# Matrix Lie Groups as 4-Dimensional Hypercomplex Manifolds with Hermitian-Norden Metrics

**Hristo Manev**\*

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**Abstract.**   There are studied Lie groups considered as almost hypercomplex Hermitian-Norden manifolds, which are integrable and have the lowest dimension four. It is established a correspondence of the derived types Lie algebras with invariant hypercomplex structures and the explicit matrix representation of their Lie groups. There are constructed examples of the considered structure of different types on some known Lie groups.

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*Key Words:* Lie group, Lie algebra, Matrix representation, Almost hypercomplex structure, Hermitian metric, Norden metric.

## 1.   Introduction

In this work we continue investigation of almost hypercomplex manifolds with Hermitian-Norden metrics. The manifold under consideration is a $4n$-dimensional manifold equipped with an almost hypercomplex structure $H$ which is a triad of anticommuting almost complex structures such that each of them is a composition of the other two structures. Moreover, it is introduced a pseudo-Riemannian metric on the manifold which is Hermitian with respect to one of the structures of $H$ and a Norden metric with respect to other two structures of $H$.

The geometry of almost hypercomplex manifolds with Hermitian-Norden metrics is studied in [7, 8, 10, 11, 14, 15, 16]. This type of manifolds are the only possible way to involve Norden-type metrics on almost hypercomplex manifolds.

In [2, 7, 12, 13, 18], there are considered Lie groups as manifolds with similar structures and metrics. Moreover, in [12], integrable hypercomplex structures with Hermitian-Norden metrics on Lie groups of dimension 4 are considered. There are constructed the corresponding five types of invariant hypercomplex structures. The different cases regarding the signature of the metric are considered.

As it is known from [5], each representation of a Lie algebra corresponds uniquely to a representation of a simply connected Lie group. This relation is one-to-one. Thus,

the knowledge of the representation of a certain Lie algebra solves the question of the representation of its Lie group.

This work establishes a correspondence between the different types Lie algebras from [12] and an explicit matrix representation of their Lie groups. The last section of the work is devoted to equipping known Lie groups with integrable hypercomplex structures and Hermitian-Norden metrics of different types.

## 2. Preliminaries

### 2.1. Almost hypercomplex manifolds with Hermitian-Norden metrics

Firstly, let us recall the notion of almost hypercomplex structure $H$ on a $4n$-dimensional manifold $\mathcal{M}$. The considered structure is a triad of anticommuting almost complex structures such that each of them is a composition of the two other structures, i.e. for $H = (J_\alpha)$ $(\alpha = 1, 2, 3)$ the following properties are valid for all cyclic permutations $(\alpha, \beta, \gamma)$ of $(1, 2, 3)$ and the identity $I$ [1, 17]:

$$J_\alpha = J_\beta \circ J_\gamma = -J_\gamma \circ J_\beta, \qquad J_\alpha^2 = -I.$$

Then $(\mathcal{M}, H)$ is called *an almost hypercomplex manifold.*

Let us equip $H$ with a metric structure, generated by a pseudo-Riemannian metric $g$ of neutral signature [7, 8]. In this study, one (resp., the other two) of the almost complex structures of $H$ acts as an isometry (resp., anti-isometries) with respect to $g$ in each tangent fibre, i.e.

$$g(\cdot, \cdot) = \varepsilon_\alpha g(J_\alpha \cdot, J_\alpha \cdot), \tag{1}$$

where
$$\varepsilon_\alpha = \begin{cases} 1, & \alpha = 1; \\ -1, & \alpha = 2; 3. \end{cases}$$

Here and further, the index $\alpha$ runs over the range $\{1, 2, 3\}$ unless otherwise is stated. Thus, there exist three $(0,2)$-tensors associated with $H$ to $g$ – a Kähler form and two metrics of Hermitian-Norden type. In the considered case, we obtain a Hermitian metric $g$ with respect to $J_1$ and a Norden metric $g$ regarding $J_2$ and $J_3$. So, we call the derived metric structure *an almost hypercomplex structure with Hermitian-Norden metrics.*

The associated 2-form $g_1$ and the associated neutral metrics $g_2$ and $g_3$ are determined by

$$g_\alpha(\cdot, \cdot) = g(J_\alpha \cdot, \cdot) = -\varepsilon_\alpha g(\cdot, J_\alpha \cdot). \tag{2}$$

The structure $(H, G) = (J_1, J_2, J_3; g, g_1, g_2, g_3)$ on $\mathcal{M}$ is called an *almost hypercomplex structure with Hermitian-Norden metrics* and the respective manifold $(\mathcal{M}, H, G)$ is called an *almost hypercomplex manifold with Hermitian-Norden metrics* ([8]).

In [8], there are introduced the fundamental tensors of a manifold $(\mathcal{M}, H, G)$ by the following three $(0, 3)$-tensors

$$F_\alpha(x, y, z) = g\big((\nabla_x J_\alpha) y, z\big) = \big(\nabla_x g_\alpha\big)(y, z),$$

where $\nabla$ is the Levi-Civita connection generated by $g$. These tensors have the basic properties

$$F_\alpha(x, y, z) = -\varepsilon_\alpha F_\alpha(x, z, y) = -\varepsilon_\alpha F_\alpha(x, J_\alpha y, J_\alpha z)$$

and the following relations are valid for all cyclic permutations $(\alpha, \beta, \gamma)$ of $(1, 2, 3)$

$$F_\alpha(x, y, z) = F_\beta(x, J_\gamma y, z) - \varepsilon_\alpha F_\gamma(x, y, J_\beta z).$$

The corresponding Lee forms $\theta_\alpha$ are defined by

$$\theta_\alpha(\cdot) = g^{kl} F_\alpha(e_k, e_l, \cdot),$$

where $\{e_1, e_2, \ldots, e_{4n}\}$ is an arbitrary basis of $T_p\mathcal{M}$, $p \in \mathcal{M}$ and $g^{ij}$ are the corresponding components of the inverse matrix of $g$.

In [8], the so-called *hyper-Kähler manifolds with Hermitian-Norden metrics* are studied. They are the almost hypercomplex manifolds with Hermitian-Norden metrics in the class $\mathcal{K}$, where $\nabla J_\alpha = 0$ for all $\alpha$. It is proved in [7] that a sufficient condition for $(\mathcal{M}, H, G)$ to be in $\mathcal{K}$ is this manifold to be of Kähler-type with respect to two of the three almost complex structures of $H$.

Let us remark that according to (1) $(\mathcal{M}, J_1, g)$ is an almost Hermitian manifold whereas $(\mathcal{M}, J_2, g)$ and $(\mathcal{M}, J_3, g)$ are almost complex manifolds with Norden metric. The basic classes of the mentioned two types of manifolds are given in [6] and [4], respectively. The special class $\mathcal{W}_0(J_\alpha)$ : $F_\alpha = 0$ of the Kähler-type manifolds belongs to any other class in the corresponding classification. For the lowest dimension, $dim\mathcal{M} = 4$, the four basic classes of almost Hermitian manifolds with respect to $J_1$ are restricted to two: $\mathcal{W}_2(J_1)$, the class of the almost Kähler manifolds, and $\mathcal{W}_4(J_1)$, the class of the Hermitian manifolds which, for this dimension 4, are determined by:

$$\mathcal{W}_2(J_1) : \ F_1(x, y, z) + F_1(y, z, x) + F_1(z, x, y) = 0;$$

$$\mathcal{W}_4(J_1) : \ F_1(x, y, z) = \frac{1}{2} \{ g(x, y)\theta_1(z) - g(x, J_1 y)\theta_1(J_1 z)$$
$$- g(x, z)\theta_1(y) + g(x, J_1 z)\theta_1(J_1 y) \}.$$

The basic classes of the almost complex manifolds with Norden metric ($\alpha = 2$ or $3$) are determined for dimension 4 as follows:

$$\mathcal{W}_1(J_\alpha) : \ F_\alpha(x, y, z) = \frac{1}{4} \{ g(x, y)\theta_\alpha(z) + g(x, J_\alpha y)\theta_\alpha(J_\alpha z)$$
$$+ g(x, z)\theta_\alpha(y) + g(x, J_\alpha z)\theta_\alpha(J_\alpha y) \};$$

$$\mathcal{W}_2(J_\alpha) : \ F_\alpha(x, y, J_\alpha z) + F_\alpha(y, z, J_\alpha x) + F_\alpha(z, x, J_\alpha y) = 0, \qquad \theta_\alpha = 0;$$

$$\mathcal{W}_3(J_\alpha) : \ F_\alpha(x, y, z) + F_\alpha(y, z, x) + F_\alpha(z, x, y) = 0.$$

Let us denote by $\mathcal{W}^0 = \{\mathcal{W} \mid \mathrm{d}\,(\theta_1 \circ J_1) = 0\}$ the class of the (locally) conformally equivalent $\mathcal{K}$-manifolds, where $\mathcal{W} = \mathcal{W}_4(J_1) \cap \mathcal{W}_1(J_2) \cap W_1(J_3)$ and a conformal transformation of the metric is given by $\overline{g} = e^{2u}g$ for a differentiable function $u$ on the manifold.

It is known that an almost hypercomplex structure $H = (J_\alpha)$ is a hypercomplex structure if the Nijenhuis tensors $[J_\alpha, J_\alpha]$, determined by

$$[J_\alpha, J_\alpha](\cdot, \cdot) = [J_\alpha\cdot, J_\alpha\cdot] - J_\alpha[J_\alpha\cdot, \cdot] - J_\alpha[\cdot, J_\alpha\cdot] - [\cdot, \cdot],$$

vanish on $\mathfrak{X}(\mathcal{M})$ for each $\alpha$ (e.g. [3]). Moreover, it is known that $H$ is hypercomplex if and only if two of $[J_\alpha, J_\alpha]$ vanish.

Then the class of 4-dimensional hypercomplex manifolds with Hermitian-Norden metrics is

$$\mathcal{HC} = \mathcal{W}_4(J_1) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2)(J_2) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2)(J_3).$$

Let us denote by $\mathcal{HC}^* = \mathcal{W}_0(J_1) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2)(J_2) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2)(J_3)$ the respective subclass of $\mathcal{HC}$.

## 3. The main result

Let $L$ be a simply connected 4-dimensional real Lie group admitting an invariant hypercomplex structure, i.e. left translations by elements of $L$ are holomorphic with respect to $J_\alpha$ for all $\alpha$. The corresponding Lie algebra of $L$, its center and its derived Lie algebra $[\mathfrak{l}, \mathfrak{l}]$ are denoted by $\mathfrak{l}$, $\mathfrak{z}$ and $\mathfrak{l}'$, respectively.

In [2], invariant hypercomplex structures $H$ on 4-dimensional real Lie groups are classified as follows

**Theorem 3.1** ([2]). *The only 4-dimensional Lie algebras admitting an integrable hypercomplex structure are the following types:*

**(hc1)** $\mathfrak{l}$ *is Abelian;*

**(hc2)** $\mathfrak{l} \cong \mathbb{R} \oplus \mathfrak{so}(3)$;

**(hc3)** $\mathfrak{l} \cong \mathfrak{aff}(\mathbb{C})$;

**(hc4)** $\mathfrak{l}$ *is the solvable Lie algebra corresponding to* $\mathbb{R}H^4$;

**(hc5)** $\mathfrak{l}$ *is the solvable Lie algebra corresponding to* $\mathbb{C}H^2$,

*where* $\mathbb{R} \oplus \mathfrak{so}(3)$ *is the Lie algebra of the Lie groups* $U(2)$ *and* $S^3 \times S^1$; $\mathfrak{aff}(\mathbb{C})$ *is the Lie algebra of the affine motion group of* $\mathbb{C}$, *the unique 4-dimensional Lie algebra carrying an Abelian hypercomplex structure;* $\mathbb{R}H^4$ *is the real hyperbolic space;* $\mathbb{C}H^2$ *is the complex hyperbolic space.*

A standard hypercomplex structure on $\mathfrak{l}$ with respect to a basis $\{e_1, e_2, e_3, e_4\}$ is defined as in [17]:

$$\begin{aligned}
J_1 e_1 &= e_2, & J_1 e_2 &= -e_1, & J_1 e_3 &= -e_4, & J_1 e_4 &= e_3; \\
J_2 e_1 &= e_3, & J_2 e_2 &= e_4, & J_2 e_3 &= -e_1, & J_2 e_4 &= -e_2; \\
J_3 e_1 &= -e_4, & J_3 e_2 &= e_3, & J_3 e_3 &= -e_2, & J_3 e_4 &= e_1.
\end{aligned} \tag{3}$$

A pseudo-Euclidian metric $g$ of neutral signature is introduced by

$$g(x, y) = x^1 y^1 + x^2 y^2 - x^3 y^3 - x^4 y^4, \tag{4}$$

where $x(x^1, x^2, x^3, x^4)$, $y(y^1, y^2, y^3, y^4) \in \mathfrak{l}$. The metric $g$ generates an almost hypercomplex structure with Hermitian-Norden metrics on $\mathfrak{l}$, according to (1) and (2).

In [12], there are considered integrable hypercomplex structures with Hermitian-Norden metrics on Lie groups of dimension 4. There are constructed and characterized different types of hypercomplex structures on Lie algebras following the Barberis classification of invariant hypercomplex structures $H$ on 4-dimensional real Lie groups, given in Theorem 3.1. Moreover, in [12], a correspondence is found between

the 4-dimensional real Lie groups for each class in the Barberis classification and the respective class of hypercomplex manifold with Hermitian-Norden metrics, given in Table 1. Some of the classes are considered separately because are distinguished the different cases concerning of the signature of $g$ on $\mathfrak{l}'$.

| (hc1) | $\mathcal{K}$ |
|---|---|
| (hc2) | $\mathcal{HC}$ |
| (hc3.1) | $\mathcal{HC}^*$ |
| (hc3.2) | $\mathcal{W}^0$ |
| (hc4.1) | $\mathcal{HC}$ |
| (hc4.2) | $\mathcal{W}^0$ |
| (hc5.1) | $\mathcal{HC}$ |
| (hc5.2) | $\mathcal{HC}$ |

Table 1

It is known (e.g. [5]) that for a real Lie algebra of finite dimension there is a corresponding connected simply connected Lie group, determined uniquely up to isomorphism. It arises the problem for the determination of the explicit matrix representation of the Lie group which is isomorphic to the Lie group equipped with an integrable hypercomplex structure corresponding to each class in Table 1.

**Theorem 3.2.** *Let $(L, H, G)$ be a 4-dimensional hypercomplex manifold with Hermitian-Norden metrics. Then the compact simply connected Lie group isomorphic to $L$, both with one and the same Lie algebra $\mathfrak{l}$, has the form*

$$e^A = E + tA + uA^2,$$

*where $E$ is the identity matrix and $A$ is the matrix representation of the corresponding Lie algebra. The matrix form of $A$ as well as the real parameters $t$ and $u$ for the different classes of $\mathfrak{l}$ admitting a structure $(H, G)$ are the following for $a, b, c, d \in \mathbb{R}$:*

$$(hc1): \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad t = 1, \quad u = 0;$$

$$(hc2): \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & d & -c \\ 0 & -d & 0 & b \\ 0 & c & -b & 0 \end{pmatrix}, \quad \Delta := b^2 + c^2 + d^2$$

- for $(b, c) \neq (0, 0)$  $\qquad t = \Delta^{-\frac{1}{2}} \sin \sqrt{\Delta},$

$$u = \Delta^{-1} \left( 1 - \cos \sqrt{\Delta} \right);$$

- for $(b, c) = (0, 0)$  $\qquad t = 1, \qquad u = 0;$

$(hc3.1):$ $\quad A = \begin{pmatrix} 0 & d & 0 & -b \\ 0 & c & 0 & a \\ 0 & -b & 0 & -d \\ 0 & -a & 0 & c \end{pmatrix}$, $\quad \Delta := a(a^2 + c^2)$

- for $a(a^2 + c^2) \neq 0$
$$t = \Delta^{-1} \left\{ -2ac(1 - e^c \cos a) + (a^2 - c^2)e^c \sin a \right\},$$
$$u = \Delta^{-1} \left\{ a(1 - e^c \cos a) + ce^c \sin a \right\};$$
- for $(a, c) = (0, 0)$ $\quad t = 1,$ $\quad u = 0;$

$(hc3.2):$ $\quad A = \begin{pmatrix} c & d & 0 & 0 \\ -d & c & 0 & 0 \\ -a & -b & 0 & 0 \\ b & -a & 0 & 0 \end{pmatrix}$, $\quad \Delta := d(c^2 + d^2)$

- for $d(c^2 + d^2) \neq 0$
$$t = \Delta^{-1} \left\{ -2cd(1 - e^c \cos d) + (d^2 - c^2)e^c \sin d \right\},$$
$$u = \Delta^{-1} \left\{ d(1 - e^c \cos d) + ce^c \sin d \right\};$$
- for $(c, d) = (0, 0)$ $\quad t = 1,$ $\quad u = 0;$

$(hc4.1):$ $\quad A = \begin{pmatrix} 0 & b & c & d \\ 0 & -a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}$,

- for $a \neq 0$ $\quad t = a^{-1},$ $\quad u = a^{-2}e^{-a};$
- for $a = 0$ $\quad t = 1,$ $\quad u = 0;$

$(hc4.2):$ $\quad A = \begin{pmatrix} -d & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \\ 0 & 0 & -d & 0 \\ a & b & c & 0 \end{pmatrix}$,

- for $d \neq 0$ $\quad t = d^{-1},$ $\quad u = d^{-2}e^{-d};$
- for $d = 0$ $\quad t = 1,$ $\quad u = 0;$

$(hc5.1):$ $\quad A = \begin{pmatrix} 0 & b & \frac{c}{2} & \frac{d}{2} \\ 0 & -a & 0 & 0 \\ 0 & \frac{d}{2} & -\frac{a}{2} & 0 \\ 0 & -\frac{c}{2} & 0 & -\frac{a}{2} \end{pmatrix}$,

- for $a \neq 0$ $\quad t = a^{-1} \left\{ e^{-a} - 4e^{-\frac{a}{2}} + 3 \right\},$
$$u = a^{-2} \left\{ 2e^{-a} - 4e^{-\frac{a}{2}} + 2 \right\};$$
- for $a = 0$ $\quad t = 1,$ $\quad u = 0;$

$(hc5.2):$ $\quad A = \begin{pmatrix} -\frac{d}{2} & 0 & -\frac{b}{2} & 0 \\ 0 & -\frac{d}{2} & \frac{a}{2} & 0 \\ 0 & 0 & -d & 0 \\ \frac{a}{2} & \frac{b}{2} & c & 0 \end{pmatrix}$,

- for $d \neq 0$ $\quad t = d^{-1} \left\{ e^{-d} - 4e^{-\frac{d}{2}} + 3 \right\},$
$$u = d^{-2} \left\{ 2e^{-d} - 4e^{-\frac{d}{2}} + 2 \right\};$$
- for $d = 0$ $\quad t = 1,$ $\quad u = 0.$

**Proof.**    Let us consider the case when $(L, H, G)$ is a 4-dimensional hypercomplex manifold with Hermitian-Norden metrics for which the corresponding 4-dimensional Lie algebra admitting a hypercomplex structure is of the class $(hc2)$. Then, according to [12], the corresponding Lie algebra of $(hc2)$ is determined by

$$[e_2, e_4] = e_3, \quad [e_4, e_3] = e_2, \quad [e_3, e_2] = e_4. \tag{6}$$

Using (6), we have that the nonzero values of the commutation coefficients $C_{ij}^k$ are

$$C_{24}^3 = C_{43}^2 = C_{32}^4 = -C_{42}^3 = -C_{34}^2 = -C_{23}^4 = 1. \tag{7}$$

According to [5], the commutation coefficients provide a matrix representation of the respective Lie algebra. This representation is obtained by the basic matrices $M_k$. Their entries are determined by

$$(M_k)_l^s = -C_{kl}^s. \tag{8}$$

Using (7) and (8), we obtain

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then we have $A = aM_1 + bM_2 + cM_3 + dM_4$ for $a, b, c, d \in \mathbb{R}$.

Let us suppose that $(b, c, d) \neq (0, 0, 0)$. The matrix representation of the considered Lie algebra is the matrix $A$, given in the respective case of Theorem 3.2. Then, the characteristic polynomial of $A$ has the form

$$P_A(\lambda) = \lambda^2(\lambda^2 + b^2 + c^2 + d^2).$$

Therefore, the eigenvalues $\lambda_k$ $(k = 1, 2, 3, 4)$ are

$$\lambda_1 = \lambda_2 = 0, \qquad \lambda_3 = -\lambda_4 = i\sqrt{\Delta},$$

where $i$ is the imaginary unit and $\Delta = b^2 + c^2 + d^2$. Hence, the corresponding linearly independent eigenvectors $p_k$ $(k = 1, 2, 3, 4)$ are

$$p_1(1, 0, 0, 0), \quad p_2(0, b, c, d), \quad p_3(0, \alpha, \beta, \gamma), \quad p_4(0, \overline{\alpha}, \overline{\beta}, \overline{\gamma}),$$

where $\alpha = c\Delta + ibd\sqrt{\Delta}$, $\beta = -b\Delta + icd\sqrt{\Delta}$, $\gamma = -i\sqrt{\Delta}(b^2 + c^2)$ and $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ are the corresponding complex conjugates. The vectors $p_k$ determine the following matrix, having $\det P = -2i\Delta^{\frac{5}{2}}(b^2 + c^2)$:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b & \alpha & \overline{\alpha} \\ 0 & c & \beta & \overline{\beta} \\ 0 & d & \gamma & \overline{\gamma} \end{pmatrix}.$$

Let us consider the first case when $\det P \neq 0$, i.e. $(b, c) \neq (0, 0)$. Then we obtain

$$
P^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & b\Delta^{-1} & c\Delta^{-1} & d\Delta^{-1} \\
0 & i\overline{\alpha}\delta & i\overline{\beta}\delta & i\overline{\gamma}\delta \\
0 & i\alpha\delta & i\beta\delta & i\gamma\delta
\end{pmatrix}
$$

for the inverse matrix of $P$, where $\delta = \frac{1}{2}\Delta^{-\frac{3}{2}}\overline{\gamma}^{-1}$. The Jordan matrix is the diagonal matrix $J = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. It is well known that $e^A = Pe^J P^{-1}$. Then, for the considered Lie algebra, we get the matrix representation of the respective Lie group as follows

$$
e^A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 - (c^2 + d^2)u & bcu + dt & bdu - ct \\
0 & bcu - dt & 1 - (b^2 + d^2)u & cdu + bt \\
0 & bdu + ct & cdu - bt & 1 - (b^2 + c^2)u
\end{pmatrix},
$$

where $t = \Delta^{-\frac{1}{2}}\sin\sqrt{\Delta}$ and $u = \Delta^{-1}\left(1 - \cos\sqrt{\Delta}\right)$. This result can be written as

$$
e^A = E + tA + uA^2.
$$

Now, let us consider the second case when $\det P = 0$, i.e. $(b, c) = (0, 0)$. Then $P$ is a non-invertible matrix. Therefore $A$ is nilpotent and $e^A$ can be computed directly from

$$
e^A = E + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^{q-1}}{(q-1)!},
$$

where $q$ stands for the nilpotency index of $A$. Then, using the form of $A$ in Theorem 3.2, we obtain that $q = 2$ in the considered case. Therefore, the matrix representation of the Lie group in this case is $e^A = E + A$. In order to generalize both cases into the formula $e^A = E + tA + uA^2$, we substitute here $t = 1$ and $u = 0$.

By similar considerations we obtain the other results in Theorem 3.2 for the rest of the classes. ∎

## 4. Equipping of known Lie groups with hypercomplex structures and Hermitian-Norden metrics

In [9] the following matrix Lie groups $G_6$, $G_8$ and $G_{10}$ are considered:

$$
G_6 = \begin{pmatrix}
1 & x & \frac{1}{2}x^2 & y \\
0 & 1 & x & z \\
0 & 0 & 1 & w \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
G_{10} = \begin{pmatrix}
1 & x & y & z \\
0 & 1 & w & v \\
0 & 0 & 1 & w \\
0 & 0 & 0 & 1
\end{pmatrix}, \tag{9}
$$

$$
G_8 = \begin{pmatrix}
\cos x & \sin x & 0 & y \\
-\sin x & \cos x & 0 & z \\
z\cos x + y\sin x & z\sin x - y\cos x & 1 & w \\
0 & 0 & 0 & 1
\end{pmatrix}, \tag{10}
$$

where $v, w, x, y, z \in \mathbb{R}$. These matrix groups and their automorphisms represent some of generalized affine symmetric spaces of infinite order.

**Theorem 4.1.** *The matrix Lie groups $G_6$, $G_8$ and $G_{10}$ admit a hypercomplex structure with Hermitian-Norden metrics of types $(hc4.1)$, $(hc3.2)$ and $(hc5.1)$, respectively.*

**Proof.** Substituting $x = z = w = 0$ and $y = d$ in $G_6$, we obtain that $G_6$ can be considered as a hypercomplex manifold with Hermitian-Norden metrics of type $(hc4.1)$, bearing in mind Theorem 3.2 for $(hc4.1)$ $(t = 1$, $u = a = b = c = 0)$.

Respectively, using Theorem 3.2 for $(hc3.2)$ $(t = 1$, $u = a = b = 0$, $c = -d = -1)$ and (10) with $y = z = w = 0$, $x = \frac{3\pi}{2} + 2k\pi$ $(k \in \mathbb{Z})$, we have that $G_8$ can be considered as a hypercomplex manifold with Hermitian-Norden metrics of type $(hc3.2)$.

In a similar way, by means of Theorem 3.2 for $(hc5.1)$ $(t = 1$, $u = a = c = d = 0)$ and substituting $y = z = w = v = 0$ and $x = b$ in $G_{10}$, we have that $G_{10}$ can be considered as a hypercomplex manifold with Hermitian-Norden metrics of type $(hc5.1)$. ∎

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Hristo Manev, Medical University Plovdiv, Faculty of Public Health, Department of Medical Informatics, Biostatistics and e-Learning, Plovdiv 4002, Bulgaria
hristo.manev@mu-plovdiv.bg