Journal of Lie Theory
Volume **30** (2020) 587–616

Semigroups and Moment Lyapunov Exponents

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Communicated by G. Ólafsson

Abstract. Let G be a noncompact semi-simple Lie group with finite center and μ a probability measure on G. We consider (i) the semigroup S_μ generated by the support of μ (with the assumption that $\mathrm{int} S_\mu \neq \emptyset$); (ii) The spectral radii r_λ of the operators $U_\lambda(\mu)$ where U_λ is a (nonunitary) representation of G induced by a real character and (iii) the moment Lyapunov exponents $\gamma(\lambda, x)$ of the i.i.d. random product on G defined by μ. The equality $r_\lambda = \gamma(\lambda, x)$ holds in many cases. We give a necessary and sufficient condition to have $S_\mu = G$ in terms of the analyticity of the map $\lambda \mapsto r_\lambda$. The condition is applied to measures obtained by solutions of invariant stochastic differential equations on G yielding a necessary and sufficient condition for the controllability of invariant control systems on G in terms of the largest eigenvalues of second order differential operators.

Mathematics Subject Classification: 22E46, 34D08, 22F30.

Key Words: Semi-simple Lie groups, semigroups, moment Lyapunov exponent, flag manifolds.

1. Introduction

Semigroups in Lie groups are a long standing research theme of Jimmie Lawson to whom this paper is dedicated. Here we consider semigroups in a noncompact semi-simple Lie group G. The results to be proved relate the theory of semigroups to the asymptotic of random products in G in an attempt to develop probabilistic and analytic methods to study semigroups in semi-simple Lie groups.

In order to be more precise let us introduce the notation and concepts involved. Fix an Iwasawa decomposition $G = KAN$ so that the maximal flag manifold of G reads $\mathbb{F} = G/P = K/M$ where $P = MAN$ is a minimal parabolic subgroup with M the centralizer of A in K.

To get the K-invariant cocycles over $\mathbb{F}$ let $\mathfrak{a}$ be the Lie algebra of A and define the maps $\mathsf{a} : G \times K \to \mathfrak{a}$ and $\rho = e^{\mathsf{a}} : G \times K \to A$ by the requirement

$$gu = ke^{\mathsf{a}(g,u)}n = k\rho(g, u)\, n \in KAN.$$

These maps are right M-invariant in the second variable hence they factor to maps (with same notation) $\mathsf{a} : G \times \mathbb{F} \to \mathfrak{a}$ and $\rho : G \times \mathbb{F} \to A$. For $\lambda \in \mathfrak{a}^*$ we write $\mathsf{a}_\lambda(g, x) = \lambda \mathsf{a}(g, x)$ and $\rho_\lambda(g, x) = e^{\mathsf{a}_\lambda(g,x)}$.

Given a probability measure μ on G form the sample space $\Omega = G^{\mathbb{N}}$ endowed with the product measure $\mu^{\mathbb{N}}$ and take the independent and identically distributed (i.i.d.)

*Supported by CNPq grant no. 303755/09-1 and Fapesp grant no. 2019/10625-4

ISSN 0949–5932 / \$2.50

sequence y_n of random variables $y_n(\mathbf{x}) = x_n$, $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ whose common law is the push-forward $(y_n)_*(\mu^{\mathbb{N}}) = \mu$. The law of the random product

$$g_n = y_n \cdots y_1$$

is the n-th convolution power μ^n. The moment Lyapunov exponents of the random product g_n depends on $x \in \mathbb{F}$ and $\lambda \in \mathfrak{a}^*$ and is defined by

$$\gamma(\lambda, x) = \lim_{n\to+\infty} \frac{1}{n} \log \int \rho_\lambda(g, x)\, \mu^n(dg) \tag{1}$$

when the limit exists. (This definition makes sense only when μ has exponential moments in the sense that $\int \rho_\lambda(g,x)\,\mu(dg) < \infty$.)

In this paper we consider the semigroup S_μ generated by the support suppμ of μ and look at the dependence of $\gamma(\lambda, x)$ on (λ, x) according to the action of S_μ on the flag manifolds of G. We assume throughout that S_μ has nonempty interior and G has finite center.

Our first result in this direction takes into account the case where S_μ is the whole group G.

Theorem 1.1. *If S_μ acts transitively on $\mathbb{F}$ (or equivalently if $S_\mu = G$) then $\gamma(\lambda, x) = \gamma(\lambda)$ is constant as a function of x. The function $\gamma(\lambda)$ is analytic and convex.*

On the other hand if S_μ is a proper semigroup then its action on the flag manifolds is described by the structure of its control sets which is summarized in the concept of flag type of a semigroup (with nonempty interior). To state the results that yield the flag type of a semigroup S let us introduce some notation regarding the flag manifolds of G.

Denote by Π the set of (restricted) roots of $\mathfrak{a}$ and let Π^+ be the set of positive roots such that $N = \exp \mathfrak{n}$ and $\mathfrak{n} = \sum_{\alpha\in\Pi^+} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is the root space of α. Let $\Sigma \subset \Pi^+$ be the corresponding set of simple roots. For $\Theta \subset \Sigma$ we denote by P_Θ the standard parabolic subgroup having parabolic subalgebra $\mathfrak{p}_\Theta = \mathfrak{p} \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_{-\alpha}$ and put $\mathbb{F}_\Theta = G/P_\Theta$ for the flag manifold defined by Θ. We endow the set of flag manifolds with the partial order where $\mathbb{F}_{\Theta_1}$ is bigger than $\mathbb{F}_{\Theta_2}$ if there is a fibration $\mathbb{F}_{\Theta_1} \to \mathbb{F}_{\Theta_2}$ or equivalently if $P_{\Theta_1} \subset P_{\Theta_2}$.

Now an invariant control set D for the S-action in a manifold is a subset such that clSx = cl$D$ for every $x \in D$ and D is maximal (by inclusion) with this property. It was proved in [15] that there is a unique invariant control set for the S action on a flag manifold $\mathbb{F}_\Theta$. We denote this invariant control set by C_Θ.

We can introduce now the flag type of a semigroup in a semi-simple Lie group. The results ensuring its existence and uniqueness were proved in [20] (see also [17], [15] and [16]).

Theorem – Definition 1.2. *Let $S \subset G$ be a proper semigroup with* int$S \neq \emptyset$. *Then there exists a unique flag manifold $\mathbb{F}_{\Theta(S)}$ satisfying the following two conditions:*

(1) *$\mathbb{F}_{\Theta(S)}$ is maximal among the flag manifolds $\mathbb{F}_\Theta$ such that the unique S-invariant control set $C_\Theta \subset \mathbb{F}_\Theta$ is contractible in the sense that for every $h \in$ int$S$ we have $h^n C_\Theta$ shrinks to a point as $n \to +\infty$.*

(2) $\mathbb{F}_{\Theta(S)}$ *is minimal among the flag manifolds* $\mathbb{F}_\Theta$ *such that* $C = \pi^{-1}(C_\Theta)$ *is the invariant control set in the maximal flag manifold* $\mathbb{F}$.

The flag manifold $\mathbb{F}_{\Theta(S)} = G/P_{\Theta(S)}$ *is called the* flag type *of the semigroup* S.

The result in the next theorem is the starting point for the relationship between moment Lyapunov exponents and flag type of semigroups. In its statement we denote by $(\mathfrak{a}^*)^+$ the positive Weyl chamber in $\mathfrak{a}^*$ w.r.t. Π^+. Its closure $\mathrm{cl}\,(\mathfrak{a}^*)^+$ is the convex cone generated by the fundamental weights $\Phi = \{\omega_1, \ldots, \omega_l\}$ defined by

$$\langle \alpha_i^\vee, \omega_j \rangle = \frac{2\langle \alpha_i, \omega_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \delta_{ij}$$

where $\Sigma = \{\alpha_1, \ldots, \alpha_l\}$ is the simple system of roots. More generally for a subset $\Theta \subset \Sigma$ we denote by $\mathfrak{a}^*_\Theta$ the subspace orthogonal to Θ (w.r.t. the Cartan-Killing form) which is spanned by $\Phi \setminus \Phi_\Theta$ where Φ_Θ is the set of fundamental ω_i such that $\alpha_i \in \Theta$. The partial chamber $(\mathfrak{a}^*_\Theta)^+$ is the convex cone

$$(\mathfrak{a}^*_\Theta)^+ = \{\theta \in \mathfrak{a}^*_\Theta : \langle \alpha, \theta \rangle > 0,\ \alpha \in \Theta\}.$$

Its closure $\mathrm{cl}\,(\mathfrak{a}^*_\Theta)^+$ is the cone generated by $\Phi \setminus \Phi_\Theta$.

Theorem 1.3. *Suppose that* S_μ *is a proper semigroup with* $\mathrm{int} S_\mu \neq \emptyset$. *Let* $C = \pi^{-1}\left(C_{\Theta(S_\mu)}\right)$ *be its invariant control set in the maximal flag manifold and* $\mathbb{F}_{\Theta(S_\mu)}$ *its flag type. Then for any* $x \in C$ *we have*

$$\lim_{p \to -\infty} \gamma(p\lambda, x) < 0 \quad \textit{if} \quad \lambda \in \left(\mathfrak{a}^*_{\Theta(S_\mu)}\right)^+.$$

This theorem yields a necessary condition for $S_\mu \neq G$ in terms of the analyticity of the spectral radii of operators associated to μ. To introduce these operators we consider the representations U_λ on function spaces on $\mathbb{F}$ defined by

$$(U_\lambda(g) f)(x) = \rho_\lambda(g, x) f(gx)$$

with f a function in $\mathbb{F}$. This is a representation by right action, that is, $U_\lambda(gh) = U_\lambda(h) \circ U_\lambda(g)$, which is a representation by the left action of the opposite group.

If the probability measure μ on G has exponential moments then we get a well defined operator $U_\lambda(\mu) = \int U_\lambda(g)\,\mu(dg)$, that is,

$$(U_\lambda(\mu) f)(x) = \int_G \rho_\lambda(g, x) f(gx)\,\mu(dg).$$

We have $U_\lambda(\mu^n) = U_\lambda(\mu)^n$ so that the moment Lyapunov exponents are given by

$$\gamma(\lambda, x) = \lim_{n \to +\infty} \frac{1}{n} \log\left(U_\lambda(\mu)^n 1\right)(x). \qquad (2)$$

Under further assumptions on μ the operators $U_\lambda(\mu)$ are compact in the Banach space of continuous function $\mathcal{C}(\mathbb{F})$ as well on $L^2(\mathbb{F})$. Hence the spectrum of each $U_\lambda(\mu)$ is enumerable and since these operators are positive the spectral radius r_λ of $U_\lambda(\mu)$ is an eigenvalue having a nonnegative eigenfunction. These facts permit to apply existing pertubation theorems of compact operators and prove analyticity of $\lambda \mapsto r_\lambda$ for λ in certain ranges.

In case $\lambda \in (\mathfrak{a}_\Theta^*)^+$ then $\rho_\lambda(g,x)$ factors to a cocycle over $\mathbb{F}_\Theta$ and we can define the representation U_λ^Θ on function spaces on $\mathbb{F}_\Theta$ by the same expression as above.

Formula (2) permits to relate the spectral radii to the moment Lyapunov exponents and prove the equality $\log r_\lambda = \gamma(\lambda, x)$ in some cases. The proof of Theorem 1.1 follow these lines. Under the assumption that $S_\mu = G$ the positive operators $U_\lambda(\mu)$ are irreducible so that each r_λ is an eigenvalue of multiplicity one yielding the equality $\log r_\lambda = \gamma(\lambda, x)$ for every $x \in \mathbb{F}$. (See Section 6 for the details. In particular irreducibility of positive operators is worked out in Proposition 6.2.)

More generally if $S_\mu \neq G$ we consider the restriction $U_\lambda^C(\mu)$ of $U_\lambda(\mu)$ to the function spaces on the invariant control set $C \subset \mathbb{F}$. Again the $U_\lambda^C(\mu)$ is irreducible so that $\log r_\lambda^C = \gamma(\lambda, x)$ now for $x \in C$ where r_λ^C is the spectral radius of $U_\lambda^C(\mu)$.

The function $\eta_\lambda^C(p) = \log r_{p\lambda}^C$ is analytic and convex. The limit in Theorem 1.3 determines the shape of η_λ^C when $\lambda \in (\mathfrak{a}_{\Theta(S_\mu)}^*)^+$. On the other hand the shape of the function $\eta_\lambda(p) = \log r_{p\lambda}$ is obtained by computing adjoints in $L^2(\mathbb{F})$.

By comparing the functions of p, $\eta_\lambda^C(p)$ and $\eta_\lambda(p)$ we obtain the following result on the nonanalyticity of $\eta_\lambda(p)$.

Theorem 1.4. *Suppose that S_μ is a proper semigroup with $\mathrm{int} S_\mu \neq \emptyset$ with flag type $\mathbb{F}_{\Theta(S_\mu)}$. Then $\lambda \mapsto \log r_\lambda$ is not analytic at $-\omega_{\Theta(S_\mu)}$ where r_λ is the spectral radius of $U_\lambda(\mu)$ on the maximal flag manifold.*

In this statement for a subset $\Theta \subset \Sigma$ we denote by ω_Θ half the sum of positive roots outside $\langle\Theta\rangle$ counted with multiplicities:

$$\omega_\Theta = \frac{1}{2} \sum_{\alpha \in \Pi^+ \setminus \langle\Theta\rangle} (\dim \mathfrak{g}_\alpha)\, \alpha.$$

As an immediate consequence of Theorems 1.1 and 1.4 we get the following necessary and sufficient condition to have $S_\mu = G$ in terms of the analyticity of the function $\lambda \mapsto r_\lambda$ whose domain is $\mathfrak{a}^*$.

Corollary 1.5. *A necessary and sufficient condition to have $S_\mu = G$ is that $\lambda \mapsto \log r_\lambda$ is an analytic function on $\mathfrak{a}^*$. In case $S_\mu \neq G$ analyticity fails at $\lambda = -\omega_{\Theta(S_\mu)}$ where $\mathbb{F}_{\Theta(S_\mu)}$ is the flag type of S_μ.*

One of our motivations to relate moment Lyapunov exponents to the flag type of semigroups was the attempt to get an indirect condition, like in this corollary, to the hard problem of deciding when a semigroup generated by a subset is the whole group.

In Section 9 we apply the previous results to the controllability problem of a control system

$$\dot{g} = X(g) + \sum_{j=1}^{m} u_j Y_j(g) \tag{3}$$

on G where $X, Y_1, \ldots, Y_m$ are right invariant vector fields and the control range is $u \in \mathbb{R}$. The controllability problem amounts to decide when the control semigroup

S generated by $\{e^{t\left(X+\sum_{j=1}^{m} u_j Y_j\right)} : t \geq 0, u \in \mathbb{R}\}$ is proper or not. To this purpose we write the corresponding stochastic differential equation

$$dg = X(g)\, dt + \sum_{j=1}^{m} Y_j(g) \circ dW_j \tag{4}$$

whose solution from the identity have transition probabilities μ_t, $t \geq 0$, that form a one-parameter semigroup under convolution because $\mu_t * \mu_s = \mu_{t+s}$. Hence for any $\lambda \in \mathfrak{a}^*$ we have the one-parameter semigroup of operators $U_\lambda(\mu_t)$ acting on function spaces on $\mathbb{F}$. Under the assumption that $X, Y_1, \ldots, Y_m$ generate the Lie algebra $\mathfrak{g}$ of G it follows that the operators $U_\lambda(\mu_t)$, $t > 0$, are compact (this is a necessary condition for controllability).

The link of the semigroup $U_\lambda(\mu_t)$ to the controllability problem comes from the well-known support theorem (see Ikeda-Watanabe [12], Chapter VI) ensuring that the control semigroup is the union

$$S = \bigcup_{t \geq 0} \mathrm{supp}\mu_t.$$

Hence by applying a continuous-time version of the previous results we can relate the controllability problem to spectral radii of $U_\lambda(\mu_t)$ and the moment Lyapunov exponents.

Now by stochastic calculus (Itô's formula) the one-parameter semigroup $U_\lambda(\mu_t)$ has the infinitesimal generator $L_\lambda = U_\lambda(L)$ where $L = X + \frac{1}{2}\sum_{j=1}^{m} Y_j^2$ and $U_\lambda(L)$ stands for the infinitesimal representation of the universal enveloping algebra of $\mathfrak{g}$ (in Section 9 we write $U_\lambda(L)$ as a second order differential operator in $\mathbb{F}$).

Under the Lie algebra rank condition (i.e., X and Y_j generate the Lie algebra $\mathfrak{g}$ of G) the infinitesimal generator $U_\lambda(L)$ has a principal eigenvalue γ_λ such that $e^{t\gamma_\lambda}$ is the spectral radius of $U_\lambda(\mu_t)$, $t>0$. Hence the previous results yield the following necessary and sufficient condition for controllability in terms of the analyticity of $\lambda \mapsto \gamma_\lambda$.

Theorem 1.6. *Assume that the invariant control system (3) satisfies the Lie algebra rank condition. Then it is controllable (that is, the control semigroup S equals G) if and only if $\lambda \mapsto \gamma_\lambda$ is analytic where γ_λ is the principal eigenvalue of the second order operator $U_\lambda(L)$.*

2. Measures

In this paper we work with *exposed measures* (*mesures étalées*) on groups as defined next. The following proposition-definition was proved in Azencott [4] (see Définition I.8 and the following proof by an application of Lemma 3.3 of Furstenberg [7]).

Proposition 2.1. *A measure μ on a Lie group G is said to be* exposed *if it satisfies one of the following equivalent conditions.*

(1) *Some convolution power μ^n of μ is not singular w.r.t. the Haar measure dg.*

(2) *There are an integer m, an open set U and $c > 0$ such that such that $\mu^m \geq c\,dg$ on U.*

It is known (and easy to prove) that μ is exposed if $\mu = \phi(g)\,dg$ has a density $\phi(g)$ w.r.t. the Haar measure dg (see [4]). Furthermore if μ is exposed then $\mathrm{int} S_\mu \neq \emptyset$ because $\mathrm{supp}\mu^m \subset S_\mu$ for every $n \geq 1$.

When G is semi-simple it is said that a Borel probability measure μ has exponential moments provided

$$\int_G \rho_\lambda(g,x)\,\mu(dg) < \infty \tag{5}$$

for every $x \in \mathbb{F}$ and $\lambda \in \mathfrak{a}^*$. Equivalently μ has exponential moments if

$$\int_G \sup_x \rho_\lambda(g,x)\,\mu(dg) < \infty$$

for all $\lambda \in \mathfrak{a}^*$. (Sketch of proof: If λ is a dominant weight then $\rho_\lambda(g,x) = \|R_\lambda(g)x\| / \|x\|$ where $\|\cdot\|$ is a norm in the the space of the finite dimensional representation R_λ defined by λ. In this case

$$\sup_x \rho_\lambda(g,x) = \|R_\lambda(g)\| \leq C \max\{\|R_\lambda(g)e_1\|, \ldots, \|R_\lambda(g)e_n\|\}$$

ensuring that the first condition implies the second. For a general λ write it as a linear combination of dominant weights.)

By the cocycle property $\rho_\lambda(gh,x) = \rho_\lambda(g,hx)\rho_\lambda(h,x)$ it follows that if μ and ν have exponential moments then their convolution $\mu * \nu$ also has exponential moments and hence any convolution power μ^n has exponential moments.

3. Flag type

In this section we prove part of Theorem 1.3. Namely we show that $\gamma(p\lambda,x)$ remains negative as $p \to -\infty$ when $\lambda \in (\mathfrak{a}^*_{\Theta(S_\mu)})^+$ and x belongs to the invariant control set. Afterwards when we relate the moment Lyapunov exponents $\gamma(\lambda,x)$ with the spectral radius of the operator $U_\lambda^C(\mu)$ the limit in Theorem 1.3 will become clear.

As in the introduction for a subset $\Theta \subset \Sigma = \{\alpha_1, \ldots, \alpha_l\}$ we let $\Phi_\Theta = \{\omega_{i_1}, \ldots, \omega_{i_j}\}$ be the set of fundamental weights with the same indices as those in Θ. Equivalently

$$\Phi \setminus \Phi_\Theta = \{\omega \in \Phi : \forall \alpha \in \Theta,\ \langle \alpha, \omega \rangle = 0\}$$

and denote by $(\mathfrak{a}^*_\Theta)^+$ the "partial chamber"

$$(\mathfrak{a}^*_\Theta)^+ = \{\beta \in \mathfrak{a}^*_\Theta : \forall \alpha \in \Sigma \setminus \Theta,\ \langle \alpha, \beta \rangle > 0\}$$

which is the interior (in $\mathfrak{a}^*_\Theta$) of the convex cone $\mathrm{cl}(\mathfrak{a}^*_\Theta)^+$ spanned by $\Phi \setminus \Phi_\Theta$.

Theorem 3.1. *Assume that μ is an exposed measure and let $\mathbb{F}_\Theta$, $\Theta =\subset \Sigma$, be the flag type of S_μ. Take $\lambda \in \mathrm{cl}\left(\mathfrak{a}^*_{\Theta(S_\mu)}\right)^+$. Then for all $x \in C$ and $p < 0$,*

$$\gamma_\lambda(p,x) = \gamma(p\lambda,x) = \lim\sup_{n \to +\infty} \frac{1}{n} \log \int_G \rho_{p\lambda}(g,x)\,\mu^n(dg) \leq 0.$$

Here $C \subset \mathbb{F}$ is the invariant control set of S_μ.

The proof of this theorem is based in the following lemma on semigroups proved first in [19]. For the sake of completeness we reproduce its proof below. In the statement of the lemma we write C_0 for the core of the invariant control set C of a semigroup S. The core is defined by

$$C_0 = \{x \in C : \exists h \in \mathrm{int} S, hx = x\}$$

and is an open set whose closure is C.

Lemma 3.2. *Let S be a semigroup whose flag type is $\mathbb{F}_\Theta$, $\Theta \subset \Sigma$. Denote by C its invariant control set in the maximal flag manifold $\mathbb{F}$. Take $\lambda \in (\mathfrak{a}_\Theta^*)^+$ and $x \in C_0$. Then there exists $c > 0$ such that $\rho_\lambda(g, x) = e^{\lambda(\mathsf{a}(g,x))} > c$ for all $g \in S$.*

This lemma yields the following estimate of the moment Lyapunov exponents for probability measures having support in S.

Corollary 3.3. *With the notation as in the lemma suppose that θ is a probability measure with support* $\mathrm{supp}\theta \subset S$. *If $\lambda \in (\mathfrak{a}_\Theta^*)^+$, $x \in C$ and $p < 0$ then*

$$\int_G e^{p\lambda(\mathsf{a}(g,x))} \theta(dg) < c^p$$

where $c > 0$ is as in the Lemma 3.2.

Proof. In fact, since $p < 0$ we have $e^{p\lambda(\mathsf{a}(g,x))} < c^p$ and hence

$$\int_G e^{p\lambda(\mathsf{a}(g,x))} \theta(dg) = \int_S e^{p\lambda(\mathsf{a}(g,x))} \theta(dg) < \int_S c^p \theta(dg) = c^p. \qquad \blacksquare$$

Theorem 3.1 follows immediately from this corollary. In fact,

$$\gamma_\lambda(p, x) = \lim \sup_{n \to +\infty} \frac{1}{n} \log \int_G e^{p\lambda(\mathsf{a}(g,x))} \mu^n(dg).$$

But the support of the n-th convolution power μ^n is contained in S_μ. Hence by the corollary the last integral is bounded above by c^p. It follows that

$$\gamma_\lambda(p, x) \leq \lim \sup_{n \to +\infty} \frac{1}{n} \log c^p = 0.$$

We check later that the $\lim\sup$ is indeed a limit and thus conclude the proof of Theorem 3.1. To prove Lemma 3.2 we assume without loss of generality that $A^+ \cap \mathrm{int} S \neq \emptyset$, which implies that origin x_0 of $\mathbb{F} = G/MAN$ belongs to the core C_0 of the invariant control set C. The convex cone

$$\Gamma_N = \{H \in \mathfrak{a} : \exists n \in N, \exists t > 0, \ e^{tH} n \in \mathrm{int} S\}$$

was considered in [20], Section 4. It is proved there that since $\mathbb{F}_\Theta$, $\Theta \subset \Sigma$, is the flag type of S then

$$\Gamma_N \subset \mathrm{cl}\left(\bigcup_{w \in \mathcal{W}_\Theta} \mathfrak{a}^+\right) \tag{6}$$

where $\mathcal{W}_\Theta$ is the subgroup of the Weyl group generated by the reflections w.r.t. the simple roots in Θ. (In [20] it is proved that if the inclusion fails then there are $w \notin \mathcal{W}_\Theta$, $H \in w\mathfrak{a}^+$ and $n \in N$ such that $g = e^H n \in \mathrm{int}S$. But this implies that x_0 is a fixed point of type w of g. Since $w \notin \mathcal{W}_\Theta$ this contradicts the fact that Θ is the flag type of S.)

From the inclusion (6) we can prove the following estimate for elements in $\mathrm{int}S$ fixing x_0.

Lemma 3.4. *Suppose that $g \in \mathrm{int}S$ is such that $gx_0 = x_0$. Then $\rho_\lambda(g, x_0) \geq 1$.*

Proof. If $gx_0 = x_0$ then we can write $g = man \in MAN$ in which case $\rho_\lambda(g, x_0) = e^{\lambda(H)}$ where $a = e^H$. We claim that $H \in C_\Theta$. In fact, we can pertub g inside $\mathrm{int}S$ and assume that m has finite order. Then for some $j \geq 1$ we have $g^j = a^j \overline{n} \in AN$, which implies, by the comments above, that $jH \in \Gamma_N \subset C_\Theta$ and hence $H \in C_\Theta$. Finally, for any $\lambda \in (\mathfrak{a}_\Theta^*)^+$ and $H' \in C_\Theta$ we have $\lambda(H') \geq 0$, which shows that $\rho_\lambda(g, x_0) = e^{\lambda(H)} \geq 1$ as claimed. ■

Proof of Lemma 3.2: Assume without loss of generality that in the statement of the lemma $x = x_0$ is the origin of $\mathbb{F} = G/MAN$. Suppose by contradiction that there exists a sequence $g_k \in S$ with $\rho_\lambda(g_k, x_0) \to 0$. It can be assumed that $g_k x \to y$ in which case $y \in C$. If $g \in S$ then $\rho_\lambda(gg_k, x) = \rho_\lambda(h, g_k \cdot x)\rho_\lambda(g_k, x) \to 0$ because the map $z \mapsto \rho_\lambda(g, z)$ is bounded. Take in particular $g \in \mathrm{int}S$ with $gy = x_0$. Then we can substitute g_k by gg_k and assume that $g_k \in \mathrm{int}S$ and $g_k x_0 \to x_0$.
Since $x_0 \in C^+$ we can choose $g_0 \in \mathrm{int}S$ with $g_0 x_0 = x_0$. We choose also a compact neighborhood W of g_0 with $W \subset \mathrm{int}S$ and hence $U = W^{-1}x_0$ is a neighborhood of x_0 in $\mathbb{F}$. By construction for every $z \in U$ there exists $h \in W$ such that $x_0 = hz$.

Write
$$r = \sup\{\rho_\lambda(h, z) : h \in W, z \in \mathbb{F}\}$$

which is finite by compactness. Now, take k large enough so that $g_k x_0 \in U$ and $\rho_\lambda(g_k, x_0) < 1/2r$. Then there exists $h \in W$ such that $hg_k x_0 = x_0$ and we have

$$\rho_\lambda(hg_k, x_0) = \rho_\lambda(h, g_k x_0)\rho_\lambda(g_k, x_0) \leq r\rho_\lambda(g_k, x_0) < \frac{r}{2r} = \frac{1}{2}.$$

But this contradicts the last lemma since $hg_k \in \mathrm{int}S$ and $hg_k x_0 = x_0$, concluding the proof of Lemma 3.2.

Example 3.5. We illustrate Lemma 3.2 with a subsemigroup of $\mathrm{Sl}(2, \mathbb{R})$. The cocycle over the projective line $\mathbb{P}^1$ of interest is when

$$\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1$$

in which case $\rho_\lambda(g, [z]) = \|gz\| / \|z\|$, $0 \neq z \in \mathbb{R}^2$. Consider the cone

$$W = \{(a, b) \in \mathbb{R}^2 : a \geq 0,\ |b| \leq a\}$$

and form the compression semigroup $S_W = \{g \in \mathrm{Sl}(2, \mathbb{R}) : gW \subset W\}$. Its attractor set in $\mathbb{P}^1$ is $C^+ = \{[(a, b)] \in \mathbb{P}^1 : (a, b) \in \mathrm{int}W\}$.

If $g \in S_W$ the $g(1,0) = (a,b) \in W$ so that $g(1,0) = h(1,0)$ where

$$h = \begin{pmatrix} 1 & 0 \\ b/a & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

Hence $h^{-1}g(1,0) = (1,0)$ and $h^{-1}g$ is upper triangular, that is, g has the form

$$g = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

with $\mu > 0$ and $|y| \leq 1$. The inclusions $g(1,-1) \in W$ and $g(1,1) \in W$ yield the inequalities $1 - x \geq 0$, $|\mu y(1-x) - \mu^{-1}| \leq \mu(1-x)$, $1 + x \geq 0$ and, finally, $|\mu y(1-x) + \mu^{-1}| \leq \mu(1+x)$. Hence $|x| \leq 1$ and

$$\mu^{-1} - \mu|y|(1-x) \leq \mu(1-x),$$

that is, $\mu^{-1} \leq \mu(1-x)(1+|y|)$. But $|y| \leq 1$ and $1 - x \leq 2$ so that $1 \leq 4\mu^2$ and $\mu \geq 1/2$. Since $g(1,0) = \mu(1,y)$ we get the lower bound

$$\|g(1,0)\| = \mu\sqrt{1+y^2} \geq \frac{1}{2}.$$

Now if $z = [(a,b)] \in C^+$ then there exists $h \in S_W$ with $z = hz_0$, $z_0 = [(1,0)]$. Hence if $g \in S_W$ then $gh \in S_W$ so that

$$\rho_\lambda(g,x) = \frac{\|ghz_0\|}{\|hz_0\|} \geq \frac{\|ghz_0\|}{\|h\|} \geq \frac{1}{2\|h\|}.$$

That is, for $z = hz_0 \in C^+$ we can take $c = 1/2\|h\|$ to get Lemma 3.2.

4. Results from operator theory

By definition of the operator $U_\lambda(\mu)$ we have $U_\lambda(\mu)^n 1(x) = \int_G \rho_\lambda(g,x)\mu^n(dg)$ where $1(x) = 1$. Hence the moment Lyapunov exponent $\gamma(\lambda,x)$ is given in terms of the operator $U_\lambda(\mu)$ by

$$\gamma(\lambda,x) = \limsup_{n\to\infty} \frac{1}{n} \log U_\lambda(\mu)^n 1(x).$$

Under suitable conditions this limit is the spectral radius (and even an eigenvalue) of $U_\lambda(\mu)$. This interpretation of the moment Lyapunov exponents open the way to apply operator theory to prove some of the main properties of $\gamma(\lambda,x)$ (e.g. $\limsup = \lim$, analyticity, etc.).

In this section we recollect results on operator theory that will be used in the proofs.

If X is a compact space we let $\mathcal{C}(X)$ be the Banach space of continuous functions on X endowed with the sup norm $\|\cdot\|_\infty$. An operator T in $\mathcal{C}(X)$ is positive if $Tf \geq 0$ whenever f is a nonnegative function in $\mathcal{C}(X)$.

The following result is the classical Krein–Rutman theorem which is a generalization to infinite dimensions of the Perron-Frobenius theorem for nonnegative matrices (see Schaefer [21]).

Theorem 4.1. *Let E be a Banach space and $W \subset E$ a convex cone such that the subspace $W - W$ is dense in E. Let $T : E \to E$ be a compact operator such that $T(W) \subset W$ and assume that its spectral radius $r(T)$ is strictly positive. Then $r(T)$ is an eigenvalue of T having an eigenvector in W.*

This theorem applies to positive operators of functional spaces ($\mathcal{C}(X)$ or $L^p(X)$) by taking W to be the cone of nonnegative functions. By writing a function f as $f = f^+ - f^-$ where $f^+ = (|f| + f)/2$ and $f^- = (|f| - f)/2$ it follows that $W - W$ is the whole functional space in each case.

A Borel subset $E \subset X$ is said to be invariant by a positive operator T on $\mathcal{C}(X)$ if $\mathrm{supp} f \subset E$ implies $\mathrm{supp}(Tf) \subset E$. The operator T is said to be irreducible if X is the only invariant set that contains the support $\mathrm{supp} f$ of a nonzero function f.

Proposition 4.2. *Let T be a positive compact operator on $\mathcal{C}(X)$. Suppose T is irreducible and $r(T) > 0$. Let f be a nonnegative eigenfunction of the spectral radius. Then* $\mathrm{supp} f = X$.

Proof. The set of functions $g \in \mathcal{C}(X)$ such that $\mathrm{supp} g \subset \mathrm{supp} f$ is the closure of the set

$$B = \{g \in \mathcal{C}(X) : \exists \alpha \geq 0,\ |g| \leq \alpha f\}.$$

If $g \in B$ then $|Tg| \leq T|g| \leq \alpha T f = \alpha\beta f$ so that $TB \subset B$. By continuity we conclude that $\mathrm{supp} Tg \subset \mathrm{supp} f$ if $\mathrm{supp} g \subset \mathrm{supp} f$. Hence irreducibility implies that $\mathrm{supp} f = X$. ∎

Still about positive operators we note that if T is bounded and positive then $\|T\|_\infty = \|T1\|_\infty$. In fact, if $\|f\|_\infty \leq 1$ then $|f| \leq 1$ hence $|Tf| \leq T|f| \leq T1$.

Proposition 4.3. *Let T be a bounded positive operator on $\mathcal{C}(X)$ and f a a nonnegative eigenfunction of T. Let β be the eigenvalue of f and assume that $\beta \neq 0$. Then $\limsup_n \log \frac{1}{n} (T^n 1)(x) \geq \log \beta$ if $f(x) > 0$. If moreover f is strictly positive then $\beta > 0$ and equals the spectral radius $r(T)$ of T. Furthermore $\lim \frac{1}{n} \log T^n 1(x) = \log \beta$ uniformly in x.*

Proof. Since T is positive we have $\beta^n f(x) = (T^n f)(x) \leq \|f\|_\infty (T^n 1)(x)$.

Hence,
$$\frac{1}{n} \log (T^n 1)(x) \geq \log \beta + \frac{1}{n} \log \frac{f(x)}{\|f\|_\infty}$$

and limit in the first statement follows.

Suppose now that f is strictly positive. We have $\beta > 0$ because $Tf = \beta f$. Put $m = \inf f > 0$ and $M = \sup f > 0$. Since T is a positive operator we have $mT^n 1 \leq T^n f \leq MT^n 1$, that is, for all $x \in X$ it holds

$$\frac{1}{M} \beta^n f(x) \leq T^n 1(x) \leq \frac{1}{m} T^n f(x). \tag{7}$$

Hence
$$\log \beta + \frac{1}{n} \log \frac{f(x)}{M} \leq \frac{1}{n} \log T^n 1(x) \leq \log \beta + \frac{1}{n} \log \frac{f(x)}{m}$$

implying that
$$\lim_{n \to \infty} \frac{1}{n} \log T^n 1(x) = \log \beta.$$

Since f is bounded $\frac{1}{n}\log\frac{f(x)}{M}$ and $\frac{1}{n}\log\frac{f(x)}{m}$ converge to 0 uniformly in x, proving the last statement. Now inequality (7) implies that

$$\frac{1}{M}\beta^n \|f\|_\infty \leq \|T^n 1\|_\infty \leq \frac{1}{m}\beta^n \|f\|_\infty .$$

But T is a positive operator hence $\|T^n\|_\infty = \|T^n 1\|_\infty$. Therefore

$$\lim_{n\to\infty}\frac{1}{n}\log\|T^n\| = \log\beta$$

so that β is the spectral radius $r(T)$ of T by the well known Gelfand formula $r(T) = \lim_{n\to\infty}\|T^n\|^{1/n}$. ■

Our analysis of the analyticity of $\lambda \mapsto \gamma(\lambda, x)$ is based on the following pertubation theorem (see Dunford-Schwartz [6], Section VII.6.7, Theorem 9).

Theorem 4.4. *Let $T(z)$, $|z| < \varepsilon$, be an analytic family of bounded operators of a complex Banach space E. Suppose that $\lambda_0 \in \mathbb{C}$ is an isolated point of the spectrum $\sigma(T(0))$ of $T(0)$ such that the corresponding spectral subspace $E(\lambda_0, T(0))$ has dimension m. Then there is an open set $U \ni \lambda_0$ and $\delta > 0$ such that if $|z| < \delta$ then the spectra of $T(z)$ in U splits into $k \leq m$ functions $\lambda_1(z), \ldots, \lambda_k(z)$ with $\lambda_i(0) = \lambda_0$ and satisfying the following properties:*

(1) *The dimension of the sum of the spectral subspaces associated to $\lambda_1(z), \ldots, \lambda_k(z)$ is equal to the dimension m of $E(\lambda_0, T(0))$.*

(2) *There exists $n \in \mathbb{N}$ such that each $\lambda_i(z)$ is a power series on the principal value of $z^{1/n}$.*

(3) *If $m = 1$ then $\lambda(z) = \lambda_1(z)$ is analytic at (a neighborhood of) 0.*

5. Compact operators

One of the main issues regarding the moment Lyapunov exponents stays in the relationship with the spectral radii r_λ of the operators $U_\lambda(\mu)$ acting on the Banach space of continuous functions $C(\mathbb{F})$ as well as in the L^2 space in $\mathbb{F}$ w.r.t. the K-invariant measure.

To apply the above mentioned results on operators a basic issue is the compactness of the operators $U_\lambda(\mu)$. For the operators in $C(\mathbb{F})$ this was proved in Guivarc'h-Raugi [8], Lemme 5.11. Its proof is based in an inequality which we derive below so that it can be applied to show that an eigenfunction in L^2 is in fact a continuous function. Assume from now on that $\mu = \phi dg$ has a density ϕ w.r.t. the Haar measure dg of G. In this case

$$\begin{aligned} U_\lambda(\mu) f(kx) &= \int_G \rho_\lambda(g, kx) f(gkx)\mu(dg) = \int_G \rho_\lambda(gk, x) f(gkx)\phi(g)\, dg \\ &= \int_G \rho_\lambda(g, x) f(gx)\phi\left(gk^{-1}\right) dg \end{aligned}$$

because dg is right invariant and the cocycle is K-invariant. Hence if f is a bounded function with norm $\|f\|_\infty = \sup|f(x)|$ then

$$\begin{aligned} |U_\lambda(\mu) f(kx) - U_\lambda(\mu) f(x)| &\leq \|f\|_\infty \int_G |\rho_\lambda(g,x)\left(\phi\left(gk^{-1}\right) - \phi(g)\right)| dg \\ &\leq \|f\|_\infty \int_G \Phi(g) |\phi\left(gk^{-1}\right) - \phi(g)| dg \end{aligned}$$

where $\Phi(g) = \sup_{x\in\mathbb{F}} \rho_\lambda(g,x)$. Because of the K-invariance of the cocycle we have $\Phi\left(gk^{-1}\right) = \Phi(g)$ for any $k \in K$. Put $F = \Phi\phi$ then the last term in the above inequality becomes

$$\|f\|_\infty \int_G |\Phi\left(gk^{-1}\right)\phi\left(gk^{-1}\right) - \Phi(g)\phi(g)| dg = \|f\|_\infty \|F \circ L_{k^{-1}} - F\|_1 . \tag{8}$$

(This integral exists because μ has exponential moments and hence $F \in L^1(G, dg)$.) Notice that $U_L(k) F = F \circ L_{k^{-1}}$ is the left regular representation of K on $L^1(G, dg)$ with norm $\|\cdot\|_1$. This representation is continuous, that is, the map $(k,g) \mapsto U_L(k) g$ is continuous. By the above inequality if $\mu = \phi(g)\, dg$ we have

1. $U_\lambda(\mu)(L^\infty(\mathbb{F})) \subset \mathcal{C}(\mathbb{F})$ as follows by the continuity of the left regular representation of K on G.
2. $U_\lambda(\mu)$ is a compact operator in $\mathcal{C}(\mathbb{F})$ for every λ. In fact the inequality shows that if B_1 is the unit ball in $\mathcal{C}(\mathbb{F})$ then $U_\lambda(\mu)(B_1)$ is bounded and equicontinuous. Hence by Arzelá-Ascoli's theorem $U_\lambda(\mu)(B_1)$ is relatively compact, that is, $U_\lambda(\mu)$ is a compact operator.

Now we discuss compactness of the operators on the L^2 spaces. For this purpose we apply the integral formula $dg = \rho_{2\omega}(g,1)\, dk\ da\ dn$ for the Iwasawa decomposition $G = KAN$ (see e.g. [10], Section I.5) to view $U_\lambda(\mu)$ as an integral operator on $L^2(K,m)$. We have,

$$\begin{aligned} U_\lambda(\mu) f(u) &= \int_G \rho_\lambda(g,u)\phi(g)\overline{f}(gu)\, dg = \int_G \rho_\lambda(gu,1)\phi(g)\overline{f}(gu)\, dg \\ &= \int_G \rho_\lambda(g,1)\phi\left(gu^{-1}\right)\overline{f}(g)\, dg \\ &= \int_K dk \int_{AN} \overline{f}(kan)\, e^{(\lambda+2\omega)\log a}\phi\left(gu^{-1}\right) da\ dn \\ &= \int_K f(k)\, dk \int_{AN} e^{(\lambda+2\omega)\log a}\phi\left(kanu^{-1}\right) da\ dn. \end{aligned}$$

Hence $U_\lambda(\mu) f(u) = \int_K Q_\lambda(u,k) f(k)\, dk$ where the kernel $Q_\lambda(u,k)$ is given by

$$\begin{aligned} Q_\lambda(u,k) &= \int_{AN} e^{(\lambda+2\omega)\log a}\phi\left(kanu^{-1}\right) da\ dn \\ &= \int_{kANu^{-1}} e^{(\lambda+2\omega)\log a}\phi(g)\, da\ dn. \end{aligned} \tag{9}$$

In the last integral the measure in $kANu^{-1}$ is the translation of the product of the Haar measures in $A \times N$.

It is well known that an integral operator in a L^2-space is compact provided the kernel Q_λ is of Hilbert-Schmidt class, that is,

$$\int Q_\lambda(u,k)^2\,du\,dk<\infty. \tag{10}$$

For example, if ϕ is continuous and has compact support then $Q_\lambda(u,k)$ is bounded and hence of Hilbert-Schmidt class.
For later reference we state the following proposition that summarizes the above discussion.

Proposition 5.1. *An operator $U_\lambda(\mu)$ is compact in $L^2(K,m)$ if the integrability condition* (10) *holds.*

Clearly the same conclusion holds for the operators on $L^2(\mathbb{F},m)$ since they are obtained by restriction to a closed subspace of $L^2(K,m)$.

When the kernel Q_λ is of Hilbert-Schmidt class we can apply Cauchy-Schwarz inequality to conclude that the functions in the image of $U_\lambda(\mu)$ are bounded.

Proposition 5.2. *Suppose that Q_λ is of Hilbert-Schmidt class. Then $U_\lambda(\mu)f$ is bounded if f belongs to $L^2(K,m)$ or $L^2(\mathbb{F},m)$.*

Proof. In fact, for any $u\in K$ we have

$$|U_\lambda(\mu)f(u)|\le\int_K|Q(u,k)f(k)|\,dk\le\|f\|_2\int_K Q(u,k)^2\,dk<\infty. \qquad \blacksquare$$

Combining the this proposition with the previous estimates we obtain the following regularity result for the eigenfunctions of $U_\lambda(\mu)$ on L^2.

Corollary 5.3. *Suppose that Q_λ is of Hilbert-Schmidt class and let f be an eigenfunction (nonzero eigenvalue) of $U_\lambda(\mu)$ in $L^2(K,m)$ or $L^2(\mathbb{F},m)$. Then f is continuous.*

Proof. By the above proposition $U_\lambda(\mu)f$ is bounded. Hence $U_\lambda(\mu)^2f$ is continuous because $U_\lambda(\mu)(L^\infty(\mathbb{F}))\subset\mathcal{C}(\mathbb{F})$. Since f is an eigenfuction with nonzero eigenvalue it follows that f is continuous as well. $\blacksquare$

The results of this section hold true also for the representations U_λ^Θ on the function spaces of a partial flag manifold $\mathbb{F}_\Theta$ if $\lambda\in\mathfrak{a}_\Theta^*$. This is because the mapping $f\mapsto f\circ\pi$ (where $\pi:\mathbb{F}\to\mathbb{F}_\Theta$ is the projection) intertwine the representations U_λ^Θ and U_λ. By this map $\mathcal{C}(\mathbb{F}_\Theta)$ can be seen as a closed subspace of $\mathcal{C}(\mathbb{F})$ etc. so that if $U_\lambda(\mu)$ is compact the same happens to $U_\lambda^\Theta(\mu)$.

6. Operators on the invariant control sets

In this section we restrict the representations U_λ^Θ to get representations of S_μ on the function spaces of the invariant control set C_Θ, $\Theta\subset\Sigma$. The restriction is possible because C_Θ is S_μ-invariant. We denote these representations of S_μ by U_λ^C without distinguishing the flag manifold $\mathbb{F}_\Theta$ in this notation.

In each case the operator $U_\lambda^C(\mu)$ is irreducible in $\mathcal{C}(C_\Theta)$ the Banach space of continuous functions on C_Θ. In what follows we rely on this irreducibility to prove analyticity as a function of λ of the spectral radius r_λ^C of $U_\lambda^C(\mu)$.

The only assumption to prove this fact is that $\mu = \phi(g)\,dg$ has a density w.r.t. the Haar measure dg of G. Under this assumption $U_\lambda^C(\mu)$ is a compact operator on $\mathcal{C}(C_\Theta)$. The proof of irreducibility of $U_\lambda^C(\mu)$ is based in the following lemma.

Lemma 6.1. *Let D be an invariant control set of S_μ (on some homogeneous space). Suppose that $x \in D$ and V is an open subset contained in D. Then there exists a measurable set $A \subset G$ with $\mu^n(A) > 0$ for some $n \geq 1$ such that $gx \in V$ for all $g \in A$.*

Proof. Since $\mathrm{cl}(S_\mu x) = D$ and V is open there exists $g \in S_\mu$ such that $gx \in V$. Hence for some open neighborhood A of g we have $hx \in V$ for every $h \in A$. Moreover if $n \geq 1$ is such that $g \in \mathrm{supp}\mu^n$ then $\mu^n(A) > 0$ proving the lemma. ■

For the next proposition recall that a measurable set $E \subset C_\Theta$ is said to be invariant by the positive operator $U_\lambda^C(\mu)$ if $\mathrm{supp}U_\lambda^C(\mu)\,h \subset E$ when $\mathrm{supp}h \subset E$ and this condition is not vacuous meaning that there exists $h \neq 0$ with $\mathrm{supp}h \subset E$. $U_\lambda^C(\mu)$ is irreducible if there is no proper invariant E.

Proposition 6.2. *For any λ the operator $U_\lambda^C(\mu)$ is irreducible on $\mathcal{C}(C_\Theta)$ where $C_\Theta \subset \mathbb{F}_\Theta$ is the invariant control set.*

Proof. Suppose by contradiction $U_\lambda^C(\mu)$ is not irreducible so that there exists an invariant $E \neq C_\Theta$. Clearly E is also invariant by any iteration $U_\lambda^C(\mu)^n$ of $U_\lambda^C(\mu)$. Take $x \in C_\Theta \setminus E$. If $f > 0$ is continuous with $\mathrm{supp}f \subset E$ (e.g. $f = |h|$ with h as in the definition) take $u > 0$ such that the open set $V_u = \{y \in \mathbb{F}_\Theta : f(y) > u\}$ is not empty. By Lemma 6.1 there exists $n \geq 1$ and a set $A \subset G$ with $\mu^n(A) > 0$ such that $gx \in V_u$ if $g \in A$. For such n and A we have

$$\begin{aligned} U_\lambda^C(\mu)^n f(x) &= \int_G \rho_\lambda(g,x)\, f(gx)\, \mu^n(dg) \\ &\geq \int_A \rho_\lambda(g,x)\, f(gx)\, \mu^n(dg) \geq u \int_A \rho_\lambda(g,x)\, \mu^n(dg) > 0 \end{aligned}$$

contradicting the fact that $x \notin \mathrm{supp}U_\lambda^C(\mu)^n f$. ■

The irreducibility combined with Proposition 4.2 ensures that the spectral radius $r_\lambda^C = r\left(U_\lambda^C(\mu)\right)$ of $U_\lambda^C(\mu)$ is an eigenvalue with nonnegative eigenfunction $f \neq 0$ such that $\mathrm{supp}f = C_\Theta$. Such an eigenfunction is strictly positive. In fact, if $f(x) = 0$ for some $x \in C_\Theta$ then

$$0 = r_\lambda^n f(x) = \int_G \rho_\lambda(g,x)\, f(gx)\, \mu^n(dg)$$

for all $n \geq 1$ so that $f(gx) = 0$ for μ^n-almost all g. This implies that $f(gx) = 0$ on Sx which is dense in C_Θ contradicting the fact that $f \neq 0$.

The following statement was proved in Guivarc'h-Raugi [8], Proposition 5.10 (see the last part of the proof).

Proposition 6.3. *Suppose that an operator $U_\lambda^\Theta(\mu)$ or $U_\lambda^C(\mu)$ has a strictly positive eigenfunction f. Then the eigenvalue of f is the spectral radius r whose eigenspace is one dimensional. Moreover there is no peripheral spectrum, that is, if β is an eigenvalue of absolute value r, then $\beta = r$.*

Now we can apply the perturbation Theorem 4.4. Since r_λ^C is an eigenvalue with multiplicity one, statement (3) of that theorem ensures that the map $\lambda \mapsto r_\lambda^C$ is analytic. Thus we get the following result that summarizes the discussions above.

Theorem 6.4. *Suppose that $\mu = \phi(g)\, dg$ has a density w.r.t. the Haar measure. Take a flag manifold $\mathbb{F}_\Theta$ and let $C_\Theta \subset \mathbb{F}$ be the unique invariant control set. If $\rho_\lambda(g,x)$ is a cocycle defined in $\mathbb{F}_\Theta$ we let the operator $U_\lambda^C(\mu)$ act on the Banach space $\mathcal{C}(C_\Theta)$ of continuous functions on C_Θ. Then we have the following properties:*

1. *$U_\lambda^C(\mu)$ is a compact, positive and irreducible operator.*
2. *The spectral radius r_λ^C of $U_\lambda^C(\mu)$ has multiplicity one and its eigenspace is generated by a strictly positive eigenfunction.*
3. *If β is an eigenvalue of $U_\lambda^C(\mu)$ with $|\beta| = r_\lambda^C$ then $\beta = r_\lambda^C$.*
4. *The map $\lambda \mapsto r_\lambda^C$ is analytic.*

Remark 6.5. The above theorem includes the case when $S_\mu = G$ as in Theorem 1.1. In this case $C_\Theta = \mathbb{F}_\Theta$ so that $r_\lambda^C = r_\lambda$ is the spectral radius of $U_\lambda^\Theta(\mu)$ acting on $\mathcal{C}(\mathbb{F}_\Theta)$. Hence the first part of Theorem 1.1 will follow after proving that $r_\lambda^C = \gamma(\lambda, x)$, $x \in C_\Theta$. ■

We conclude this section with the following comments about the spectral radii and moment Lyapunov exponents on the invariant control sets in different flag manifolds where ρ_λ is defined.

If $\lambda \in \mathfrak{a}_\Theta^*$ so that the cocycle ρ_λ is well defined in $\mathbb{F}_\Theta$ then ρ_λ is also defined in the flag manifolds $\mathbb{F}_{\Theta_1}$ with $\Theta_1 \subset \Theta$. Let $\pi : \mathbb{F}_{\Theta_1} \to \mathbb{F}_\Theta$ be the canonical projection. By equivariance we have $\rho_\lambda(g, \pi x) = \rho_\lambda(g, x)$ if $x \in \mathbb{F}_{\Theta_1}$ and $g \in G$ (where we use the same notation ρ_λ for the cocycles over $\mathbb{F}_{\Theta_1}$ and $\mathbb{F}_\Theta$). Also, $U_\lambda^{\Theta_1}(\mu)(f \circ \pi) = \left(U_\lambda^\Theta(\mu) f\right) \circ \pi$ if f is a function on $\mathbb{F}_\Theta$ and $\pi(C_{\Theta_1}) = C_\Theta$. Hence if f is a strictly positive eigenfunction of $U_\lambda^\Theta(\mu)$ on C_Θ then $f \circ \pi$ is an eigenfunction of $U_\lambda^{\Theta_1}(\mu)$ with the same eigenvalue and strictly positive on C_{Θ_1}. It follows that the spectral radii of both operators on the invariant control sets C_Θ and C_{Θ_1} are the same.

7. Operators on the flag manifolds

Unless $S_\mu = G$ the operators $U_\lambda^\Theta(\mu)$ are not irreducible on $\mathcal{C}(\mathbb{F}_\Theta)$ because C_Θ is a proper invariant set if S_μ is a proper semigroup.

Let r_λ^Θ be the spectral radius of $U_\lambda^\Theta(\mu)$ acting on the space $\mathcal{C}(\mathbb{F}_\Theta)$ of continuous functions. In case $G = \mathrm{Sl}(d, \mathbb{R})$ it was proved in Guivarc'h-Raugi [8] that for λ in a certain range r_λ is a principal eigenvalue and hence in this range $\lambda \mapsto r_\lambda$ is analytic (see [8], Proposition 5.10, Théorème 5.13 and Corollaire 5.14).

Here we extend those results to arbitrary semi-simple Lie groups and get a range of analyticity of $\lambda \mapsto r_\lambda^\Theta$. Even in the case of $G = \mathrm{Sl}(d, \mathbb{R})$ we enlarge the range covered by [8].

We assume throughout this section that the kernel Q_λ defined in (9) is of Hilbert-Schmidt class. This assumption is required to have compactness of the operators in the L^2 spaces.

Recall that we denote by $(\mathfrak{a}_{\Theta}^{*})^{+}$ the partial chamber which is the interior (in $\mathfrak{a}_{\Theta}^{*}$) of the cone spanned by the fundamental weights $\Phi \setminus \Phi_{\Theta}$.

Theorem 7.1. *Suppose that $\mu = \phi(g)\,dg$ has a density with respect the Haar measure and the kernel Q_{λ} defined in (9) is of Hilbert-Schimidt class. Let ω_{Θ} be half the sum of the positive roots outside $\langle\Theta\rangle$ counted with multiplicities. Then $U_{\lambda}^{\Theta}(\mu)$ has a strictly positive eigenfunction ϕ_{λ} on $\mathbb{F}_{\Theta}$ with eigenvalue r_{λ}^{Θ} if we have $\lambda \in \Gamma_{\Theta} = -\omega_{\Theta} + (\mathfrak{a}_{\Theta}^{*})^{+}$.*

The proof is based on the construction of operators $T_{\lambda}^{\Theta} : \mathcal{C}(\mathbb{F}_{\Theta})^{*} \to \mathcal{C}(\mathbb{F}_{\Theta})$ that intertwine the dual representations $U_{\lambda}^{\Theta}(\mu)^{*}$ and $U_{\lambda}^{\Theta}(\mu)$. The operator has the form

$$T_{\lambda}^{\Theta}\nu(x) = \int_{\mathbb{F}_{\Theta}} \Delta_{\lambda}^{\Theta}(x,y)\,\nu(dy)$$

where ν is a measure and the kernel $\Delta_{\lambda}^{\Theta}(x,y)$ is positive but in general not bounded or even integrable w.r.t. m_{Θ}. The set of those λ such that $\Delta_{\lambda}^{\Theta}(x,y)$ is m_{Θ}-integrable is given by known integral formulas.

By applying the intertwining operator we will get the strictly positive eigenfunction of the theorem in the form $\phi_{\lambda} = T_{\lambda}^{\Theta}\nu_{\lambda}$ with $\nu = fm_{\Theta}$ a probability eigen measure of $U_{\lambda}^{\Theta}(\mu)^{*}$ such that f is continuous. This works for $\lambda \in \Gamma_{\Theta}$ because this is the range of integrability of $\Delta_{\lambda}^{\Theta}(x,y)$.

To get an eigen measure ν_{λ} with continuous density we use the fact that the representation $U_{-\lambda-2\omega_{\Theta}}^{\Theta}(\mu^{-1})$ is the adjoint of $U_{\lambda}^{\Theta}(\mu)$ in $L^{2}(\mathbb{F}_{\Theta}, m_{\Theta})$. The conditions on μ in Theorem 7.1 are required to have compactness of $U_{-\lambda-2\omega_{\Theta}}^{\Theta}(\mu^{-1})$.

Proposition 7.2. *Under the conditions of Theorem* 7.1 *there exists an eigen measure ν_{λ} (in $\mathcal{C}(\mathbb{F}_{\Theta})^{*}$) of $U_{\lambda}^{\Theta}(\mu)^{*}$ with eigenvalue r_{λ}^{Θ}. The measure $\nu_{\lambda} = fm_{\Theta}$ has a continuous density f w.r.t. the K-invariant measure m_{Θ}.*

Proof. The measure $\mu^{-1} = \iota_{*}\mu$ where $\iota(g) = g^{-1}$ satisfies the same conditions as μ. Hence $U_{-\lambda-2\omega_{\Theta}}(\mu^{-1})$ is a compact operator in both spaces $\mathcal{C}(\mathbb{F}_{\Theta})$ and $L^{2}(\mathbb{F}_{\Theta}, m_{\Theta})$.

Because of the Krein-Rutman Theorem 4.1 the spectral radius, say $s_{-\lambda-2\omega_{\Theta}} > 0$, of $U_{-\lambda-2\omega_{\Theta}}(\mu^{-1})$ on $L^{2}(\mathbb{F}_{\Theta}, m_{\Theta})$ is an eigenvalue having a nonnegative eigenfunction. Now by Corollary 5.3 an eigenfunction of $U_{\lambda}^{\Theta}(\mu)$ in $L^{2}(\mathbb{F}_{\Theta}, m_{\Theta})$ is continuous and hence an eigenfunction in $\mathcal{C}(\mathbb{F}_{\Theta})$. This shows that $s_{-\lambda-2\omega_{\Theta}} \le r_{-\lambda-2\omega_{\Theta}}^{\Theta}$. The reverse inequality holds because $\mathcal{C}(\mathbb{F}_{\Theta}) \subset L^{2}(\mathbb{F}_{\Theta}, m_{\Theta})$ so that $s_{-\lambda-2\omega_{\Theta}} = r_{-\lambda-2\omega_{\Theta}}^{\Theta}$. The same argument shows that $s_{\lambda} = r_{\lambda}^{\Theta}$ (obvious notation). Hence

$$r_{\lambda}^{\Theta} = s_{\lambda} = s_{-\lambda-2\omega_{\Theta}} = r_{-\lambda-2\omega_{\Theta}}^{\Theta}$$

because $U_{-\lambda-2\omega_{\Theta}}(\mu^{-1})$ is the $L^{2}(\mathbb{F}_{\Theta}, m_{\Theta})$-adjoint of $U_{\lambda}(\mu)$. In conclusion $f_{\lambda}m_{\Theta}$ is an eigen measure of $U_{\lambda}(\mu)^{*}$ if $f_{\lambda} \in L^{2}(\mathbb{F}_{\Theta}, m_{\Theta})$ is a nonnegative eigenfunction of $U_{-\lambda-2\omega_{\Theta}}(\mu^{-1})$. ■

Remark 7.3. In [8] the existence of eigen measures is ensured by an application of Schauder-Tychonoff fixed point theorem and without the restrictive assumptions of Theorem 7.1. Here we insisted in the $L^{2}(\mathbb{F}_{\Theta}, m_{\Theta})$-adjoint approach of the above proposition to make sure that the density f_{λ} of the eigen measure is continuous. ■

We proceed to the definition of the kernel $\Delta_{\lambda}^{\Theta}(x,y)$ of the operator $T_{\lambda}:\mathcal{C}(\mathbb{F})^{*}\to\mathcal{C}(\mathbb{F})$:

Proposition 7.4. *Assume $\lambda\in\mathfrak{a}_{\Theta}^{*}$ to be given. Then there is an analytic function $\Delta_{\lambda}:\mathbb{F}_{\Theta}\times\mathbb{F}_{\Theta}\to\mathbb{R}_{+}\cup\{+\infty\}$ satisfying the following properties:*

(1) $\Delta_{\lambda}(kx,ky)=\Delta_{\lambda}(x,y)$ *for all* $k\in K$ *and* $x,y\in\mathbb{F}_{\Theta}$.

(2) $\Delta_{\lambda}(\overline{n}x_{0},x_{0})=\rho_{\lambda}(\overline{n},x_{0})^{-1}$ *if* x_{0} *is the origin of* $\mathbb{F}_{\Theta}$ *and* $\overline{n}\in N^{-}$. *Hence* $\Delta_{\lambda}(x,y)>0$.

(3) *If* $g\in G$ *and* $x,y\in\mathbb{F}_{\Theta}$ *then*

$$\rho_{\lambda}(g,x)\,\Delta_{\lambda}(gx,y)=\Delta_{\lambda}\left(x,g^{T}y\right)\rho_{\lambda}\left(g^{T},y\right) \tag{11}$$

where $g^{T}=\theta\left(g^{-1}\right)$ *is the transposition with respect to the Cartan involution* θ.

(4) $\Delta_{\lambda+\mu}(x,y)=\Delta_{\lambda}(x,y)\,\Delta_{\mu}(x,y)$.

(5) $\Delta_{\lambda}(x,y)<+\infty$ *and continuous if* $\lambda\in(\mathfrak{a}_{\Theta}^{*})^{+}$.

Proof. Assume first that λ is a dominant weight and consider the flag manifold $\mathbb{F}_{\lambda}=\mathbb{F}_{\Theta_{\lambda}}$ where $\Theta_{\lambda}=\{\alpha\in\Sigma:\langle\lambda,\alpha\rangle=0\}$. Since $\lambda\in\mathrm{cl}\,(\mathfrak{a}_{\Theta}^{*})^{+}$ we have $\Theta\subset\Theta_{\lambda}$ so that there is a fibration $\pi_{\lambda}:\mathbb{F}_{\Theta}\to\mathbb{F}_{\lambda}$.

The flag manifold $\mathbb{F}_{\lambda}$ realizes as the projective orbit $G\cdot V_{\lambda}$ of the λ-weight space V_{λ} in the space $V(\lambda)$ of the representation R_{λ} with highest weight λ. The cocycle ρ_{λ} factors to $\mathbb{F}_{\lambda}$ and is given by

$$\rho_{\lambda}(g,x)=\frac{\|gv\|}{\|v\|}$$

where $\|\cdot\|$ is the norm of the K-invariant inner product and v is a generator of the line $x\in\mathbb{F}_{\lambda}=G\cdot V_{\lambda}$. This being so define

$$\Delta_{\lambda}^{\Theta_{\lambda}}(x,y)=|\cos\theta(x,y)|\qquad\Delta_{\lambda}^{\Theta}=\Delta_{\lambda}^{\Theta_{\lambda}}\circ(\pi_{\lambda}\times\pi_{\lambda})$$

where $\theta(x,y)$ is the angle between representatives of $x,y\in\mathbb{F}_{\lambda}\subset\mathbb{P}(V(\lambda))$. Property (1) is satisfied by $\Delta_{\lambda}^{\Theta}$ because $k\in K$ is an isometry. To see (2) take $v_{0}\in V_{\lambda}$ with $\|v_{0}\|=1$ and let b_{0} be the origin of $\mathbb{F}_{\lambda}$ which is identified with $V_{\lambda}\in\mathbb{P}(V(\lambda))$. For $\overline{n}\in N^{-}$ write $v=\overline{n}v_{0}=R_{\lambda}(\overline{n})\,v_{0}$. Let $X\in\mathfrak{n}^{-}$ be such that $\overline{n}=e^{X}$. Then

$$v=e^{X}v_{0}=v_{0}+\sum_{k\geq1}\frac{1}{k!}R_{\lambda}(X)^{k}\,v_{0}.$$

For each $k\geq1$ we have $R_{\lambda}(X)^{k}\,v_{0}\in\sum_{\mu\neq\lambda}V_{\mu}$ where the V_{μ} are the weight spaces. Since the weight spaces are orthogonal to each other we get $v=\overline{n}v_{0}=v_{0}+v_{1}$ with $\langle v_{0},v_{1}\rangle=0$. Hence if $\theta=\theta(\overline{n}b_{0},b_{0})$ then

$$\cos\theta=\frac{\langle v,v_{0}\rangle}{\|v\|}=\frac{\langle v_{0}+v_{1},v_{0}\rangle}{\|v\|}=\frac{1}{\|v\|}.$$

It follows that $\rho_{\lambda}(\overline{n},x_{0})=\|\overline{n}v_{0}\|=1/\cos\theta=1/\Delta_{\lambda}^{\Theta_{\lambda}}(\overline{n}b_{\Theta},b_{\Theta})$, which proves (2) for $\Delta_{\lambda}^{\Theta_{\lambda}}$. By equivariance of π_{λ} the equality (2) holds for $\Delta_{\lambda}^{\Theta}$ as well.

To get (11) let v and w be representatives of x and y, respectively. Then

$$\begin{aligned}\rho_{\lambda}(g,x)\,\Delta_{\lambda}^{\Theta}(gx,y)&=\frac{\|gv\|}{\|v\|}\frac{\langle gv,w\rangle}{\|gv\|\cdot\|w\|}=\frac{\langle v,g^{T}w\rangle}{\|v\|\cdot\|g^{T}w\|}\frac{\left\|g^{T}w\right\|}{\|w\|}\\&=\Delta_{\lambda}^{\Theta}\left(x,g^{T}y\right)\rho_{\lambda}\left(g^{T},y\right).\end{aligned}$$

This concludes the definition of Δ_λ^Θ when λ is a dominant weight. Analyticity of Δ_λ^Θ follows by (2).

Now, for a general $\lambda \in \mathfrak{a}_\Theta^*$ we can write $\lambda = r_1\omega_1 + \cdots + r_k\omega_k$ where $\{\omega_1, \ldots, \omega_k\}$ are the fundamental weights in $\Phi \setminus \Phi_\Theta$. If we put

$$\Delta_\lambda = \Delta_{\omega_1}^{r_1} \cdots \Delta_{\omega_l}^{r_l}$$

then the properties in the statement are satisfied because $\rho_\lambda = \rho_{\omega_1}^{r_1} \cdots \rho_{\omega_l}^{r_l}$. ■

The following lemma about integrability of ρ_λ is well known. The restriction in Theorem 7.1 that $\lambda \in \Gamma_\Theta$ is based on this integrability.

Lemma 7.5. *Let ω_Θ be half the sum of the positive roots outside $\langle\Theta\rangle$ counted with multiplicities. If $\theta \in (\mathfrak{a}_\Theta^*)^+$ we have*

$$\int_{N^-} e^{-(\theta+\omega_\Theta)\mathsf{a}(\overline{n},x_0)} d\overline{n} < \infty.$$

Proof. See Helgason [10], Theorem 6.14. Actually the integral is the value of the c-function of Harish-Chandra on $-i\theta$. ■

As a consequence of this lemma and (2) of Proposition 7.4 we get integrability of Δ_λ^Θ for λ in the cone Γ_Θ.

Lemma 7.6. *Denote by m_Θ the K-invariant measure on $\mathbb{F}_\Theta$. Then for every $x \in \mathbb{F}_\Theta$, if $\lambda \in -\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$, we have*

$$\int_{\mathbb{F}_\Theta} \Delta_\lambda^\Theta (y, x)\, m_\Theta (dy) < \infty.$$

Proof. By K-invariance of Δ_λ^Θ it suffices to prove integrability when $x = x_0$ is the origin of $\mathbb{F}_\Theta$. By Proposition 7.4 (2) we have

$$\int_{\mathbb{F}_\Theta} \Delta_\lambda^\Theta (y, x_0)\, m\,(dy) = \int_{N^-} \rho_\lambda (\overline{n}, x_0)^{-1} e^{-2\omega_\Theta \mathsf{a}(\overline{n},x_0)} d\overline{n}$$

where we are using the integral formula

$$\int_{\mathbb{F}_\Theta} f(x)\, m\,(dx) = \int_{N^-} f(\overline{n}x_0)\, e^{-2\omega_\Theta \mathsf{a}(\overline{n},x_0)} d\overline{n}$$

(see [10], Theorem I.5.20). The right hand side of the above equality is

$$\int_{N^-} e^{-(\lambda+2\omega_\Theta)\mathsf{a}(\overline{n},x_0)} d\overline{n}$$

which is finite if $\theta = \lambda + \omega \in (\mathfrak{a}_\Theta^*)^+$, by the above lemma. That is $\Delta_\lambda^\Theta (y, x_0)$ is integrable if λ belongs to the cone $\Gamma_\Theta = -\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$. ■

Having this estimate we can finish the proof of Theorem 7.1. Define as before

$$T_\lambda \nu (x) = \int_{\mathbb{F}_\Theta} \Delta_\lambda^\Theta (y,x) \, \nu (dx)$$

if the integral exists where ν is a measure in $\mathbb{F}_\Theta$.
By (11) we have $U_\lambda (g) (T_\lambda \nu) = T_\lambda \left(U_\lambda^\Theta \left(g^T \right)^* \nu \right)$ if $g \in G$ and hence

$$U_\lambda^\Theta (\mu) (T_\lambda \nu) = T_\lambda \left(U_\lambda^\Theta \left(\mu^T \right)^* \nu \right), \tag{12}$$

that is, the operator T_λ intertwines the operators $U_\lambda^\Theta \left(\mu^T \right)^*$ and $U_\lambda (\mu)$.
As μ the measure μ^T satisfies the same hypothesis of Theorem 7.1. Hence by Proposition 7.2 the adjoint $U_\lambda \left(\mu^T \right)^*$ admits eigen measure $\nu_\lambda = f_\lambda m_\Theta$. By the integrability Lemma 7.6 we have a well defined function ϕ_λ defined by

$$\phi_\lambda (y) = T_\lambda (f_\lambda m_\Theta) (y) = \int_{\mathbb{F}_\Theta} \Delta_\lambda^\Theta (y,x) \, f_\lambda (x) \, m_\Theta (dx)$$

if $\lambda \in \Gamma_\Theta = -\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$. Since T_λ intertwines the representations we have that ϕ_λ is an eigenfunction of $U_\lambda^\Theta (\mu)$. For each $x \in \mathbb{F}_\Theta$ $\Delta_\lambda^\Theta (y,x)$ is strictly positive except in a set of m_Θ-measure 0. Hence ϕ_λ is a strictly positive eigenfunction of $U_\lambda^\Theta (\mu)$. Hence by Proposition 4.3 the eigenvalue of ϕ_λ is the spectral radius r_λ^Θ of $U_\lambda^\Theta (\mu)$ concluding the proof of Theorem 7.1.

Now as in the case of the operators in the invariant control set we apply Proposition 6.3 to see that r_λ^Θ is an eigenvalue with multiplicity one if $\lambda \in -\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$ because by Theorem 7.1 $U_\lambda^\Theta (\mu)$ has a strictly positive eigenfunction.
Since the spectral radius r_λ^Θ, $\lambda \in -\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$ has multiplicity one we can apply the perturbation Theorem 4.4 (3) to conclude analyticity of r_λ^Θ.

Corollary 7.7. *Let r_λ^Θ be the spectral radius of $U_\lambda^\Theta (\mu)$. Then the map $\lambda \mapsto r_\lambda^\Theta$ is analytic on $-\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$ if μ satisfies the conditions of Theorem* 7.1.

Now we take $\mu^{-1} = \iota_* (\mu)$ in place of μ. Denote by s_λ the spectral radius of $U_\lambda^\Theta (\mu^{-1})$ on $\mathcal{C} (\mathbb{F}_\Theta)$. If μ satisfies the conditions of Theorem 7.1 the same happens to μ^{-1}. Under these conditions the map $\lambda \mapsto r_\lambda$ is analytic on $-\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$. Moreover, the spectral radius of $U_\lambda^\Theta (\mu^{-1})$ on $L^2 (\mathbb{F}_\Theta, m_\Theta)$ is s_λ as well. Now the adjoint of $U_\lambda^\Theta (\mu^{-1})$ on $L^2 (\mathbb{F}_\Theta, m_\Theta)$ is $U_{-\lambda-2\omega_\Theta}^\Theta (\mu)$. Hence $s_\lambda = r_{-\lambda-2\omega_\Theta}^\Theta$. By this symmetry we enlarge the domain of analyticity of r_λ^Θ.

Corollary 7.8. *The map $\lambda \mapsto r_\lambda^\Theta$ is analytic on $-\omega_\Theta - (\mathfrak{a}_\Theta^*)^+$ if μ satisfies the conditions of Theorem* 7.1.

Proof. In fact, $\lambda \in -\omega_\Theta - (\mathfrak{a}_\Theta^*)^+$ if and only if $-\lambda - 2\omega_\Theta \in -\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$. ■

Finally if $\lambda \in \mathfrak{a}_\Theta^*$ so that the cocycle ρ_λ is well defined in $\mathbb{F}_\Theta$ then ρ_λ is also defined in the flag manifolds $\mathbb{F}_{\Theta_1}$ with $\Theta_1 \subset \Theta$. Then by the same arguments as in the comments at the end of Section 6 if $\lambda \in -\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$ then there is a strictly positive eigenfunction ϕ_λ for $U_\lambda^\Theta (\mu)$ and it follows that $\phi_\lambda \circ \pi$ is a strictly positive eigenfunction of $U_\lambda^{\Theta_1} (\mu)$ with the same eigenvalue. Therefore for $\lambda \in -\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$ we have $r_\lambda^{\Theta_1} = r_\lambda^\Theta$.

8. Moment Lyapunov exponent and spectral radius

The relationship between a moment Lyapunov exponent $\gamma(\lambda, x)$ and a spectral radius r_λ (or r_λ^C) comes from the immediate observation that

$$U_\lambda(\mu)^n 1(x) = \int_G \rho_\lambda(g,x)\,\mu^n(dg)$$

so that $\gamma(\lambda, x) = \limsup_{n\to\infty} \frac{1}{n} \log U_\lambda(\mu)^n 1(x)$. In these equalities we can take $U_\lambda^C(\mu)$ in place of $U_\lambda(\mu)$ in case x belongs to an invariant control set.

On the other hand we have the general Proposition 4.3 ensuring that the spectral radius of T is given by $\lim_{n\to\infty} T1(x)$ if T is a positive operator having a strictly positive eigenfunction. This fact and the results of the previous section permit to prove analyticity properties of $\gamma(\lambda, x)$ (as a function of λ) ensuring by pass that the lim sup in the definition is in fact a limit.

Given a flag manifold $\mathbb{F}_\Theta$ take as before $\lambda \in \mathfrak{a}_\Theta^*$ so that $\rho_\lambda(g,x)$ is well defined on $\mathbb{F}_\Theta$. We let $C_\Theta \subset \mathbb{F}_\Theta$ be the unique invariant control set of S_μ (assumed to have nonempty interior) and consider the operators $U_\lambda^C(\mu)$ and $U_\lambda^\Theta(\mu)$ acting on $\mathcal{C}(C_\Theta)$ and $\mathcal{C}(\mathbb{F}_\Theta)$, respectively. As before the spectral radius of $U_\lambda^C(\mu)$ (respectively $U_\lambda^\Theta(\mu)$) is denoted by r_λ^C (respectively r_λ^Θ).

By piecing together previous results we get the following main theorem about regularity properties of $\gamma(\lambda, x)$ as a function of λ and its relationship to the spectral radii r_λ^Θ and r_λ^C.

Theorem 8.1. *Assume that $\mu = \phi(g)\,dg$ has a density w.r.t. to the Haar measure and admits exponential moments. For $\lambda \in \mathfrak{a}_\Theta^*$ the following statements hold true:*

(1) *For every $x \in C_\Theta$ we have*

$$\gamma(\lambda, x) = \lim_{n\to\infty} \frac{1}{n} \log \int_G \rho_\lambda(g,x)\,\mu^n(dg) = \log r_\lambda^C$$

where the limit is uniform for $x \in C_\Theta$. If $x \in C_\Theta$ then the function $\lambda \mapsto \gamma(\lambda, x)$ is analytic.

(2) *Suppose in addition that $\phi \in L^3(G, dg)$. If $\lambda \in -\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$ then we have $\gamma(\lambda, x) = \log r_\lambda^\Theta$ for all $x \in \mathbb{F}_\Theta$. The* lim sup *in the definition of $\gamma(\lambda, x)$ is a limit which is uniform for $x \in \mathbb{F}_\Theta$.*

(3) *If $\lambda \in -\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$ then $r_\lambda^\Theta = r_\lambda^C$ (in case $\phi \in L^3(G, dg)$) and $\gamma(\lambda, \cdot) = \log r_\lambda^\Theta$ is constant in $\mathbb{F}_\Theta$.*

Proof. Is a direct consequence of Proposition 4.3 since the operators $U_\lambda^C(\mu)$ as well as $U_\lambda^\Theta(\mu)$ (if $\lambda \in -\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$) admit strictly positive eigenfunctions in C_Θ and $\mathbb{F}_\Theta$ respectively. Analyticity is a consequence of Corollary 7.7. The equality between r_λ^Θ and r_λ^C in the range $\lambda \in -\omega_\Theta + (\mathfrak{a}_\Theta^*)^+$ holds because $r_\lambda^\Theta = e^{\gamma(\lambda,x)} = r_\lambda^C$ if $x \in C_\Theta$. ■

Remark 8.2. By Corollary 7.8 the spectral radius r_λ^Θ is analytic as a function of λ in the cone $-\omega_\Theta - (\mathfrak{a}_\Theta^*)^+$ as well (under the assumptions of the theorem). However in this range it is not claimed an equality of r_λ^Θ with $\gamma(\lambda, x)$ for some x. ■

The next step is to use properties of the moment Lyapunov exponents $\gamma(\lambda, x)$ to describe the shape of functions $p \mapsto r_{p\lambda}^{\Theta}$ in the cases when the equality $\gamma(\lambda, x) = r_{\lambda}^{\Theta}$ holds as ensured by Theorem 8.1.

The first property of $\gamma(\lambda, x)$ to be considered is that for a fixed x the function $\lambda \in \mathfrak{a}_{\Theta}^{*} \mapsto \gamma(\lambda, x) \in \mathbb{R}$ is convex, that is, $\gamma(t\lambda_1 + s\lambda_2, x) \leq t\gamma(\lambda_1, x) + s\gamma(\lambda_2, x)$, $t, s \geq 0$ and $t + s = 1$. This follows by a direct application of a Hölder inequality.

As to the second property of the moment Lyapunov exponents note that we have $0 \in -\omega_{\Theta} + (\mathfrak{a}_{\Theta}^{*})^{+}$ since $\omega_{\Theta} \in (\mathfrak{a}_{\Theta}^{*})^{+}$. Hence $\lambda \mapsto \log r_{\lambda}$ is analytic at 0. The derivative of this function were computed in Arnold [2] (see also Arnold-Oeljeklaus-Pardoux [3], Lemma 2.1 and Boujerol-Lacroix [5], Lemma 5.2) as the sample Lyapunov exponent. Below we reproduce this result in our framework. To do that define

$$\Lambda = \lim_{n \to +\infty} \frac{1}{n} \mathsf{a}(g_n, x) \in \mathfrak{a} \tag{13}$$

where as in the introduction g_n is the random product $g_n = y_n \cdots y_1$ and $y_n : G^{\mathbb{N}} \to G$ are the $\mu^{\mathbb{N}}$-random variables given by the n-th coordinate.

Classical results on Lyapunov exponents ensure that the limit in (13) exists $\mu^{\mathbb{N}}$-almost surely and is constant as a function of x as well so that the notation Λ in (13) is consistent (see [8] and the Multiplicative Ergodic Theorem version of [1] for an approach in our framework).

Proposition 8.3. *Write $R(\lambda) = \log r_{\lambda}$. Then its differential at 0 is $dR_0(\lambda) = \lambda(\Lambda)$.*

Proof. Take $x \in \mathbb{F}$, $\lambda \in \mathfrak{a}^{*}$ and $p > 0$ in a neighborhood of 0. Then by Jensen inequality we have

$$\int \log \rho_{p\lambda}(g, x) \mu^{n}(dg) \leq \log \int \rho_{p\lambda}(g, x) \mu^{n}(dg)$$

and hence
$$\frac{p}{n} \int \lambda \mathsf{a}(g, x) \mu^{n}(dg) \leq \frac{1}{n} \log \int e^{\lambda \mathsf{a}(g,x)} \mu^{n}(dg).$$

By the ergodic theorem the limit of the left hand side is $p\lambda(\Lambda)$ while in the right hand side is $R(p\lambda) = \gamma_{p\lambda}(x)$. Hence $\lambda(\Lambda) \leq R(p\lambda)/p$. Applying the same reasoning to $-p$ we obtain

$$\frac{R(-p\lambda)}{-p} \leq \lambda(\Lambda) \leq \frac{R(p\lambda)}{p}.$$

Since R is analytic at 0 we can then take the limit as $p \to 0$ to conclude that $dR_0(\lambda) = \lambda(\Lambda)$. ■

To get the shape of some functions $p \mapsto r_{p\lambda}$ we will use the following regularity of Λ.

Proposition 8.4. *If μ is an exposed measure then Λ belongs to the Weyl chamber $\mathfrak{a}^{+}$.*

Proof. See Théorème 2.6 in [8] where this result is proved for more general measures than the exposed ones. ■

Notation: For $\lambda \in \mathfrak{a}^{*}$ we define the functions $f_{\lambda}, f_{\lambda}^{C} : \mathbb{R} \to \mathbb{R}$ by $f_{\lambda}(p) = \log r_{p\lambda}$ and $f_{\lambda}(p) = \log r_{p\lambda}^{C}$. In these definitions we leave implicit the measure μ fixed in advance.

We write $\overline{f}_{\lambda}$ and $\overline{f}_{\lambda}^{C}$ for the functions corresponding to the measure $\mu^{-1} = \iota_{*}(\mu)$. In the notation for these functions we do not specify the flag manifolds with a superscript Θ because in the case to be considered below the spectral radius of the operators $U_{\lambda}^{\Theta}(\mu)$ are the same as those of the corresponding operators $U_{\lambda}(\mu)$ in the maximal flag manifold (see the comments at the end of both sections 6 and 7).

Notice that the functions $\lambda \mapsto \log r_{\lambda}$ are convex in the cases, specified by Theorem 8.1, where the logarithm of a spectral radius is equal to a moment Lyapunov exponent $\gamma(\lambda, x)$. Convexity combined with the previous proved properties allow to determine the shape of the functions $\lambda \mapsto \log r_{\lambda}$. The next proposition shows the possibilities for the operators in the invariant control set.

In the sequel we denote by $\mathfrak{c}^{+}$ the open cone of those $\lambda \in \mathfrak{a}^{*}$ that are strictly positive on the Weyl chamber $\mathfrak{a}^{+}$:

$$\mathfrak{c}^{+} = \{\lambda \in \mathfrak{a}^{*} : \lambda(H) > 0,\ H \in \mathfrak{a}^{+}\}$$

Proposition 8.5. *Suppose that* $\lambda \in \mathfrak{c}^{+}$. *Then for the convex analytic function* $f_{\lambda}^{C} : \mathbb{R} \to \mathbb{R}$ *we have* $f_{\lambda}^{C}(p) > 0$ *if* $p > 0$, $\lim_{p\to+\infty} f_{\lambda}^{C}(p) = +\infty$ *and there are the possibilities:*

(1) $\left(f_{\lambda}^{C}\right)'(p) > 0$ *for all* $p \in \mathbb{R}$ *in which case* $\lim_{p\to-\infty} f_{\lambda}^{C}(p) < 0$.

(2) $\left(f_{\lambda}^{C}\right)'$ *is not strictly positive. Then there exists a unique global minimum* $p_0 < 0$, $\left(f_{\lambda}^{C}\right)'' > 0$ *and* $\lim_{p\to-\infty} f_{\lambda}^{C}(p) = +\infty$.

Proof. If λ is strictly positive on $\mathfrak{a}^{+}$ then $\left(f_{\lambda}^{C}\right)'(0) = \lambda(\Lambda) > 0$ by propositions 8.3 and 8.4. Hence by convexity $\left(f_{\lambda}^{C}\right)'(p) \geq \lambda(\Lambda)$ if $p > 0$ which implies that $f_{\lambda}^{C}(p) > 0$ if $p > 0$ and $\lim_{p\to+\infty} f_{\lambda}^{C}(p) = +\infty$.

If $\left(f_{\lambda}^{C}\right)'$ is strictly positive then f_{λ}^{C} is increasing and since $f_{\lambda}^{C}(0) = 0$ we have $\lim_{p\to-\infty} f_{\lambda}^{C}(p) = \inf f_{\lambda}^{C} < 0$. On the other if $\left(f_{\lambda}^{C}\right)'$ is not strictly positive then it is not constant because $\left(f_{\lambda}^{C}\right)'(0) > 0$. It follows that $\left(f_{\lambda}^{C}\right)'$ is strictly increasing. For otherwise there would be $p_1 < p_2$ with $\left(f_{\lambda}^{C}\right)'(p_1) = \left(f_{\lambda}^{C}\right)'(p_2)$ and hence in $[p_1, p_2]$ since $\left(f_{\lambda}^{C}\right)'$ is nondecreasing which would imply by analyticity that $\left(f_{\lambda}^{C}\right)'$ is constant contrary to the assumption. Hence there exists a unique p_0 (necessarily < 0) with $\left(f_{\lambda}^{C}\right)'(p_0) = 0$ and $\left(f_{\lambda}^{C}\right)'(p) < 0$ if $p < p_0$. This implies $\lim_{p\to-\infty} f_{\lambda}^{C}(p) = +\infty$. ■

Now we can put in evidence Theorem 3.1 that relates the flag type of S_{μ} to the moment Lyapunov exponents to conclude the proof of Theorem 1.3 by proving the nonanlyticity statement of Theorem 1.4. First we note that for any $\Theta \subset \Sigma$ we have $\mathrm{cl}\left(\mathfrak{a}_{\Theta}^{*}\right)^{+} \subset \mathfrak{c}^{+}$ since the cone $\mathrm{cl}\left(\mathfrak{a}_{\Theta}^{*}\right)^{+}$ is generated by fundamental weights and any such weight is strictly positive on the Weyl chamber $\mathfrak{a}^{+}$. Therefore any $\lambda \in \mathrm{cl}(\mathfrak{a}_{\Theta(S_{\mu})}^{*})^{+}$ satisfies the condition of the above Proposition 8.5. By Theorem 3.1 if $\lambda \in \mathrm{cl}(\mathfrak{a}_{\Theta(S_{\mu})}^{*})^{+}$ then $f_{\lambda}^{C}(p)$ is negative as $p \to -\infty$ hence f_{λ}^{C} falls in the first case of the above proposition. It follows that $\lim_{p\to-\infty} f_{\lambda}^{C}(p) < 0$ if $\lambda \in \mathrm{cl}(\mathfrak{a}_{\Theta(S_{\mu})}^{*})^{+}$ as claimed in Theorem 1.3.

Now we take $\lambda = \omega_{\Theta(S_{\mu})}$ and prove Theorem 1.4 by comparing the functions $f_{\omega_{\Theta(S_{\mu})}}^{C}(p)$ and $f_{\omega_{\Theta(S_{\mu})}}(p)$. We have $\omega_{\Theta(S_{\mu})} \in \mathrm{cl}\left(\mathfrak{a}_{\Theta(S_{\mu})}^{*}\right)^{+}$ so that $f_{\omega_{\Theta(S_{\mu})}}^{C} : \mathbb{R} \to \mathbb{R}$ is strictly increasing with $f_{\omega_{\Theta(S_{\mu})}}^{C}(0) = 0$.

On the other hand if $p > -1$ then $p\omega_{\Theta(S_\mu)}$ falls in the range of Theorem 8.1 (3).

Hence
$$r^{\Theta(S_\mu)}_{p\omega_{\Theta(S_\mu)}} = r^C_{p\omega_{\Theta(S_\mu)}} = \gamma\left(p\omega_{\Theta(S_\mu)}, x\right)$$

for every $x \in \mathbb{F}_{\Theta(S_\mu)}$ if $p > -1$. In this range both operators $U^{\Theta(S_\mu)}_{p\omega_{\Theta(S_\mu)}}$ and $U_{p\omega_{\Theta(S_\mu)}}$ have strictly positive eigenfunctions so that $r^{\Theta(S_\mu)}_{p\omega_{\Theta(S_\mu)}} = r_{p\omega_{\Theta(S_\mu)}}$ (see the end of Section 7). Since the spectral radii of the operators in the invariant control sets in $\mathbb{F}$ and $\mathbb{F}_{\Theta(S_\mu)}$ also coincide we conclude that

$$f_{\omega_{\Theta(S_\mu)}}(p) = f^C_{\omega_{\Theta(S_\mu)}}(p) \qquad p > -1.$$

Notice that if $f_{\omega_{\Theta(S_\mu)}}$ is analytic in $\mathbb{R}$ then we have $f_{\omega_{\Theta(S_\mu)}} = f^C_{\omega_{\Theta(S_\mu)}}$ everywhere since $f^C_{\omega_{\Theta(S_\mu)}}$ is already analytic.

Proof of Theorem 1.4: The nonanalyticity of $f_{\omega_{\Theta(S_\mu)}}$ at $p = -1$ will follow if we prove that $\log r_{p\omega_{\Theta(S_\mu)}} > \log r^C_{p\omega_{\Theta(S_\mu)}}$ if $p < -1$. To this purpose we prove that $\log r^{\Theta(S_\mu)}_{p\omega_{\Theta(S_\mu)}} > \log r^C_{p\omega_{\Theta(S_\mu)}}$, $p < -1$, for the spectral radius $r^{\Theta(S_\mu)}_{p\omega_{\Theta(S_\mu)}}$ of the operator $U^{\Theta(S_\mu)}_{p\omega_{\Theta(S_\mu)}}$ at the flag type of S_μ. Now in $L^2(\mathbb{F}_{\omega_{\Theta(S_\mu)}})$ we have the adjoint $\left(U^{\Theta(S_\mu)}_{p\omega_{\Theta(S_\mu)}}(\mu)\right)^* = U^{\Theta(S_\mu)}_{-(2+p)\omega_{\Theta(S_\mu)}}(\mu^{-1})$ so that if we denote by $s^{\Theta(S_\mu)}_\lambda$ the spectral radius of $U^{\Theta(S_\mu)}_\lambda(\mu^{-1})$ we have $r^{\Theta(S_\mu)}_{p\omega_{\Theta(S_\mu)}} = s^{\Theta(S_\mu)}_{-(2+p)\omega_{\Theta(S_\mu)}}$. By the above facts applied to μ^{-1} instead of μ we have that on the interval $(-1, +\infty)$ the function $p \mapsto \log s^{\Theta(S_\mu)}_{p\omega_{\Theta(S_\mu)}}$ is strictly increasing has limit $+\infty$ as $p \to +\infty$. Hence on $(-\infty, -1)$ the function $p \mapsto \log r^{\Theta(S_\mu)}_{p\omega_{\Theta(S_\mu)}}$ is strictly decreasing and has limit $+\infty$ as $p \to -\infty$. Now $f^C_{\omega_{\Theta(S_\mu)}}(p) = \log r^C_{p\omega_{\Theta(S_\mu)}}$ is strictly increasing and since $f^C_{\omega_{\Theta(S_\mu)}}(-1) = \log r^{\Theta(S_\mu)}_{-\omega_{\Theta(S_\mu)}}$ we conclude that $f^C_{\omega_{\Theta(S_\mu)}}(p) < \log r^{\Theta(S_\mu)}_{p\omega_{\Theta(S_\mu)}}$ for every $p < -1$. As to the spectral radius in the maximal flag manifold we have by equivariance of the projection $\mathbb{F} \to \mathbb{F}_{\Theta(S_\mu)}$ that $r^{\Theta(S_\mu)}_{p\omega_{\Theta(S_\mu)}}$ is a positive eigenvalue of $U_{p\omega_{\Theta(S_\mu)}}(\mu)$ so that $r^{\Theta(S_\mu)}_{p\omega_{\Theta(S_\mu)}} \leq r_{p\omega_{\Theta(S_\mu)}}$. Therefore $f_{\omega_{\Theta(S_\mu)}}(p) > f^C_{\omega_{\Theta(S_\mu)}}(p)$ if $p < -1$. It follows that $f_{\omega_{\Theta(S_\mu)}}$ is not analytic at $p = -1$ for otherwise we would have $f_{\omega_{\Theta(S_\mu)}}(p) = f^C_{\omega_{\Theta(S_\mu)}}(p)$ in a neighborhood of -1 since these functions are equal if $p > -1$. This concludes the proof of Theorem 1.4. ■

The inequality $f_{\omega_{\Theta(S_\mu)}}(p) > f^C_{\omega_{\Theta(S_\mu)}}(p)$ if $p < -1$ ensured in this proof has the following consequence about the moment Lyapunov exponents.

Corollary 8.6. *If $p < -1$ then the moment Lyapunov exponent $\gamma\left(p\omega_{\Theta(S_\mu)}, x\right)$ is not constant as a function of x.*

Proof. For any $x \in C$ we have $\gamma\left(p\omega_{\Theta(S_\mu)}, x\right) = f^C_{\omega_{\Theta(S_\mu)}}(p)$. On the other hand by Proposition 4.3 there exists $y \in \mathbb{F}$ such that $\gamma\left(p\omega_{\Theta(S_\mu)}, x\right) \geq f_{\omega_{\Theta(S_\mu)}}(p)$. Hence $\gamma\left(p\omega_{\Theta(S_\mu)}, \cdot\right)$ is not constant. ■

9. Stochastic differential equations

An invariant stochastic differential equation in G gives rise to a semigroup (under convolution) of measures μ_t given by the transition probabilities $p_t(1,\cdot)$ of the solution starting at the identity. The semigroups $U_\lambda(\mu_t)$ have infinitesimal generators that are differential operators on smooth functions. The moment Lyapunov exponents can be read of from the principal eigenvalues of the infinitesimal generators.

Let
$$dg = X(g)\,dt + \sum_{j=1}^{m} Y_j(g)\circ dW_j \tag{14}$$

be a stochastic differential equation in G where X, Y_1, ..., Y_m are right invariant vector fields and $(W_1,\ldots,W_m)$ is a Brownian motion. The equation generates a Markov process in G with transition probabilities $p_t(g,\cdot)$, $t\geq 0$ and $g\in G$, satisfying $p_t(g,\cdot) = p_t(1,\cdot)\,g$ because of the right invariance of the vector fields. Write $\mu_t = p_t(1,\cdot)$. By the Markov property $\mu_{t+s} = \mu_t * \mu_s$. In this continuous-time setting the moment Lyapunov exponents are given by

$$\gamma(\lambda,x) = \lim_{t\to+\infty}\frac{1}{t}\log\int_G \rho_\lambda(g,x)\,\mu_t(dg).$$

Standard arguments show that these exponents coincide with the discrete-time exponents of a measure μ_{t_0}, $t_0>0$, (e.g. $t_0=1$).

The following preparatory facts on control and stochastic systems are proved the same way as in Arnold-Oeljeklaus-Pardoux [3]. Although [3] consider only systems in the projective spaces the control and stochastic results are easily extended to our set up. We leave the details of this extension to the forthcoming text [18].

The support theorem ensures that the semigroup generated by the supports of the measures μ_t, $t>0$, is the closure of the control semigroup S of the associated control system
$$\dot{g} = X(g) + \sum_{j=1}^{m} u_j Y_j(g) \tag{15}$$

which is the semigroup generated by

$$\Big\{\exp t\Big(X + \sum_{j=1}^{m} u_j Y_j\Big) : t\geq 0, u_j\in\mathbb{R}\Big\}.$$

It is well known that $\mathrm{int}S\neq\emptyset$ if and only if $X, Y_1,\ldots,Y_m$ generates $\mathfrak{g}$, that is, if the control system satisfies the Lie algebra rank condition. Moreover since $\mathfrak{g}$ is semisimple the ideal generated by $Y_1,\ldots,Y_m$ (which in general has codimension ≤ 1) coincides with $\mathfrak{g}$. This implies (by hypoellipticity) that a transition probability μ_t, $t>0$, has a smooth density with respect to the Haar measure of G.

It follows that under the Lie algebra rank condition the semigroups S and S_{μ_t}, $t>0$, have nonempty interior. These semigroups have the same flag type because the invariant control set of S_{μ_t} on a flag manifold $\mathbb{F}_\Theta$ is the support of a unique invariant measure of μ_t, which is the invariant measure for the Markov process induced on $\mathbb{F}_\Theta$.

The basic assumption that the measures μ_t have exponential moments can be proved as an application of Girsanov Theorem (see [3]). The condition that the density ϕ of $\mu = \phi(g)\,dg$ belongs to $L^3(G, dg)$ was used before to ensure that an eigenfunction in $L^2(\mathbb{F}_\Theta, m_\Theta)$ is continuous. Here this condition can be overcome by an application of Hörmander Theorem ensuring that an eigenfunction of an hypoellyptic operator is smooth (see the approach in [3]).

Having these preparations we can concentrate on the operators $U_\lambda^\Theta(\mu_t)$ on the spaces $\mathcal{C}(\mathbb{F}_\Theta)$ and $L^2(\mathbb{F}_\Theta, m_\Theta)$. Under the Lie algebra rank condition $t \mapsto U_\lambda^\Theta(\mu_t)$ is a semigroup of compact operators either on $\mathcal{C}(\mathbb{F}_\Theta)$ or $L^2(\mathbb{F}_\Theta, m_\Theta)$. Define the invariant second order operator L on G by

$$L = X + \frac{1}{2}\sum_{j=1}^{m} Y_j^2.$$

By Itô's formula the infinitesimal generator of the semigroup $U_\lambda^\Theta(\mu_t)$ is $L_\lambda^\Theta = U_\lambda^\Theta(L)$ where we denote also by $U_\lambda^\Theta(\cdot)$ the infinitesimal representation of $\mathfrak{g}$ as well of its extension to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. The domain of L_λ^Θ contains the subspace of smooth functions. In what follows we derive the expression of L_λ^Θ as a second order differential operator on the smooth functions on $\mathbb{F}_\Theta$.

If $X \in \mathfrak{g}$ and f is a smooth function on $\mathbb{F}_\Theta$ then

$$U_\lambda^\Theta(X)\,f(x) = \frac{d}{dt}U_\lambda^\Theta\left(e^{tX}\right)f(x)_{|t=0} = \widetilde{X}f(x) + X\mathsf{a}_\lambda(1, x)$$

where $\widetilde{X}$ the vector field induced by X on $\mathbb{F}_\Theta$ and $X\mathsf{a}_\lambda(g, x) = \lambda X\mathsf{a}(g, x)$ with $X\mathsf{a}(g, x) = \frac{d}{dt}\mathsf{a}\left(e^{tX}g, x\right)_{|t=0}$.

For simplicity of notation we abbreviate the derivatives of $\mathsf{a}(g, x)$ as follows:

Notation: If $X \in \mathfrak{g}$ and $\lambda \in \mathfrak{a}^*$ then we put

(1) $q_X(x) = X\mathsf{a}(1, x)$ so that $X\mathsf{a}_\lambda(1, x) = \lambda(q_X(x))$.

(2) $r_Y(x) = \widetilde{Y}q_Y(x)$.

With this notation we have $U_\lambda^\Theta(X) = \widetilde{X} + \lambda(q_X)$. So that for $Y \in \mathfrak{g}$ we have

$$U_\lambda^\Theta\left(Y^2\right) = U_\lambda^\Theta(Y)^2 = \widetilde{Y}^2 + \lambda(q_Y)\,\widetilde{Y} + \lambda(r_Y) + (\lambda(q_Y))^2.$$

Therefore the infinitesimal generator of the semigroup $U_\lambda^\Theta(\mu_t)$ acting on smooth functions is given by the second order operator

$$L_\lambda^\Theta = \widetilde{L} + \frac{1}{2}\sum_{j=1}^{m}\lambda\left(q_{Y_j}\right)\widetilde{Y}_j + \lambda(q_X) + \frac{1}{2}\sum_{j=1}^{m}\lambda\left(r_{Y_j}\right) + \frac{1}{2}\sum_{j=1}^{m}\left(\lambda\left(q_{Y_j}\right)\right)^2 \qquad (16)$$

where $\widetilde{L} = \widetilde{X} + \frac{1}{2}\sum_{j=1}^{m}\widetilde{Y}_j^2$.

Lemma 9.1. *If $X, Y \in \mathfrak{g}$ are right invariant vector fields then*

(1) $X\mathsf{a}(gh, x) = X\mathsf{a}(g, hx)$ *for all* $g, h \in G$. *In particular* $X\mathsf{a}(g, x) = X\mathsf{a}(1, gx)$.

(2) $YX\mathsf{a}(gh, x) = YX\mathsf{a}(g, hx)$.

(3) $YX\mathsf{a}(1,x) = \widetilde{Y}X\mathsf{a}(1,x)$.

(4) $X\mathsf{a}(g,x) = 0$ *if* $X \in \mathfrak{k}$.

Proof. By the cocycle property

$$\frac{d}{dt}\mathsf{a}\left(e^{tX}gh,x\right)_{|t=0} = \frac{d}{dt}\mathsf{a}\left(e^{tX}g,hx\right)_{|t=0} + \frac{d}{dt}\mathsf{a}(h,x)_{|t=0}$$

which shows (1). Item (2) is a direct consequence of (1). Now

$$Y(X\mathsf{a}(1,x)) = \frac{d}{dt}X\mathsf{a}\left(e^{tY},x\right)_{|t=0} = \frac{d}{dt}X\mathsf{a}\left(1,e^{tY}x\right)_{|t=0}$$

by (1). The last term is $\widetilde{Y}X\mathsf{a}(1,x)$ showing (3). Finally, $\mathsf{a}\left(e^{tX}g,x\right) = \mathsf{a}(g,x)$ if $X \in \mathfrak{k}$ so that $X\mathsf{a}(g,x) = 0$. ■

In the next lemma we compute explicitly the partial derivatives of $\mathsf{a}(g,x)$ in terms of the Cartan $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and Iwasawa $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ decompositions. If $Z \in \mathfrak{g}$ we write its components as $Z = Z_{\mathfrak{k}} + Z_{\mathfrak{a}} + Z_{\mathfrak{n}} \in \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Lemma 9.2. *For the cocycle $\mathsf{a}(g,x)$ in the maximal flag manifold $\mathbb{F}$ we have the following formulas where $x = kx_0$ and x_0 is the origin of $\mathbb{F} = G/MAN$.*

(1) $q_X(x) = X\mathsf{a}(1,x) = \left(\mathrm{Ad}(k^{-1})X\right)_{\mathfrak{a}}$.

(2) $YX\mathsf{a}(1,x) = \widetilde{Y}X\mathsf{a}(1,x) = \left(\mathrm{Ad}(k^{-1})\left[\left(\mathrm{Ad}(k^{-1})Y\right)_{\mathfrak{k}}, X\right]\right)_{\mathfrak{a}}$.

(3) $X^2\mathsf{a}(1,x) = \left(\mathrm{Ad}(k^{-1})\left[\left(\mathrm{Ad}(k^{-1})X\right)_{\mathfrak{k}}, X\right]\right)_{\mathfrak{a}}$.

(4) *If $Y = A + Z \in \mathfrak{k} \oplus \mathfrak{s}$ is the Cartan decomposition of Y then $q_Y = q_Z$ and $r_Y = r_Z + \widetilde{A}q_Z$.*

(5) *Let $p_{\mathfrak{a}} : \mathfrak{s} \to \mathfrak{a}$ be the orthogonal projection w.r.t the Cartan-Killing form. Then $q_X(x) = p_{\mathfrak{a}}(\mathrm{Ad}(k^{-1})X)$.*

(6) *If $k \in K$ and $X \in \mathfrak{g}$ then $q_X(kx) = q_{\mathrm{Ad}(k^{-1})X}(x)$ and $r_X(kx) = r_{\mathrm{Ad}(k^{-1})X}(x)$.*

Proof. We consider the Iwasawa decomposition $e^{tX}k = k_t h_t n_t$ with $k_0 = k$ and $h_0 = n_0 = 1$. We have $\log h_t = \mathsf{a}\left(e^{tX},x\right)$ so that $X\mathsf{a}(1,x) = h_0'$. On the other hand

$$X(k) = \frac{d}{dt}(k_t h_t n_t)_{|t=0} = k_0' + kh_0' + kn_0'.$$

This last equality shows that $k^{-1}X(k) = k^{-1}k_0' + h_0' + n_0'$. Since $k^{-1}k_0' \in \mathfrak{k}$, $h_0' \in \mathfrak{a}$ and $n_0' \in \mathfrak{n}$ it follows that h_0' is the $\mathfrak{a}$-component of $k^{-1}X(k) = \mathrm{Ad}(k^{-1})X$. This shows that $X\mathsf{a}(1,x) = \left(\mathrm{Ad}(k^{-1})X\right)_{\mathfrak{a}}$ as claimed in (1).

By Lemma 9.1(3) we get $YX\mathsf{a}(1,x) = \widetilde{Y}X\mathsf{a}(1,x)$ and (2) by derivating $\left(\mathrm{Ad}(k^{-1})X\right)_{\mathfrak{a}}$ as a function of k. Take the Iwasawa decomposition $e^{tY}k^{-1} = u_t h_t n_t \in KAN$ with $u_0 = k^{-1}$ and $h_0 = n_0 = 1$. Then the derivative in the direction of $\widetilde{Y}$ is

$$\frac{d}{dt}\mathrm{Ad}(u_t)_{|t=0} = d(\mathrm{Ad})_{k^{-1}}(u_0').$$

Since $u_0' \in T_{k^{-1}}K$ we have $u_0' = k^{-1}A$ with $A \in \mathfrak{k}$, that is, $u_0' = \frac{d}{dt}\left(k^{-1}e^{tA}\right)_{|t=0}$. So

$$\frac{d}{dt}\mathrm{Ad}(u_t)_{|t=0} = \frac{d}{dt}\mathrm{Ad}\left(k^{-1}e^{tA}\right)_{|t=0} = \mathrm{Ad}\left(k^{-1}\right)\mathrm{ad}(A).$$

To find $A \in \mathfrak{k}$ take the derivative of $e^{tY}k^{-1} = u_t h_t n_t$ to get

$$Y\left(k^{-1}\right) = u_0' + kh_0' + kn_0'$$

and hence $\left(\mathrm{Ad}\left(k^{-1}\right)Y\right)_{\mathfrak{k}} = ku_0' = A$. Summarizing,

$$YX\mathsf{a}\left(1,x\right) = \widetilde{Y}X\mathsf{a}\left(1,x\right) = \mathrm{Ad}\left(k^{-1}\right)\left[\left(\mathrm{Ad}\left(k^{-1}\right)Y\right)_{\mathfrak{k}}, X\right]_{\mathfrak{a}}$$

showing (2). Item (3) is a special case of (2) and (4) is a consequence of $q_A = 0$.

To derive (5) take $Z \in \mathfrak{s}$ and formulate its Iwasawa decomposition in the form $Z = Z_{\mathfrak{k}} + Z_{\mathfrak{a}} + Z_{\mathfrak{n}} \in \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$. Since $Z = \left(Z - \theta Z\right)/2$ we have $Z = Z_{\mathfrak{a}} + \left(Z_{\mathfrak{n}} - \theta Z_{\mathfrak{n}}\right)/2$. But $Z_{\mathfrak{a}}$ and $\left(Z_{\mathfrak{n}} - \theta Z_{\mathfrak{n}}\right)/2$ are orthogonal w.r.t. the Cartan-Killing form. Hence $p_{\mathfrak{a}}\left(Z\right) = Z_{\mathfrak{a}}$ which implies (5).

Finally, if $k \in K$ and $X \in \mathfrak{g}$ then by K-invariance we get

$$\mathsf{a}\left(e^{tX}, kx\right) = \mathsf{a}\left(k^{-1}e^{tX}k, x\right) = \mathsf{a}\left(e^{t\mathrm{Ad}(k)X}, x\right)$$

so that

$$q_X\left(kx\right) = \frac{d}{dt}\mathsf{a}\left(e^{t\mathrm{Ad}(k)X}, x\right)_{t=0} = q_{\mathrm{Ad}(k^{-1})X}\left(x\right).$$

A similar computation yields $r_X\left(kx\right) = r_{\mathrm{Ad}(k^{-1})X}\left(x\right)$. ■

9.1. An example with $S = G$

Consider the driftless stochastic differential equation

$$dg = \sum_{j=1}^{m} Y_j\left(g\right)\circ dW_j \tag{17}$$

where $\{Y_1, \ldots, Y_m\}$ is an orthonormal basis of $\mathfrak{s}$ where $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{s}$ is a Cartan decomposition. If $\mathfrak{g}$ has only noncompact factors then Lie algebra generated by $\mathfrak{s}$ is the whole $\mathfrak{g}$ so that in this case the above system satisfies the Lie algebra rank condition. Since the equation (17) has no drift the control semigroup is a group with nonempty interior and hence is the connected group G.

The infinitesimal generators for the above equation are

$$2L_{\lambda}^{\Theta} = \sum_{j=1}^{m}\widetilde{Y}^2 + \sum_{j=1}^{m}\lambda\left(q_{Y_j}\right)\widetilde{Y}_j + \sum_{j=1}^{m}\lambda\left(r_{Y_j}\right) + \sum_{j=1}^{m}\left(\lambda\left(q_{Y_j}\right)\right)^2. \tag{18}$$

To compute the coefficients of these operators we note first that they are the same for any orthonormal basis. This is because the vector fields appear quadratically in L_{λ}^{Θ} and the orthonormal bases are obtained from each other by an orthogonal matrix. To illustrate the proof of this independence let us check that the term $\sum_{j=1}^{m}\left(\lambda\left(q_{Y_j}\right)\right)^2$ is the same for another orthonormal basis $\{Z_1, \ldots, Z_m\}$ with $Z_j = \sum_{i=1}^{m} a_{ij}Y_i$.

We have

$$\left(\lambda\left(q_{Z_j}\right)\right)^2 = \sum_{i=1}^{m} a_{ij}^2\left(\lambda\left(q_{Y_i}\right)\right)^2 + \sum_{r,s=1}^{m} a_{rj}a_{sj}\lambda\left(q_{Y_r}\right)\lambda\left(q_{Y_s}\right)$$

so that

$$\sum_{j=1}^{m}\left(\lambda\left(q_{Z_j}\right)\right)^2 = \sum_{i=1}^{m}\left(\sum_{j=1}^{m} a_{ij}^2\right)\left(\lambda\left(q_{Y_i}\right)\right)^2 + \sum_{r,s=1}^{m}\left(\sum_{j=1}^{m} a_{rj}a_{sj}\right)\lambda\left(q_{Y_r}\right)\lambda\left(q_{Y_s}\right)$$

which is equal to $\sum_{i=1}^{m}\left(\lambda\left(q_{Y_i}\right)\right)^2$ because the matrix $\left(a_{ij}\right)_{i,j}$ is orthogonal. The other terms are treated the same way. Now we can compute the functions in (18).

Proposition 9.3. *The zero order terms of* (18) *are constant and given by*

(1) $\sum_{j=1}^{m}\left(\lambda\left(q_{Y_j}\right)\right)^2=\|\lambda\|^2$ *and*

(2) $\sum_{j=1}^{m}\lambda\left(r_{Y_j}\right)=2\langle\lambda,\omega_\Theta\rangle$.

Proof. We have $q_{Y_j}(kx)=q_{\mathrm{Ad}(k^{-1})Y_j}(x)$ and $r_X(kx)=r_{\mathrm{Ad}(k^{-1})X}(x)$ if $k\in K$ (see Lemma 9.2 (6)). Since $\{\mathrm{Ad}(k^{-1})Y_1,\ldots,\mathrm{Ad}(k^{-1})Y_m\}$ is also an orthonormal basis of $\mathfrak{s}$ the functions are K-invariant and hence constant.

To get the first formula let $H_\lambda\in\mathfrak{a}$ be defined by $\lambda(\cdot)=\langle H_\lambda,\cdot\rangle$ so that if $p_{\mathfrak{a}}:\mathfrak{s}\to\mathfrak{a}$ is the orthogonal projection then $\lambda(p_{\mathfrak{a}}(Z))=\langle H_\lambda,Z\rangle$. Hence by item (5) of Lemma 9.2 we have

$$\lambda\left(q_{Y_j}\right)(x)=\langle H_\lambda,\mathrm{Ad}\left(k^{-1}\right)Y_j\rangle\quad x=kx_0.$$

In particular $\lambda\left(q_{Y_j}\right)(x_0)=\langle H_\lambda,Y_j\rangle$ so that

$$\sum_{j=1}^{m}\left(\lambda\left(q_{Y_j}\right)\right)^2=\sum_{j=1}^{m}\langle H_\lambda,Y_j\rangle^2=\|H_\lambda\|^2=\|\lambda\|^2.$$

To compute $\sum_{j=1}^{m}\lambda\left(r_{Y_j}\right)$ at the origin x_0 we choose a special orthonormal basis, namely it is the union of orthonormal bases on the subspaces $\mathfrak{z}_\Theta\cap\mathfrak{s}$ and $(\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha})\cap\mathfrak{s}$ with α running through the set of positive roots outside $\langle\Theta\rangle$. If $Z\in\mathfrak{z}_\Theta$ then $r_Z(x_0)=0$ because x_0 is a singularity of Z. Hence the elements of the basis of $\mathfrak{z}_\Theta$ do not contribute to the sum $\sum_{j=1}^{m}\lambda\left(r_{Y_j}\right)$.

On the other hand take $X\in(\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha})\cap\mathfrak{s}$ with $\|X\|=1$ and write $X=X_\alpha+Y_\alpha$ with $X_\alpha\in\mathfrak{g}_\alpha$ and $Y_\alpha\in\mathfrak{g}_{-\alpha}$. Since $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_{-\alpha}$ and $\theta(\mathfrak{g}_\alpha)=\mathfrak{g}_{-\alpha}$ we have

(1) $1=\|X\|=2\langle X_\alpha,Y_\alpha\rangle$ which implies that $[X_\alpha,Y_\alpha]=\frac{1}{2}H_\alpha$.

(2) $\theta(X_\alpha)=Y_\alpha$ and $\theta(Y_\alpha)=X_\alpha$. This implies that $A\in\mathfrak{k}$ if $A=-X_\alpha+Y_\alpha$.

Now $\widetilde{X}(x_0)=\widetilde{Y}_\alpha(x_0)=\widetilde{A}(x_0)$, so $r_X(x_0)=\widetilde{A}q_X(x_0)$. Hence if H_λ is as above then

$$\lambda(r_X)(x_0)=\frac{d}{dt}\langle H_\lambda,\mathrm{Ad}\left(e^{-tA}\right)X\rangle_{t=0}=\langle H_\lambda,[X,A]\rangle.$$

But $[X,A]=[X_\alpha+Y_\alpha,-X_\alpha+Y_\alpha]=2[X_\alpha,Y_\alpha]=H_\alpha$ and we get $\lambda(r_X)(x_0)=\langle H_\lambda,H_\alpha\rangle$ if $X\in(\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha})\cap\mathfrak{s}$ and $\|X\|=1$. Therefore the contribution of the basis in $(\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha})\cap\mathfrak{s}$ to the sum $\sum_{j=1}^{m}\lambda\left(r_{Y_j}\right)$ is

$$\langle H_\lambda,H_\alpha\rangle\dim\mathfrak{g}_\alpha=\langle\lambda,\dim\mathfrak{g}_\alpha\alpha\rangle.$$

Adding over the positive roots outside $\langle\Theta\rangle$ we get $\sum_{j=1}^{m}\lambda\left(r_{Y_j}\right)=2\langle\lambda,\omega_\Theta\rangle$ after recalling that ω_Θ is half the sum of such roots counted with multiplicities. ■

These computations permit to write down the operators L_λ^Θ for the above stochastic equation.

Corollary 9.4. *For the equation* (17) *an operator* L_λ^Θ *is the sum of a differential operator plus a constant. Precisely:*

$$L_\lambda^\Theta=\frac{1}{2}D+\langle\lambda,\omega_\Theta\rangle+\frac{1}{2}\|\lambda\|^2,\quad\textit{where}\quad D=\sum_{j=1}^{m}\widetilde{Y}^2+\sum_{j=1}^{m}\lambda\left(q_{Y_j}\right)\widetilde{Y}_j.$$

As a consequence we have that the constant function 1 is an eigenfunction of L_λ^Θ for any λ and Θ. The corresponding eigenvalue is

$$\Gamma_\lambda = \langle \lambda, \omega_\Theta \rangle + \frac{\|\lambda\|^2}{2}.$$

This is the principal eigenvalue of L_λ^Θ because the eigenfunction 1 is strictly positive.

Remark 9.5. It was claimed in Arnold-Oeljeklaus-Pardoux [3] that a function $f_\lambda(p)$ given for the norm cocycle over the projective space is analytic in $\mathbb{R}$ (see [3], Lemma 2.4). The proof of this claim in [3] is incomplete. It exhibits a power series for f_λ but it is not checked that it has infinite convergence radius. ■

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Received August 14, 2019
and in final form February 1, 2020