colon s_C(w) \leq s_C(v)\}$ is closed for every $v \in E$. Hence (b) follows from (a). ■

Definition 2.8. For a convex subset C in the real linear space E, we define its *recession cone*

$$\lim(C) := \{x \in E\colon C + x \subseteq C\} \quad \text{and} \quad H(C) := \lim(C) \cap -\lim(C).$$

Then $\lim(C)$ is a convex cone and $H(C)$ a linear subspace. ■

Lemma 2.9. ([51, Lemma 2.9]) *If $\emptyset \neq C \subseteq E$ is an open or closed convex subset, then the following assertions hold:*

(i) $\lim(C) = \lim(\overline{C})$ *is a closed convex cone.*

(ii) $v \in \lim(C)$ *if and only if $t_j c_j \to v$ for a net with $t_j \geq 0$, $t_j \to 0$ and $c_j \in C$.*

(iii) *If $c \in C$ and $d \in E$ satisfy $c + \mathbb{R}_+ d \subseteq C$, then $d \in \lim(C)$.*

(iv) $H(C) = \{0\}$ *if and only if C contains no affine lines.*

(v) $B(C)^\star = \lim(C)$ *and* $B(C)^\perp = H(C)$.

Remark 2.10. (a) If $\dim E < \infty$, then a closed convex subset $C \subseteq E'$ is semi-equicontinuous if and only if it contains no affine lines, i.e., if the cone $\lim(C)$ is pointed. This follows from $\lim(C) = B(C)^\star$ (Lemma 2.9(v)).
(b) If E is a locally convex space and $C \subseteq E'$ is equicontinuous, then the recession cone $\lim(C)$ is trivial because half-lines are not equicontinuous. ■

Example 2.11. We consider the real Hilbert space

$$E := \ell^2 = \Big\{(a_n)_{n\in\mathbb{N}}\colon a_n \in \mathbb{R},\ \sum_{n=1}^\infty a_n^2 < \infty\Big\}$$

and identify E' with ℓ^2. Then $C := \{x \in E'\colon \|x\|_\infty \leq 1\}$ is a weak-$*$-closed convex subset. The cone $B(C)$ coincides with the subspace $\ell^1 \subseteq \ell^2$, hence has no interior points. In particular C is not semi-equicontinuous, although $\lim(C) = \{0\}$ and C contains no affine lines (Remark 2.10(a)).

This example is also interesting in the context of Remark 2.6. The continuous inclusion $\iota\colon \ell^1 \to \ell^2$ induces an inclusion $\iota'\colon \ell^2 \hookrightarrow \ell^\infty \cong (\ell^1)'$. Now $\iota'(C)$ is contained in the unit ball of ℓ^∞, hence is equicontinuous on ℓ^1, although C itself is not equicontinuous on ℓ^2. ■

Example 2.12. (a) If $\mathcal{E}$ is a real Hilbert space, then the closed unit ball $C := \{\xi \in \mathcal{E}\colon \|\xi\| \leq 1\}$ is a weakly compact convex set with $\lim(C) = \{0\}$.

(b) If $\mathcal{E}$ is a real Hilbert space, then $\mathcal{L} := \mathbb{R}\times\mathcal{E}\times\mathbb{R}$ carries the Lorentzian bilinear form

$$[\beta((z,x,t),(z',x',t')) := zt' + z't - \langle \xi, \eta\rangle,$$

and

$$\mathcal{L}_+ := \{\mathbf{x} := (z,x,t)\colon z \geq 0, \beta(\mathbf{x},\mathbf{x}) \geq 0\}$$

is a self-dual closed convex cone. For every $m > 0$, the hyperboloid

$$H := \{\mathbf{x} \in \mathcal{L}_+\colon \beta(\mathbf{x},\mathbf{x}) \geq m^2\}$$

is a weakly closed convex subset with $\lim(H) = \mathcal{L}_+$ (cf. Proposition A.9). The closed convex subset

$$P := \mathcal{L}_+ \cap (\mathbb{R} \times \mathcal{E} \times \{1\}) = \{\mathbf{x} = (z,x,1)\colon z \geq \tfrac{1}{2}\|x\|^2\}$$

is a paraboloid. It is the epigraph of the function $\|x\|^2/2$ on $\mathcal{E}$, and its recession cone satisfies

$$\lim(P) \subseteq \mathcal{L}_+ \cap (\mathbb{R} \times \mathcal{E} \times \{0\}) = \mathbb{R}_+(1,0,0),$$

hence coincides with the ray $\mathbb{R}_+(1,0,0)$.[1] ■

Definition 2.13. For a convex subset $C \subseteq E$ and $x \in C$, we write

$$L_x(C) := \overline{\mathbb{R}_+(C-x)}$$

for the *subtangent cone of* C *in* x. Then the restriction of a linear functional $\alpha \in E'$ to C takes a minimal value in x if and only if $\alpha \in L_x(C)^\star$. In particular, $L_x(C)^\star \subseteq B(C)$. ■

2.2. Fixed points

In this subsection we collect several results on the existence of fixed points for affine group actions. We start with the result for the most general environment, where the group has to be compact (cf. [51, Prop. 2.11]).

Proposition 2.14. *Let K be a compact group acting continuously on the complete locally convex space E by the representation $\pi\colon K \to \mathrm{GL}(E)$.*

(i) *If $\Omega \subseteq E$ is an open or closed K-invariant convex subset, then Ω is invariant under the fixed point projection $p(v) := \int_K \pi(k)v\, d\mu_K(k)$, where μ_K is a normalized Haar measure on K. In particular, $p(\Omega) = \Omega \cap E^K$.*

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[1] In view of the preceding examples, we may call a weak-$*$-closed convex subset $C \subseteq E'$ *elliptic* if $\lim(C) = \{0\}$, *parabolic* if $\lim(C) = \mathbb{R}_+\alpha$ for some $\alpha \neq 0$, and *hyperbolic* if $\lim(C)$ separates the points of E.

(ii) *If $C \subseteq E'$ is a weak-$*$-closed convex K-invariant subset, then C is invariant under the adjoint $p'(\lambda)v := \lambda(p(v)) = \int_K \lambda(\pi(k)v)\,d\mu_K(k)$. In particular, $p'(C) = C \cap (E')^K$.*

Remark 2.15. (a) The preceding proposition can also be used to study fixed point projections for non-compact groups. Let us assume that G contains a directed family $(K_j)_{j\in J}$ of compact subgroups for which the union $\bigcup_{j\in J} K_j$ is dense in G. If G acts continuously on E and $C \subseteq E'$ is a weak-$*$-compact G-invariant subset, then we obtain with Proposition 2.14(ii) for each $j \in J$ and $\lambda \in C$ that $p_j'(\lambda) := \int_{K_j} \lambda \circ \pi(k)\,d\mu_{K_j}(k) \in C$. As C is weak-$*$-compact, the net $(p_j'(\lambda))_{j\in J}$ has a cluster point in C which is invariant under every K_j, hence also under G.

(b) Proposition 2.14(i) does not extend to the situation of (a). Here a typical example arises from the action of the group $S_\infty = S_{(\mathbb{N})}$ of all finite permutations on $E = \mathbb{R}^{(\mathbb{N})}$. Then $\Omega := \{x \in \mathbb{R}^{(\mathbb{N})} \colon \sum_j x_j > 0\}$ is an open invariant convex cone which contains no fixed point, whereas its dual cone $\Omega^\star = (\mathbb{R}_+)^{\mathbb{N}}$ contains constant functions which are fixed points. ■

Kakutani's Fixed Point Theorem below ([72, Thm. 5.11]) weakens the assumption on the group but strengthens the assumptions on the invariant subset.

Theorem 2.16. (Kakutani's Fixed Point Theorem) *Let E be a locally convex space and let $G \subseteq \mathrm{GL}(E)$ be an equicontinuous group. Then each G-invariant compact convex subset of E contains a G-fixed point.*

The following proposition is a bridge between equicontinuity and compactness.

Proposition 2.17. *Let E be a locally convex space and let $G \subseteq \mathrm{GL}(E)$ be an equicontinuous group for which E is the union of finite-dimensional G-invariant subspaces. Then the following assertions hold:*

(i) *G is equicontinuous on the dual space E' with respect to the weak-$*$-topology.*

(ii) *There exists a continuous representation $\rho\colon K \to \mathrm{GL}(E)$ of a compact group K for which $G \subseteq \rho(K)$ is dense in the topology of pointwise convergence.*

Kakutani's Theorem applies to G-invariant weak-$*$-compact convex subsets $C \subseteq E'$ (cf. also Remark 2.15(a)), but, in view of (ii), one can also apply Proposition 2.14 to the compact group K.

Proof. (i) On E' the weak-$*$-topology coincides with the topology of uniform convergence on compact absolutely convex subsets of E which are contained in a finite-dimensional subspace. Our assumptions imply that E is the union of G-invariant compact absolutely convex subsets and the polars of these sets form a fundamental systems of G-invariant 0-neighborhoods in E'.

(ii) Consider E as a direct limit $E = \varinjlim E_j$ of finite-dimensional G-invariant subspaces. Then the representations (ρ_j, E_j) of G on the E_j yield a homomorphism $G \to \prod_{j\in I} \mathrm{GL}(E_j), g \mapsto (\rho_j(g))_{j\in I}$, and since each group $\rho_i(G)$ is relatively compact, we obtain a homomorphism of G into the compact group $\prod_{i\in I} \overline{\rho_i(G)}$. Let $K := \overline{G} \subseteq \mathrm{GL}(E)$ denote the closure with respect to the topology of pointwise convergence.

Then K also preserves all subspaces E_i, so that it may be considered as a subgroup of the compact group $\prod_{i\in I} \overline{\rho_i(G)}$, in which it is the closure of the image of G in this group, hence compact. That the action of K on E is continuous follows from the equicontinuity of K and the fact that all its orbit maps are continuous. ■

The following proposition is an example of a result not covered by these general tools. It can be proved by direct arguments:

Proposition 2.18. *Let J be a set. Then the following assertions hold:*

(a) *Every open convex cone $\Omega \subseteq \ell^\infty(J)$ invariant under the natural action of the full permutation group S_J contains a fixed point, i.e., a constant function.*

(b) *For a non-empty open $S_{(J)}$-invariant cone $\Omega \subseteq \mathbb{R}^{(J)}$, we have:*

  (i) *There exist disjoint finite subsets $\emptyset \neq F_1, F_2 \subseteq J$ and $a, b > 0$ with $ae_{F_1} - be_{F_2} \in \Omega$, where $e_F := \sum_{j\in F} e_j$.*

  (ii) *If Ω is proper, then the summation functional $\chi(x) := \sum_{j\in J} x_j$ satisfies $\chi \in \Omega^\star \cup -\Omega^\star$.*

  (iii) *$\Omega^\star \subseteq \ell^\infty(J, \mathbb{R})$.*

(c) *$\lambda \in \mathbb{R}^J \cong (\mathbb{R}^{(J)})'$ has a semi-equicontinuous orbit under $S_{(J)}$ if and only if it is bounded, and then its orbit is equicontinuous.*

Proof. (a) is [53, Lemma 3.5].

(b) (i): As Ω is open, there exists an element $x \in \Omega$ with $\chi(x) \neq 0$. If $F \supseteq \mathrm{supp}(x)$ is finite, then averaging over S_F shows that some multiple of e_F is contained in Ω. If $e_F \in \Omega$, then we put $F_1 := F$ and observe that, for any sufficiently small $b > 0$ and any finite subset $F_2 \subseteq J$ disjoint from F_1, the element $e_{F_1} - be_{F_2}$ is contained in Ω.

(b)(ii): If χ is not contained in $\Omega^\star$ or $-\Omega^\star$, then there exist $x, y \in \Omega$ with $\chi(x) < 0 < \chi(y)$. Then the argument in (i) provides a finite non-empty subset $F \subseteq J$ with $\pm e_F \in \Omega$. Hence $0 = e_F - e_F \in \Omega$ implies that $\Omega = \mathbb{R}^{(J)}$ is not proper.

(b)(iii): Let $\lambda \in \Omega^\star$ and choose F_1, F_2 and $a, b > 0$ as in (i) with $x := ae_{F_1} - be_{F_2} \in \Omega$. Applying λ to the $S_{(J\setminus F_2)}$-orbit of x, it follows that λ is bounded from below. We likewise see that λ is bounded from above by evaluating on the $S_{(J\setminus F_1)}$-orbit of x.

(c): Clearly, the orbit of a bounded function $\lambda\colon J \to \mathbb{R}$ is equicontinuous, hence in particular semi-equicontinuous. If, conversely, $S_{(J)}\lambda$ is semi-equicontinuous, then $B(S_{(J)}\lambda)^0 \subseteq \mathbb{R}^{(J)}$ is a non-empty open invariant cone. This cone does not change if we add a constant function to λ. Adding a suitable constant $c \in \mathbb{R}$, we see that the cone $(\mathcal{O}_{\lambda+c})^\star$ has interior points, and thus (b)(iii) implies that λ is bounded. ■

From the Bruhat-Tits Fixed Point Theorem (Theorem A.13), we obtain:

Proposition 2.19. *Let $\sigma\colon G \to E \rtimes \mathrm{O}(E)$ define an affine isometric action of the group G on the euclidean space E and $\emptyset \neq \Omega \subseteq E$ be an invariant open or closed convex subset. Then the following assertions hold:*

(i) *If E is complete and Ω contains a bounded G-orbit, then it contains a fixed point.*

(ii) *Suppose that the affine subspace E^G of fixed points is non-empty and complete. Then the orthogonal projection $p\colon E \to E^G$ satisfies $p(\Omega) = \Omega \cap E^G$.*

Proof. (i) By Lemma A.4, every proper open invariant convex subset of E is exhausted by closed invariant convex subsets. Hence it suffices to prove the assertion if Ω is closed. Then Ω is a Bruhat-Tits space with respect to the induced metric (Example A.14), so that the assertion follows from the Bruhat-Tits Theorem A.13.

(ii) As in (i), it suffices to consider the case where Ω is closed. As E^G is complete, we have $E = E^G \oplus (E^G)^\perp$. Let $\widehat{E}$ denote completion of E, extend the G-action to $\widehat{E}$, and write $\widehat{\Omega}$ for the closure of Ω in $\widehat{E}$. Then $\widehat{E} = E^G \oplus \widehat{(E^G)^\perp}$ shows that $\widehat{E}^G = E^G$.

We have to show that $c \in \Omega$ implies $p(c) \in \Omega$. The subset $C := p^{-1}(p(c)) \cap \widehat{\Omega} \subseteq \widehat{E}$ is a closed convex G-invariant subset containing c. As G fixes $p(c)$, the G-orbit of c is bounded, so that C contains a fixed point by (i). Since $p(c)$ is the only G-fixed point in $p^{-1}(p(c))$, it follows that $p(c) \in C \subseteq \widehat{\Omega}$. The closedness of Ω in E implies that $\widehat{\Omega} \cap E = \Omega$, so that we obtain $p(c) \in E^G \cap \widehat{\Omega} = E^G \cap \Omega$. ∎

2.3. Coxeter groups

In [22] we studied convex hulls of orbits of Coxeter groups to apply these results to orbit projections in infinite-dimensional Lie algebras (see [55, 36] for some concrete applications). Here we extend some of these results, so that they fit better the typical requirements in the Lie algebra context.

Definition 2.20. (a) Let V be a finite-dimensional real vector space. A *reflection data on* V consists of a finite family $(\alpha_s)_{s\in S}$ of linear functionals on V and a family $(\alpha_s^\vee)_{s\in S}$ of elements of V satisfying

$$\alpha_s(\alpha_s^\vee) = 2 \quad \text{for} \quad s \in S.$$

Then $r_s(v) := v - \alpha_s(v)\alpha_s^\vee$ is a reflection on V. We write $\mathcal{W} := \langle r_s \colon s \in S\rangle \subseteq \mathrm{GL}(V)$ for the subgroup generated by these reflections, which is a Coxeter group ([79], [26]). The cone $K := \{v \in V \colon (\forall s \in S)\, \alpha_s(v) \geq 0\}$ is called the *fundamental chamber.*

(b) A reflection data is called a *linear Coxeter system* (cf. [79]) if

(LCS1) K has interior points.

(LCS2) $(\forall s \in S)\, \alpha_s \notin \mathrm{cone}(\{\alpha_t \colon t \neq s\})$.

(LCS3) $(\forall w \in \mathcal{W} \setminus \{e\})\, wK^0 \cap K^0 = \emptyset$. ∎

Theorem 2.21. ([79, Thm. 2]) *For a linear Coxeter system* $(V, (\alpha_s)_{s\in S}, (\alpha_s^\vee)_{s\in S})$, *the following assertions hold:*

(i) *The subset* $\mathcal{T} := \mathcal{W}K$ *is a convex cone.*

(ii) $\mathcal{W}$ *acts discretely on the interior* $\mathcal{T}^0$ *of* $\mathcal{T}$, *i.e., point stabilizers are finite,* $\mathcal{T}^0/\mathcal{W}$ *is Hausdorff, and every* $x \in \mathcal{T}^0$ *has a neighborhood* U *with* $wU \cap U = \emptyset$ *for* $w \in \mathcal{W} \setminus \{e\}$.

(iii) *An element* $x \in K$ *is contained in* $\mathcal{T}^0$ *if and only if the stabilizer* $\mathcal{W}_x$ *is finite.*

The cone $\mathcal{T}$ is called the *Tits cone* of the linear Coxeter system. We write

$$\mathrm{co}(v) := \mathrm{conv}(\mathcal{W}v) \quad \text{and} \quad \overline{\mathrm{co}}(v) := \overline{\mathrm{conv}}(\mathcal{W}v) \tag{6}$$

for the (closed) convex hull of a $\mathcal{W}$-orbit.

We define two cones

$$C_S := \operatorname{cone}\{\alpha_s^\vee\colon s\in S\}\subseteq V, \qquad C_S^\vee := \operatorname{cone}\{\alpha_s\colon s\in S\}\subseteq V^*.$$

A *reflection in* $\mathcal{W}$ is an element conjugate to some r_s, $s\in S$. Any reflection can be written as $r_\alpha(v)=v-\alpha(v)\alpha^\vee$ with $\alpha\in V^*$ and $\alpha^\vee\in V$, where α belongs to the set $\Delta:=\mathcal{W}\{\alpha_s\colon s\in S\}$ of *roots* and $\alpha^\vee\in\Delta^\vee:=\mathcal{W}\{\alpha_s^\vee\colon s\in S\}$ is a *coroot*. To $v\in E$, we associate the cone

$$C_v := \operatorname{cone}\{\alpha^\vee\colon \alpha(v)>0\}.$$

We also recall from [22, Rem. 2.5] that the subset $\Delta^+ := \Delta\cap K^\star$ of positive roots satisfies $\Delta=\Delta^+\dot\cup -\Delta^+$ and

$$\Delta^+\subseteq C_S^\vee. \tag{7}$$

Lemma 2.22. *Let $C\neq\emptyset$ be an open convex cone in the finite-dimensional real vector space V and $\Gamma\subseteq\{g\in\mathrm{GL}(V)\colon gC=C\}$ be a subgroup such that $|\det(\gamma)|=1$ for all $\gamma\in\Gamma$. Then*

$$\overline{\operatorname{conv}}(\Gamma.v)\subseteq C \quad \textit{for all} \quad v\in C.$$

Proof. From [43, Thm. V.5.4] we know that there exists a smooth Γ-invariant convex function $\varphi\colon C\to(0,\infty)$ such that $x_n\to x\in\partial C$ and $x_n\in C$ implies $\varphi(x_n)\to\infty$. Therefore all sublevel sets $\{\varphi\leq c\}$, $c>0$, are convex subsets of C which are closed in V. As φ is Γ-invariant, the assertion follows from $\operatorname{conv}(\Gamma v)\subseteq\{\varphi\leq\varphi(v)\}$. ■

Lemma 2.23. *If $v\in\mathcal{T}$ and $w\in C_v$, then there exists an $\varepsilon>0$ such that $v-\varepsilon w\in\operatorname{co}(v)$.*

Proof. We write $w=\sum_{j=1}^k c_j\alpha_j^\vee$ with $c_j>0$ and $\alpha_j(v)>0$. Then $r_{\alpha_j}(v)=v-\alpha_j(v)\alpha_j^\vee\in\operatorname{co}(v)$ implies that

$$v-\sum_{j=1}^k d_j\alpha_j(v)\alpha_j^\vee\in\operatorname{co}(v) \quad \text{for} \quad d_j\geq 0, \sum_j d_j\leq 1.$$

For $t>0$ we compare this expression with

$$v_t := v-tw = v-\sum_{j=1}^k tc_j\alpha_j^\vee = v-\sum_{j=1}^k t\frac{c_j}{\alpha_j(v)}\alpha_j(v)\alpha_j^\vee,$$

and find that $v_t\in\operatorname{co}(v)$ if $t\sum_{j=1}^k\frac{c_j}{\alpha_j(v)}\leq 1$. This prove the lemma. ■

Lemma 2.24. *For $v\in K$, the following assertions hold:*

(i) *$C_v\subseteq C_S$ with equality for $v\in K^0$.*

(ii) *If $v\in\mathcal{T}^0$, then the stabilizer $\mathcal{W}_v$ is finite and $\bigcap_{w\in\mathcal{W}_v} wC_S=C_v$. This implies in particular that C_v is closed and that*

$$\bigcap_{w\in\mathcal{W}_v} w(v-C_S)=v-C_v \quad \textit{and} \quad \bigcap_{w\in\mathcal{W}} w(v-C_v)=\bigcap_{w\in\mathcal{W}} w(v-C_S). \tag{8}$$

Proof. (i) If $\alpha(v) > 0$ for some $v \in K$ and $\alpha \in \Delta$, then $\alpha \in \Delta^+ \subseteq C_S^\vee$ by (7). Write $\alpha = w\alpha_s \in \Delta$ for some $w \in \mathcal{W}$ and $s \in S$. Then $\alpha^\vee = w\alpha_s^\vee \in C_S$ follows from $\alpha = w\alpha_s \in C_S^\vee$ by [22, Thm. 1.10]. This proves $C_v \subseteq C_S$. If $v \in K^0$, then $\alpha_s(v) > 0$ for every $s \in S$, so that $C_S \subseteq C_v$ and thus $C_S = C_v$.

(ii) As $C_v \subseteq C_S$ for $v \in K$, and C_v is $\mathcal{W}_v$-invariant, it follows immediately that $C_v \subseteq \bigcap_{w\in\mathcal{W}_v} wC_S$.

Let $S_v := \{s \in S \colon \alpha_s(v) = 0\}$ and observe that $(V, (\alpha_s)_{s\in S_v}, (\alpha_s^\vee)_{s\in S_v})$ is also a linear Coxeter system ([22, Rem 1.5]). We know $\mathcal{W}_v = \langle r_s \colon s \in S_v\rangle$ from [22, Prop. 1.12]. If $\alpha \in \Delta^+$ vanishes on v, then $r_\alpha \in \mathcal{W}_v$, and thus $\alpha \in \mathcal{W}_v\{\alpha_s \colon s \in S_v\}$. We conclude that $\alpha^\vee \in C_{S_v}$. This shows that $C_S = C_v + C_{S_v}$, and this leads to

$$\bigcap_{w\in\mathcal{W}_v} wC_S \subseteq \bigcap_{w\in\mathcal{W}_v} w(C_v + C_{S_v}) = C_v,$$

where the last equality follows from [43, Cor. V.2.10], applied with $x = 0$. The first equality in (8) now follows from $w(v - C_S) = v - wC_S$ for $w \in \mathcal{W}_v$.

Finally, the second equality in (8) follows from

$$\bigcap_{w\in\mathcal{W}} w(v - C_v) = \bigcap_{w\in\mathcal{W}} \bigcap_{w'\in\mathcal{W}_v} ww'(v - C_S) = \bigcap_{w\in\mathcal{W}} w(v - C_S). \qquad \blacksquare$$

Theorem 2.25. (Convexity Theorem for Coxeter groups) *For $v \in \mathcal{T}$, we have*

$$\overline{\mathrm{co}}(v) \cap \mathcal{T} = \mathcal{T} \cap \bigcap_{w\in\mathcal{W}} w(v - \overline{C_v}). \tag{9}$$

If $v \in \mathcal{T}^0$, then C_v is closed, $\overline{\mathrm{co}}(v) \subseteq \mathcal{T}^0$, and

$$\overline{\mathrm{co}}(v) = \mathcal{T} \cap \bigcap_{w\in\mathcal{W}} w(v - C_v) = \mathcal{T} \cap \bigcap_{w\in\mathcal{W}} w(v - C_S). \tag{10}$$

For the special case where $v \in K^0$, this theorem is contained in [21, Satz 3.21], which was never published. For the case where $\mathcal{W}$ is finite, it is contained in [43, Prop. V.2.9].

Proof. From [22, Thm. 2.7] we obtain $\mathrm{co}(v) \subseteq v - C_v$, hence $\overline{\mathrm{co}}(v) \subseteq v - \overline{C_v}$. Since $\mathrm{co}(v)$ is $\mathcal{W}$-invariant, it follows that $\overline{\mathrm{co}}(v) \subseteq \bigcap_{w\in\mathcal{W}} w(v - \overline{C_v})$. Intersecting with $\mathcal{T}$ thus proves "$\subseteq$" in (9).

To verify "$\supseteq$", we use $\mathcal{T} = \mathcal{W}K$ to see that it suffices to show that, for $v \in K$, any $u \in K \cap (v - \overline{C_v})$ is contained in $\overline{\mathrm{co}}(v)$.

Case 1: $u \in v - C_v$. Consider the line segment

$$\gamma\colon [0,1] \to K, \qquad \gamma(t) := v + t(u - v).$$

Then $\gamma^{-1}(\overline{\mathrm{co}}(v)) = [0, c]$ for some $c \in [0,1]$ and we have to show that $c = 1$.

As $u, v \in K$, we have $\gamma(t) \in K$ for $0 \leq t \leq 1$. Hence $\alpha(u), \alpha(v) \geq 0$ for $\alpha \in \Delta \cap K^\star$, and if $\alpha(v) > 0$, then $\alpha(\gamma(t)) > 0$ for $0 \leq t < 1$. This implies that $C_v \subseteq C_{\gamma(t)}$ for $t < 1$.

Suppose that $c < 1$. For $v' := \gamma(c) \in \overline{\mathrm{co}}(v)$ we then obtain

$$u - v' = (1 - c)(u - v) \in -C_v \subseteq -C_{v'}.$$

By Lemma 2.23, there exists $\varepsilon \in (0,1)$ such that $v' + \varepsilon(u - v') \in \overline{\mathrm{co}}(v') \subseteq \overline{\mathrm{co}}(v)$. As

$$v' + \varepsilon(u - v') = v + c(u - v) + \varepsilon(1 - c)(u - v) = \gamma(c + \varepsilon(1 - c)),$$

this contradicts the maximality of c, so that we have shown that $c = 1$ and hence that $(v - C_v) \cap K \subseteq \overline{\mathrm{co}}(v)$.

Case 2: $u \in v - (\overline{C_v} \setminus C_v)$.

Case a: If $v - \overline{C_v}$ intersects K^0, then $(v - \overline{C_v}) \cap K^0$ is dense in $(v - \overline{C_v}) \cap K$, and since $v - C_v$ is dense in $v - \overline{C_v}$, the set $(v - C_v) \cap K^0 \subseteq (v - C_v) \cap K$ is also dense in $(v - \overline{C_v}) \cap K$. In Case 1 we have seen that $(v - C_v) \cap K \subseteq \overline{\mathrm{co}}(v)$, so that we also obtain $(v - \overline{C_v}) \cap K \subseteq \overline{\mathrm{co}}(v)$.

Case b: $(v - \overline{C_v}) \cap K^0 = \emptyset$: By definition of K^0, this is equivalent to the existence of an $s \in S$ with $\alpha_s(v - \overline{C_v}) \subseteq (-\infty, 0]$. As $\mathrm{co}(v) \subseteq v - C_v$ ([22, Thm. 2.17]), this entails $\alpha_s \in -\,\mathrm{co}(v)^\star$, so that $r^*_{\alpha_s}\alpha_s = -\alpha_s$ further leads to $\alpha_s \in \mathrm{co}(v)^\perp$.

We now reduce this case to the first one, for a smaller linear Coxeter system, writing

$$S = S' \dot\cup S'' \quad \text{with} \quad S' := \{s \in S \colon \alpha_s(C_v) \neq \{0\}\}, \quad S'' := \{s \in S \colon \alpha_s(C_v) = \{0\}\}.$$

By [22, Rem 1.5], $(V, (\alpha_s)_{s\in S'}, (\alpha_s^\vee)_{s\in S'})$ is also a linear Coxeter system. For $t \in S'$ and $s \in S''$ we have $\alpha_s(\alpha_t^\vee) = 0$ and by [22, Prop. 1.4] also $\alpha_t(\alpha_s^\vee) = 0$. This implies that r_s commutes with r_t. For

$$\mathcal{W}' := \langle r_s \colon s \in S' \rangle \quad \text{and} \quad \mathcal{W}'' := \langle r_s \colon s \in S'' \rangle$$

we therefore obtain $\mathcal{W} = \mathcal{W}'\mathcal{W}''$, where both subgroups commute, and $\mathcal{W}''$ fixes all points in $\mathrm{co}(v)$. In particular, we obtain

$$\mathrm{co}(v) = \mathrm{co}'(v) := \mathrm{conv}(\mathcal{W}'v).$$

Next we observe that $\overline{C_v} = \overline{\mathrm{cone}}(v - \mathrm{co}(v)) = \overline{C_v'}$ and

$$K' := \{v \in V \colon (\forall s \in S')\, \alpha_s(v) \geq 0\} \supseteq K,$$

so that it suffices to show that $(v - \overline{C_v'}) \cap K' \subseteq \overline{\mathrm{co}}'(v)$. By definition of S', no α_s, $s \in S'$, vanishes on C_v', so that this follows from Case 2a.

We thus obtain $(v - \overline{C_v}) \cap K \subseteq \overline{\mathrm{co}}(v)$ in all cases, and this proves (9).

If, in addition, $v \in \mathcal{T}^0$, then Lemma 2.22 and $|\det(r_s)| = 1$ for $s \in S$ imply that $\overline{\mathrm{co}}(v) \subseteq \mathcal{T}^0$, so that (10) follows from Lemma 2.24(ii). ■

Examples 2.26. For more details on typical classes of Coxeter groups, we refer to [26].

(a) For finite Coxeter groups we have $\mathcal{T} = E$. They arise in particular from finite-dimensional Lie groups as Weyl groups (Definition 3.6 and [43, Prop. VII.2.10]). A typical example is the action of the symmetric group S_n on the euclidean space $\mathbb{R}^n$, for $\alpha_j(x) = x_{j+1} - x_j$ and $\alpha_j^\vee = e_{j+1} - e_j$, $j = 1, \ldots, n-1$.

(b) For affine Coxeter groups, the space E' carries a positive semidefinite $\mathcal{W}$-invariant form degenerate along a one-dimensional subspace $\mathbb{R}\lambda$, so that $\mathcal{W}$ fixes λ. Then the hyperplane $H = \lambda^{-1}(1) \subseteq E$ is $\mathcal{W}$-invariant and the interior $\mathcal{T}^0 = \lambda^{-1}(\mathbb{R}_+^\times)$ of the Tits cone is an open half space. In this case $\mathcal{W}$ can be viewed as a group generated by affine euclidean reflections on H.

(c) Other important classes of Coxeter groups preserve a Lorentzian form and can be viewed as isometry groups of hyperbolic spaces ([79], [30]). ■

3. The Lie algebra context

3.1. The problems

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We are aiming at a systematic analysis of $\mathrm{Ad}^*(G)$-invariant semi-equicontinuous subsets $C \subseteq \mathfrak{g}'$, resp., $\mathrm{Ad}(G)$-invariant continuous positively homogeneous convex functions on open cones in $\mathfrak{g}$. By the Duality Theorem 2.4 and the subsequent remark, it basically suffices to study semi-equicontinuous invariant subsets $C \subseteq \mathfrak{g}'$ and open invariant cones in $\mathfrak{g}$, resp., $\mathfrak{g} \oplus \mathbb{R}$. This leads to the following problems:

(P1) Classify the semi-equicontinuous coadjoint orbits $\mathcal{O}_\lambda := \mathrm{Ad}^*(G)\lambda \subseteq \mathfrak{g}'$.

(P2) Classify open invariant convex cones in $\mathfrak{g}$.

Semi-equicontinuous weak-$*$-closed convex subsets $C \subseteq \mathfrak{g}'$ are locally compact (with respect to the weak-$*$-topology) and every $x \in B(C)^0$ defines a proper evaluation function $\eta_x \colon C \to \mathbb{R}, \alpha \mapsto \alpha(x)$. In particular, there exists an extreme point λ minimizing η_x ([49, Prop. 6.13]). We then have

$$\lambda(x) = \min \mathcal{O}_\lambda(x) = \min \lambda(\mathcal{O}_x) \quad \text{for} \quad \mathcal{O}_x = \mathrm{Ad}(G)x. \tag{min}$$

For $y \in \mathfrak{g}$, the smooth function $f(t) := \lambda(e^{t\,\mathrm{ad}\,y}x)$ then satisfies $0 = f'(0) \leq f''(0)$, which leads to the necessary conditions

$$\lambda([x,y]) = 0 \quad \text{and} \quad \lambda([y,[y,x]]) \geq 0 \quad \text{for} \quad y \in \mathfrak{g}. \tag{pos}$$

Here, elements $x \in \mathfrak{g}$ which are extreme points in $\mathrm{conv}(\mathcal{O}_x)$ are most interesting.

(P3) Determine natural conditions under which (pos) implies (min).

Unitary representations: If $(\pi, \mathcal{H})$ is a smooth unitary representation of G, then its support functional s_π (see (1) in the introduction), coincides with the support functional of the range of the momentum map on the projective space of $\mathcal{H}^\infty$:

$$\Phi_\pi \colon \mathbb{P}(\mathcal{H}^\infty) \to \mathfrak{g}' \quad \text{with} \quad \Phi_\pi([v])(x) = \frac{\langle v, -i\partial\pi(x)v\rangle}{\langle v, v\rangle} \quad \text{for } [v] = \mathbb{C}v.$$

The representation is semibounded if and only if the range of Φ_π is semi-equicontinuous. Then the *momentum set* $I_\pi := \overline{\mathrm{conv}}(\mathrm{im}\,\Phi_\pi)$ also has this property.

(P4) Calculate the momentum set I_π for irreducible semibounded representations π as concretely as possible. It is enough to do this for irreducible ones because we have direct integral decompositions for semibounded representations (cf. [58]), so that [43, Thm. X.6.16] on the momentum set of direct integrals reduces the problem to the irreducible case. As we shall see below, one may expect that

$$I_\pi = \overline{\mathrm{conv}}(\mathcal{O}_\lambda) \quad \text{for some} \quad \lambda \in \mathfrak{g}' \tag{11}$$

in all cases where the representation has an extremal weight λ (see [51, §5.2] and Examples 3.18 and 3.19).

Double extensions and projective unitary representations: In the context of unitary representations, the examples arising in physics [69, 71, 76, 51] often lead to projective unitary representations $(\mathtt{d}\pi, \mathcal{H}^\infty)$ of a semidirect product Lie algebra $\mathfrak{g}^\sharp = \mathfrak{g} \rtimes_D \mathbb{R}$, where $D \in \mathrm{der}(\mathfrak{g})$ and $\mathbf{d} := (0,1)$ satisfies the *positive energy condition* $-i\mathtt{d}\pi(\mathbf{d}) \geq 0$ (see also [55], [28, §8]). Here $\mathtt{d}\pi(x) := \partial\pi(x)|_{\mathcal{H}^\infty}$ and $\mathcal{H}^\infty$ is the space of smooth vectors.

Lifting to a unitary representation of a central extension, we thus obtain a Lie algebra of the form

$$\widehat{\mathfrak{g}} = (\mathbb{R} \oplus_\omega \mathfrak{g}) \rtimes \mathbb{R}\mathbf{d}, \qquad \mathbf{d} = (0,0,1), \tag{12}$$

with Lie bracket

$$[(z,x,t),(z',x',t')] = (\omega(x,x') + t\delta(x') - t'\delta(x), [x,x'] + tDx' - t'Dx, 0), \tag{13}$$

where $\omega\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a Lie algebra 2-cocycle and $[\mathbf{d},(z,x)] = (\delta(x), Dx)$ is a lift of the derivation D to the central extension $\tilde{\mathfrak{g}} = \mathbb{R} \oplus_\omega \mathfrak{g} = \mathbb{R}\mathbf{c} \oplus \mathfrak{g}$. Here δ, D and ω are related by the equation

$$\omega(Dx,x') + \omega(x,Dx') = \delta([x,x']) \quad \text{for} \quad x,x' \in \mathfrak{g}. \tag{14}$$

Definition 3.1. We call $\widehat{\mathfrak{g}}$ the *double extension* defined by the pair (ω, D) (if $\delta = 0$), resp., by (ω, D, δ). ■

As we may assume that $\lambda := (1,0,0) \in \widehat{\mathfrak{g}}'$ is contained in the momentum set (cf. [27, Prop. 15]), we then obtain

$$\inf \lambda(\mathcal{O}_{\mathbf{d}}) = \inf \mathcal{O}_\lambda(\mathbf{d}) > -\infty. \tag{15}$$

A particularly interesting case arises if there exists an eigenvector v_0 of $-i\mathtt{d}\pi(\mathbf{d})$ corresponding to the minimal spectral value 0. Then $\lambda := \Phi_\pi([v_0])$ satisfies the minimality condition $\inf \mathcal{O}_\lambda(\mathbf{d}) = \lambda(\mathbf{d}) = 0$, and the necessary conditions from (pos) take the form

$$\delta = 0 \quad \text{and} \quad \omega(Dx,x) \geq 0 \quad \text{for} \quad x \in \mathfrak{g}. \tag{16}$$

In particular the symmetric bilinear form $\omega(Dx,y)$ is positive semidefinite. For oscillator algebras ($\mathfrak{g}$ abelian), these conditions were the starting point for the classification results in [61] (see Subsection 5.2 for details). Accordingly, one has to explore the structural implications of (16) for other classes of Lie algebras. This leads to our last general problem:

(P5) Classify the pairs (ω, D) satisfying (14) with $\delta = 0$ and the *positive energy condition* (PEC):

$$\inf \mathcal{O}_\lambda(\mathbf{d}) = \lambda(\mathbf{d}) = 0. \tag{PEC}$$

Determine for which of these pairs (ω, D) the coadjoint orbit $\mathcal{O}_\lambda$ is semi-equicontinuous.

The positive energy condition is very natural because it is often satisfied in examples related to physics, where $-i\mathtt{d}\pi(\mathbf{d})$ corresponds to a Hamilton operator ([51], [69]). For some concrete results concerning the case where $\widehat{\mathfrak{g}}$ is locally finite, we refer to [35].

Remark 3.2. If the convex hull of the adjoint orbit $\mathcal{O}_{\mathbf{d}}$ has interior points in the hyperplane $\mathbb{R} \times \mathfrak{g} \times \{1\}$, then the cone generated by $\mathcal{O}_{\mathbf{d}}$ has interior points, and therefore $\mathcal{O}_{\mathbf{d}}^\star$ consists of semi-equicontinuous orbits (cf. Lemma A.15 for such situations). ■

3.2. Root decompositions

Definition 3.3. (a) We call an abelian subalgebra $\mathfrak{t}$ of the locally convex Lie algebra $\mathfrak{g} = \mathbf{L}(G)$ an *elliptic Cartan subalgebra* if $\mathfrak{t}$ is maximal abelian, the root spaces

$$\mathfrak{g}_{\mathbb{C}}^\alpha := \{x \in \mathfrak{g}_{\mathbb{C}} \colon (\forall h \in \mathfrak{t}_{\mathbb{C}})[h, x] = \alpha(h)x\}$$

span a dense subspace $\mathfrak{g}_{\mathbb{C}}^{\mathrm{alg}} \subseteq \mathfrak{g}_{\mathbb{C}}$, and

$$\alpha(\mathfrak{t}) \subseteq i\mathbb{R} \quad \text{for every root} \quad \alpha \in \Delta := \{\alpha \in \mathfrak{t}_{\mathbb{C}}^* \setminus \{0\} \colon \mathfrak{g}_{\mathbb{C}}^\alpha \neq \{0\}\}.$$

(b) Since $[\mathfrak{g}_{\mathbb{C}}^\alpha, \mathfrak{g}_{\alpha}^\beta] \subseteq \mathfrak{g}_{\mathbb{C}}^{\alpha+\beta}$, the subspace $\mathfrak{g}_{\mathbb{C}}^{\mathrm{alg}}$ is a Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and it intersects $\mathfrak{g}$ in the dense subalgebra $\mathfrak{g}^{\mathrm{alg}} := \mathfrak{g}_{\mathbb{C}}^{\mathrm{alg}} \cap \mathfrak{g}$. We shall always **assume** that the canonical projection $\mathfrak{g}^{\mathrm{alg}} \to \mathfrak{t}$ extends to a continuous projection $p_{\mathfrak{t}} \colon \mathfrak{g} \to \mathfrak{t}$, so that we can identify $\mathfrak{t}'$ with the subspace $(\ker p_{\mathfrak{t}})^\perp \subseteq \mathfrak{g}'$ of all functionals vanishing on all root spaces. We write $p_{\mathfrak{t}'} \colon \mathfrak{g}' \to \mathfrak{t}'$ for the restriction map.

(c) If $\sigma \colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ denotes the complex conjugation with respect to $\mathfrak{g}$, we write $x^* := -\sigma(x)$ for $x \in \mathfrak{g}_{\mathbb{C}}$, so that $\mathfrak{g} = \{x \in \mathfrak{g}_{\mathbb{C}} \colon x^* = -x\}$. We then have $x_\alpha^* \in \mathfrak{g}_{\mathbb{C}}^{-\alpha}$ for $x_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$. ■

Remark 3.4. If $\mathfrak{g}$ carries a positive definite $e^{\mathrm{ad}\,\mathfrak{t}}$-invariant form κ and $\mathfrak{t}$ is complete with respect to the induced scalar product, then $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp$ (even if $\mathfrak{g}$ itself is not complete). Therefore we have a natural orthogonal projection $p_{\mathfrak{t}} \colon \mathfrak{g} \to \mathfrak{t}$ and for every open or closed invariant subset $\Omega \subseteq \mathfrak{g}$, we have $p_{\mathfrak{t}}(\Omega) = \Omega \cap \mathfrak{t}$ by Proposition 2.19(ii). The completeness requirement for $\mathfrak{t}$ is automatic if $\dim \mathfrak{t} < \infty$, but there are also interesting situations where it is satisfied for infinite-dimensional $\mathfrak{t}$ but $\mathfrak{g}$ itself is not complete. Typical examples are the twisted loop algebras with values in Hilbert-Lie algebras (see Subsection 7.3). ■

In the present context it is of particular interest to understand adjoint orbits of elements $x \in \mathfrak{t}$ and coadjoint orbits of elements $\lambda \in \mathfrak{t}' \cong [\mathfrak{t}, \mathfrak{g}]^\perp$. This can be done in terms of their projections onto $\mathfrak{t}$, resp., $\mathfrak{t}'$.

The finite-dimensional Lie subalgebras arising from root vectors come in four types ([51, Lemma C.2]):

Lemma 3.5. *For $0 \neq x_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$ the subalgebra $\mathfrak{g}_{\mathbb{C}}(x_\alpha) := \mathrm{span}\{x_\alpha, x_\alpha^*, [x_\alpha, x_\alpha^*]\}$ is σ-invariant and of one of the following types:*

- (A) *The abelian type: $[x_\alpha, x_\alpha^*] = 0$, i.e., $\mathfrak{g}_{\mathbb{C}}(x_\alpha)$ is two-dimensional abelian.*
- (N) *The nilpotent type: $[x_\alpha, x_\alpha^*] \neq 0$ and $\alpha([x_\alpha, x_\alpha^*]) = 0$, i.e., $\mathfrak{g}_{\mathbb{C}}(x_\alpha)$ is a three-dimensional Heisenberg algebra.*
- (S) *The simple type: $\alpha([x_\alpha, x_\alpha^*]) \neq 0$, i.e., $\mathfrak{g}_{\mathbb{C}}(x_\alpha) \cong \mathfrak{sl}_2(\mathbb{C})$. In this case we distinguish the two cases:*
- (CS) *$\alpha([x_\alpha, x_\alpha^*]) > 0$, i.e., $\mathfrak{g}_{\mathbb{C}}(x_\alpha) \cap \mathfrak{g} \cong \mathfrak{su}_2(\mathbb{C})$, and*
- (NS) *$\alpha([x_\alpha, x_\alpha^*]) < 0$, i.e., $\mathfrak{g}_{\mathbb{C}}(x_\alpha) \cap \mathfrak{g} \cong \mathfrak{su}_{1,1}(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{R})$.*

Definition 3.6. We call a root $\alpha \in \Delta$ *compact* if there exists an element $x_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$ with $\alpha([x_\alpha, x_\alpha^*]) > 0$ and if the corresponding $\mathfrak{sl}_2(\mathbb{C})$-subalgebra $\mathfrak{g}_{\mathbb{C}}(x_\alpha)$ acts in a locally finite way on $\mathfrak{g}_{\mathbb{C}}^{\mathrm{alg}}$. Then $\dim \mathfrak{g}_{\mathbb{C}}^\alpha = 1$ ([44, Prop. I.6]) and there exists a unique element $\alpha^\vee \in i\mathfrak{t} \cap [\mathfrak{g}_{\mathbb{C}}^\alpha, \mathfrak{g}_{\mathbb{C}}^{-\alpha}]$ with $\alpha(\alpha^\vee) = 2$. We write $\Delta_k \subseteq \Delta$ for the subset of compact roots. The linear endomorphism

$$r_\alpha \colon \mathfrak{t} \to \mathfrak{t}, \quad r_\alpha(x) := x - \alpha(x)\alpha^\vee = x + (i\alpha)(x) i\alpha^\vee$$

is called the corresponding reflection and

$$\mathcal{W} := \mathcal{W}(\mathfrak{g}, \mathfrak{t}) := \langle r_\alpha \colon \alpha \in \Delta_k \rangle \subseteq \mathrm{GL}(\mathfrak{t})$$

is called the *Weyl group.* ■

The following proposition provides useful information for the analysis of invariant cones and orbit projections.

Proposition 3.7. *For $x \in \mathfrak{t}$, $x_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$ and $\lambda \in \mathfrak{t}'$, the following assertions hold:*

(i) $p_{\mathfrak{t}}(e^{\mathbb{R}\,\mathrm{ad}(x_\alpha - x_\alpha^*)}x) = x + \begin{cases} \mathbb{R}^+\alpha(x)[x_\alpha^*, x_\alpha] & \text{for } \alpha([x_\alpha, x_\alpha^*]) \le 0 \\ [-1,0]\alpha(x)\alpha^\vee & \text{for } \alpha([x_\alpha, x_\alpha^*]) > 0. \end{cases}$

(ii) $p_{\mathfrak{t}'}(e^{\mathbb{R}\,\mathrm{ad}^*(x_\alpha - x_\alpha^*)}\lambda) = \lambda + \begin{cases} \mathbb{R}^+\lambda([x_\alpha^*, x_\alpha])\alpha & \text{for } \alpha([x_\alpha, x_\alpha^*]) \le 0 \\ [-1,0]\lambda(\alpha^\vee)\alpha & \text{for } \alpha([x_\alpha, x_\alpha^*]) > 0. \end{cases}$

Proof. (i) is an immediate consequence of [43, Lemma VII.2.9], and (ii) follows from (i) and $p_{\mathfrak{t}'}(\lambda)(e^{\mathrm{ad}\, y}x) = \lambda(p_{\mathfrak{t}}(e^{\mathrm{ad}\, y}x))$. ■

Proposition 3.7 easily shows that root vectors of type (A) lead to degeneracies with respect to invariant convex sets:

Proposition 3.8. *Let $0 \neq x_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$ with $[x_\alpha, x_\alpha^*] = 0$.*

(i) *If $C \subseteq \mathfrak{g}$ is an open invariant convex subset intersecting $\mathfrak{t}$, then $i(x_\alpha + x_\alpha^*) \in \lim(C)$.*

(ii) *If $\mathcal{O}_\lambda \subseteq \mathfrak{g}'$ is semi-equicontinuous with $B(\mathcal{O}_\lambda)^0 \cap \mathfrak{t} \neq \emptyset$, then $i(x_\alpha + x_\alpha^*) \in \mathcal{O}_\lambda^\perp$.*

Note that (ii) implies that semi-equicontinuous coadjoint orbits lie in a proper weak-$*$-closed hyperplane.

Proof. (i) Let $x \in C \cap \mathfrak{t}$. Then $y := x_\alpha - x_\alpha^*$ and $z := i(x_\alpha + x_\alpha^*)$ satisfy $(\mathrm{ad}\, y)^2 x = 0$, so that

$$e^{\mathbb{R}\,\mathrm{ad}\, y}x = x + \mathbb{R}[x, y] = x + \mathbb{R} i\alpha(x) z.$$

Since α does not vanish on the open subset $C \cap \mathfrak{t}$, it follows that $\pm z \in \lim(C)$.

(ii) Since $\mathfrak{t} \cap B(\mathcal{O}_\lambda)^0$ is open, it contains an element x with $\alpha(x) \neq 0$. For y, z as above and $\mu \in \mathcal{O}_\lambda$, we now have

$$\mu(e^{\mathbb{R}\,\mathrm{ad}\, y}x) = \mu(x) + \mathbb{R} i\alpha(x)\mu(z).$$

As $\mathcal{O}_\lambda(x)$ is bounded from below, it follows that $\mu(z) = 0$, and thus $z \in \mathcal{O}_\lambda^\perp$. ■

To exclude the pathologies described in the preceding proposition, we introduce the following concept:

Definition 3.9. (a) We say that $\mathfrak{g}$ has *cone potential* if type (A) in Lemma 3.5 does not occur, i.e., $0 \neq x_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$ implies $[x_\alpha, x_\alpha^*] \neq 0$ ([43, Def. VII.2.22]).

(b) We call a root $\alpha \in \Delta$ *non-compact* if there exists a non-zero $x_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$ with $\alpha([x_\alpha, x_\alpha^*]) \leq 0$. We write $\Delta_p \subseteq \Delta$ for the set of non-compact roots. ■

The following theorem shows that, in central aspects, the action of the exponential $T = \exp \mathfrak{t}$ of an elliptic Cartan subalgebra $\mathfrak{t}$ behaves very much like a compact group. We may therefore expect that much of the finite-dimensional machinery developed in [43] carries over.

Theorem 3.10. *Let $\mathfrak{t} \subseteq \mathfrak{g}$ be an elliptic Cartan subalgebra for which there exists a continuous $\mathfrak{t}$-equivariant linear projection $p_{\mathfrak{t}} \colon \mathfrak{g} \to \mathfrak{t}$.*

(i) *If $\emptyset \neq C \subseteq \mathfrak{g}$ is open invariant and convex, then $p_{\mathfrak{t}}(C) = C \cap \mathfrak{t}$.*

(ii) *If $\emptyset \neq C \subseteq \mathfrak{g}'$ is semi-equicontinuous weak-$*$-closed and convex, then $p_{\mathfrak{t}'}(C) = C \cap \mathfrak{t}'$.*

Proof. (i) Let $T := \exp(\mathfrak{t}) \subseteq G$ be the connected subgroup corresponding to $\mathfrak{t}$. If $x \in \mathfrak{g}^{\text{alg}}$, then the orbit $\text{Ad}(T)x$ is finite-dimensional and its closure coincides with the orbit of a compact torus group T_1. Hence Proposition 2.14, applied to the T-action on finite-dimensional invariant subspaces, implies that $p_{\mathfrak{t}}(C \cap \mathfrak{g}^{\text{alg}}) \subseteq C$. As $C \cap \mathfrak{g}^{\text{alg}}$ is dense in C, we obtain $p_{\mathfrak{t}}(C) \subseteq \overline{C \cap \mathfrak{t}}$ from the continuity of $p_{\mathfrak{t}}$. Since $p_{\mathfrak{t}} \colon \mathfrak{g} \to \mathfrak{t}$ is a projection, the summation map $\mathfrak{t} \oplus \ker(p_{\mathfrak{t}}) \to \mathfrak{g}$ is a topological isomorphism, so that $p_{\mathfrak{t}}$ is also open. Hence $p_{\mathfrak{t}}(C)$ is an open subset of $\overline{C \cap \mathfrak{t}}$. Lemma A.2(ii) now show that $p_{\mathfrak{t}}(C) \subseteq C \cap \mathfrak{t}$. The inclusion $C \cap \mathfrak{t} \subseteq p_{\mathfrak{t}}(C)$ is trivial.

(ii) Applying (i) to the open invariant cone $B(C)^0$, we find an element $x_0 \in B(C)^0 \cap \mathfrak{t}$. Then all subsets

$$C_r := \{\lambda \in C \colon \lambda(x_0) \leq r\}, \quad r \in \mathbb{R},$$

are weak-$*$-compact ([49, Prop. 6.13]) and $\text{Ad}(T)$-invariant.

Let $R \colon \mathfrak{g}' \to (\mathfrak{g}^{\text{alg}})'$ denote the restriction map. As C_r is weak-$*$-compact, its image $R(C_r)$ is also weak-$*$-compact. The action of $\text{Ad}(T)$ on $E := \mathfrak{g}^{\text{alg}}$ satisfies the assumptions of Proposition 2.17, so that Kakutani's Theorem implies that, for every $\lambda \in C_r$, the subset $\overline{\text{conv}}(\text{Ad}(T)'\lambda)$ contains a fixed point, hence an element of $\mathfrak{t}'$. This implies that $p_{\mathfrak{t}'}(\lambda) = \lambda|_{\mathfrak{t}} \in C$. ■

Combining Theorem 3.10 with Proposition 3.7, we obtain the following two corollaries:

Corollary 3.11. *If $C \subseteq \mathfrak{g}$ is an open invariant convex subset, then $C_{\mathfrak{t}} := C \cap \mathfrak{t} = p_{\mathfrak{t}}(C)$ is $\mathcal{W}$-invariant and satisfies*

$$i\alpha(C_{\mathfrak{t}}) \cdot i[x_\alpha, x_\alpha^*] \subseteq \lim(C) \quad \textit{for} \quad \alpha([x_\alpha, x_\alpha^*]) \leq 0.$$

In particular, if $\lim(C)$ is pointed and $\mathfrak{g}$ has cone potential, then $i\Delta_p \subseteq C_{\mathfrak{t}}^\star \cup -C_{\mathfrak{t}}^\star$.

Corollary 3.12. *If $C \subseteq \mathfrak{g}'$ is a weak-$*$-closed convex subset, then $C_{\mathfrak{t}'} := C \cap \mathfrak{t}' = p_{\mathfrak{t}'}(C)$ is $\mathcal{W}$-invariant with*

$$\langle C, i[x_\alpha, x_\alpha^*]\rangle \cdot i\alpha \subseteq \lim(C_{\mathfrak{t}'}) \quad \textit{for} \quad \alpha([x_\alpha, x_\alpha^*]) \leq 0.$$

If $\lim(C_{\mathfrak{t}'})$ is pointed, then $i[x_\alpha, x_\alpha^*] \in C^\star \cup -C^\star$, and $i[x_\alpha, x_\alpha^*] \in C^\star$ implies $i\alpha \in \lim(C_{\mathfrak{t}'})$.

Remark 3.13. Assume that $\mathfrak{g}$ has cone potential.
(a) Any open invariant convex cone $C \subseteq \mathfrak{g}$ intersects $\mathfrak{t}$ by Theorem 3.10. If $\overline{C}$ is pointed, then Corollary 3.11 implies that C specifies a $\mathcal{W}$-invariant *positive system of non-compact roots*

$$\Delta_p^+ := \{\alpha \in \Delta_p \colon i\alpha \in C_{\mathfrak{t}}^\star\} = \Delta_p \cap -iC_{\mathfrak{t}}^\star \quad \text{with} \quad \Delta_p = \Delta_p^+ \dot\cup - \Delta_p^+.$$

This in turn defines two cones:

$$C_{\min}(\Delta_p^+) := \overline{\mathrm{cone}}(\{i[x_\alpha, x_\alpha^*] \colon x_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha, \alpha \in \Delta_p^+\}), \qquad C_{\max}(\Delta_p^+) := (i\Delta_p^+)^\star. \quad (17)$$

Corollary 3.11 now entails that

$$C_{\min}(\Delta_p^+) \subseteq \overline{C_{\mathfrak{t}}} \subseteq C_{\max}(\Delta_p^+). \quad (18)$$

(b) Suppose that $\lambda \in \mathfrak{t}'$ is such that $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda \subseteq \mathfrak{g}'$ is semi-equicontinuous and put $C_\lambda := \overline{\mathrm{conv}}(\mathcal{O}_\lambda)$ (weak-$*$ closure). We further assume that $\mathcal{O}_\lambda^\perp = \{0\}$. Then $B(\mathcal{O}_\lambda)^0 = B(C_\lambda)^0 \subseteq \mathfrak{g}$ is an open invariant convex cone with $B(\mathcal{O}_\lambda)^\star = \lim(C_\lambda)$ (Lemma 2.9). As in (a), we choose the positive system Δ_p^+ such that

$$i\Delta_p^+ = (i\Delta_p) \cap B(\mathcal{O}_\lambda)^\star = (i\Delta_p) \cap \lim(C_\lambda)$$

and obtain the relation

$$\mathrm{conv}(\mathcal{W}\lambda) + \mathrm{cone}(i\Delta_p^+) \subseteq p_{\mathfrak{t}'}(\overline{\mathrm{conv}}(\mathcal{O}_\lambda)). \quad (19)$$

In Remark 3.17 below we shall see how representation theoretic arguments can be used to obtain the converse inclusion. In any case we see that

$$B(\mathcal{O}_\lambda) \cap \mathfrak{t} \subseteq B(\mathcal{W}\lambda) \cap (i\Delta_p^+)^\star = B(\mathcal{W}\lambda) \cap C_{\max}(\Delta_p^+). \quad (20)$$

In particular, the cone $C_{\max}(\Delta_p^+)$ needs to have interior points if $\mathcal{O}_\lambda$ is semi-equicontinuous, and the Weyl group orbit $\mathcal{W}\lambda \subseteq \mathfrak{t}$ has to be semi-equicontinuous too. ■

Remark 3.14. Let $\mathfrak{t} \subseteq \mathfrak{g}$ be an elliptic Cartan subalgebra and $T := \exp \mathfrak{t}$. For $\alpha \in \Delta$, we consider the character $\chi_\alpha(\exp x) := e^{i\alpha(x)}$ of T and the subgroup $\mathcal{Q} \subseteq \widehat{T} := \mathrm{Hom}(T, \mathbb{T}) \hookrightarrow \mathfrak{t}'$, generated by these characters. We consider $\mathcal{Q}$ as a discrete group, so that

$$S := \widehat{\mathcal{Q}} = \mathrm{Hom}(\mathcal{Q}, \mathbb{T})$$

is a compact abelian group. The adjoint representation defines a homomorphism

$$A \colon T \to S, \quad A(\exp x)_\beta := e^{i\beta(x)}$$

whose range separates the character group $\widehat{S} \cong \mathcal{Q}$ and therefore is dense in S ([24, Thm. 7.64]).

Each $s = (s_\beta)_{\beta\in\Delta} \in S$ defines an automorphism $\psi(s) \in \mathrm{Aut}(\mathfrak{g}^{\mathrm{alg}}) \cong \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}^{\mathrm{alg}})^\sigma$ by $\psi(s)x_\alpha = \chi_\alpha(s)x_\alpha$ for $x_\alpha \in \mathfrak{g}_{\mathbb{C}}$

(a) If $\mathfrak{g} = \mathfrak{g}^{\mathrm{alg}}$, then the topology of pointwise convergence coincides on $A(T)$ with the product topology induced from S, so that S is the closure of $A(T)$ in the topology of pointwise convergence. This group preserves all weak-$*$-closed $\mathrm{Ad}(T)$-invariant convex subsets of the algebraic dual $(\mathfrak{g}^{\mathrm{alg}})^*$ (Lemma 2.7).

(b) If $\mathfrak{g}$ is strictly larger than $\mathfrak{g}^{\mathrm{alg}}$, then it is not clear that all elements of S act by automorphisms on $\mathfrak{g}$. However, in all concrete examples we are aware of, this is the case. For example if $\mathfrak{g} = \mathfrak{u}_p(\mathcal{H})$ and $(e_j)_{j\in J}$ is an orthonormal basis, then $\mathfrak{t} \cong i\ell^p(J,\mathbb{R})$ is an elliptic Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{g}^{\mathrm{alg}} = \mathfrak{u}(J,\mathbb{C})$. Here $S \cong \mathbb{T}^J/\mathbb{T}$, where $\mathbb{T} \subseteq \mathbb{T}^J$ represents the constant functions $J \to \mathbb{T}$. The compact group S acts continuously on $\mathfrak{u}_p(\mathcal{H})$ (cf. Remark 5.6). ■

3.3. The connection with unitary representations

Unitary representations can provide information on invariant convex subsets through their momentum sets and support functionals.

Proposition 3.15. *If $(\pi,\mathcal{H})$ is a unitary representation of G, then*

$$C_\pi := \{x \in \mathfrak{g} \colon -i\partial\pi(x) \geq 0\} = I_\pi^\star$$

is a closed convex invariant cone in $\mathfrak{g}$. If there exists an element $\mathbf{c} \in \mathfrak{g}$ with $\partial\pi(\mathbf{c}) = i\mathbf{1}$ and π is semibounded, then C_π has interior points.

Proof. The definition of the momentum set (Subsection 3.1) implies that $C_\pi = I_\pi^\star$, so that the $\mathrm{Ad}(G)$-invariance of I_π implies that the closed convex cone C_π is also invariant.
For the second assertion, suppose that $s_\pi \leq m$ on an open subset $U \subseteq \mathfrak{g}$. Then $-m\mathbf{c} + U \subseteq C_\pi$ shows that C_π has interior points. ■

Remark 3.16. (Connections with Lie supergroups) For a unitary representation of a (possible infinite-dimensional) Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$, we have $-i\mathrm{d}\pi(x)^2 \geq 0$ for odd elements $x \in \mathfrak{g}_{\overline{1}}$ ([57]). Therefore

$$C_\pi := \{x \in \mathfrak{g}_{\overline{0}} \colon -i\partial\pi(x) \geq 0\}$$

is a closed convex invariant cone in the even part $\mathfrak{g}_{\overline{0}}$ which contains all brackets $[x,x]$, $x \in \mathfrak{g}_{\overline{1}}$. In many situations this cone has interior points, so that a classification of open invariant cones in $\mathfrak{g}_{\overline{0}}$ is of central importance to understand the unitary representations of the Lie superalgebra $\mathfrak{g}$ ([5], [60]). ■

Remark 3.17. (a) Let $\iota\colon H \to G$ be an injective morphism of connected Lie groups and identify $\mathfrak{h}$ by the tangent map $\mathbf{L}(\iota)$ with a Lie subalgebra of $\mathfrak{g}$. If $(\pi,\mathcal{H})$ is a semibounded unitary representation for which there exists an $x_0 \in \mathfrak{h} \cap B(I_\pi)^0$, then $\pi|_H$ is a semibounded representation of H with momentum set

$$I_{\pi|_H} = I_\pi|_{\mathfrak{h}}.$$

If the spectrum of the restriction $\pi|_H$ is known, this representation theoretic information can be used to derive information on the momentum set I_π and on coadjoint G-orbits (see [53, 55] for examples).

In particular, we obtain for every $\lambda \in I_\pi$ the relation

$$\mathcal{O}_\lambda|_{\mathfrak{h}} \subseteq I_{\pi|_H}. \tag{21}$$

(b) If H is abelian, then every semibounded representation $(\pi, \mathcal{H})$ of H is given by a spectral measure P on $\mathfrak{h}'$ ([49, §7]) and $I_\pi = \overline{\mathrm{conv}}(\mathrm{supp}(P))$, so that, in the context of (a),

$$\mathcal{O}_\lambda|_{\mathfrak{h}} \subseteq \overline{\mathrm{conv}}(\mathrm{supp}(\pi|_H)). \tag{22}$$

This information is particularly useful if one has explicit information on the spectrum of $\pi|_H$. ■

Examples 3.18. An important special case arises for $H = T = \exp(\mathfrak{t})$, $\mathfrak{t}$ an elliptic Cartan subalgebra, if $\pi|_T$ is a direct sum of weight spaces

$$\mathcal{H}_{i\mu}(\mathfrak{t}) = \{v \in \mathcal{H} \colon (\forall x \in \mathfrak{t})\, \mathtt{d}\pi(x)v = i\mu(x)v\}, \quad \mu \in \mathcal{P}_\pi := \{\nu \in \mathfrak{t}' \colon \mathcal{H}_{i\nu}(\mathfrak{t}) \neq \{0\}\}.$$

If $\quad \overline{\mathrm{conv}}(\mathrm{supp}(\pi|_T)) = \overline{\mathrm{conv}}(\mathcal{W}\lambda) \quad$ for a weight $\quad \lambda \in \mathfrak{t}'$, (23)

then we obtain from (22) $\quad \mathcal{O}_\lambda|_{\mathfrak{t}} \subseteq \overline{\mathrm{conv}}(\mathcal{W}\lambda).$ (24)

This situation arises in particular if:

(a) $\Delta = \Delta_k$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{\mathrm{alg}}$ is a locally finite Lie algebra, and π_λ is a unitary representation with extremal weight $i\lambda$ ([41], [46, §3]).

(b) $\mathfrak{g} = \mathfrak{g}^{\mathrm{alg}}$ is a unitary real form of the Kac-Moody Lie algebra $\mathfrak{g}_{\mathbb{C}}$, i.e., all root vectors are of type (N) or (CS), and π_λ is a unitary representation with highest weight $i\lambda$ with respect to some positive system Δ^+ ([30, Ch. 11]). These representations exists if and only if $i\lambda(\check{\alpha}) \in \mathbb{N}_0$ for all simple roots $\alpha \in \Delta_+$.

(c) $\mathfrak{g} = \mathfrak{g}^{\mathrm{alg}}$ is a unitary form of a locally affine Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and π_λ is a unitary representation with extremal weight $i\lambda$ not vanishing in the central element $\mathbf{c}$ (see [52, Thms. 4.10/11] for details). These Lie algebras arise in particular as dense subalgebras of twisted loop algebras $\mathcal{L}_\varphi^\tau(\mathfrak{k}_{\mathbb{C}})$ (see Subsection 7.3).

In all these cases, combining (24) with the trivial inclusion $\mathcal{W}\lambda \subseteq \mathcal{O}_\lambda \cap \mathfrak{t}' \subseteq p_{\mathfrak{t}'}(\mathcal{O}_\lambda)$ leads to

$$p_{\mathfrak{t}'}(\overline{\mathrm{conv}}(\mathcal{O}_\lambda)) = \overline{\mathrm{conv}}(\mathcal{W}\lambda) \tag{25}$$

for the extremal weight $i\lambda$ of the unitary representation π. ■

Example 3.19. If $\mathfrak{g} = \mathfrak{g}^{\mathrm{alg}}$ is locally finite and not all roots are compact, then the situation is more complicated. For the corresponding unitary highest weight representations π_λ (see [45] for a classification) we obtain under the assumption that π_λ has discrete kernel that

$$\mathrm{conv}(\mathcal{P}_{\pi_\lambda}) = \mathrm{conv}(\mathcal{W}\lambda) + i\,\mathrm{cone}(\Delta_p^+), \tag{26}$$

which leads to $\quad p_{\mathfrak{t}'}(\overline{\mathrm{conv}}(\mathcal{O}_\lambda)) = \overline{\mathrm{conv}(\mathcal{W}\lambda) + i\,\mathrm{cone}(\Delta_p^+)}.$ (27)

Problem 3.20. We define a quasi-order on $\mathfrak{g}'$ by

$$\mu \prec \lambda \quad \text{if} \quad \overline{\mathrm{conv}}(\mathcal{O}_\mu) \subseteq \overline{\mathrm{conv}}(\mathcal{O}_\lambda)$$

and say that λ *majorizes* μ if $\mu \prec \lambda$.

(P6) Let $\mathfrak{t} \subseteq \mathfrak{g}$ be an elliptic Cartan subalgebra. Give an explicit description of the quasi-order $\prec$ on $\mathfrak{t}' \subseteq \mathfrak{g}'$.

If all roots are compact, this this is closely related to the order relation

$$\mu \prec_{\mathcal{W}} \lambda \quad \text{if} \quad \overline{\mathrm{conv}}(\mathcal{W}\mu) \subseteq \overline{\mathrm{conv}}(\mathcal{W}\lambda).$$

See Example 5.3, Remark 5.4, Remark 5.6 and Problem 5.7 for related issues. ■

4. Finite-dimensional Lie algebras

In this section we recall the key results on invariant convex cones in finite-dimensional Lie algebras.

4.1. Compact Lie algebras

The best behaved examples are *compact* Lie algebras $\mathfrak{g}$, i.e., Lie algebras of a compact Lie group G. Then $\mathfrak{g}$ is finite-dimensional and carries an $\mathrm{Ad}(G)$-invariant scalar product κ. The map $\mathfrak{g} \to \mathfrak{g}', x \mapsto x^* := \kappa(x, \cdot)$ is a G-equivariant equivalence between adjoint and coadjoint representation. Moreover, every x is an extreme point of $\mathrm{conv}(\mathcal{O}_x)$ because the orbit lies in a euclidean sphere.
We collect the main results in the following theorem:

Theorem 4.1. *Let $\mathfrak{g}$ be a compact Lie algebra, $\mathfrak{t} \subseteq \mathfrak{g}$ be maximal abelian and $p_{\mathfrak{t}} \colon \mathfrak{g} \to \mathfrak{t}$ be the projection along $[\mathfrak{t}, \mathfrak{g}]$. Then the following assertions hold:*

(i) *Every adjoint orbit $\mathcal{O}_x = \mathrm{Ad}(G)x \subseteq \mathfrak{g}$ intersects $\mathfrak{t}$ in an orbit of the Weyl group $\mathcal{W}$, which is a finite Coxeter group.*

(ii) $p_{\mathfrak{t}}(\mathcal{O}_x) = \mathrm{conv}(\mathcal{W}x)$ *for $x \in \mathfrak{t}$* (Kostant's Convexity Theorem).

(iii) *The map $C \mapsto C \cap \mathfrak{t}$ defines a bijection from $\mathrm{Ad}(G)$-invariant open/closed convex subsets of $\mathfrak{g}$ to $\mathcal{W}$-invariant open/closed convex subsets of $\mathfrak{t}$.*

(iv) *For $x \in \mathfrak{t}$, the cone $L_x(\mathrm{conv}\,\mathcal{W}x)$* (Definition 2.13) *is generated by the elements $-\alpha(x)\alpha^\vee$, $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$.*

(v) *If $\lambda([x, y]) = 0$ and $\lambda([y, [y, x]]) \geq 0$ hold for all $y \in \mathfrak{g}$, then $\lambda(x) = \min \lambda(\mathcal{O}_x)$.*

Proof. (i) [19, Thm. 12.2.2, Lemma 12.2.16]; (ii) is contained in [32], and (iii) in [40, Thm. III.17].

(iv) follows from the fact that $\mathcal{W}$ is a finite Coxeter group with $\mathcal{T} = \mathfrak{t}$ and [22, Thm. 2.7] (see also Theorem 2.25).

(v) For $z^* = \kappa(z, \cdot)$, the first part of condition (pos) leads to $[x, z] = 0$ so that both lie in a maximal abelian subalgebra $\mathfrak{t}$, and the second part of (pos) can be analyzed in terms of the root decomposition (Definition 3.3) which implies (v). ■

Assertions (iii) and (iv) in Theorem 4.1 are the key to a complete classification of convex $\mathcal{W}$-invariant subsets of $\mathfrak{t}$ and $\mathrm{Ad}(G)$-invariant convex subsets of $\mathfrak{g}$, carried out in [40, 43].

Example 4.2. For $G = \mathrm{U}_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{u}_n(\mathbb{C})$, we chose the subspace $\mathfrak{t} \cong i\mathbb{R}^n$ of diagonal matrices as an elliptic Cartan subalgebra and

$$p_\mathfrak{t} \colon \mathfrak{u}_n(\mathbb{C}) \to i\mathbb{R}^n, \quad p_\mathfrak{t}(x) = \mathrm{diag}(x_{11}, \ldots, x_{nn})$$

is the corresponding projection. If $x = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathfrak{t}$, then the adjoint orbit $\mathcal{O}_x := \{gxg^{-1} \colon g \in \mathrm{U}_n(\mathbb{C})\}$ is the set of all skew-hermitian matrices with the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, and the Schur-Horn Convexity Theorem asserts that

$$p_\mathfrak{t}(\mathcal{O}_x) = \mathrm{conv}(\{(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}) \colon \sigma \in S_n\}) = \mathrm{conv}(S_n.\lambda) \tag{28}$$

is the convex hull of the orbit of x under the symmetric group $S_n \cong \mathcal{W}$. ■

Remark 4.3. For a compact Lie algebra $\mathfrak{g}$, Proposition 2.14 implies in particular that every open invariant cone $\Omega \subseteq \mathfrak{g}$ intersects the center $\mathfrak{z}(\mathfrak{g})$. In particular, $\mathfrak{g}$ contains no non-trivial open invariant cones if $\mathfrak{z}(\mathfrak{g}) = \{0\}$, i.e., if $\mathfrak{g}$ is semisimple. If, conversely, $0 \neq z \in \mathfrak{z}(\mathfrak{g})$, then there exist open convex $\mathrm{Ad}(G)$-invariant neighborhoods U of z not containing 0, and then $\Omega := \bigcup_{t>0} tU$ is a proper open invariant cone (cf. Lemma A.5(b)). ■

4.2. Non-compact finite-dimensional Lie algebras

In this subsection we assume that $\mathfrak{g}$ is finite-dimensional and that $\mathfrak{t} \subseteq \mathfrak{g}$ is a compactly embedded (=elliptic) Cartan subalgebra. The existence of $\mathfrak{t}$ is ensured by the assumption that the subset $\mathfrak{g}'_{\mathrm{seq}} \subseteq \mathfrak{g}'$ of functionals λ with semi-equicontinuous orbit $\mathcal{O}_\lambda$ separates the points of $\mathfrak{g}$ ([43, Thm. VII.3.28]). If G is a connected Lie group with $\mathfrak{g} = \mathbf{L}(G)$, then we write $T = \exp(\mathfrak{t})$ for the subgroup corresponding to $\mathfrak{t}$.

For $\lambda \in \mathfrak{t}'$, the orbit $\mathcal{O}_\lambda \subseteq \mathfrak{g}'$ is semi-equicontinuous if and only if it is admissible in the sense of [43] because it is automatically closed ([43, Thm. VIII.1.8]). The Lie algebra $\mathfrak{g}$ is said to be *quasihermitian* if $\mathfrak{z}_\mathfrak{g}(\mathfrak{z}(\mathfrak{k})) = \mathfrak{k}$ holds for the compact Lie subalgebra $\mathfrak{k}$ with $\mathfrak{k}_\mathbb{C} = \mathfrak{t}_\mathbb{C} + \sum_{\alpha \in \Delta_k} \mathfrak{g}_\mathbb{C}^\alpha$. This condition is always satisfied if $\mathfrak{g}'$ is generated by semi-equicontinuous orbits ([43, Thm. VII.3.28]).

Theorem 4.4. (Convexity theorem for semi-equicontinuous orbits – finite-dim. case) *Suppose that $\mathfrak{g}$ is quasihermitian. For $\lambda \in \mathfrak{t}'$, the coadjoint orbit $\mathcal{O}_\lambda$ is semi-equicontinuous if and only if there exists a $\mathcal{W}$-invariant positive system Δ_p^+ with $\lambda \in C_{\min}(\Delta_p^+)^\star$, and then $p_{\mathfrak{t}'}(\mathcal{O}_\lambda)$ is convex with*

$$p_{\mathfrak{t}'}(\mathcal{O}_\lambda) \subseteq \mathrm{conv}(\mathcal{W}\lambda) + \mathrm{cone}(i\Delta_p^+). \tag{29}$$

If, in addition, $\mathcal{O}_\lambda^\perp = \{0\}$, then equality holds in (29).

Proof. The first part and (29) follow from [43, Def. VII.2.6, Thm. VIII.1.19]. If $\mathcal{O}_\lambda$ spans $\mathfrak{g}'$, then (19) in Remark 3.13(b) implies equality in (29). ■

For adjoint orbits we have ([43, Thm. VIII.1.36]):

Theorem 4.5. (Convexity theorem for adjoint orbits – finite-dimensional case) *Suppose that $\mathfrak{g}$ is finite-dimensional and $\mathfrak{t} \subseteq \mathfrak{g}$ is an elliptic Cartan subalgebra. If Δ_p^+ is a $\mathcal{W}$-invariant positive system of non-compact roots, then*

$$p_\mathfrak{t}(\mathcal{O}_x) \subseteq \mathrm{conv}(\mathcal{W}x) + C_{\min}(\Delta_p^+) \quad \textit{for} \quad x \in C_{\max}.$$

These convexity theorems lead to complete information on invariant convex subsets (cf. Remark 3.17(c)). The main tool is the action of the finite Coxeter group $\mathcal{W} = \mathcal{W}(\mathfrak{g}, \mathfrak{t})$, and the determination of the cones $C_{\max}(\Delta_p^+)$ for the different $\mathcal{W}$-invariant positive systems in Δ_p.

Remark 4.6. (a) If $\lambda \in \mathfrak{g}'$ is contained in the algebraic interior of an invariant semi-equicontinuous subset $C \subseteq \mathfrak{g}'$, i.e., $C - \lambda$ is absorbing in the linear space $C-C$ (λ is strictly admissible in the terminology of [43]), then $\mathcal{O}_\lambda$ intersects $\mathfrak{t}'$ ([43, Thm. VIII.1.8, Lemma VIII.1.27]), $\mathrm{conv}(\mathcal{O}_\lambda)$ is closed and $\mathcal{O}_\lambda = \mathrm{Ext}(\mathrm{conv}(\mathcal{O}_\lambda))$ is the set of its extreme points ([43, Prop. VIII.1.30]). Not every coadjoint orbit in $\mathfrak{g}'_{\mathrm{seq}}$ intersects $\mathfrak{t}' \cong [\mathfrak{t}, \mathfrak{g}]^\perp$, as the nilpotent orbits in $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ show.

(b) For every irreducible semibounded representation $(\pi, \mathcal{H})$, there exists a $\lambda \in \mathfrak{t}' \cong [\mathfrak{t}, \mathfrak{g}]^\perp$ such that the set $\mathrm{Ext}(I_\pi)$ of extreme points of I_π is a single coadjoint orbit $\mathcal{O}_\lambda$ and $I_\pi = \mathrm{conv}(\mathcal{O}_\lambda)$ ([43, Thm. X.4.1]). Further $\mathcal{H}$ decomposes into weight spaces $\mathcal{H}_{i\alpha}(T)$, $\alpha \in \mathfrak{t}'$. The corresponding weight set $\mathcal{P}_\pi \subseteq \mathfrak{t}'$ has the property that

$$\mathcal{O}_\lambda|_{\mathfrak{t}} = \mathrm{conv}(\mathcal{P}_\pi) \quad \text{and} \quad \lambda \in \mathrm{Ext}(\mathrm{conv}(\mathcal{P}_\pi)),$$

i.e., λ *is an extremal weight* (cf. Example 3.19). ■

In the finite-dimensional context, many convexity theorems have been proved by symplectic techniques using convexity properties of momentum maps. We do not expect this to work in the infinite-dimensional context, where we are mostly interested in inclusions such as in (24) or (29) and not in equalities (see in particular [7, 6]). We therefore put a stronger emphasis on functional analytic arguments using convex sets.

5. Infinite-dimensional Lie algebras

For infinite-dimensional Lie algebras, only very particular results concerning problems (P1-6) are known. They show certain common patterns, but so far no systematic theory has been developed to create a unifying picture. We now briefly discuss several classes of infinite-dimensional Lie algebras and their invariant convex cones.

5.1. Nilpotent and 2-step solvable Lie algebras

If $x \in \mathfrak{g}$ satisfies $(\mathrm{ad}\, x)^2 = 0$, then $\mathrm{Ad}^*(\exp tx)\lambda = \lambda \circ e^{-t\,\mathrm{ad}\, x} = \lambda - t(\lambda \circ \mathrm{ad}\, x)$ for $\lambda \in \mathfrak{g}'$ shows that the orbits of the corresponding one-parameter group in $\mathfrak{g}'$ are either trivial of affine lines. If $\mathcal{O}_\lambda$ is semi-equicontinuous, the orbit must be trivial (cf. Proposition 3.8). A closer inspection of this simple observation leads to:

Theorem 5.1. *Suppose that $\mathfrak{g}$ is either nilpotent or 2-step solvable, i.e., $[\mathfrak{g}, \mathfrak{g}]$ is abelian. Then $\mathfrak{g}'_{\mathrm{seq}} \subseteq [\mathfrak{g}, \mathfrak{g}]^\perp$, i.e., all semi-equicontinuous coadjoint orbits are trivial. If there exists a pointed invariant cone $W \subseteq \mathfrak{g}$ with non-empty interior, then $\mathfrak{g}$ is abelian.*

Proof. The first assertion follows from [61, Thm. 1.5]. For the second assertion, we note that $W^\star \subseteq \mathfrak{g}'_{\mathrm{seq}}$, and since W is pointed, $W^\star$ separates the points of $\mathfrak{g}$. As the coadjoint action is trivial on $\mathfrak{g}'_{\mathrm{seq}}$, the adjoint action on $\mathfrak{g}$ must be trivial as well. This means that $\mathfrak{g}$ is abelian. ■

5.2. Oscillator algebras

In ([61, Thms. 2.8, 3.2, Prop. 3.4]), we determine all semi-equicontinuous coadjoint orbits in *oscillator algebras*, i.e., double extensions $\mathfrak{g} = \mathfrak{g}(V, \omega, D) = (\mathbb{R} \oplus_\omega V) \rtimes_D \mathbb{R}$ of an abelian Lie algebra V, where (V, ω) is a symplectic vector space and $D \in \mathfrak{sp}(V, \omega)$ (Definition 3.1):

Theorem 5.2. *For an oscillator algebra, the following are equivalent:*

(i) $\mathfrak{g}'_{\rm seq} \neq \emptyset$.

(ii) *$q(x,y) := \omega(Dx, y)$ or $\omega(x, Dy)$ is positive definite, all functionals $i_x\omega = \omega(x, \cdot)$ are q-continuous, and the corresponding quadratic form $x \mapsto \|i_x\omega\|_q^2$ is continuous.*

(iii) *$\mathfrak{g}$ is a Lorentzian double extension $\mathfrak{g}(V, \kappa, D)$ of a locally convex euclidean vector space (V, κ) on which D is a κ-skew-symmetric derivation and $\omega(v, w) = \kappa(v, Dw)$.*

If these conditions are satisfied, then $\lambda = (z^*, \alpha, t^*) \in \mathfrak{g}'_{\rm seq}$ if and only if $z^* \neq 0$ and $\alpha|_{D(V)}$ is κ-bounded. If, in addition, $D(V)$ is dense in V, then the κ-boundedness of α on $D(V)$ implies its κ-boundedness.

5.3. Kac-Moody Lie algebras

In [31, Thm. 2(b)] Kac and Peterson generalized Kostant's Convexity Theorem to symmetrizable Kac-Moody algebras, and this implies the Atiyah-Pressley Convexity Theorem for loop groups [4, Thm. 1]. Concretely, let $\mathfrak{t} \subseteq \mathfrak{g}$ be an elliptic Cartan subalgebra such that $\mathfrak{g} = \mathfrak{g}^{\rm alg}$, $\mathfrak{g}_\mathbb{C}$ is a symmetrizable Kac-Moody algebra, and all root vectors are of type (N) or (CS) (cf. Lemma 3.5), i.e., $\mathfrak{g}$ is a unitary real form of $\mathfrak{g}_\mathbb{C}$. For every element x in the Tits cone $\mathcal{T} \subseteq \mathfrak{t}$ (Subsection 2.3), it asserts that

$$p_\mathfrak{t}(\mathcal{O}_x) = \mathrm{conv}(\mathcal{W}x) \quad \text{for} \quad x \in \mathcal{T},$$

where $\mathcal{W} = \mathcal{W}(\mathfrak{g}, \mathfrak{t})$ is a finitely generated Coxeter group (Definition 3.6, [30, Ch. 6]). This result applies in particular to twisted loop algebras with finite-dimensional compact target groups (cf. Subsection 7.3).

Since $\mathfrak{g}$ carries a non-degenerate invariant symmetric bilinear form, we likewise obtain for coadjoint orbits $\mathcal{O}_\lambda$, $\lambda \in \mathcal{T}'$ (the Tits cone in $\mathfrak{t}'$), that

$$p_{\mathfrak{t}'}(\mathcal{O}_\lambda) = \mathrm{conv}(\mathcal{W}\lambda) \quad \text{for} \quad \lambda \in \mathcal{T}'.$$

We now sketch a rather general representation theoretic argument that provides the inclusion "$\subseteq$" of this convexity theorem. If $\lambda \in \mathfrak{t}'$ is dominant integral, i.e., $i\lambda(\alpha^\vee) \in \mathbb{N}_0$ for all simple roots $\alpha \in \Pi$, then it is a highest weight of a unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ (see [31]), so that the discussion in Example 3.18(b) implies that

$$p_{\mathfrak{t}'}(\mathcal{O}_\lambda) \subseteq \overline{\mathrm{conv}}(\mathcal{W}\lambda) \subseteq \lambda + i\,\mathrm{cone}(\Pi). \tag{30}$$

Approximating general elements $\lambda \in \mathcal{T}'$ by positive multiples of integral ones, we see that (30) holds for every $\lambda \in (\mathcal{T}')^0$. The Convexity Theorem for Coxeter groups, combined with Lemma 2.22, now yields

$$p_{\mathfrak{t}'}(\mathcal{O}_\lambda) \subseteq \mathcal{T}' \cap \bigcap_{w \in \mathcal{W}} w(\lambda + i\,\mathrm{cone}(\Pi)) = \overline{\mathrm{conv}}(\mathcal{W}\lambda). \tag{31}$$

5.4. Lie algebras of vector fields

In this subsection we briefly comment on two results on Lie algebras of vector fields.

Vector fields on the circle. Let $\mathfrak{g} = \mathcal{V}(\mathbb{S}^1) = C^\infty(\mathbb{S}^1)\partial_\theta$ be the Lie algebra of smooth vector fields on the circle $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$, where $\mathbf{d} := \frac{\partial}{\partial\theta}$ denotes the generator of the right rotations. Then

$$W := \Big\{ f\frac{\partial}{\partial\theta} \colon f > 0 \Big\}$$

is an open invariant cone and $\pm W$ are the only open invariant cones in this Lie algebra. Based on this observation and a specific Convexity Theorem, we classify in [51, §8] all semi-equicontinuous coadjoint orbits of the Virasoro algebra and the Lie algebra $\mathcal{V}(\mathbb{S}^1)$ of smooth vector fields on the circle. Here Lemma A.15 applies with $\mathbf{d}$ as above. It implies that $\pm W$ are the only proper open invariant cones in $\mathcal{V}(\mathbb{S}^1)$ ([51, Thm. 8.3]) and $W^\star \cup -W^\star$ is the set of semi-equicontinuous coadjoint orbits ([51, Prop. 8.4]).

In the Virasoro algebra $\mathfrak{vir}$, a central extension $\mathfrak{vir} = \mathbb{R}\mathbf{c} \oplus_\omega \mathcal{V}(\mathbb{S}^1)$, the situation is more complicated. Here the open invariant cones are classified by a sign and an angle between $\pi/2$ and π ([51, Thm. 8.15]).

The diffeomorphism group of an annulus. In [7, 6] a convexity theorem for the diffeomorphism group of the annulus $M := [0,1] \times \mathbb{S}^1$ is obtained. Here $\mathfrak{g}$ is the corresponding Lie algebra of smooth functions with the Poisson bracket and $\mathfrak{t} \subseteq \mathfrak{g}$ is the abelian subalgebra of radial functions $f(r,\theta) = f(r)$. In this context several completion procedures are required to identify the natural generalization of the Weyl group, which in this context leads to the semigroup $\overline{\mathcal{W}}$ of measure preserving maps on $[0,1]$ and the closed convex hulls of its orbits in $L^1([0,1])$ ([10], [73, 74]); see also Problem 5.7). This exhibits an interesting analogy with the situation discussed in Remark 5.6 below, where we show that the monoid $\mathrm{Isom}(\mathcal{H})$ acts on orbit closures in the dual spaces $\mathfrak{u}_p(\mathcal{H})'$ and $\mathfrak{u}(\mathcal{H})'$.

5.5. Unitary Lie algebras

The full unitary Lie algebra of a Hilbert space. Let $\mathcal{H}$ be a complex Hilbert space with an orthonormal basis $(e_j)_{j\in J}$, so that $\mathcal{H} \cong \ell^2(J,\mathbb{C})$. For the full unitary group $G = \mathrm{U}(\mathcal{H})$, a generalization of the Schur-Horn Theorem (Example 4.2) was obtained by A. Neumann [62, 66]. We write $\mathfrak{t} \cong i\ell^\infty(J,\mathbb{R}) \subseteq \mathfrak{g} = \mathfrak{u}(\mathcal{H})$ for the subalgebra of diagonal operators with respect to the orthonormal basis. Although it is maximal abelian, it is not an elliptic Cartan subalgebra in the sense of Definition 3.3 because the subalgebra $\mathfrak{g}^{\mathrm{alg}}$ is not dense in $\mathfrak{g}$ with respect to the operator norm. Nevertheless, we have a natural projection on diagonal matrices:

$$p_\mathfrak{t} \colon \mathfrak{u}(\mathcal{H}) \to \mathfrak{t}, \qquad p_\mathfrak{t}(x)_j = \langle e_j, xe_j\rangle.$$

The permutation group S_J naturally acts on $\mathcal{H}$ by $\sigma e_j := e_{\sigma(j)}$, normalizing $\mathfrak{t}$, which gives rise to a *big Weyl group* $\overline{\mathcal{W}} \cong S_J \subseteq \mathrm{GL}(\mathfrak{t})$. Neumann's result asserts that

(a) $\overline{p_\mathfrak{t}(\mathcal{O}_x)} = \overline{\mathrm{conv}}(S_J x)$ for $x \in \mathfrak{t}$.

(b) Every open or closed invariant subset $\Omega \subseteq \mathfrak{u}(\mathcal{H})$ is determined by the S_J-invariant open subset $\Omega \cap \mathfrak{t}$ (see [64, Thm. 4.33] for closed subsets and use Lemma A.4 for open ones).

If $x \in \mathfrak{u}(\mathcal{H})$ is not diagonalizable, then Neumann also describes $\overline{p_{\mathfrak{t}}(\mathcal{O}_x)}$ in terms of the essential spectrum of x. The norm continuous unitary representations $(\pi_k)_{k\in\mathbb{N}}$ of $\mathrm{U}(\mathcal{H})$ on the spaces $\Lambda^k(\mathcal{H})$ provide a sequence of invariant convex functions by their support functionals s_{π_k} (see (1) in the introduction). To evaluate these functionals on diagonal operators, we observe that the restrictions of π_k to the diagonal subgroup T is diagonalizable with the weights

$$x_F := \sum_{j\in F} x_j, \quad \text{where} \quad F \subseteq J, |F| = k, \quad x_j := \langle e_j, xe_j\rangle.$$

For $x \in \ell^\infty(J,\mathbb{R})$ and the corresponding element $\mathrm{diag}(x) \in i\mathfrak{t}$, we then have

$$s_k(x) := s_{\pi_k}(-i\,\mathrm{diag}(x)) = \sup\{x_F \colon F \subseteq J, |F| = k\}. \tag{32}$$

Neumann then shows that, if J is countable, then the support functionals s_{π_k} suffice to determine closed convex hulls of adjoint orbits ([64, Thm. 4.37], [62, Prop. 2.8]):

$$\overline{\mathrm{conv}}(\mathcal{O}_x) = \{y \in \mathfrak{u}(\mathcal{H}) \colon (\forall k \in \mathbb{N})\ s_{\pi_k}(y) \le s_{\pi_k}(x),\ s_{\pi_k}(-y) \le s_{\pi_k}(-x)\} \tag{33}$$

$$\overline{\mathrm{conv}}(S_J x) = \{y \in \ell^\infty(J,\mathbb{R}) \colon (\forall k \in \mathbb{N})\ s_k(y) \le s_k(x), s_k(-y) \le s_k(-x)\}. \tag{34}$$

Example 5.3. If $\mathcal{H} = \mathbb{C}^n$, then we likewise obtain finitely many invariant convex functions $s_k, k = 1,\ldots,n$, on $\mathbb{R}^n$, such that

$$\mathrm{conv}(S_n x) = \{y \in \mathbb{R}^n \colon s_k(y) \le s_k(x), k = 1,\ldots,n-1; s_n(y) = s_n(x)\}.$$

If x is decreasing, then $s_k(x) = x_1 + \cdots + x_k$, so that one obtains the classical Hardy-Littlewood-Pólya inequalities describing convex hulls of S_n-orbits in $\mathbb{R}^n$ (cf. (28), [73]). This idea is fundamental in Neumann's generalization of the Schur-Horn-Kostant Theorem to the pair $(\mathfrak{g},\mathfrak{t}) = (\mathfrak{u}(\mathcal{H}), i\ell^\infty(J,\mathbb{R}))$. ■

Remark 5.4. Each s_k extends by the same formula to a lower semicontinuous function $s_k \colon \mathbb{R}^J \to (-\infty,\infty]$ on the full space of real-valued functions on J. These functions are invariant under the full permutation group S_J. If $F \subseteq J$ is a k-element subset, then

$$s_k(\lambda) = \sup\langle\lambda, S_J e_F\rangle = \sup\langle\lambda, S_{(J)} e_F\rangle \quad \text{for} \quad e_F := \sum_{j\in F} e_F \in \mathbb{R}^{(J)} \cong (\mathbb{R}^J)'.$$

It is easy to see that $s_k(\lambda) < \infty$ if and only if λ is bounded from above. Accordingly, $s_k(\pm\lambda) < \infty$ is equivalent to the boundedness of λ, and by Proposition 2.18 this is equivalent to the (semi-)equicontinuity of $S_{(J)}\lambda$, resp., $S_J\lambda$. ■

Neumann also generalized Kostant's Convexity Theorem to the full orthogonal and symplectic Lie algebras of bounded operators. We refer to [66] for details.

From Proposition 2.18(a) one derives in particular that every open invariant cone in $\mathfrak{u}(\mathcal{H})$ intersects the center $i\mathbf{1}$ ([53, Thm. 5.6], see also Remark 4.3) and that, for a real Hilbert space $\mathcal{H}$ of dimension > 2 and a quaternionic Hilbert space $\mathcal{H}$, all open invariant cones in $\mathfrak{o}(\mathcal{H})$, resp., $\mathfrak{u}_{\mathbb{H}}(\mathcal{H})$ are trivial. These are interesting analogs of the corresponding observations concerning the finite-dimensional compact Lie algebras $\mathfrak{u}_n(\mathbb{K})$.

Unitary Lie algebras of compact operators. One may also consider the the unitary Banach-Lie groups $\mathrm{U}_p(\mathcal{H})$, $1 \leq p \leq \infty$, whose Lie algebras are the Banach spaces $\mathfrak{u}_p(\mathcal{H})$ of skew-hermitian operators of Schatten class p with $\|x\|_p = \mathrm{tr}(|x|^p)^{1/p}$. For these Lie algebras we have:

Proposition 5.5. *If $\mathcal{H}$ is an infinite-dimensional complex Hilbert space, then $\mathfrak{z}(\mathfrak{u}_p(\mathcal{H})) = \{0\}$, and $\mathfrak{u}_p(\mathcal{H})$ contains non-trivial open invariant cones if and only if $p = 1$.*

Proof. That $\mathfrak{z}(\mathfrak{u}_p(\mathcal{H})) = \{0\}$ follows from the fact that any operator commuting with all rank-one operators is a multiple of $\mathbf{1}$, which is not a compact operator.

For $p = 1$, the trace functional is continuous, hence defines an invariant open half space. To obtain a pointed cone, consider the subset

$$\Delta := \{x \in \mathrm{Herm}_1(\mathcal{H})\colon x \geq 0, \mathrm{tr}\, x = 1\}$$

which is bounded, $\|\cdot\|_1$-closed, convex and $\mathrm{U}(\mathcal{H})$-invariant, and its distance from 0 is 1. Hence Lemma A.5(b) applies to any open convex set of the form $\Omega := \Delta + B_r(0)$, $0 < r < 1$, and shows that $\mathbb{R}_+^\times \Omega$ is a pointed open invariant cone.

For $1 < p < \infty$, let $\Omega \subseteq \mathfrak{u}_p(\mathcal{H})$ be an open invariant cone. Intersecting with diagonal operators with respect to an orthonormal basis, we obtain an open cone $W \subseteq \ell^p(J, \mathbb{R})$, invariant under the action of the group $S_{(J)}$ of finite permutations. To see that $0 \in W$, it suffices to assume that $J = \mathbb{N}$. As W is open, it contains an element x with finite support in $\mathbb{N}$. For every $y \in \ell^p(\mathbb{N})$ with finite support disjoint from $\mathrm{supp}(x)$, there exists an $\varepsilon > 0$ with $x \pm \varepsilon y \in W$. For $k \in \mathbb{N}$, pick permutations $\sigma_1, \ldots, \sigma_k \in S_{(\mathbb{N})}$ fixing all elements in $\mathrm{supp}(y)$, and for which the subsets $\sigma_j(\mathrm{supp}\, x)$ are mutually disjoint. Then

$$\pm\varepsilon y + \frac{1}{k}\sum_{j=1}^k \sigma_j x \in W \quad \text{and} \quad \frac{1}{k}\Big\|\sum_{j=1}^k \sigma_j x\Big\|_p = \frac{1}{k}(k\|x\|_p^p)^{1/p} = k^{-1+\frac{1}{p}}\|x\|_p,$$

and this expression converges to 0 for $k \to \infty$. This implies that $\pm y \in \overline{W}$. By permutation invariance, it follows that $H(\overline{W})$ contains all elements of finite support, and hence that $\overline{W} = \ell^p(\mathbb{N})$. This in turn implies $W = \ell^p(\mathbb{N})$ (Lemma A.2).

For $p = \infty$ we argue similarly with $\frac{1}{k}\big\|\sum_{j=1}^k \sigma_j x\big\|_\infty \leq \frac{1}{k}$. ■

Remark 5.6. (a) The Banach-Lie algebras $\mathfrak{u}_p(\mathcal{H})$ exhibit an interesting infinite-dimensional feature, namely that the unitary group $\mathrm{U}(\mathcal{H})_s$, i.e., the group $\mathrm{U}(\mathcal{H})$, endowed with the strong operator topology, acts on these Lie algebras by conjugation as automorphisms:

$$\mathrm{Ad}\colon \mathrm{U}(\mathcal{H})_s \times \mathfrak{u}_p(\mathcal{H}) \to \mathfrak{u}_p(\mathcal{H}), \quad (g, X) \mapsto gXg^{-1},$$

and this action by isometries is continuous. As the subgroup $\mathrm{U}_p(\mathcal{H})$ is dense in $\mathrm{U}(\mathcal{H})_s$, the subgroup $\mathrm{Ad}(\mathrm{U}_p(\mathcal{H})) \subseteq \mathcal{L}(\mathfrak{u}_p(\mathcal{H}))$ is dense in $\mathrm{Ad}(\mathrm{U}(\mathcal{H}))$. By Lemma 2.7, it follows that every weak-$*$-closed convex subset $C \subseteq \mathfrak{u}_p(\mathcal{H})'$ is $\mathrm{Ad}(\mathrm{U}(\mathcal{H}))'$-invariant and thus, by the Duality Theorem 2.4, also every open invariant cone, and further every continuous positively homogeneous convex function on such a cone.

(b) One can even go one step further by observing that the closure of $\mathrm{U}(\mathcal{H})$ in the space $\mathcal{L}(\mathcal{H})$ with respect to the strong operator topology is the monoid

$$\mathrm{Isom}(\mathcal{H}) = \{A \in \mathcal{L}(\mathcal{H}) \colon A^*A = \mathbf{1}\}$$

of isometries.[2] This monoid also acts continuously on $\mathfrak{u}_p(\mathcal{H})$ by Lie algebra endomorphisms via

$$\mathrm{Ad}(g)X := gXg^*,$$

and this action by contractions is continuous with respect to the strong operator topology on $\mathrm{Isom}(\mathcal{H})$. Here we use that

$$gXYg^* = gXg^*gYg \quad \text{for} \quad g \in \mathrm{Isom}(\mathcal{H}).$$

The same argument as under (a) now shows that every $\mathrm{Ad}(\mathrm{U}_p(\mathcal{H}))$-invariant weak-$*$-closed convex subset of $\mathfrak{u}_p(\mathcal{H})$ is invariant under $\mathrm{Isom}(\mathcal{H})$.
If $\mathcal{H} = L^2(X, \mathfrak{S}, \mu)$, then each $\mathfrak{S}$-measurable map $\varphi \colon X \to X$ with $\varphi_*\mu = \mu$ defines an isometry on $\mathcal{H}$. Hence this monoid acts in particular by the adjoint action on $\mathfrak{u}_p(\mathcal{H})$ (cf. Subsection 5.4 and Remark 5.8 below). If X is finite and non-atomic, [10, Thm. 5] implies that the strong closure of the group of invertible measure preserving transformations is the monoid of measure preserving transformations.
(c) On the level of diagonal matrices, we have the action of the Weyl group $\mathcal{W} \cong S_{(J)}$ on $\ell^p(J, \mathbb{R})$, and this action extends to a continuous action of the monoid $\mathrm{Inj}(J)$ that has the same invariant weak-$*$-closed convex subsets in $\ell^p(J, \mathbb{R})' \cong \ell^q(J, \mathbb{R})$, where p and q are related by $p^{-1} + q^{-1} = 1$. In particular, $S_{(J)}$ and the much larger monoid $\mathrm{Inj}(J)$ have the same open invariant cones and the same invariant lower semicontinuous positively homogeneous convex functions on $\mathfrak{u}_p(\mathcal{H})$. Note that $\mathrm{Inj}(J)$ corresponds to the measure preserving maps on J for $\mathfrak{S} = 2^J$ and the counting measure $\mu = \sum_{j \in J} \delta_j$. ■

Problem 5.7. Suppose that μ is σ-finite, so that $L^\infty(X, \mathfrak{S}, \mu)$ is the dual space of $L^1(X, \mathfrak{S}, \mu)$. Is it possible to describe the weak-$*$-closures of orbits of the group $\Gamma = \mathrm{Meas}(X, \mathfrak{S}, \mu)$ of (measurably invertible) measure preserving transformations in $L^\infty(X, \mathfrak{S}, \mu)$? Note that, for every $E \in \mathfrak{S}$ of finite measure, we obtain a lower semicontinuous invariant convex function by

$$s_E(f) := \sup_{\gamma \in \Gamma} \int_{\gamma(E)} f \, d\mu.$$

For the special case where X is countable, $\mathfrak{S} = 2^X$ and μ is the counting measure, this specializes to the situation of A. Neumann's Theorem as in (34). ■

Remark 5.8. In [73, 74] J.V. Ryff studies the case $X = (0,1)$, endowed with Lebesgue measure μ. We write $\mathcal{M}$ for the monoid of measure preserving transformations of $(0,1)$. For every function $f \in L^1(0,1)$, the probability measure $f_*\mu$ defines a right-continuous decreasing function

$$f^* \colon (0,1) \to \mathbb{R}, \quad f^*(s) := \sup\{y \in \mathbb{R} \colon \mu(\{f > y\}) > s\}$$

[2] Since each $A \in \mathrm{Isom}(\mathcal{H})$ is determined by the orthonormal family $f_j := Ae_j$ and we may w.l.o.g. assume that $f_j = e_{\sigma(j)}$ for an injection $\sigma \colon J \to J$, the density of $\mathrm{U}(\mathcal{H})$ in $\mathrm{Isom}(\mathcal{H})$ follows from the density of the monoid $\mathrm{Inj}(J)$ of injections in the group $S_J = \mathrm{Bij}(J)$ of all bijections with respect to the topology of pointwise convergence.

with the property that $f^* \in L^1(0,1)$ with

$$\int_0^s f \leq \int_0^s f^* \quad \text{for} \quad 0 < s < 1 \quad \text{and} \quad \int_0^1 f = \int_0^1 f^*. \tag{35}$$

For $f, g \in L^1(0,1)$, we define $g \prec f$ (f *majorizes* g) if

$$\int_0^s g^* \leq \int_0^s f^* \quad \text{for} \quad 0 < s < 1 \quad \text{and} \quad \int_0^1 f = \int_0^1 f^*. \tag{36}$$

For each $f \in L^1(0,1)$, the set $\Omega(f) := \{g \in L^1(0,1) \colon g \prec f\}$ of all functions majorized by f then has the following properties:

- It is a weakly compact convex subset of $L^1(0,1)$ ([73, Thms. 2,3]).
- $\mathrm{Ext}(f) = \mathcal{M}.f^* = \{f \circ \sigma \colon \sigma \in \mathcal{M}\}$ is the set of all functions equimeasurable with f (i.e., $f^* = g^*$), in particular $\Omega(f) = \Omega(f^*)$ ([73, Thm. 1], [74, Thm. 1]).

We thus obtain $$\Omega(f) = \overline{\mathrm{conv}}(\mathcal{M}.f),$$

and that this set is characterized by the generalizations of the Hardy-Littlewood-Pólya inequalities given by (36). ■

Operator algebras. In the context of operator algebras, results resembling those for the full unitary group have been obtained recently for type II factors and projections of unitary orbits onto maximal abelian subalgebras [1, 2]. See also [16] for a general analysis of closed convex hulls of unitary orbits in von Neumann algebras. In this context one encounters maximal abelian subalgebras

$$\mathfrak{t} \subseteq \mathfrak{g} := \mathfrak{u}(\mathcal{A}) := \{x \in \mathcal{A} \colon x^* = -x\}$$

of the type $L^\infty(X, \mathfrak{S}, \mu)$ and "Weyl groups" acting by measure preserving transformations on this space (see Problem 5.7).

In a C^*-algebra $\mathcal{A}$ one often considers abelian subalgebras $\mathcal{T} \subseteq \mathcal{A}$ and a continuous contraction $E \colon \mathcal{A} \to \mathcal{T}$ (a *conditional expectation*), i.e., a map preserving positivity and satisfying the equivariance condition

$$E(D_1 A D_2) = D_1 E(A) D_2 \quad \text{for} \quad A \in \mathcal{A}, D_1, D_2 \in \mathcal{T}.$$

Let $\mathcal{O}_X = \{UXU^* \colon U \in \mathrm{U}(\mathcal{A})\}$ denote the adjoint orbit of $X = -X^* \in \mathfrak{u}(\mathcal{A})$. If $E \colon \mathcal{A} \to \mathcal{T}$ is a conditional expectation, then we expect a relation of the type

$$E(\overline{\mathrm{conv}}(\mathcal{O}_X)) = \overline{\mathrm{conv}}(\mathcal{W}X) \quad \text{for} \quad X \in \mathcal{T},$$

where $\mathcal{W} \subseteq \mathrm{GL}(\mathcal{T})$ is an analog of the Weyl group.

Example 5.9. A particularly important example is the surjective conditional expectation $E \colon \mathcal{M} \to Z(\mathcal{M})$ of a W^*-algebra $\mathcal{M}$ with values in its center (cf. [75]). In this case the action of $\mathcal{W}$ is trivial and

$$\{E(X)\} = E(\overline{\mathrm{conv}}(\mathcal{O}_X)) = \overline{\mathrm{conv}}(\mathcal{O}_X) \cap Z(\mathcal{M}),$$

so that the convexity theorem holds in a trivial way.

The situation becomes more interesting if we project onto a maximal abelian subalgebra which is not central; see [1, 2] for type II algebras. In this context one obtains interesting generalizations of the convex functionals s_{π_k} from (32). Using a finite trace τ, one obtains the $\mathrm{U}(\mathcal{A})$-invariant continuous convex/concave functionals ([2, §4]):

$$U_t(a) := \sup\{\tau(ap) \colon p^2 = p = p^*, \tau(p) = t\},$$
$$L_t(a) := \inf\{\tau(ap) \colon p^2 = p = p^*, \tau(p) = t\}, \qquad t \geq 0. \qquad \blacksquare$$

Remark 5.10. If $\mathcal{M}$ is a von Neumann algebra of operators on a separable Hilbert space and $\mathcal{T} \subseteq \mathcal{M}$ maximal abelian, then $\mathcal{T} \cong L^\infty(X, \mathfrak{S}, \mu)$ for a finite measure μ, with predual $\mathcal{T}_* = L^1(X, \mathfrak{S}, \mu)$. The majorization order on real functions in $\mathcal{T}_*$ is expected to have natural connections with the theory of decreasing rearrangements (cf. Remark 5.8). $\blacksquare$

5.6. Projective limits

Let $\mathfrak{g} = \varprojlim \mathfrak{g}_j$ be a projective limit of locally convex Lie algebras $\mathfrak{g}_j$ and $W \subseteq \mathfrak{g}$ an open invariant cone ([23]). If $x \in W$, then $W - x$ is a 0-neighborhood in $\mathfrak{g}$, hence contains the kernel of a projection $p_j \colon \mathfrak{g} \to \mathfrak{g}_j$. This implies that $x + \ker p_j \subseteq W$, so that $\ker p_j \subseteq H(W)$ (Lemma 2.9(iii)). Therefore W is the inverse image of an open invariant cone in one of the Lie algebras $\mathfrak{g}_j$. In particular, projective limits with non-injective connecting maps never contain pointed open invariant cones.

6. Direct limits

In this section we describe some special features of direct limit Lie algebras, first in the context of root decompositions and then for more general situations.

6.1. Direct limits of finite-dimensional Lie algebras

First we consider the case, where $\mathfrak{t} \subseteq \mathfrak{g}$ is an elliptic Cartan subalgebra and $\mathfrak{g} = \mathfrak{g}^{\mathrm{alg}}$ is locally finite, i.e., every finite subset of $\mathfrak{g}$ generates a finite-dimensional Lie subalgebra. Then $\mathfrak{g}$ is the union of subalgebras $\mathfrak{g}_F$ with

$$\mathfrak{g}_{F,\mathbb{C}} = \mathfrak{t}_\mathbb{C} + \sum_{\alpha \in \Delta_F} \mathfrak{g}^\alpha_{F,\mathbb{C}}, \qquad \Delta_F \subseteq \Delta \ \text{ finite }, \qquad \dim \mathfrak{g}^\alpha_{F,\mathbb{C}} < \infty.$$

As $\Delta_F^\perp \subseteq \mathfrak{t}_\mathbb{C}$ is of finite codimension, these subalgebras can be analyzed with the methods from [43], outlined in Section 4.

Examples 6.1. (a) A particularly interesting case arises for $\Delta = \Delta_k$, i.e., when all roots are compact. One can show that the simple locally finite Lie algebras of this type are $\mathfrak{su}(J, \mathbb{C})$ and $\mathfrak{u}(J, \mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{H}$, for an infinite set J (see [78], [53] for details).
Let $\lambda \in \mathfrak{t}'$. Applying Kostant's Convexity Theorem (Theorem 4.1(ii)) to the subalgebras $\mathfrak{g}_F$, it follows that

$$p_{\mathfrak{t}'}(\mathcal{O}_\lambda) = \mathrm{conv}(\mathcal{W}\lambda).$$

If $\mathcal{O}_\lambda$ is semi-equicontinuous, the relation $\mathfrak{g} = \mathrm{Ad}(G)\mathfrak{t}$ implies that $B(\mathcal{O}_\lambda)^0$ intersects $\mathfrak{t}$, so that $\mathcal{W}\lambda \subseteq \mathfrak{t}'$ is also semi-equicontinuous (Remark 3.13(b)).

If $\mathfrak{g}$ is one of the three infinite-dimensional Lie algebras $\mathfrak{u}(J,\mathbb{K})$, $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$, Lemma 6.2 below implies that $\mathcal{W}\lambda$ is semi-equicontinuous if and only if $\lambda(\Delta^\vee)$ is bounded.

(b) If not all roots are compact, then $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ with

$$\mathfrak{k}_\mathbb{C}=\mathfrak{t}_\mathbb{C}+\sum_{\alpha\in\Delta_k}\mathfrak{g}_\mathbb{C}^\alpha \quad \text{and} \quad \mathfrak{p}_\mathbb{C}=\sum_{\alpha\in\Delta_p}\mathfrak{g}_\mathbb{C}^\alpha.$$

The semi-equicontinuity of $\mathcal{O}_\lambda$ for $\lambda\in\mathfrak{t}'$ implies in particular that $\mathcal{O}_\lambda^K:=\operatorname{Ad}'(K)\lambda$ is semi-equicontinuous. If $\mathfrak{k}$ is a direct sum of a center and finitely many simple summands, then Lemma 6.2 easily implies that $\mathcal{O}_\lambda^K$ is semi-equicontinuous if and only if $\lambda(\Delta_k^\vee)$ is bounded. Here a key point is that the $\mathcal{W}$-action on Δ_k has only finitely many orbits.

A typical examples is the real form $\mathfrak{g}=\mathfrak{u}(J_1,J_2,\mathbb{C})\subseteq\mathfrak{g}_\mathbb{C}=\mathfrak{gl}(J,\mathbb{C})$, where $J=J_1\dot\cup J_2$. Then $\mathfrak{t}=i\mathbb{R}^{(J)}$ and the Weyl group is $\mathcal{W}\cong S_{(J_1)}\times S_{(J_2)}\subseteq S_{(J)}$, acting by permutations. Up to sign, the only $\mathcal{W}$-invariant positive system of non-compact roots is

$$\Delta_p^+=\{\varepsilon_j-\varepsilon_k\colon j\in J_1,k\in J_2\},\qquad \text{so that} \qquad C_{\max}(\Delta_p^+)=\{ix\colon x|_{J_2}\leq x|_{J_1}\}.$$

If J_1 and J_2 are both infinite, we obtain the cone

$$C_{\max}(\Delta_p^+)=\{-ix\colon x|_{J_2}\leq 0\leq x|_{J_1}\},$$

which has no interior points. ■

Lemma 6.2. *For $\mathfrak{g}=\mathfrak{u}(J,\mathbb{K})$, $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$ and an elliptic Cartan subalgebra $\mathfrak{t}$ obtained from a root decomposition of $\mathfrak{g}_\mathbb{C}$, the following are equivalent for $\lambda\in\mathfrak{t}'$:*

(i) $\mathcal{O}_\lambda\subseteq\mathfrak{g}'$ *is semi-equicontinuous.*

(ii) $\mathcal{O}_\lambda\subseteq\mathfrak{g}'$ *is equicontinuous.*

(iii) $\mathcal{W}\lambda\subseteq\mathfrak{t}'$ *is semi-equicontinuous.*

(iv) $\mathcal{W}\lambda\subseteq\mathfrak{t}'$ *is equicontinuous.*

(v) $\lambda(\Delta^\vee)$ *is bounded.*

Proof. The equivalences of (i) and (iii) and of (ii) and (iv) follow easily from Kostant's Convexity Theorem because $\mathfrak{g}=\operatorname{Ad}(G)\mathfrak{t}$ (cf. Examples 6.1(a) and Theorem 4.1). For a suitably chosen set J', we can identify $\mathfrak{t}$ with $i\mathbb{R}^{(J')}$, and then $\mathcal{W}$ contains the subgroup $S_{(J')}$ of finite permutations on J' and the elements $e_j-e_k, j\neq k\in J'$, are coroots.

If $\mathcal{W}\lambda$ is semi-equicontinuous, then Proposition 2.18(c) implies that $\lambda\in\mathfrak{t}'\cong i\mathbb{R}^{J'}$ is bounded. This implies (v) and from the concrete description of the Weyl group in all cases, we immediately see that $\mathcal{W}\lambda$ is equicontinuous. That (v) implies (iv) follows easily from the fact that by evaluating λ on the coroots e_j-e_k, it follows that λ is bounded. ■

6.2. Direct limits of compact Lie algebras

In this subsection we assume that $\Delta=\Delta_k$. We have already seen in Example 6.1(a) that, for $\mathfrak{g}=\mathfrak{u}(J,\mathbb{K})$, any $\lambda\in\mathfrak{t}'\cong\mathbb{R}^J$ corresponding to a semi-equicontinuous coadjoint orbit is bounded.

This is reflected by the following proposition, which represents the information we have on $\mathfrak{g}$ itself. The convex subset

$$B_{\mathfrak{t}} := \mathrm{conv}(i\Delta^\vee \cup -i\Delta^\vee) \subseteq \mathfrak{t}$$

is symmetric and invariant under the Weyl group $\mathcal{W}$. Since it is also generating, Kostant's Convexity Theorem (Theorem 4.1(ii)), applied to finite-dimensional subalgebras, implies that $B := \mathrm{Ad}(K)B_{\mathfrak{t}}$ is a symmetric convex invariant generating subset with $B \cap \mathfrak{t} = B_{\mathfrak{t}}$. Therefore

$$\|x\|_{\max} := \inf\{t > 0 \colon x \in tB\}$$

is an invariant norm on $\mathfrak{g}$ with $\|i\alpha^\vee\|_{\max} \leq 1$ for each $\alpha \in \Delta$. The following lemma shows that $\|\cdot\|_{\max}$ is a maximal invariant norm.

Proposition 6.3. *If $\mathfrak{g}$ is $\mathfrak{sl}(J,\mathbb{C})$ or $\mathfrak{u}(J,\mathbb{K})$ for $\mathbb{K} = \mathbb{R},\mathbb{H}$, and $\|\cdot\|$ is an invariant norm on $\mathfrak{g}$, then*

$$c := \sup_{\alpha\in\Delta} \|i\alpha^\vee\| < \infty \quad \textit{and} \quad \|x\| \leq c\|x\|_{\max} \quad \textit{for all } x \in \mathfrak{g}.$$

Proof. Since $\mathfrak{g}$ is simple, there exists at most two $\mathcal{W}$-orbits in Δ and α and β lie in the same orbit if and only if $\|\alpha\| = \|\beta\|$ ([25, §10.4, Lemma C]). This shows that c is finite.

As every element $x \in \mathfrak{g}$ is conjugate under G to an element of $\mathfrak{t}$ (Theorem 4.1), it suffices to consider elements $x \in \mathfrak{t}$. Then $\|y\|_{\max} \leq 1$ implies $\|y\| \leq c$. We conclude that $\|\cdot\| \leq c\|\cdot\|_{\max}$. ∎

Example 6.4. To determine the maximal norm $\|\cdot\|_{\max}$ more explicitly, we consider a set J and the simple Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(J,\mathbb{C})$ endowed with its canonical involution $x^* = \overline{x}^\top$, so that $\mathfrak{g} = \mathfrak{su}(J,\mathbb{C}) := \{x \in \mathfrak{sl}(J,\mathbb{C}) \colon x^* = -x\}$.

Let $E_{ij} = (\delta_{ik}\delta_{j\ell})_{k,\ell\in J} \in \mathbb{C}^{J\times J}$ be the matrix with one entry 1 in the position (i,j). For $\alpha = \varepsilon_i - \varepsilon_j$ we have $\alpha^\vee = E_{ii} - E_{jj}$ and $\|i\alpha^\vee\|_1 = 2$, where $\|\cdot\|_1$ is the trace norm. We claim that $\|x\|_1 = 2\|x\|_{\max}$ on all of $\mathfrak{g}$. Since all coroots are conjugate under the Weyl group $\mathcal{W} \cong S_{(J)}$, Prop. 6.3 shows that $\|x\|_1 \leq 2\|x\|_{\max}$ for all $x \in \mathfrak{g}$.

If J is an infinite set, then one readily verifies that the extreme points of the set

$$S := \Big\{x \in \mathbb{R}^{(J)} \colon \sum_j x_j = 0, \sum_j |x_j| \leq 2\Big\} \tag{37}$$

consist of the points of the form $e_i - e_j$, $i \neq j$. In fact, if $x = (x_j)_{j\in J} \in S$ has two positive entries x_i and x_j, then $0 < x_i, x_j < 2$ implies the existence of an $\varepsilon > 0$ such that $\|x\|_1 = \|x + t(e_i - e_j)\|_1 = 2$ for $|t| < \varepsilon$, so that it cannot be an extreme point. Therefore at most one entry x_i is positive, and likewise one entry x_j is negative. From $x_j = -x_i$ we conclude that $x_i = 1 = -x_j$, i.e., $x = e_i - e_j$. Therefore

$$S = \mathrm{conv}\{e_i - e_j \colon i \neq j\}.$$

Letting J be arbitrary, this argument still shows that $2\|x\|_{\max} = \|x\|_1$ for $x \in \mathfrak{t}$, hence for $x \in \mathfrak{g}$. ∎

Corollary 6.5. *If $\mathfrak{g}$ is $\mathfrak{su}(J,\mathbb{C})$ or $\mathfrak{u}(J,\mathbb{K})$ for $\mathbb{K} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$ and an infinite set J, then the following assertions hold:*

(i) *For every invariant norm $\|\cdot\|$ on $\mathfrak{g}$, there exists a $C > 0$ with $\|x\| \leq C\|x\|_1$, where $\|x\|_1 = \mathrm{tr}(|x|)$ denotes the trace norm.*

(ii) *For $\mathcal{H} := \ell^2(J,\mathbb{K})$, every norm-continuous representation $\rho\colon \mathfrak{g} \to \mathcal{L}(E)$ on a Banach space E extends to a representation of the Banach-Lie algebra completion $\mathfrak{su}_1(\mathcal{H})$ for $\mathfrak{g} = \mathfrak{su}(J,\mathbb{C})$, and to $\mathfrak{u}_1(\mathcal{H})$ for $\mathfrak{g} = \mathfrak{u}(J,\mathbb{K})$.*

(iii) *Every $\mathrm{Ad}(G)$-invariant convex subset of $\mathfrak{g}$ which is open with respect to the finest locally convex topology is also open with respect to $\|\cdot\|_1$.*

Proof. (i) follows from Proposition 6.3 for the case where $\mathfrak{g}$ is simple. For $\mathfrak{g} = \mathfrak{u}(J,\mathbb{C})$ we write $\mathfrak{g} \cong \mathfrak{su}(J,\mathbb{C}) \rtimes \mathbb{R}$ and observe that the open $\|\cdot\|$-unit ball B intersects $\mathfrak{su}(J,\mathbb{C})$ in a $\|\cdot\|_1$-open subset. Since $\mathfrak{su}(J,\mathbb{C})$ has finite codimension, B contains a $\|\cdot\|_1$-ball, and this implies (i).

(ii) follows from (i).

(iii) First we assume that $\mathfrak{g} \neq \mathfrak{u}(J,\mathbb{C})$, so that $\mathfrak{g}$ is simple. Let $\Omega \subseteq \mathfrak{g}$ be an invariant convex subset which is open for the finest locally convex topology on $\mathfrak{g}$, i.e., $\Omega - x_0$ is absorbing for each $x_0 \in \Omega$. Since $\mathfrak{g}$ is a union of simple compact Lie algebras $\mathfrak{g}_F$, $F \subseteq J$ finite, we have $\mathfrak{g}_F \cap \Omega \neq \emptyset$ for such a subset F. Then $\emptyset \neq \Omega \cap \mathfrak{z}(\mathfrak{g}_F) \subseteq \{0\}$ (Remark 4.3) shows that $0 \in \Omega$. Hence $\Omega \cap -\Omega$ is the open unit ball for an invariant norm, hence contains an interior point with respect to $\|\cdot\|_1$ and is therefore $\|\cdot\|_1$-open by Lemma A.7.

Now we consider the case $\mathfrak{g} = \mathfrak{u}(J,\mathbb{C})$. Using Konstant's Convexity Theorem, we have to show that every $S_{(J)}$-invariant open convex subset of $\mathbb{R}^{(J)}$ in ℓ^1-open, resp., contains an ℓ^1-inner point (Lemma A.7). As in the proof of Proposition 2.18(b), we find a finite subset $F \subseteq J$ and $a \neq 0$ with $x_0 := ae_F \in \Omega$. Then $\Omega - x_0$ is an open 0-neighborhood invariant under the group $S_{(J\setminus F)}$. This implies that $(\Omega - x_0) \cap \mathbb{R}^{(J\setminus F)}$ is $\|\cdot\|_1$-open because it contains the $S_{(J\setminus F)}$-orbit of some element of the form $e_j - e_k$, $j \neq k \in J \setminus F$ (Lemma A.7). As $\mathbb{R}^{(J\setminus F)}$ has finite codimension in $\mathbb{R}^{(J)}$, it follows that Ω is also open with respect to $\|\cdot\|_1$. ■

The maximal invariant norm $\|\cdot\|_{\max}$ on $\mathfrak{g}$ leads to a to a minimal Banach completion of the Lie algebra $\mathfrak{g}$. In this sense, Example 6.4 shows that, for any infinite-dimensional Hilbert space over $\mathbb{K} = \mathbb{R},\mathbb{C},\mathbb{H}$, the subspace $\mathfrak{u}_1(\mathcal{H})$ of the Schatten ideal $B_1(\mathcal{H})$ on the Hilbert space $\mathcal{H} = \ell^2(J,\mathbb{K})$, is the minimal Banach completion of the Lie algebra $\mathfrak{u}(J,\mathbb{K})$ with respect to any $\mathrm{U}(J,\mathbb{K})$-invariant norm.

6.3. Euclidean Lie algebras

An extremely important tool for studying invariant convex subsets of a Lie algebra $\mathfrak{g}$ are invariant symmetric bilinear forms. There are many interesting infinite-dimensional Lie algebras $\mathfrak{g}$ which carry an invariant positive definite form κ. We then call the pair $(\mathfrak{g},\kappa)$ a *euclidean Lie algebra*.

Examples 6.6. (a) Compact Lie algebras (cf. Subsection 4.1).

(b) Hilbert-Lie algebras (Definition 6.10), such as the Lie algebra $\mathfrak{u}_2(\mathcal{H})$ of skew-hermitian Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$ over $\mathbb{K} = \mathbb{R},\mathbb{C},\mathbb{H}$, with $\kappa(x,y) = -\mathrm{tr}_{\mathbb{R}}(xy)$.

(c) Mapping Lie algebras, such as $C^\infty(M,\mathfrak{k})$ for a compact Lie algebra $\mathfrak{k}$ and a compact smooth manifold M. Here every positive measure μ on M leads to an invariant form on $C^\infty(M,\mathfrak{k})$ by

$$\kappa(\xi,\eta) := \int_M \kappa_{\mathfrak{k}}(\xi(m),\eta(m))\,d\mu(m).$$

Every vector field $X \in \mathcal{V}(M)$ whose flow preserves μ defines a κ-skew derivation. Here the case $M = \mathbb{S}^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ is of particular importance. In this case

$$\kappa(\xi,\eta) := \frac{1}{2\pi}\int_0^{2\pi} \kappa_{\mathfrak{k}}(\xi(t),\eta(t))\,dt$$

and a vector field on $\mathbb{S}^1$ preserves the measure if and only if it is constant, i.e., generates rigid rotations.

(d) Poisson bracket algebras, such as $(C^\infty(M),\{\cdot,\cdot\})$ for a compact symplectic manifold (M,ω), where we put $\kappa(F,H) := \int_M FH\omega^n$. Here every symplectic vector field defines a κ-skew derivation. ■

The following proposition applies to the increasing sequence of the Lie algebras $\mathfrak{su}_n(\mathbb{K})$ for $\mathbb{K} = \mathbb{R},\mathbb{C},\mathbb{H}$, but also to more complicated sequences, such as $\mathfrak{su}_{2^n}(\mathbb{K})$. In particular, we do not require the existence of an elliptic Cartan subalgebra.

Proposition 6.7. *Let $\mathfrak{g}$ be a locally finite Lie algebra which is the directed union of compact simple Lie algebras. Then $\mathfrak{g}$ has an invariant scalar product which is unique up to a positive constant.*

Proof. The existence of an invariant symmetric bilinear form, which is invariant under all derivations, is shown in [47, Prop. II.1]. Let $0 \neq x \in \mathfrak{g}$ and $\mathfrak{g}_0 \subseteq \mathfrak{g}$ be a finite-dimensional compact simple Lie algebra containing x and an element y with $\kappa(x,y) \neq 0$. Then κ restricts to a non-zero invariant symmetric bilinear form on $\mathfrak{g}_0$, hence is either positive or negative definite. Replacing κ by $-\kappa$ if necessary, we may assume that κ is positive definite on $\mathfrak{g}_0$. Then it is also positive definite on all simple compact subalgebras $\mathfrak{g}_1$ containing $\mathfrak{g}_0$, and since $\mathfrak{g}$ is exhausted by such subalgebras, κ is positive definite.

The uniqueness assertion follows immediately from the corresponding uniqueness for simple compact Lie algebras. ■

Examples 6.8. (a) For the Lie algebra $\mathfrak{g} = \mathfrak{su}(\mathbb{N},\mathbb{C})$, the increasing union of the simple compact Lie algebras $\mathfrak{g}_n = \mathfrak{su}_n(\mathbb{C})$, a natural invariant scalar product is given by $\kappa(x,y) = -\operatorname{tr}(xy)$. The corresponding norm is the Hilbert-Schmidt norm, and by completion we obtain the simple Hilbert-Lie algebra $\mathfrak{u}_2(\ell^2(\mathbb{N}))$. In particular, the bracket extends to the completion.

(b) For the union $\mathfrak{g} := \mathfrak{su}_{2^\infty}(\mathbb{C})$ of the Lie algebras $\mathfrak{g}_n = \mathfrak{su}_{2^n}(\mathbb{C})$, defined by the embeddings $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, a compatible sequence of scalar products is given by $\kappa_n(x,y) = -2^{-n}\operatorname{tr}(xy)$. In this case the Lie bracket is not continuous with respect to the norm defined by the corresponding scalar product on $\mathfrak{g}$. ■

Problem 6.9. (a) Does Proposition 6.3 generalize in a suitable way to Lie algebras $\mathfrak{g}$ which are direct limits of simple compact ones but which do not contain an elliptic Cartan subalgebra?

(b) Is it true that every direct limit of compact Lie algebras has an invariant scalar product? By Proposition 6.7 this is true for direct limits of simple compact Lie algebras. ■

Definition 6.10. A *Hilbert-Lie algebra* is a euclidean Lie algebra which is complete with respect to the scalar product. ■

Proposition 6.11. *If $\mathfrak{g}$ is a Hilbert-Lie algebra, then each non-empty open invariant convex subset $\Omega \subseteq \mathfrak{g}$ intersects the center.*

Proof. If $\Omega = \mathfrak{g}$, there is nothing to show because $0 \in \Omega$. We may therefore assume that $\Omega \neq \mathfrak{g}$. Then Ω is a proper open convex subset, and since the group $\Gamma := \langle e^{\operatorname{ad}\mathfrak{g}} \rangle$ of inner automorphisms of $\mathfrak{g}$ acts isometrically on $\mathfrak{g}$ preserving Ω, the closed convex subsets

$$\Omega_c := \{x \in \Omega \colon d_\Omega(x) \leq c\}$$

are Γ-invariant (Lemma A.4). Each set Ω_c is a Bruhat-Tits space with respect to the induced Hilbert metric from $\mathfrak{g}$ and since Γ acts on this space isometrically with bounded orbits, the Bruhat-Tits Theorem A.13 implies the existence of a fixed point $z \in \Omega_c \subseteq \Omega$. Now it only remain to observe that $\mathfrak{z}(\mathfrak{g})$ is the set of fixed points for the action of Γ on $\mathfrak{g}$. ■

Corollary 6.12. *A Hilbert-Lie algebra contains a proper open invariant cone if and only if its center is non-trivial.*

Proof. That the existence of an open invariant cone implies that $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$ follows from Proposition 6.11.

If, conversely, there exist a non-zero $z_0 \in \mathfrak{z}(\mathfrak{g})$, and for $0 < r < \|z_0\}$, the open ball $B_r(z_0)$ is invariant with $0 \notin \overline{B_r(z_0)}$. Then Lemma A.5(b) implies that $\mathbb{R}_+^\times B_r(z_0)$ is a pointed open invariant cone. ■

7. Double extensions

In this final section we take a closer look at double extensions (Definition 3.1). This is motivated in particular by their natural emergence from covariant positive energy representations (cf. [27] and [28, §8]).

7.1. Lorentzian double extensions

In addition to invariant scalar products, invariant *Lorentzian forms* on Lie algebras are also very useful to analyze convex hulls of adjoint and coadjoint orbits. They arise naturally from double extensions of euclidean Lie algebras. If $(\mathfrak{g}, \kappa)$ is a euclidean Lie algebra (Subsection 6.3), and $\widehat{\mathfrak{g}} = (\mathfrak{g}, \kappa, D)$ is a double extension defined by the cocycle $\omega_D(x, y) = \kappa(Dx, y)$, where

$$D \in \operatorname{der}(\mathfrak{g}, \kappa) := \{D \in \operatorname{der}(\mathfrak{g}) \colon (\forall x, y \in \mathfrak{g})\ \kappa(Dx, y) + \kappa(x, Dy) = 0\}$$

is a κ-skew-symmetric derivation on $\mathfrak{g}$, then

$$\beta((z,x,t),(z',x',t')) := zt' + tz' - \kappa(x,x') \tag{38}$$

defines an invariant Lorentzian form on $\widehat{\mathfrak{g}}$ ([38]). We call such double extensions *Lorentzian.* On $\widehat{\mathfrak{g}}$ we have an action Ad of the Lie group

$$G^\sharp := G \rtimes_{\alpha^G} \mathbb{R},$$

where $\alpha^G \colon \mathbb{R} \to \mathrm{Aut}(G)$ is a smooth action for which the corresponding action on $\mathfrak{g}$ is generated by the derivation D.

Definition 7.1. In the following we write $\mathbf{c} := (1,0,0)$, $\mathbf{d} := (0,0,1)$ and, accordingly, $\mathbf{c}^* := (1,0,0)$ and $\mathbf{d}^* := (0,0,1)$ for the corresponding elements in $\widehat{\mathfrak{g}}'$. Note that $\mathbf{c} \in \widehat{\mathfrak{g}}$ and $\mathbf{d}^* \in \widehat{\mathfrak{g}}'$ are G-invariant. For $x \in \mathfrak{g}$ we put

$$\|x\| := \|x\|_\kappa := \sqrt{\kappa(x,x)}.$$

The *β-dual of* $\widehat{\mathfrak{g}}$ is the subset $\widehat{\mathfrak{g}}^\beta$ of $\widehat{\mathfrak{g}}'$, consisting of all linear functionals whose restriction to $\mathfrak{g}$ is continuous with respect to the κ-topology, i.e., contained in the Hilbert completion of the euclidean space $(\mathfrak{g}, \kappa)$. ■

Since β is a Lorentzian, the open subset

$$W := \{\mathbf{x} = (z,x,t) \in \mathfrak{g} \colon t > 0, \beta(\mathbf{x},\mathbf{x}) > 0\} \subseteq (0,\infty) \times \mathfrak{g} \times (0,\infty)$$

is an open convex invariant cone and $\chi(\mathbf{x}) := \beta(\mathbf{x},\mathbf{x})^{-1}$ defines on W a smooth strictly convex invariant function which tends to infinity at the boundary (Proposition A.9 and Example 2.12).

Proposition 7.2. *The following assertions hold:*

(a) $\Omega := W - \mathbb{R}_+\mathbf{c} = \{(z,x,t) \colon t > 0\}$ *is an open invariant half space bounded by the hyperplane ideal* $\tilde{\mathfrak{g}} = \mathbb{R}\mathbf{c} + \mathfrak{g}$.

(b) *For each* $x \in \Omega$, *we have* $x \in \mathrm{Ext}(\overline{\mathrm{conv}}(\mathcal{O}_x))$.

(c) *If* $\lambda = (z^*, \alpha, t^*) \in \widehat{\mathfrak{g}}^\beta$ *with* $z^* > 0$ *and* $\mathbf{x} = (z,x,t)$ *with* $t > 0$, *then* $\lambda(\mathcal{O}_x) = \mathcal{O}_\lambda(x)$ *is bounded from below.*

(d) $W^\star$ *is contained in the* β*-dual* $\widehat{\mathfrak{g}}^\beta$ *of* $\widehat{\mathfrak{g}}$.

(e) *If* $\lambda \in \widehat{\mathfrak{g}}^\beta$ *satisfies* $\lambda(\mathbf{c}) = z^* \neq 0$, *then its orbit* $\mathcal{O}_\lambda$ *is semi-equicontinuous.*

Proof. (a) For $\mathbf{x} = (z,x,t)$ with $t > 0$ we have $\beta(\mathbf{x},\mathbf{x}) = 2zt - \kappa(x,x)$. For $z_0 := \frac{1}{2t}\kappa(x,x)$ and $\mathbf{x}_0 := (z_0,x,t)$ we thus obtain

$$\mathbf{x} = \mathbf{x}_0 + (z - z_0)\mathbf{c} \quad \text{and} \quad \beta(\mathbf{x}_0,\mathbf{x}_0) = 2z_0t - \kappa(x,x) = 0,$$

so that $\mathbf{x}_0 \in \partial W$, and for $z_1 > z_0$ we get $\mathbf{x}_1 := (z_1,x,t) \in W$.

(b) Since $\mathbf{c}$ is fixed by the adjoint action, $\mathcal{O}_{\mathbf{x}} = \mathcal{O}_{\mathbf{x}_0} + (z - z_0)\mathbf{c}$. The orbit $\mathcal{O}_{\mathbf{x}_0}$ lies in the set $\big\{(\kappa(y,y)/2t, y, t) \colon y \in \mathfrak{g}\big\}$, which is the graph of the strictly convex function $f(y) := \kappa(y,y)/2t = \|y\|^2/2t$ on $\mathfrak{g}$. Therefore

$$\overline{\mathrm{conv}}(\mathcal{O}_{\mathbf{x}_0}) \subseteq \{(z,y,t) \colon y \in \mathfrak{g}, z \geq f(y)\}$$

and thus $\mathcal{O}_{\mathbf{x}_0} \subseteq \{(f(y),y,t) \colon y \in \mathfrak{g}\} = \mathrm{Ext}(\{(z,y,t) \colon y \in \mathfrak{g}, z \geq f(y)\})$ implies $\mathcal{O}_{\mathbf{x}_0} \subseteq \mathrm{Ext}(\overline{\mathrm{conv}}(\mathcal{O}_{\mathbf{x}_0}))$.

(c) In view of (a), we may w.l.o.g. assume that $\mathbf{x} = \mathbf{x}_0 \in \partial W$. Then

$$\lambda(f(y), y, t) = tt^* + \alpha(y) + f(y)z^* = tt^* + \alpha(y) + z^* \frac{\|y\|^2}{2t}$$

is bounded below on $\mathfrak{g}$ because $|\alpha(y)| \leq \|\alpha\| \cdot \|y\|$ and $z^*/t > 0$.

(d) If $\lambda = (z^*, \alpha, t^*) \in W^\star$, then λ is bounded on the set $\big((\mathbf{c}+\mathbf{d})-W\big)\cap(W-(\mathbf{c}+\mathbf{d}))$. This set consists of all elements $\mathbf{x}_0 = (z, x, t)$, satisfying $(1\pm z, \pm x, 1\pm t) \in W$, which means that $|z| < 1$, $|t| < 1$ and $\|x\|^2 < 2(1-|z|)(1-|t|)$. Therefore α is bounded on an open ball in $\mathfrak{g}$, hence continuous with respect to the κ-topology, i.e., $\lambda \in \widehat{\mathfrak{g}}^\beta$.

(e) Write $\lambda = (z^*, x^*, t^*)$ and observe that

$$\beta^*((z^*, x^*, t^*), (\tilde{z}^*, \tilde{x}^*, \tilde{t}^*)) := z^*\tilde{t}^* + \tilde{z}^* t^* - \kappa(x^*, \tilde{x}^*)$$

defines an invariant Lorentzian form on the β-dual. If $z^* = \lambda(\mathbf{c}) \neq 0$, then

$$\beta^*(\lambda + t\mathbf{d}^*, \lambda + t\mathbf{d}^*) = \beta^*(\lambda, \lambda) + 2t\beta^*(\lambda, \mathbf{d}^*) = \beta^*(\lambda, \lambda) + 2tz^*,$$

so that $t := -\frac{1}{2z^*}\beta^*(\lambda, \lambda)$ leads for $\lambda_0 := \lambda + t\mathbf{d}^*$ to $\beta^*(\lambda_0, \lambda_0) = 0$. Now $\mathcal{O}_\lambda = \mathcal{O}_{\lambda_0} - t\mathbf{d}^*$ and since $\lambda_0 \in W^\star$ follows from (d), $\mathcal{O}_\lambda$ is semi-equicontinuous because $W^\star$ is the dual of an open cone (Example 2.3(b)). ■

We now turn to global aspects of double extensions. Assume that $\mathfrak{g} = \mathbf{L}(G)$ for a 1-connected Lie group G and that $\alpha^G \colon \mathbb{R} \to \mathrm{Aut}(G)$ is a smooth action for which the corresponding action on the Lie algebra $\alpha_t := \mathbf{L}(\alpha_t^G)$ satisfies $\alpha'(0) = D$. We may thus form the semidirect product group $G^\sharp := G \rtimes_{\alpha^G} \mathbb{R}$ for which there exists a smooth action on the central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}^\sharp := \mathfrak{g} \rtimes_D \mathbb{R}$ by $\mathbb{R}$.

The action of $\mathbb{R}$ is simply given by

$$\mathrm{Ad}(\exp s\mathbf{d})(z, x, t) = (z, \alpha_s(x), t).$$

The G-action on $\mathfrak{g}^\sharp = \mathfrak{g} \rtimes \mathbb{R}\mathbf{d}$ has the form

$$\mathrm{Ad}(g)(x, t) := (\mathrm{Ad}(g)x + t\gamma(g), t)$$

for the cocycle
$$\gamma \colon G \to \mathfrak{g}, \quad \gamma(g) = \mathrm{Ad}(g)\mathbf{d} - \mathbf{d}. \tag{39}$$

For the central extension $\tilde{\mathfrak{g}} = \mathbb{R}\mathbf{c} \oplus \mathfrak{g}$, the pairing

$$\tilde{\mathfrak{g}} \times \mathfrak{g}^\sharp \to \mathbb{R}, \quad \big((z, x), (x', t')\big) \mapsto zt'$$

is $\mathrm{Ad}(G)$-invariant, so that the G-action on $\tilde{\mathfrak{g}}$ takes the form

$$\mathrm{Ad}(g)(z, x) = \Big(z - \kappa(x, \gamma(g^{-1})), \mathrm{Ad}(g)x\Big) = \Big(z + \kappa(\mathrm{Ad}(g)x, \gamma(g)), \mathrm{Ad}(g)x\Big),$$

where we use that $\gamma(g^{-1}) = -\mathrm{Ad}(g)^{-1}\gamma(g)$. Using the fact that the function $\beta(\mathbf{x}, \mathbf{x})$ on $\widehat{\mathfrak{g}}$ is $\mathrm{Ad}(G)$-invariant, we obtain:

Proposition 7.3. *For $g \in G$, $z, t \in \mathbb{R}$ and $x \in \mathfrak{g}$, we have*

$$\mathrm{Ad}_{\mathfrak{g}}(g)(z, x, t) = \Big(z - \kappa(\gamma(g^{-1}), x) + \frac{t}{2}\kappa(\gamma(g), \gamma(g)), \mathrm{Ad}(g)x + t\gamma(g), t\Big),$$

and in particular,

$$\mathrm{Ad}_{\mathfrak{g}}(g)\mathbf{d} = \Big(\frac{1}{2}\kappa(\gamma(g), \gamma(g)), \gamma(g), 1\Big) = \Big(\frac{1}{2}\|\gamma(g)\|^2, \gamma(g), 1\Big).$$

The coadjoint action is given by

$$\mathrm{Ad}^*_{\mathfrak{g}}(g^{-1})(z^*, x^*, t^*) = (z^*, x^*, t^*) \circ \mathrm{Ad}_{\mathfrak{g}}(g)$$
$$= \Big(z^*, x^* \circ \mathrm{Ad}_{\mathfrak{g}}(g) - z^*\kappa(\gamma(g)^{-1}, \cdot), \frac{z^*}{2}\kappa(\gamma(g), \gamma(g)) + x^*(\gamma(g)) + t^*\Big).$$

Remark 7.4. As $\mathbf{d}$ is fixed by $\mathrm{Ad}(\exp \mathbb{R}\mathbf{d})$, the cocycle $\gamma\colon G \to \mathfrak{g}, \gamma(g) = \mathrm{Ad}(g)\mathbf{d} - \mathbf{d}$ is equivariant with respect to the action of $\exp(\mathbb{R}\mathbf{d})$ on G and $\mathfrak{g}$, respectively. It follows in particular that the smooth function $G \to \mathbb{R}$, $g \mapsto \kappa(\gamma(g), \gamma(g))$ is $\exp(\mathbb{R}\mathbf{d})$-invariant. We also see that $\mathrm{Ad}(G)\mathbf{d} = \mathrm{Ad}(G^\sharp)\mathbf{d}$ is an adjoint orbit in $\widehat{\mathfrak{g}}$. ■

Examples 7.5. (a) Important examples arise from Hilbert-Lie algebras $\mathfrak{k}$ (Definition 6.10) with the positive definite invariant scalar product $\kappa_{\mathfrak{k}}$,

$$\mathfrak{g} = C^\infty(\mathbb{S}^1, \mathfrak{k}), \quad \kappa(\xi, \eta) := \frac{1}{2\pi}\int_0^{2\pi} \kappa_{\mathfrak{k}}(\xi(t), \eta(t))\, dt, \quad \mathbb{S}^1 \cong \mathbb{R}/2\pi\mathbb{Z}.$$

Then $D\xi := \xi'$ is a derivation leaving κ invariant. If K is a 1-connected Lie group with Lie algebra $\mathfrak{k}$, then $G := C^\infty(\mathbb{S}^1, K)$ is a 1-connected Lie group with Lie algebra $\mathfrak{g}$ ([59]). The action of $\mathbb{R}$ on $\mathfrak{g}$ is given by

$$\alpha_t(\xi)(s) := \xi(t+s) \quad \text{and} \quad \alpha_t^G(f)(s) := f(t+s).$$

This implies that $$\gamma(g)(s) = -\frac{d}{dt}\Big|_{t=0} g(t+s)g(s)^{-1} = -\delta^r(g)_s,$$

so that $\gamma = \delta^r$ is the right logarithmic derivative.

(b) A more general class of examples arises from principal K-bundles (P, M, q, K), where K is a compact group, and a volume form μ on the compact manifold M.

Then $\mathfrak{g} := \mathfrak{gau}(P) \cong \{\xi \in C^\infty(P, \mathfrak{k})\colon (\forall p \in P)(\forall k \in K)\, \xi(pk) = \mathrm{Ad}(k)^{-1}\xi(p)\}$ carries an invariant scalar product given by

$$\kappa(\xi, \eta) := \int_M \tilde\kappa_{\mathfrak{k}}(\xi, \eta) \cdot \mu,$$

where we use that the function $\kappa_{\mathfrak{k}}(\xi, \eta)$ on P is constant on the K-orbits, hence factors through a smooth function $\tilde\kappa_{\mathfrak{k}}(\xi, \eta)$ on M.

If $X \in \mathcal{V}(M, \mu)$ is a vector field with $\mathcal{L}_X\mu = 0$, then any lift $\tilde X \in \mathfrak{aut}(P) = \mathcal{V}(P)^K$ defines a skew-symmetric derivation of $(\mathfrak{g}, \kappa)$ and we may form the corresponding double extension.

(c) An interesting example is the following: We consider a left invariant vector field X on $M = \mathbb{T}^2$ whose orbits are dense. Then $\ker D = \{0\}$ on $\mathfrak{g}^0 := C^\infty(\mathbb{T}^2, \mathbb{R})$.

Similar examples arise from (irrational) quantum tori. ■

7.2. Double extensions of Hilbert-Lie algebras

Example 7.6. Let $\mathcal{H}$ be a complex Hilbert space and $\mathfrak{g} = \mathfrak{u}_2(\mathcal{H})$ be the Lie algebra of skew-hermitian Hilbert-Schmidt operators.

Each continuous cocycle $\omega \in Z^2(\mathfrak{g}, \mathbb{R})$ can be written as

$$\omega(x,y) = \operatorname{tr}([d,x]y)$$

for some bounded skew hermitian operator $d \in \mathfrak{u}(\mathcal{H})$ ([14]). Then $Dx := [d,x]$ defines a continuous derivation of $\mathfrak{g}$ preserving the invariant scalar product defined by $(x,y) := -\operatorname{tr}(xy)$. From that we easily derive that the cocycle $\gamma\colon G = \mathrm{U}_2(\mathcal{H}) \to \mathfrak{g}$ is given by

$$\gamma(g) = gdg^{-1} - d.$$

If $d \in \mathbb{R}i\mathbf{1} + \mathfrak{u}_2(\mathcal{H})$, then the cocycle ω is trivial and γ is obviously bounded.

If, conversely, γ is bounded, then the Bruhat-Tits Theorem A.13 implies that the affine isometric action of G on $\mathfrak{g}$ defined by $g * x := \operatorname{Ad}(g)x + \gamma(g)$ has a fixed point $x \in \mathfrak{g}$. This means that

$$g(d+x)g^{-1} = d + x \quad \text{for each} \quad g \in \mathrm{U}_2(\mathcal{H}),$$

and hence that $d + x \in \mathbb{C}\mathbf{1}$, which leads to $d \in \mathfrak{u}_2(\mathcal{H}) + \mathbb{R}i\mathbf{1}$. We conclude that each non-trivial 2-cocycle ω leads to an unbounded cocycle γ.

If $\gamma(G)$ is semi-equicontinuous, then $B(\operatorname{im}(\gamma))^0$ is an open invariant cone in $\mathfrak{u}_2(\mathcal{H})$, hence equal to the whole space (Proposition 6.11) and Remark 2.2 further entails that γ is bounded. ∎

Remark 7.7. Consider the Lorentzian double extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g} = \mathfrak{u}_2(\mathcal{H})$, defined by the skew-hermitian bounded operator d (Example 7.6). We want to classify semi-equicontinuous coadjoint orbits in $\widehat{\mathfrak{g}}'$ and open invariant cones in $\widehat{\mathfrak{g}}$.

Beyond the general observations that hold for all Lorentzian double extensions (Proposition 7.2), this seems rather difficult. However, according to [67, Satz 1], there exists a skew-hermitian Hilbert-Schmidt operator s such that $d' := d + s$ is diagonalizable. Then the double extensions defined by d and d' are isomorphic, so that we may assume that d is diagonalizable. Then the centralizer $\mathfrak{g}^d := \{x \in \mathfrak{g} \colon [d,x] = 0\}$ contains a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and

$$\widehat{\mathfrak{t}} = \mathbb{R}\mathbf{c} \oplus \mathfrak{t} \oplus \mathbb{R}\mathbf{d}$$

is an elliptic Cartan subalgebra of $\widehat{\mathfrak{g}}$. In the corresponding root system all roots are compact and $\mathfrak{g}^{\mathrm{alg}}$ is a direct limit of compact Lie algebras. We refer to [35] for a concrete analysis of this situation. ∎

7.3. Twisted loop algebras

Let K be a connected Lie group whose Lie algebra $\mathfrak{k}$ is a simple Hilbert-Lie algebra. We fix $\Phi \in \operatorname{Aut}(K)$ with $\Phi^N = \mathrm{id}_K$, and write $\varphi = \mathbf{L}(\Phi) \in \operatorname{Aut}(\mathfrak{k})$. For $\tau := \frac{2\pi}{N}$, the corresponding *twisted loop group (of period τ)* is

$$\mathcal{L}_\Phi^\tau(K) := \{\xi \in C^\infty(\mathbb{R}, K) \colon (\forall x \in \mathbb{R})\ \xi(x+\tau) = \Phi^{-1}(\xi(x))\} \tag{40}$$

with Lie algebra

$$\mathcal{L}_\varphi^\tau(\mathfrak{k}) := \{\xi \in C^\infty(\mathbb{R}, \mathfrak{k}) \colon (\forall x \in \mathbb{R})\ \xi(x+\tau) = \varphi^{-1}(\xi(x))\}, \tag{41}$$

where $\varphi = \mathbf{L}(\Phi) \in \mathrm{Aut}(\mathfrak{k})$ is the automorphism of $\mathfrak{k}$ induced by Φ. We have a natural action of $\mathbb{R}$ on $\mathcal{L}_\Phi(K)$ by

$$\alpha_t(\xi)(x) = \xi(x+t) \quad \text{and} \quad D\xi = \frac{d}{dt}\Big|_{t=0}\alpha_t(\xi) = \xi'. \tag{42}$$

It satisfies $\alpha_{2\pi} = \mathrm{id}_{\mathcal{L}_\Phi(K)}$, hence factors through an action of $\mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{T}$.

We write κ for an invariant scalar product on $\mathfrak{k}$ normalized in such a way that $\kappa(i\check{\alpha}, i\check{\alpha}) = 2$ for long roots α. Accordingly, we obtain the double extension

$$\widehat{\mathcal{L}}^\tau_\varphi(\mathfrak{k}) := (\mathbb{R} \oplus_\omega \mathcal{L}^\tau_\varphi(\mathfrak{k})) \rtimes_D \mathbb{R} \quad \text{with} \quad D\xi = \xi',$$

where
$$\omega(\xi, \eta) := \frac{c}{2\pi\tau}\int_0^\tau \kappa(\xi'(t), \eta(t))\, dt$$

for some $c \in \mathbb{Z}$ (the *central charge*).

Let $\mathfrak{t}^\circ \subseteq \mathfrak{k}^\varphi$ be a maximal abelian subalgebra. Then $\mathfrak{z}_\mathfrak{k}(\mathfrak{t}^\circ)$ is an elliptic Cartan subalgebra of $\mathfrak{k}$ by [55, Lemma D.2], and

$$\mathfrak{t} = \mathbb{R}\mathbf{c} \oplus \mathfrak{t}^\circ \oplus \mathbb{R}\mathbf{d} \tag{43}$$

is an elliptic Cartan subalgebra of $\widehat{\mathcal{L}}^\tau_\varphi(\mathfrak{k})$. The corresponding set of roots $\Delta \subseteq i\mathfrak{t}'$ can be identified with the set of pairs (α, n), where

$$(\alpha, n)(z, h, s) := (0, \alpha, n)(z, h, s) = \alpha(h) + isn, \quad n \in \mathbb{Z}, \alpha \in \Delta_n.$$

Here $\Delta_n \subseteq i(\mathfrak{t}^\circ)^*$ is the set of $\mathfrak{t}^\circ$-weights in the φ-eigenspace

$$\mathfrak{k}^n_\mathbb{C} = \{x \in \mathfrak{k}_\mathbb{C} \colon \varphi^{-1}(x) = e^{in\tau}x\}.$$

In this case all root vectors are of type (N) or (CS). If $\mathfrak{k}$ is finite-dimensional, then $\widehat{\mathfrak{g}}^{\mathrm{alg}}_\mathbb{C}$ is an affine Kac-Moody algebra, and in general it is a direct limit of affine Kac-Moody Lie algebras. For these Lie algebras, we have the following convexity theorem for coadjoint orbits. It is obtained by applying direct limit arguments to the topological version of the Kac-Peterson Convexity Theorem in Subsection 5.3. We refer to P. Helmreich's thesis [15] for details.

Theorem 7.8. (Convexity Theorem for twisted loop groups)
For $\lambda \in \widehat{\mathfrak{t}}'$ *with* $\lambda(\mathbf{c}) \neq 0$, *we have* $p_{\mathfrak{t}'}(\overline{\mathrm{conv}}\mathcal{O}_\lambda) = \overline{\mathrm{conv}}(\mathcal{W}\lambda)$.

Problem 7.9. Classify the adjoint orbits of elements of the form $(z, x, t) \in \mathfrak{g}$ with $t \neq 0$. ■

7.4. Non-Lorentzian double extensions

To deal with double extensions of Hilbert-Lie algebras and of twisted loop algebras, one has to determine to which extent the rich geometric information available for Lorentzian double extensions is still available for non-Lorentzian ones. Here a promising approach is to consider a Lie algebra $\mathfrak{g}$ with a positive definite form $\kappa_\mathfrak{g}$ and two $\kappa_\mathfrak{g}$-skew derivations $D_1, D_2 \in \mathrm{der}(\mathfrak{g}, \kappa_\mathfrak{g})$. We further assume that the derivation

$[D_1, D_2]$ is inner. One can show that in this case the Lorentzian double extension $\widehat{\mathfrak{g}}_1$ defined by (ω_{D_1}, D_1) carries a κ_1-skew symmetric derivation $\tilde{D}_2$ obtained by lifting D_2, so that we can form a double extension $\widehat{\widehat{\mathfrak{g}}}$ of $\widehat{\mathfrak{g}}_1$ defined by $(\omega_{\tilde{D}_2}, \tilde{D}_2)$. This results in a so-called *bidouble extension* which carries an invariant symmetric bilinear form which is non-Lorentzian of index 2 but which contains two hyperplane ideals that are central extensions of Lorentzian double extensions. This geometric structure should be explored with respect to its impact on closed convex hulls of adjoint and coadjoint orbits.

A. Appendices

A.1. Some facts on convex sets

The following observation shows that semi-equicontinuous convex sets share many important properties with compact ones (cf. [49, Prop. 6.13]):

Proposition A.1. *Let $C \subseteq E'$ be a non-empty weak-$*$-closed convex semi-equicontinuous subset and $v \in B(C)^0$. Then C is weak-$*$-locally compact, the function $\eta_v \colon C \to \mathbb{R}, \eta_v(\alpha) := \alpha(v)$ is proper, and there exists an extreme point $\alpha \in C$ with $\alpha(v) = \min\langle C, v\rangle$.*

The following lemma ([9, Cor. II.2.6.1]) is often useful:

Lemma A.2. *For a convex subset C of a locally convex space E, the sets C^0 and $\overline{C}$ are convex. If $C^0 \neq \emptyset$, then $\overline{C}$ and C have the same interior and $\overline{C^0} = \overline{C}$.*

From the Hahn-Banach separation theorem, we obtain immediately:

Proposition A.3. *If $D \subseteq E$ is a convex cone, then $\overline{D} = (D^\star)^\star$, and if $C \subseteq E'$ is a convex cone, then $(C^\star)^\star = \overline{C}$ is its weak-$*$-closure.*

The following lemma is often useful to obtain results on open convex subsets of normed spaces from those on closed ones. If $\dim E < \infty$, then the characteristic function

$$\varphi_C(x) := \int_{B(C)} e^{-\alpha(x)+\inf \alpha(C)}\, d\mu_{E^*}(\alpha) \tag{44}$$

of an open convex subset $C \subseteq E$ has similar properties, but it is invariant under all affine automorphisms of C preserving Lebesgue measure $\mu_{E'}$ ([43, Thm. V.5.4]).

Lemma A.4. *Let $p \colon E \to \mathbb{R}_+$ be a seminorm on the real vector space E and $U \subseteq E$ be a proper p-open convex subset. We write $B_r^p(x) := \{y \in E \colon p(x-y) < r\}$ for the open p-ball of radius r and $d_{\partial U}(x) := \sup\{r \geq 0 \colon B_r^p(x) \subseteq U\}$ for the distance of x from ∂U with respect to p. Then*

$$d_U^p \colon U \to \mathbb{R}, \quad d_U^p(x) := d_{\partial U}(x)^{-1}$$

is a continuous convex function on U with $\lim_{x_n \to x \in \partial U} d_U^p(x_n) = \infty$. In particular, all the set $U_c := \{x \in U \colon d_U^p(x) \leq c\}$ are closed convex subsets of E.

Proof. To see that d_U^p is convex, let $B := B_1^p(0)$ denote the open p-unit ball in E. For $x, y \in U$ we have $x + \frac{1}{d_U^p(x)}B$, $y + \frac{1}{d_U^p(y)}B \subseteq U$. For $0 < \lambda < 1$, $z = \lambda x + (1-\lambda)y$ and $0 < p(v) < s := \frac{\lambda}{d_U^p(x)} + \frac{1-\lambda}{d_U^p(y)}$, let $v_0 := \frac{v}{s}$. Then $v_0 \in B$, so that

$$\begin{aligned} z + v &= \lambda x + (1-\lambda)y + \Big(\frac{\lambda}{d_U^p(x)} + \frac{1-\lambda}{d_U^p(y)}\Big)v_0 \\ &\in \lambda\Big(x + \frac{1}{d_U^p(x)}B\Big) + (1-\lambda)\Big(y + \frac{1}{d_U^p(y)}B\Big) \subseteq U. \end{aligned}$$

Hence $d_{\partial U}^p(z) \geq \frac{\lambda}{d_U^p(x)} + \frac{1-\lambda}{d_U^p(y)}$. We conclude from the convexity and the antitony of the function $r \mapsto \frac{1}{r}$ for $r > 0$ that $d_U^p(z) \leq \lambda d_U^p(x) + (1-\lambda)d_U^p(y)$, i.e. that d_U^p is a convex function.

The continuity of d_U^p on U follows from the continuity of the distance function $d_{\partial U}^p$ with respect to the semi-metric defined by p. If $x_n \to x \in \partial U$, then $d_{\partial U}^p(x_n) \to 0$, so that $d_U^p(x_n) \to \infty$. ■

Lemma A.5. (Constructing open cones) *Let E be a locally convex space.*

(a) *Let $H \subseteq E$ be a closed hyperplane not containing 0, $E_0 := H - H$, and let $\emptyset \neq \Omega \subseteq H$ be a relatively open convex subset. Then $W := \mathbb{R}_+^\times \Omega$ is an open convex cone in E with*

$$\overline{W} \cap E_0 = \lim(\Omega) \quad \text{and} \quad H(W) = H(\Omega).$$

In particular, W is pointed if and only if Ω contains no affine lines.

(b) *If Ω is a bounded open convex subset of the real locally convex space E with $0 \notin \overline{\Omega}$, then $\mathbb{R}_+^\times \Omega$ is a pointed open cone.*

Proof. (a) The convexity of W follows immediately from the convexity of Ω. Let $\lambda\colon E \to \mathbb{R}$ be the unique continuous linear functional with $\lambda(H) = \{1\}$. Then $E_0 = \ker \lambda$, the map

$$\mathbb{R}_+^\times \times H \to \lambda^{-1}(\mathbb{R}_+^\times), \quad (t, v) \mapsto tv$$

is a homeomorphism, and W is the image of the open subset $\mathbb{R}_+^\times \times \Omega$, hence open.

Let $w \in \overline{W} \cap E_0$ and pick a net $t_j w_j \to w$ with $w_j \in \Omega$. Then we have $t_j = \lambda(t_j w_j) \to \lambda(w) = 0$, so that Lemma 2.9(ii) shows that $w \in \lim(\Omega)$. If, conversely, $0 \neq w \in \lim(\Omega)$ and $x \in \Omega$, then $x + \mathbb{R}_+ w \subseteq \Omega$, and therefore $\frac{1}{n}(x + nw) \to w$ implies that $w \in \overline{W} \cap E_0$. This proves that $\overline{W} \cap E_0 = \lim(\Omega)$.

From $\lambda \in W^\star$ we derive $H(W) \subseteq \ker \lambda = E_0$, so that

$$H(W) = H(\overline{W}) = \overline{W} \cap (-\overline{W}) \cap E_0 = \lim(\Omega) \cap -\lim(\Omega) = H(\Omega).$$

(b) To see that $W := \mathbb{R}_+^\times \Omega$ is a convex cone, we observe that, for $x, y \in \Omega$ and $t, s > 0$, we have

$$tx + sy = (t+s)\Big(\frac{t}{t+s}x + \frac{s}{t+s}y\Big) \in (t+s)\Omega \subseteq W.$$

To see that W is pointed, we use the Hahn-Banach-Separation Theorem and $0 \notin \overline{\Omega}$ to find a continuous linear functional $\lambda \in E'$ with $\inf \lambda(\Omega) = 1$. Let $w \in \overline{W}$ and

$t_jx_j \to w$ with $x_j \in \Omega$ and $t_j > 0$. Then $t_j \le t_j\lambda(x_j) \to \lambda(w)$. By passing to a suitable subnet, we may thus assume that $t_j \to t_0 \ge 0$. If $t_0 = 0$, then the boundedness of Ω implies that $w = 0$. If $t_0 > 0$, then $x_j \to t_0^{-1}w \in \overline{\Omega}$ and $\lambda(w) = t_0 \lim \lambda(x_j) \ge t_0$. This shows that $\overline{W} \setminus \{0\} \subseteq \lambda^{-1}(\mathbb{R}_+^\times)$, and hence that $\overline{W} \cap -\overline{W} = \{0\}$. ■

Lemma A.6. (Enlarging semi-equicontinuous subsets by cones) *Let E be a locally convex space, $\Omega \subseteq E$ be a non-empty open convex cone and let $C \subseteq E'$ be a weak-$*$-closed convex subset with $\Omega \subseteq B(-C)$. Then $C - \Omega^\star$ is weak-$*$-closed semi-equicontinuous, and a convex subset $C_1 \subseteq E'$ is contained in $C - \Omega^\star$ if and only if $s_{C_1}|_\Omega \le s_C|_\Omega$.*

Proof. The convex subset $C_2 := C - \Omega^\star \subseteq E'$. It clearly satisfies $s_{C_2} = s_C$ on Ω, so that any subset $C_1 \subseteq C_2$ satisfies $s_{C_1} \le s_{C_2} = s_C$ on Ω.

Assume, conversely, that $s_{C_1} \le s_C$ holds on Ω. From Proposition A.1 we know that C and $\Omega^\star$ are weak-$*$-locally compact, and for every $x \in \Omega \subseteq B(-C)$, the evaluation maps $\eta_x \colon C \to \mathbb{R}$ and $\eta_x \colon -\Omega^\star \to \mathbb{R}$ are proper and bounded from above. This implies that the function

$$f \colon C \times \Omega^\star \to \mathbb{R}, \quad (\lambda, \beta) \mapsto (\lambda - \beta)(x)$$

is also proper and bounded from above. It follows that the subset $C_2 = C - \Omega^\star \subseteq E'$ is weak-$*$-closed, and since $s_{C_2} = s_C$ holds on Ω, we see that C_2 is semi-equicontinuous.

For any $y \in B(-C_2)$, the evaluation map η_y must be bounded from above on the convex cone $-\Omega^\star$, which implies that $y \in (\Omega^\star)^\star = \overline{\Omega}$. This proves that $\Omega \subseteq B(-C_2) \subseteq \overline{\Omega}$, and since $B(-C_2)^0$ is open and convex, we obtain $\Omega = B(-C_2)^0$ from Lemma A.2. Now (3) in the proof Theorem 2.4 yields

$$\begin{aligned} C - \Omega^\star = C_2 &= \{\lambda \in E' \colon (\forall v \in B(-C_2)^0)\ \lambda(v) \le s_{C_2}(v)\} \\ &= \{\lambda \in E' \colon (\forall v \in \Omega)\ \lambda(v) \le s_C(v)\}, \end{aligned}$$

and our assumption thus implies that $C_1 \subseteq C - \Omega^\star$. ■

Lemma A.7. *Let (E, τ) be a locally convex space and $\tau' \subseteq \tau$ be a coarser locally convex topology on E. If $\Omega \subseteq E$ is τ-open and convex and contains a τ'-inner point, then Ω is also τ'-open.*

Proof. Let $x_0 \in \Omega$ be τ'-inner and $U \subseteq \Omega$ be an open convex τ'-neighborhood of x_0. If $x_1 \in \Omega$ is any other point, there exists an $\varepsilon > 0$ with $x_2 := x_1 + \varepsilon(x_1 - x_0) = (1+\varepsilon)x_1 - \varepsilon x_0 \in \Omega$ because Ω is τ-open. Then

$$x_1 = \frac{1}{1+\varepsilon}(\varepsilon x_0 + x_2) \in \frac{\varepsilon}{1+\varepsilon}U + \frac{1}{1+\varepsilon}x_2 \subseteq \Omega$$

shows that Ω is also a τ'-neighborhood of x_1. ■

Some Lorentzian geometry. Let E be a locally convex space and $\beta \colon E \times E \to \mathbb{R}$ be a *Lorentzian form*, i.e., β is symmetric bilinear, and there exists $v_0 \in E$ with $\beta(v_0, v_0) > 0$ such that β is negative definite on $v_0^\perp$.

Note that this implies that the same holds for any $v \in E$ with $\beta(v,v) > 0$ (cf. [8, Cor. IV.7.5]). Then $\{v\colon \beta(v,v) > 0\}$ is the union of two open convex cones, and

$$W := \{v \in E\colon \beta(v,v) > 0, \beta(v_0, v) > 0\}$$

is one of them.

Lemma A.8. (Inverse CS inequality) *For $\beta(v,v) \geq 0$ and $x \in E$ we have*

$$\beta(v,v)\beta(x,x) \leq \beta(x,v)^2.$$

Proof. If $\beta(v,v) = 0$, there is nothing to show. So we assume that $\beta(v,v) > 0$ and we may w.l.o.g. assume that $\beta(v,v) = 1$. Writing $E = \mathbb{R}v \oplus v^\perp$ and, accordingly, $x = \lambda v + w$ with $\lambda = \beta(v,x)$, we have

$$\beta(v,v)\beta(x,x) = \beta(x,x) = \lambda^2 + \beta(w,w) \leq \lambda^2 = \beta(x,v)^2$$

because β is negative definite on $v^\perp$. ■

Proposition A.9. *The smooth function $\chi\colon W \to \mathbb{R}, \chi(v) := \beta(v,v)^{-1}$ has the following properties:*

(i) $\chi(v+w) \leq \chi(v)$ *for $v, w \in W$.*

(ii) *If $v_n \to v \in \partial W$, then $\chi(v_n) \to \infty$.*

(iii) χ *is strictly convex.*

Proof. (i) We have to show that, for $v, w \in W$, we have

$$\beta(v+w, v+w) = \beta(v,v) + \beta(w,w) + 2\beta(v,w) \geq \beta(v,v).$$

Since $\beta(w,w) > 0$, it suffices to see that $\beta(v,w) > 0$. To verify this inequality, pick $v_0 \in W$ with $\beta(v_0,v_0) = 1$ and write $E = \mathbb{R}v_0 \oplus v_0^\perp$. Accordingly, we have $v = \lambda v_0 + v'$ and $w = \mu v_0 + w'$ with $\lambda, \mu > 0$, so that

$$\beta(v,w) = \lambda\mu + \beta(v',w') \quad \text{and} \quad 0 < \beta(v,v) = \lambda^2 + \beta(v',v').$$

Applying the Cauchy-Schwarz inequality to the restriction of $-\beta$ to $v_0^\perp$, we get

$$\beta(v',w')^2 \leq \beta(v',v')\beta(w',w') < \lambda^2\mu^2,$$

which leads to $\beta(v,w) > 0$.

(ii) is a trivial consequence of the definition.

(iii) For $v \in W$ we have

$$(\partial_x\chi)(v) = -\frac{2}{\beta(v,v)^2}\beta(v,x), \quad \text{and} \quad (\partial_x^2\chi)(v) = \frac{2}{\beta(v,v)^3}\Big(2\beta(v,x)^2 - \beta(v,v)\beta(x,x)\Big).$$

From the inverse Cauchy-Schwarz inequality (Lemma A.8) we derive

$$\beta(x,x)\beta(v,v) \leq \beta(x,v)^2,$$

so that $\partial_x^2\chi(v) \geq 0$ for every $x \in E$. The relation $(\partial_x^2\chi)(v) = 0$ implies

$$\beta(v,x)^2 + \big(\beta(v,x)^2 - \beta(v,v)\beta(x,x)\big) = 0,$$

and since both summands are non-negative, we obtain $\beta(v,x) = 0$. Then the relation $-\beta(v,v)\beta(x,x) = 0$ further leads to $\beta(x,x) = 0$. As β is negative definite on $v^\perp$, we finally obtain $x = 0$. ■

A.2. The Bruhat-Tits Theorem

Definition A.10. Let (X,d) be a metric space. We say that (X,d) satisfies the *semi parallelogram law* if, for $x_1, x_2 \in X$, there exists a point $z \in X$ such that, for each $x \in X$, we have

$$d(x_1,x_2)^2 + 4d(x,z)^2 \leq 2d(x,x_1)^2 + 2d(x,x_2)^2. \tag{SPL}$$

Every Euclidean space satisfies the parallelogram law if z is chosen to be the midpoint of x_1 and x_2, and this trivially implies the semi parallelogram law. Accordingly, the point z occurring in the preceding definition plays the role of a "midpoint" of x_1 and x_2.

Remark A.11. Suppose that $(E, \|\cdot\|)$ is a normed space endowed with the metric $d(x,y) := \|x-y\|$. When does (E,d) satisfy the semi parallelogram law? This is the case if the norm is defined by a positive definite symmetric bilinear form $\langle \cdot,\cdot \rangle$ via $\|x\| := \sqrt{\langle x,x \rangle}$. If, conversely, the semi parallelogram law is satisfied, then we obtain with $x_1 = -x_2 = a$ and $x = b$ for all $a,b \in E$ the relation

$$4\|a\|^2 + 4\|b\|^2 \leq 2\|a+b\|^2 + 2\|a-b\|^2,$$

which in turn leads to

$$2\|a+b\|^2 + 2\|a-b\|^2 \leq \|(a+b)+(a-b)\|^2 + \|(a+b)-(a-b)\|^2 = 4\|a\|^2 + 4\|b\|^2,$$

and therefore to the parallelogram law

$$2\|a\|^2 + 2\|b\|^2 = \|a+b\|^2 + \|a-b\|^2 \quad \text{for all} \quad a,b \in E.$$

By a theorem of P. Jordan and J. von Neumann ([29]), each normed satisfying the parallelogram law is euclidean. In fact,

$$\langle a,b \rangle := \frac{\|a+b\|^2 - \|a-b\|^2}{4}$$

is a positive definite symmetric bilinear form with $\|a\| = \sqrt{\langle a,a \rangle}$. Therefore the euclidean spaces are precisely the normed spaces satisfying the semi parallelogram law. ■

Definition A.12. A *Bruhat-Tits space* is a complete metric space (X,d) satisfying the semi parallelogram law. ■

We write $\mathrm{Aut}(X,d)$ for the group of *automorphisms of the metric space* (X,d). For an elementary proof of the following theorem we refer to [34].

Theorem A.13. (Bruhat-Tits Fixed Point Theorem) *Let X be a Bruhat-Tits space and $G \subseteq \mathrm{Aut}(X,d)$ a subgroup with a bounded orbit. Then G has a fixed point.*

Example A.14. If E is a real Hilbert space and $C \subseteq E$ a closed convex subset, then C is a Bruhat-Tits space with respect to the euclidean metric. ■

A.3. A general lemma

The following lemma ([51, Lemma 6.15]) captures the spirit of various constructions of invariant convex cones in Lie algebras in a quite natural way. F.i., it applies to finite-dimensional simple algebras as well as $\mathcal{V}(\mathbb{S}^1)$ ([51, §8]).

Lemma A.15. *Suppose that the element $\mathbf{d} \in \mathfrak{g}$ has the following properties:*

(a) *The interior $W_{\min}$ of the invariant convex cone generated by $\mathcal{O}_d = \mathrm{Ad}(G)\mathbf{d}$ is non-empty and different from $\mathfrak{g}$.*

(b) *There exists a continuous linear projection $p_{\mathbf{d}} \colon \mathfrak{g} \to \mathbb{R}\mathbf{d}$ which preserves every open and closed convex invariant subset.*

(c) *There exists an element $x \in \mathfrak{g}$ for which $p_{\mathbf{d}}(\mathcal{O}_x)$ is unbounded.*

Then each non-empty open invariant cone W contains $W_{\min}$ or $-W_{\min}$, and for $\lambda \in \mathfrak{g}'$ the following are equivalent

(i) $\lambda \in \mathfrak{g}'_{\mathrm{seq}}$*, i.e., $\mathcal{O}_\lambda$ is semi-equicontinuous.*

(ii) *$\mathcal{O}_\lambda(d)$ is bounded from below or above.*

(iii) $\lambda \in W_{\min}^\star \cup -W_{\min}^\star$.

Typical classes of Lie algebras to which this lemma applies are finite-dimensional hermitian simple Lie algebras with $\mathfrak{z}(\mathfrak{k}) = \mathbb{R}\mathbf{d}$ ([43]) and $\mathcal{V}(\mathbb{S}^1)$ with the generator $\mathbf{d}$ of rigid rotations of the circle (cf. Subsection 5.4). For further applications we refer to the restricted metaplectic group discussed in [51, §9.2].

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