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The Length and Depth of Real Algebraic Groups

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Abstract. Let G be a connected real algebraic group. An unrefinable chain of G is a chain of subgroups $G = G_0 > G_1 > ... > G_t = 1$ where each G_i is a maximal connected real subgroup of G_{i-1}. The maximal (respectively, minimal) length of such an unrefinable chain is called the length (respectively, depth) of G. We give a precise formula for the length of G, which generalises results of Burness, Liebeck and Shalev on complex algebraic groups and also on compact Lie groups. If G is simple then we bound the depth of G above and below, and in many cases we compute the exact value. In particular, the depth of any simple G is at most 9.

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Key Words: Length, depth, real algebraic groups.

1. Introduction

A real algebraic group is *connected* if it connected in the Zariski topology. Let G be a connected real algebraic group. An *unrefinable* chain of length t of G is a chain of real subgroups $G = G_0 > G_1 > ... > G_t = 1$ where each G_i is a maximal connected real subgroup of G_{i-1}. The *length* $l(G)$ (resp. *depth* $\lambda(G)$) of G is the maximal (resp. minimal) length of such an unrefinable chain. The corresponding notions for connected complex algebraic groups are denoted by $l_{\mathbb{C}}$ and $\lambda_{\mathbb{C}}$. Let $G(\mathbb{C})$ denote the complexification of G and let $R(G)$ be the radical of G.

In this paper we study the length and depth of real algebraic groups. These invariants were first introduced for finite groups in the 1960s (see [5] and the references therein for a comprehensive summary). More recently, length and depth have been introduced and studied for algebraic groups over algebraically closed fields in [3] and for compact real Lie groups in [4]. In this paper we generalise the latter results by looking at all real algebraic groups.

In Theorem 1.1 we obtain a precise formula for the length of any connected reductive real algebraic group. In addition, we bound the depth of any ($\mathbb{R}$-)simple real algebraic group in Theorem 1.2. To prove Theorems 1.1 and 1.2 we use the exact values for the length and depth of any simple complex algebraic group and any simple compact Lie group as computed in [3] and [4] respectively.

Henceforth let G be a connected reductive real algebraic group. The semisimple quotient $G/R(G)$ is $\mathbb{R}$-isomorphic to the derived subgroup G'. By Corollary 7.11

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of [11], we can decompose G as a commuting product $\big(\prod_{i=1}^{m} G_i\big)T^k$ where each G_i is simple and T^k is a torus of dimension k. The *rank* $r(G)$ of G is the dimension of a maximal torus of G and the *real rank* $r_{\mathbb{R}}(G)$ of G is the dimension of a maximal split real torus of G. The *semisimple rank* and the *semisimple real rank* of G refer to the quantities $r(G')$ and $r_{\mathbb{R}}(G')$ respectively. Up to conjugacy, there exists a unique maximal compact subgroup K of G and a unique compact form G_c of $G(\mathbb{C})$ (refer to §2.1.1).

Let S be a maximal split real torus of G and let T be a maximal real torus of G that contains S. Let Φ be the root system of G with respect to T, let Φ_0 be the subsystem of Φ that vanishes on S and let $W := W(\Phi)$. Let Δ be a base of Φ and let Φ^+ be the corresponding subset of positive roots of Φ. Let Δ_0 be the subset of Δ that vanishes on S. The derived subgroup $C_G(S)'$ is a compact real subgroup of G called the *(semisimple) anisotropic kernel* of G.

As described in §2.3 of [19], there is a natural action $*$ of the Galois group $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}_2$ on Δ that stabilises Δ_0. The orbits of Γ in $\Delta \setminus \Delta_0$ are called *distinguished*. The *index* $\mathcal{S}(G)$ of G is the data consisting of Δ, Δ_0 and the $*$-action of Γ on Δ. We illustrate $\mathcal{S}(G)$ using a *Tits-Satake diagram*, which is constructed by taking the Dynkin diagram of G, blackening each vertex in Δ_0 and linking all of the vertices in each Γ-orbit of Δ with a solid gray bar.

Let I be any $*$-invariant subset of Δ that contains Δ_0. Associated to I is a parabolic subsystem Φ_I of Φ with base I, a standard parabolic real subgroup P_I of G and a standard Levi real subgroup L_I of G where Φ_I is the root system of L_I with respect to T (see §14.17 and §21.11 of [1]). If $I = \Delta_0$ then $\Phi_{\Delta_0} = \Phi_0$, P_{Δ_0} and $L_{\Delta_0} = C_G(S)$ are called *minimal*.

Our arguments and results are independent of the choice of isogeny type. Theorem 1.1 gives a formula for the length of any connected reductive real algebraic group. To obtain the explicit values of this formula we need the results of [4] (see §2.2) to compute the length of the compact group $C_G(S)'$.

Theorem 1.1. *Let G be a connected reductive real algebraic group. Then*

$$l(G) = |\Phi^+| - |\Phi_0^+| + r(G) + r_{\mathbb{R}}(G') - r\big(C_G(S)'\big) + l\big(C_G(S)'\big)\ .$$

In particular, the values of $l(G)$ for $G(\mathbb{C})$ a simple complex group are given in Tables 1 *and* 2 *.*

In Theorem 1.2 we bound the depth of any simple real algebraic group G. The case where G is compact has already been done in [4]. If $G(\mathbb{C})$ is a simple complex group then the depth $\lambda_{\mathbb{C}}\big(G(\mathbb{C})\big)$ has been computed in [3]. In particular, $3 \le \lambda_{\mathbb{C}}\big(G(\mathbb{C})\big) \le 6$. The roman numeral notation used for exceptional G is standard in the literature, for example see Figure 6.2 of [11].

Theorem 1.2. *Let G be a simple real algebraic group.*
Then $\lambda_{\mathbb{C}}\big(G(\mathbb{C})\big) - 1 \le \lambda(G) \le 9$. Moreover

(i) *If G is quasisplit or compact then $\lambda(G) = \lambda_{\mathbb{C}}\big(G(\mathbb{C})\big) - 1$.*

(ii) *For G exceptional* $$\lambda(G) = \begin{cases} 3 & \text{if } G = GI,\ FI,\ EV \text{ or } EVIII \\ 4 & \text{if } G = FII,\ EI,\ EII,\ EIV \text{ or } EVI\ . \\ 5 & \text{if } G = EIII,\ EVII \text{ or } EIX \end{cases}$$

(iii) *For G classical*

- *if* $G = \mathrm{SL}_n(\mathbb{H})$ $(n > 1)$ or $\mathrm{SO}(2k+1,1)$ $(k > 3)$ *then* $\lambda(G) = 4$,
- *if* $G = \mathrm{SO}^*(2k)$ $(k \geq 4)$ *then* $4 \leq \lambda(G) \leq 6 - \zeta_k$,
- *if* $G = \mathrm{Sp}(p,q)$ $(p \geq q > 0)$ *then* $4 \leq \lambda(G) \leq 6 - \delta_{pq} - \delta_{1q}$,
- *if* $G = \mathrm{SO}(p,q)$ $(p \geq q > 0)$ *then* $\lambda(G) \leq 8 - \eta_{pq}$, *and*
- *if* $G = \mathrm{SU}(p,q)$ $(p \geq q > 0)$ *then* $\lambda(G) \leq 9 - \eta_{pq}$

where

$$\zeta_k := \begin{cases} 1 & \text{if } k \neq 7 \text{ and } k \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \eta_{pq} := \begin{cases} 3 & \text{if } p-q = 3 \text{ or } q = 1 \\ 2 & \text{if } p-q = 4 \text{ or } q = 2 \\ 0 & \text{if } q = 7 \text{ and } p \geq 14 \text{ is even} \\ 1 & \text{otherwise.} \end{cases}$$

The upper bounds of $\lambda(G)$ *for* G *classical are given in Table* 1 *and the values of* $\lambda(G)$ *for* G *exceptional are given in Table* 2*.*

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2. Preliminaries

In Section 2.1 we introduce and characterise the notion of a real form of a connected reductive complex algebraic group. We also present Komrakov's classification of reductive maximal connected subgroups of real forms of simple complex algebraic groups. In Sections 2.2 and 2.3 respectively we present some results about the length and depth of real and complex connected algebraic groups.

2.1. Real and complex algebraic groups

Let X be a complex algebraic group. A real algebraic group G is a *real form* of X if $G(\mathbb{C})$ is $\mathbb{C}$-isomorphic to X. In this paper, we consider a real form of X to be a subgroup of X.

2.1.1. Real forms

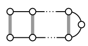
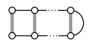
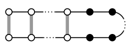
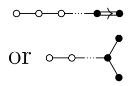
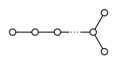
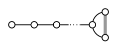
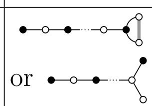
In this subsection we let X be a connected reductive complex algebraic group. There is a bijective correspondence (up to conjugacy) between real forms of X, holomorphic involutions of X and antiholomorphic involutions of X.

Proposition 2.1 (Problems 2.3.27, 3.1.9, 3.1.10 and Theorem 2.3.6 of [15])**.** *Let* G *be a real form of* X*. Then there is a unique antiholomorphic involution* σ_G *of* X *that fixes* G *pointwise. Conversely, let* σ *be an antiholomorphic involution of* X*. Then the fixed point set* X^σ *is a real form of* X*.*

Theorem 2.2 (Weyl, Theorems 5.2.8 and 5.2.9 of [15])**.** *There exists a real form of* X *that is compact as a Lie group. Any two compact real forms of* X *are conjugate as subgroups of* X*.*

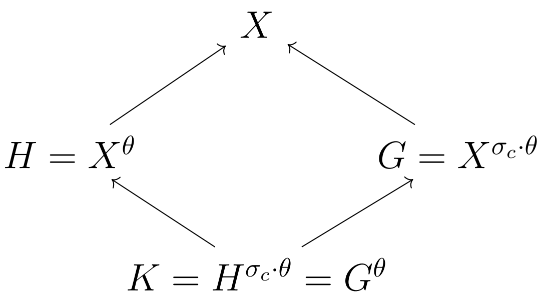
Henceforth we fix a compact real form G_c of X. Let σ_c be the unique antiholomorphic involution of X that satisfies $G_c = X^{\sigma_c}$.

Table 1: Length and depth of the classical real algebraic groups

G	$r(G)$	$r_{\mathbb{R}}(G)$	$\mathcal{S}(G)$	$C_G(S)'$	$l(G)$	K°	$\lambda(G)$
$\mathrm{SL}_{n+1}(\mathbb{R})$ $(n \geq 1)$	n	n		1	$\frac{n}{2}(n+5)$	$\mathrm{SO}(n+1)$	2 if $n=1$ 3 if $n=2$ 5 if $n=6$ 4 otherwise
$\mathrm{SL}_n(\mathbb{H})$ $(n>1)$	$2n-1$	$n-1$		$\mathrm{SL}(2)^n$	$2(n^2+n-1)$	$\mathrm{Sp}(n)$	4
$\mathrm{SU}(p,p)$ $(p>1)$	$2p-1$	p		1	$2p^2+2p-1$	$\mathrm{U}(p)\times\mathrm{SU}(p)$	4
$\mathrm{SU}(p,p-1)$ $(p>1)$	$2p-2$	$p-1$		1	$2(p^2-1)$	$\mathrm{U}(p)\times\mathrm{SU}(p-1)$	3 if $p=2$ 5 if $p=4$ 4 otherwise
$\mathrm{SU}(p,q)$ $(p>q+1)$	$p+q-1$	q		$\mathrm{SU}(p-q)$	$2(pq+p-1)$	$\mathrm{U}(p)\times\mathrm{SU}(q)$	$\leq 9-\eta_{pq}$
$\mathrm{SO}(p,q)$ $(p>q+2)$	$\lfloor\frac{p+q}{2}\rfloor$	q		$\mathrm{SO}(p-q)$	$pq+p-1$ $+\lfloor\frac{p-q}{4}\rfloor$	$\mathrm{SO}(p)\times\mathrm{SO}(q)$	$\leq 8-\eta_{pq}$
$\mathrm{SO}(p,p-1)$ $(p\geq 3)$	$p-1$	$p-1$		1	p^2-1	$\mathrm{SO}(p)\times\mathrm{SO}(p-1)$	3 if $p\neq 4$ 4 if $p=4$
$\mathrm{SO}(p,p)$ $(p\geq 4)$	p	p		1	p^2+p	$\mathrm{SO}(p)\times\mathrm{SO}(p)$	4
$\mathrm{SO}(p,p-2)$ $(p\geq 5)$	$p-1$	$p-2$		1	p^2-p-1	$\mathrm{SO}(p)\times\mathrm{SO}(p-2)$	4
$\mathrm{SO}^*(2k)$ $(k\geq 4)$	k	$\lfloor\frac{k}{2}\rfloor$		$\mathrm{SU}(2)^{\lfloor\frac{k}{2}\rfloor}$	$k^2+\lfloor\frac{k}{2}\rfloor$	$\mathrm{U}(k)$	$\leq 6-\zeta_k$
$\mathrm{Sp}_{2n}(\mathbb{R})$ $(n>1)$	n	n		1	$n(n+2)$	$\mathrm{U}(n)$	3
$\mathrm{Sp}(p,q)$ $(p\geq q)$	$p+q$	q		$\mathrm{SU}(2)^q\times$ $\mathrm{Sp}(p-q)$	$4pq+3p-1$ $+\delta_{pq}$	$\mathrm{Sp}(p)\times\mathrm{Sp}(q)$	$\leq 6-\delta_{pq}-\delta_{1q}$ 3 if $q=0$

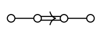
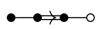
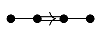
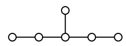
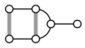
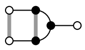
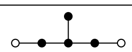
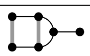
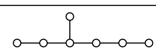
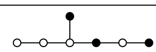
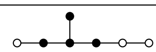
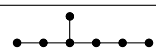
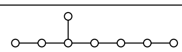
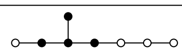
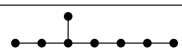
Theorem 2.3 (Cartan, Theorems 5.1.4 and 5.2.3 of [15]). *Any holomorphic involution of X has a conjugate θ that commutes with σ_c. The map $\theta \mapsto \sigma_c\cdot\theta$ defines a bijection from the set of* $\mathrm{Aut}(X)$*-conjugacy classes of holomorphic involutions of X to the set of* $\mathrm{Aut}(X)$*-conjugacy classes of antiholomorphic involutions of X.*

So let θ be a holomorphic involution of X that commutes with σ_c and denote $H := X^\theta$. Then θ stabilises the real form $G := X^{\sigma_c\cdot\theta}$ of X and $G^\theta = H^{\sigma_c\cdot\theta} =: K$ is a real form of H. We illustrate this in the following commutative diagram.



By Theorem 5.3.3 of [15], K is a maximal compact subgroup of G and any maximal compact subgroup of G is conjugate to K.

Table 2: Length and depth of the exceptional real algebraic groups

G	$r(G)$	$r_{\mathbb{R}}(G)$	$\mathcal{S}(G)$	$C_G(S)'$	$l(G)$	K°	$\lambda(G)$
GI	2	2		1	10	$(A_1)_c^2$	3
$(G_2)_c$	2	0		$(G_2)_c$	5	$(G_2)_c$	3
FI	4	4		1	32	$(C_3A_1)_c$	3
FII	4	1		$(B_3)_c$	24	$(B_4)_c$	4
$(F_4)_c$	4	0		$(F_4)_c$	11	$(F_4)_c$	3
EI	6	6		1	48	$(C_4)_c$	4
EII	6	4		1	46	$(A_5A_1)_c$	4
$EIII$	6	2		$(A_3)_c$	41	$(D_5)_c\mathbb{T}$	5
EIV	6	2		$(D_4)_c$	37	$(F_4)_c$	4
$(E_6)_c$	6	0		$(E_6)_c$	13	$(E_6)_c$	4
EV	7	7		1	77	$(A_7)_c$	3
EVI	7	4		$(A_1)_c^3$	74	$(D_6A_1)_c$	4
$EVII$	7	3		$(D_4)_c$	66	$(E_6)_c\mathbb{T}$	5
$(E_7)_c$	7	0		$(E_7)_c$	17	$(E_7)_c$	3
$EVIII$	8	8		1	136	$(D_8)_c$	3
EIX	8	4		$(D_4)_c$	125	$(E_7A_1)_c$	5
$(E_8)_c$	8	0		$(E_8)_c$	20	$(E_8)_c$	3

If H has maximal rank in X then G is an *inner* form of X. Otherwise, G is an *outer* form of X. These definitions are consistent with those in §2.2.4 of [16], if we take the compact real form G_c to be the basepoint of the pointed set of real forms of X.

Proposition 2.4 (§2.2.4 of [16]). *There exists a unique (up to conjugacy) split form G_s of X.*

2.1.2. Maximal connected subgroups of simple real algebraic groups

Any complex algebraic group X can be considered as a real algebraic group $X_{\mathbb{R}}$ of twice the dimension in a process called *realification* (see §2.3.5 of [15]). The complexification of $X_{\mathbb{R}}$ is X^2.

Proposition 2.5 (*Theorem* 5.1.1 *of [15]*). *Let G be a simple real algebraic group. Then either $G = X_{\mathbb{R}}$ for some simple complex group X or G is a real form of a simple complex group.*

The following result of Komrakov is taken from Tables $3-62$ of [12]. However, this source contains a few minor errors, which we correct using Theorem 1 of [18].

Theorem 2.6 (Komrakov, [12]). *Let G be a real form of a simple complex algebraic group such that G is neither split nor compact. Let M be a reductive maximal connected real subgroup of G. If G is classical then the possibilities for*

$M < G$ are listed up to conjugacy in $\mathrm{Aut}(G)$ *in Table 3 except for the cases where $M(\mathbb{C})$ is a simple group that acts irreducibly on the natural module of $G(\mathbb{C})$. If G is exceptional then the possibilities for $M < G$ are listed up to conjugacy in* $\mathrm{Aut}(G)$ *in Table* 5 *in Section* 3.

Table 3: Reductive maximal connected subgroups of non-split classical real algebraic groups

G	M
$\mathrm{SU}(p,q)$ $(p \geq q)$	$\mathrm{SU}(p_1,q_1) \times \mathrm{SU}(p_2,q_2) \times \mathbb{T}$ for $p = p_1 + p_2$, $q = q_1 + q_2$, $(p_1+q_1)(p_2+q_2) \neq 0$ $\mathrm{SU}(p_1,q_1) \otimes \mathrm{SU}(p_2,q_2)$ where $p = p_1q_2 + p_2q_1$ and $q = p_1p_2 + q_1q_2$ $\mathrm{SL}_n(\mathbb{C})_{\mathbb{R}}$ where $p = n(n+1)/2$ and $q = n(n-1)/2$
$\mathrm{SL}_n(\mathbb{H})$ $(n \geq 2)$	$\mathrm{SL}_n(\mathbb{C})_{\mathbb{R}} \times \mathbb{T}$ $\mathrm{SL}_{n_1}(\mathbb{R}) \otimes \mathrm{SL}_{n_2}(\mathbb{H})$ where $n = n_1n_2$
$\mathrm{SO}(p,q)$ $(p > q)$	$\mathrm{SO}(p_1,q_1) \times \mathrm{SO}(p_2,q_2)$ for $p = p_1 + p_2$, $q = q_1 + q_2$, $(p_1+q_1)(p_2+q_2) \neq 0$ $\mathrm{SU}(p/2,q/2) \times \mathbb{T}$ where p and q are even $\mathrm{SO}(p_1,q_1) \otimes \mathrm{SO}(p_2,q_2)$ where $p = p_1q_2 + p_2q_1$ and $q = p_1p_2 + q_1q_2$ $\mathrm{Sp}(p_1,q_1) \otimes \mathrm{Sp}(p_2,q_2)$ where $p = 4(p_1q_2 + p_2q_1)$ and $q = 4(p_1p_2 + q_1q_2)$ $\mathrm{SO}_n(\mathbb{C})_{\mathbb{R}}$ where $p = n(n+1)/2$ and $q = n(n-1)/2$ $\mathrm{Sp}_{2n}(\mathbb{C})_{\mathbb{R}}$ where $p = n(2n+1)$ and $q = n(2n-1)$
$\mathrm{SO}^*(2n)$ $(n \geq 3)$	$\mathrm{SO}^*(2n_1) \times \mathrm{SO}^*(2n_2)$ where $n = n_1 + n_2$ $\mathrm{SO}_n(\mathbb{C})_{\mathbb{R}}$ $\mathrm{SU}(p,q) \times \mathbb{T}$ where $p + q = n$ $\mathrm{Sp}_{2m}(\mathbb{R}) \otimes \mathrm{Sp}(p,q)$ where $n = 2m(p+q)$ $\mathrm{SO}^*(2m) \otimes \mathrm{SO}(p,q)$ where $n = m(p+q)$
$\mathrm{Sp}(p,q)$ $(p \geq q)$	$\mathrm{Sp}(p_1,q_1) \times \mathrm{Sp}(p_2,q_2)$ for $p = p_1 + p_2$, $q = q_1 + q_2$, $(p_1+q_1)(p_2+q_2) \neq 0$ $\mathrm{SU}(p,q) \times \mathbb{T}$ $\mathrm{Sp}(p_1,q_1) \otimes \mathrm{SO}(p_2,q_2)$ where $p = p_1q_2 + p_2q_1$ and $q = p_1p_2 + q_1q_2$ $\mathrm{Sp}_{2n_1}(\mathbb{R}) \otimes \mathrm{SO}^*(2n_2)$ for $p = q = n_1n_2$ $\mathrm{Sp}_{2n}(\mathbb{C})_{\mathbb{R}}$ where $p = q = n$

The following observation is trivial but useful.

Remark 2.7. Let M be a proper real subgroup of a real algebraic group G. Then $M(\mathbb{C})$ is a proper complex subgroup of $G(\mathbb{C})$.

Lemma 2.8. *Let X be a connected reductive complex algebraic group and let H be a connected reductive complex subgroup of X. Let G be a real form of X and let M be a real form of H that is contained in G. If H is maximal connected in X then M is maximal connected in G.*

Proof. Let O be a connected proper real subgroup of G that contains M. Then $O(\mathbb{C})^\circ$ is a connected proper subgroup of X that contains H by Remark 2.7 and so $O(\mathbb{C})^\circ = H$ by maximality. That is, O is a real form of (a finite extension of) H that contains M. Hence $O = M$ (again by Remark 2.7). ∎

Note that the converse to Lemma 2.8 does not hold. For example, there is a maximal connected copy of $\mathrm{PGL}_2(\mathbb{R}) \times G_2(\mathbb{C})_{\mathbb{R}}$ that is contained in the split form of E_8 and yet $A_1(G_2)^2 < F_4G_2 < E_8$ as complex groups.

Proposition 2.9. *Let $G = X_{\mathbb{R}}$ for some simple complex group X. A subgroup M of G is maximal connected if and only if M is maximal parabolic, M is a real form of X or $M = H_{\mathbb{R}}$ for some reductive maximal connected complex subgroup H of X.*

Proof. Let M be a non-parabolic maximal connected real subgroup of G. Then M is reductive by Corollary 3.3 of [2]. It follows from Proposition 2.1 that there exists a unique antiholomorphic involution σ of $G(\mathbb{C}) \cong X^2$ that satisfies $G = G(\mathbb{C})^\sigma$ and $M = M(\mathbb{C})^\sigma$. Let $(x, y) \in G(\mathbb{C}) \cong X^2$. Then $\sigma(x, y) = \big(\sigma_0(y), \sigma_0(x)\big)$ for some antiholomorphic involution σ_0 of X where X^{σ_0} is compact. Let θ be the holomorphic involution of $G(\mathbb{C})$ that sends $(x, y) \mapsto (y, x)$. Then $G(\mathbb{C})^\theta \cong X$ is diagonally embedded in $G(\mathbb{C})$. Observe that θ commutes with σ and hence θ stabilises both G and M.

Let O be a connected reductive σ-stable proper complex subgroup of $G(\mathbb{C})$ that contains $M(\mathbb{C})$. Taking fixed points under σ gives us $M \leq O^\sigma < G$ and so $O^\sigma = M$ by maximality. Then $O = M(\mathbb{C})$ since O^σ is a real form of O (this would fail if O was parabolic). That is, $M(\mathbb{C})$ is maximal among connected reductive σ-stable complex subgroups of $G(\mathbb{C})$. Then either $M(\mathbb{C}) \cong X$ is diagonally embedded in $G(\mathbb{C})$ or $M(\mathbb{C}) \cong H^2$ for some reductive maximal connected complex subgroup H of X where σ acts on $M(\mathbb{C})$ by swapping the two copies of H. Hence either M is a real form of X or $M = H_{\mathbb{R}}$.

Conversely, if M is a real form of X then $M(\mathbb{C})$ is a diagonally embedded maximal connected subgroup of $G(\mathbb{C})$ and hence M is maximal connected in G by Lemma 2.8. If $M = H_{\mathbb{R}}$ for $H < X$ as stated above then M cannot be contained in a parabolic subgroup of G nor in a real form of X. So again M is maximal connected in G. ∎

Proposition 2.10. *Let G be a simple real algebraic group and let K be a maximal compact subgroup of G. Then K° is a maximal connected subgroup of G.*

Proof. Recall the setup from §2.1.1. Let σ_G be the unique antiholomorphic involution of $G(\mathbb{C})$ that satisfies $G = G(\mathbb{C})^{\sigma_G}$. Let θ be a holomorphic involution of $G(\mathbb{C})$ that commutes with σ_G and that acts on G with fixed points $G^\theta = K$. Denote $H := G(\mathbb{C})^\theta$.

By Proposition 2.5, either $G = X_{\mathbb{R}}$ for some simple complex group X or $G(\mathbb{C})$ is a simple complex group. If $G(\mathbb{C})$ is a simple complex group then we consider two subcases: when H° is maximal connected in G and when H° is not maximal connected in G.

We first consider the case where $G = X_{\mathbb{R}}$ for some simple complex group X. Recall that θ acts on $G(\mathbb{C}) = X^2$ by swapping the two copies of X and so K is isomorphic to the compact form of X. Hence K is maximal connected in G by Proposition 2.9.

Henceforth assume that $G(\mathbb{C})$ is a simple complex group. Let Φ be the root system of $G(\mathbb{C})$ (with respect to some maximal torus), let Δ be a base of Φ and let α_0 be the highest root of Φ.

If H° is maximal connected in G then the result follows from Lemma 2.8 since K° is a real form of H°.

It remains to consider the subcase where H° is not maximal connected in G. We use Table 4.3.1 of [9] (which classifies involutions of simple complex groups) and Theorem

19.1 of [14] (which classifies maximal connected subgroups of simple complex groups) to check that H° is conjugate to a standard Levi subgroup L_I of $G(\mathbb{C})$ where $I = \Delta \setminus \{\alpha\}$ for some simple root α with a coefficient of 1 in α_0. That is, we can take $H^\circ < G(\mathbb{C})$ to be one of the following maximal rank subgroups.

$G(\mathbb{C})$	A_n	B_n	C_n	D_n	D_n	E_6	E_7
H°	$A_k A_{n-k-1} T^1$	$B_{n-1} T^1$	$A_{n-1} T^1$	$A_{n-1} T^1$	$D_{n-1} T^1$	$D_5 T^1$	$E_6 T^1$

Assume (for a contradiction) that M is a connected proper real subgroup of G that strictly contains K°. Then $M(\mathbb{C})^\circ$ is a connected proper subgroup of maximal rank in $G(\mathbb{C})$ that strictly contains H° by Remark 2.7. But H° is maximal among connected reductive subgroups of $G(\mathbb{C})$ by Corollary 13.7 and Theorem 13.12 of [14] (Borel, de Siebenthal). So $M(\mathbb{C})$ is conjugate to P_I.
Finally, we observe that K° contains a compact real torus T_c of dimension $r(G)$ since H° has maximal rank in $G(\mathbb{C})$. Then T_c is contained in some Levi subgroup $\mathcal{L} = \mathcal{L}' Z(\mathcal{L})^\circ \cong M/R_u(M)$ of M where $Z(\mathcal{L})^\circ \cong \mathbb{R}^\times$ is split. But this is a contradiction as $r(\mathcal{L}') < r(G)$. ■

2.2. Length

Theorem 2.11 (Theorem 1 of [3]). *Let G be a connected reductive complex algebraic group. Let B be a Borel subgroup of G. Then $l_{\mathbb{C}}(G) = \dim(B) + r(G') = |\Phi^+| + r(G) + r(G')$.*

Lemma 2.12 (additivity). *Let G be a connected real or complex algebraic group with $G/R(G) = \prod_{i=1}^{m} G_i$ where each G_i is simple. Then $l(G) = \dim\big(R(G)\big) + \sum_{i=1}^{m} l(G_i)$.*

Proof. First observe that $l\big(R(G)\big) = \dim\big(R(G)\big)$ by Lemma 2.2 of [3] (which holds over both $\mathbb{R}$ and $\mathbb{C}$) since $R(G)$ is soluble. The result then follows from Lemma 2.1(ii) of [3] as $G/R(G)$ is semisimple. ■

We denote the compact real form of $\mathrm{Sp}_{2n}(\mathbb{C})$ by $\mathrm{Sp}(n)$ (not $\mathrm{Sp}(2n)$ as in [4]).

Theorem 2.13 (Theorem 1 of [4]). *The length of each compact simple Lie group G is as follows.*

G	$\mathrm{SU}(n)$	$\mathrm{Sp}(n)$	$\mathrm{SO}(n)$	G_2	F_4	E_6	E_7	E_8
$l(G)$	$2n-2$	$3n-1$	$n + \lfloor \frac{n}{4} \rfloor - 1$	5	11	13	17	20

In the following lemma we find the lower bound for $l(G)$ as stated in Theorem 1.1. We use notation that is taken from the introduction.

Lemma 2.14. *Let G be a connected reductive real algebraic group. Then $l(G) \geq \Lambda_G$ where*

$$\Lambda_G := |\Phi^+| - |\Phi_0^+| + r(G) + r_{\mathbb{R}}(G') - r\big(C_G(S)'\big) + l\big(C_G(S)'\big).$$

In particular, if G is split then $C_G(S)'$ is trivial and so $l(G) \geq |\Phi^+| + r(G) + r(G')$.

Proof. We construct an unrefinable chain

$$G > P_{\Delta\backslash\mathcal{O}_1} > P_{\Delta\backslash(\mathcal{O}_1\cup\mathcal{O}_2)} > \ldots > P_{\Delta_0}$$

of parabolic real subgroups of G by removing distinguished orbits $\mathcal{O}_j$ (one by one, in any order) from Δ until we attain Δ_0. This chain has length $r_{\mathbb{R}}(G')$ and $l(P_{\Delta_0}) = \dim\big(R(P_{\Delta_0})\big) + l\big(C_G(S)'\big)$ by Lemma 2.12.

The $\mathbb{R}$-dimension of any real algebraic group is equal to the $\mathbb{C}$-dimension of its complexification. So $\dim\big(C_G(S)'\big) = |\Phi_0| + r\big(C_G(S)'\big)$ by Theorem 8.17(b) of [14] and $\dim(P_{\Delta_0}) = |\Phi^+ \cup \Phi_0| + r(G) = |\Phi^+| + |\Phi_0^+| + r(G)$. Hence

$$\dim\big(R(P_{\Delta_0})\big) = \dim(P_{\Delta_0}) - \dim\big(C_G(S)'\big) = |\Phi^+| - |\Phi_0^+| + r(G) - r\big(C_G(S)'\big)$$

and we are done. ■

Proposition 2.15. *Let G be a non-compact connected reductive real algebraic group where G' is non-trivial. Let G_c (resp. G_s) denote the compact (resp. split) form of $G(\mathbb{C})$. Then $l(G_c) < l(G) \leq l(G_s) = l_{\mathbb{C}}\big(G(\mathbb{C})\big)$.*

Proof. Let M be a maximal connected real subgroup of G. Then $M(\mathbb{C})^\circ$ is a connected (but not necessarily maximal) proper complex subgroup of $G(\mathbb{C})$ by Remark 2.7. By complexifying an unrefinable chain of G of maximal length we observe that $l(G) \leq l_{\mathbb{C}}\big(G(\mathbb{C})\big)$. But $l_{\mathbb{C}}\big(G(\mathbb{C})\big) = |\Phi^+| + r(G) + r(G') \leq l(G_s)$ by Theorem 2.11 and Lemma 2.14 and so $l(G) \leq l(G_s) = l_{\mathbb{C}}\big(G(\mathbb{C})\big)$.

For the lower bound, we write G as a commuting product $\big(\prod_{i=1}^m G_i\big)T^k$ where $m \geq 1$, each G_i is simple and T^k is a torus of dimension k. Then $l(G) = k + \sum_{i=1}^m l(G_i)$ by Lemma 2.12. So to show that $l(G) > l(G_c)$ for any non-compact G, it suffices to consider only the cases where G is simple.

We first consider the cases where G is a non-compact real form of a simple classical complex group. For each case we use Theorem 2.13, Lemma 2.14 and Table 1 to check that $l(G) \geq \Lambda_G > l(G_c)$.

Let G be a non-compact real form of $\mathrm{SL}_n(\mathbb{C})$. Then

$$l(G) \geq \Lambda_G = \begin{cases} \frac{(n-1)(n+4)}{2} & \text{if } G = \mathrm{SL}_n(\mathbb{R}) \\ 2(k^2+k-1) & \text{if } G = \mathrm{SL}_k(\mathbb{H}) \ (n = 2k > 2) \\ 2(pq+p-1) + \delta_{pq} & \text{if } G = \mathrm{SU}(p,q) \ \binom{p \geq q > 0,}{p+q = n} \end{cases}$$

and $\Lambda_G > 2n-2 = l\big(\mathrm{SU}(n)\big)$. Similarly, let G be a non-compact real form of $\mathrm{SO}_n(\mathbb{C})$. Then

$$l(G) \geq \Lambda_G = \begin{cases} k^2 + \lfloor \frac{k}{2} \rfloor & \text{if } G = \mathrm{SO}^*(2k) \ (n = 2k \geq 8) \\ pq+p-1+\delta_{pq} + \lfloor \frac{p-q}{4} \rfloor & \text{if } G = \mathrm{SO}(p,q) \ \binom{p \geq q > 0,}{p+q = n} \end{cases}$$

and $\Lambda_G > n + \lfloor \frac{n}{4} \rfloor - 1 = l\big(\mathrm{SO}(n)\big)$. If G is a non-compact real form of $\mathrm{Sp}_{2n}(\mathbb{C})$ then

$$l(G) \geq \Lambda_G = \begin{cases} n(n+2) & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}) \\ 4pq+3p-1+\delta_{pq} & \text{if } G = \mathrm{Sp}(p,q) \ \binom{p \geq q > 0,}{p+q = 2n} \end{cases}$$

and $\Lambda_G > 3n-1 = l\big(\mathrm{Sp}(n)\big)$.

Next, we consider the cases where G is a non-compact real form of an exceptional complex group. Once again we use Theorem 2.13 and Lemma 2.14 to check that $l(G) \geq \Lambda_G > l(G_c)$, where the values of Λ_G can be found in Table 2.

Finally, let $G = X_{\mathbb{R}}$ where X is a simple complex group with root system Φ_X. Then $G_c = (X_c)^2$ where X_c denotes the compact form of X . It is easy to check that $2l(X_c) \leq l_{\mathbb{C}}(X) + r(X)$ for any simple complex X using Theorems 2.11 and 2.13. Hence

$$l(G) \geq \Lambda_G = 2|\Phi_X^+| + 3r(X) > |\Phi_X^+| + 3r(X) = l_{\mathbb{C}}(X) + r(X) \geq 2l(X_c) = l(G_c)$$

by Theorem 2.11 and Lemma 2.14. ■

2.3. Depth

The first lemma is stated in [3] over $\mathbb{C}$, but the proof also works over $\mathbb{R}$.

Lemma 2.16 (Lemma 2.5 of [3]). *Let $G = NH$ be a connected real or complex algebraic group, where N and H are non-trivial connected proper subgroups of G and N is normal in G. Then $\lambda(G) \geq \lambda(H) + 1$.*

Corollary 2.17. *Let $G = \left(\prod_{i=1}^m G_i\right)T^k$ be a connected reductive real or complex algebraic group where each G_i is simple and T^k is a torus of dimension k. If $m = 0$ then $\lambda(G) = k$. If $m \geq 1$ then*

$$\lambda(G) = \lambda(G') + k \geq \max_{i=1,\ldots,m} \left\{\lambda(G_i)\right\} + m - 1 + k\,.$$

Proof. It follows immediately from Lemma 2.16 that $\lambda(G) = \lambda(G') + k$. Now assume that $m \geq 1$. We show that

$$\lambda(G') \geq \max_{i=1,\ldots,m} \left\{\lambda(G_i)\right\} + m - 1$$

by induction on m. If $m = 1$ then we are done. Let $i_0 \leq m$ be a positive integer that satisfies $\lambda(G_{i_0}) \geq \lambda(G_i)$ for all $1 \leq i \leq m$. If $m > 1$ then take some $j \neq i_0$ and so

$$\lambda(G') \geq \lambda(G'/G_j) + 1 \geq \max_{i=1,\ldots,m} \left\{\lambda(G_i)\right\} + m - 1$$

by Lemma 2.16 and the inductive hypothesis. ■

Corollary 2.18. *Let $G = (G_0)^m$ where G_0 is a simple real or complex algebraic group and $m \geq 1$. Then $\lambda(G) = \lambda(G_0) + m - 1$.*

Proof. We induct on m. If $m = 2$ then there exists a diagonally embedded copy of G_0 that is maximal connected in G and so $\lambda(G) \leq \lambda(G_0) + 1$. If $m > 2$ then $\lambda(G) \leq \lambda(G/G_0) + 1 = \lambda(G_0) + m - 1$ by the inductive hypothesis. Then we are done by Corollary 2.17. ■

Theorem 2.19 (Theorem 4 of [3]). *Let G be a simple complex algebraic group. Then*

$$\lambda_{\mathbb{C}}(G) = \begin{cases} 3 & \text{if } G = A_1 \\ 5 & \text{if } G = A_r \ (r \geq 3, r \neq 6), B_3, D_r \text{ or } E_6 \\ 6 & \text{if } G = A_6 \\ 4 & \text{in all other cases} \end{cases}.$$

Theorem 2.20 (Theorem 6 of [4]). *Let G be a compact simple real algebraic group. Then $\lambda(G) = \lambda_{\mathbb{C}}\big(G(\mathbb{C})\big) - 1$.*

3. Proof of Theorem 1.1

Let G be a connected reductive real algebraic group. We already have the lower bound

$$l(G) \geq |\Phi^+| - |\Phi_0^+| + r(G) + r_{\mathbb{R}}(G') - r\big(C_G(S)'\big) + l\big(C_G(S)'\big) =: \Lambda_G$$

from Lemma 2.14. So to prove Theorem 1.1 it suffices to check that $l(M) < \Lambda_G$ for every maximal connected subgroup M of G. Our proof is by induction on $l(G)$ and we compute $l(M)$ using Lemma 2.12 (*additivity*), the inductive hypothesis and Tables 1 and 2.

An outline of the proof is as follows. We first let M be a maximal parabolic subgroup of G and check that $l(M) < \Lambda_G$ by applying the inductive hypothesis to the (reductive) Levi subgroup of M. Then we show that it suffices to consider only the cases where G is simple (and neither split nor compact) and M is reductive. Finally, we apply the inductive hypothesis to check that $l(M) < \Lambda_G$ for each of the following cases:

- Case (A): $G = X_{\mathbb{R}}$ for some simple complex group X.
- Case (B): $G(\mathbb{C}) = \mathrm{SL}_d(\mathbb{C})$ for $d \geq 3$ and $M(\mathbb{C}) = \mathrm{SO}_d(\mathbb{C})$ or $\mathrm{Sp}_d(\mathbb{C})$ (if d is even).
- Case (C): All remaining cases where $G(\mathbb{C})$ is a simple classical group and $M(\mathbb{C})$ is a simple group that acts irreducibly on the natural module of $G(\mathbb{C})$.
- Case (D): All remaining M in classical G (which are listed in Table 3).
- Case (E): All remaining M in exceptional G (which are listed in Table 5).

Proof. Let M be a maximal connected (real) subgroup of G. The complexification $M(\mathbb{C})^\circ$ is a connected (but not necessarily maximal) proper complex subgroup of $G(\mathbb{C})$. If G is split then $l(M) \leq l_{\mathbb{C}}\big(M(\mathbb{C})\big) < l_{\mathbb{C}}\big(G(\mathbb{C})\big) = l(G)$ by Proposition 2.15. The case where G is compact has been done in [4]. So we may assume that G is neither split nor compact. In particular, G is not of type A_1.

We first consider the case where M is a parabolic subgroup of G. Let $M = P_I$ where $I = \Delta \setminus \mathcal{O}$ for some distinguished orbit $\mathcal{O}$ of $\Delta \setminus \Delta_0$. The anisotropic kernel of $(L_I)'$ is $C_G(S)'$ and the root system is Φ_I. We compute

$$\begin{aligned}
l(P_I) &= \dim\big(R(P_I)\big) + l\big((L_I)'\big) && \text{(Lemma 2.12)}\\
&= |\Phi^+| - |\Phi_I^+| + r(G) - r\big((L_I)'\big) + l\big((L_I)'\big) && \text{(pf. of Lemma 2.14)}\\
&= |\Phi^+| - |\Phi_0^+| + r(G) + r_{\mathbb{R}}\big((L_I)'\big) - r\big(C_G(S)'\big) + l\big(C_G(S)'\big) && \text{(induct on } l\big((L_I)'\big))\\
&\leq l(G) - r_{\mathbb{R}}(G') + r_{\mathbb{R}}\big((L_I)'\big) && \text{(Lemma 2.14)}\\
&\leq l(G) - 1.
\end{aligned}$$

Henceforth, by Proposition 21.12 of [1], we can assume that M is not parabolic. Then M is reductive by Corollary 3.3 of [2]. Recall that we can decompose G

as a commuting product $\left(\prod_{i=1}^{m} G_i\right)T^a$ where each G_i is simple and T^a is a torus of dimension a. Since M is maximal connected in G, one of the following three possibilities must occur.

(i) If $M = \left(\prod_{i=1}^{m} G_i\right)T^{a-1}$ then $l(M) = l(G) - 1$ by Lemma 2.12.

(ii) If $M = \left(\prod_{i\neq j} G_i\right)M_jT^a$ for some j (where M_j is a reductive maximal connected subgroup of G_j) then $l(M) = l(G) - l(G_j) + l(M_j)$ again by Lemma 2.12.

(iii) Otherwise, if $M = \left(\prod_{i\neq j,k} G_i\right)G_j^D T^a$ for some $j \neq k$ such that G_j is isogenous to G_k (where $G_j^D \cong G_j$ is embedded diagonally in G_jG_k) then $l(M) = l(G) - l(G_j) < l(G)$ by Lemma 2.12.

To see this, consider the projection of M to each of the factors G_i for $i = 1, ..., m$ and T^a. If M projects non-surjectively to some factor then it is of type (i) or (ii). Otherwise, the intersection of M with $\prod_{i=1}^{m} G_i$ is a product of diagonal subgroups and so, by maximality, M is of type (iii).

So to show $l(M) < l(G)$ it suffices to consider only the cases where G is a simple real group.

Case A: We first assume that $G = X_{\mathbb{R}}$ for some simple complex group X.

Let Φ_X be the root system of X. Then Φ is the union of two perpendicular root systems of type Φ_X. Applying Lemma 2.14 gives us $l(G) \geq |\Phi^+| + r(G) + r_{\mathbb{R}}(G) = 2|\Phi_X^+| + 3r(X)$. Recall that M is a non-parabolic maximal connected subgroup of G. By Proposition 2.9, either $M = H_{\mathbb{R}}$ for some reductive maximal connected complex subgroup H of X (with root system Φ_H) or M is a real form of X. If $M = H_{\mathbb{R}}$ then

$$|\Phi_H^+| + r(H) + r(H') = l_{\mathbb{C}}(H) < l_{\mathbb{C}}(X) = |\Phi_X^+| + 2r(X) \tag{1}$$

by Theorem 2.11 and so

$$\begin{aligned} l(M) &= 2|\Phi_H^+| + 2r(H) + r(H') && \text{(inductive hypothesis)} \\ &< 2|\Phi_X^+| + 3r(X) && \text{(Equation 1)} \\ &\leq l(G). && \text{(Lemma 2.14)} \end{aligned}$$

Similarly, if M is a real form of X then

$$l(M) \leq l_{\mathbb{C}}(X) = |\Phi_X^+| + 2r(X) < 2|\Phi_X^+| + 3r(X) \leq l(G).$$

Henceforth, by Proposition 2.5, we can assume that G is a real form of a simple complex group.

Case B: We next consider the cases where $G(\mathbb{C}) = \mathrm{SL}_d(\mathbb{C})$ for $d \geq 3$ and $M(\mathbb{C}) = \mathrm{SO}_d(\mathbb{C})$ or $\mathrm{Sp}_d(\mathbb{C})$ (if d is even).

A non-split real form of $\mathrm{SL}_d(\mathbb{C})$ is isomorphic to either $\mathrm{SL}_k(\mathbb{H})$ (if $d = 2k$) or to $\mathrm{SU}(p,q)$ for some $p+q = d$. A real form of $\mathrm{SO}_d(\mathbb{C})$ is isomorphic to either $\mathrm{SO}^*(2k)$ (if $d = 2k$) or to $\mathrm{SO}(p,q)$ for some $p+q = d$. Similarly, a real form of $\mathrm{Sp}_d(\mathbb{C})$ is isomorphic to either $\mathrm{Sp}_d(\mathbb{R})$ or to $\mathrm{Sp}(p,q)$ for some $p+q = d/2$.

We first assume that $G = \mathrm{SL}_k(\mathbb{H})$ for $d = 2k$. Then $l(G) \geq 2(k^2 + k - 1)$ by Lemma 2.14 and Table 1. By Proposition 2.15, the length of M is maximised when

M is split in which case $l(M) = l_{\mathbb{C}}\big(M(\mathbb{C})\big)$. Using Theorem 2.11, we check that $l_{\mathbb{C}}\big(\mathrm{SO}_{2k}(\mathbb{C})\big) = k^2 + k < 2(k^2 + k - 1)$ and $l_{\mathbb{C}}\big(\mathrm{Sp}_{2k}(\mathbb{C})\big) = k^2 + 2k < 2(k^2 + k - 1)$. Hence $l(M) \leq l_{\mathbb{C}}\big(M(\mathbb{C})\big) < l(G)$ for all possible $M < G$, as required.

Now let $G = \mathrm{SU}(p, q)$ for $p \geq q$ and $p + q \geq 3$. Then $l(G) \geq 2(pq + p - 1) + \delta_{pq}$ by Lemma 2.14 and Table 1. In lieu of precise knowledge about which real forms of $M(\mathbb{C})$ embed in G (which would take some work to establish) we check that $l(M) < l(G)$ for all real forms M of $M(\mathbb{C})$ that satisfy $r_{\mathbb{R}}(M) \leq r_{\mathbb{R}}(G) = q$.

If $M = \mathrm{SO}(p', q')$ for $p' \geq q'$ and $p' + q' = p + q$ then $q' = r_{\mathbb{R}}(M) \leq r_{\mathbb{R}}(G) = q$. So

$$\begin{aligned} l(M) &= p'q' + p' - 1 + \delta_{p'q'} + \lfloor (p' - q')/4 \rfloor && \text{(inductive hypothesis, Table 1)} \\ &< 2(p'q' + p' - 1) + \delta_{p'q'} \leq 2(pq + p - 1) + \delta_{pq} \\ &\leq l(G). && \text{(Lemma 2.14, Table 1)} \end{aligned}$$

Next let $M = \mathrm{Sp}(p', q')$ for $p' \geq q'$ and $2(p' + q') = p + q$. Then $q' = r_{\mathbb{R}}(M) \leq q$, $p \leq 2p' + q'$ and

$$l(M) = 4p'q' + 3p' - 1 + \delta_{p'q'} < 2\big((2p' + q')q' + (2p' + q') - 1\big) + \delta_{p'q'} \leq 2(pq + p - 1) + \delta_{pq}$$

by the inductive hypothesis and Lemma 2.14.

If we take $M = \mathrm{Sp}_{2k}(\mathbb{R})$ for $p + q = 2k$ then $k = r_{\mathbb{R}}(M) \leq q$ and so $p = q = k$.

Then $$l(M) = k^2 + 2k < 2k^2 + 2k - 1 \leq l(G)$$

by the inductive hypothesis and Lemma 2.14.

Finally, if $M = \mathrm{SO}^*(2k)$ for $p + q = 2k$ then $\lfloor k/2 \rfloor = r_{\mathbb{R}}(M) \leq q$, $p \leq \lceil 3k/2 \rceil$ and

$$l(M) = k^2 + \lfloor k/2 \rfloor < 2\big(\lceil 3k/2 \rceil \lfloor k/2 \rfloor\big) + \lfloor (p + q)/4 \rfloor \leq 2pq + p/2 \leq 2(pq + p - 1) + \delta_{pq}$$

by the inductive hypothesis and Lemma 2.14.

Case C: Now consider the (remaining) cases where $G(\mathbb{C})$ is a classical group and $M(\mathbb{C})$ is a simple group that acts irreducibly on the natural module of $G(\mathbb{C})$.

Let V be a complex vector space of dimension $d > 1$ equipped with either the zero form or a non-degenerate bilinear form (symmetric or skew-symmetric). Let $G(\mathbb{C}) = \mathrm{Cl}(V)$ be the group of isometries of V with determinant 1. Let $M(\mathbb{C})$ be a connected simple complex group that acts irreducibly on V but that is not isomorphic to either $\mathrm{SO}_d(\mathbb{C})$, $\mathrm{Sp}_d(\mathbb{C})$ or to $\mathrm{SL}_d(\mathbb{C})$. Let λ_V be the highest weight of V as an irreducible $M(\mathbb{C})$-module. For any given $M(\mathbb{C})$, Lemmas 3.5 and 3.6 of [4] give a lower bound $N = N\big(M(\mathbb{C})\big)$ for the dimension d of V.

Table 4: The values of $N = N\big(M(\mathbb{C})\big)$ for $M(\mathbb{C})$ a simple complex algebraic group

$M(\mathbb{C})$	$\mathrm{SL}_k(\mathbb{C})$				$\mathrm{Sp}_{2k}(\mathbb{C})$		$\mathrm{SO}_k(\mathbb{C})$		G_2	F_4	E_6	E_7	E_8
	$k=$ 2	3	4	$k > 4$	$k=2$	$k > 2$	$7 \leq k \leq 14$, $k \neq 8$	$k = 8$, $k > 14$					
N	4	6	10	$\frac{k}{2}(k-1)$	10	$\frac{k}{2}(k-1)-1$	$2^{\lfloor \frac{k-1}{2} \rfloor}$	$\frac{k}{2}(k-1)$	7	26	27	56	248

Assume (for a contradiction) that $l(M) \geq l(G)$. For a given $M(\mathbb{C})$, by Proposition 2.15, the length of M is maximised when M is split in which case $l(M) = l_{\mathbb{C}}\big(M(\mathbb{C})\big)$.

Similarly, by Proposition 2.15 and Theorem 2.13, the length of G is minimised when $G = \mathrm{SO}(N)$ in which case $l(G) = N + \lfloor N/4 \rfloor - 1$. So we can exclude all $M(\mathbb{C})$ that satisfy the inequality $l_{\mathbb{C}}\big(M(\mathbb{C})\big) < N + \lfloor N/4 \rfloor - 1$. For example, if $M(\mathbb{C}) = E_8(\mathbb{C})$ then $N(M) = 248$ and $l(M) \le l_{\mathbb{C}}\big(E_8(\mathbb{C})\big) = 136 < 248 + \lfloor 248/4 \rfloor - 1 = 309 \le l(G)$ using Theorem 2.11. Using Table 4, this argument excludes the cases where $M(\mathbb{C})$ is isogenous to $\mathrm{SL}_n(\mathbb{C})$ for $n \ge 17$, $\mathrm{Sp}_n(\mathbb{C})$ for $n \ge 4$, $\mathrm{SO}_n(\mathbb{C})$ for $n \ne 7, 9, 10, 12$ or $E_8(\mathbb{C})$.

We first assume that $G(\mathbb{C}) = \mathrm{SO}_d(\mathbb{C})$ or $\mathrm{Sp}_d(\mathbb{C})$ which, by Lemma 78 of [17], occurs if and only if $\lambda_V = -w_0\lambda_V$ (where w_0 denotes the longest element of the Weyl group W). By Tables $A.6 - A.53$ of [13], if $\lambda_V = -w_0\lambda_V$ and $l_{\mathbb{C}}\big(M(\mathbb{C})\big) \ge d + \lfloor d/4 \rfloor - 1 = l\big(\mathrm{SO}(d)\big)$ then $(M(\mathbb{C}), d)$ must be one of $(A_5, 20)$, $(B_3, 8)$, $(B_4, 16)$, $(D_6, 32)$, $(G_2, 7)$, $(F_4, 26)$ or $(E_7, 56)$.

If $(M(\mathbb{C}), d)$ is $(A_5, 20)$, $(D_6, 32)$ or $(E_7, 56)$ then $G(\mathbb{C}) = \mathrm{Sp}_d(\mathbb{C})$ by Lemma 79 of [17]. We can exclude each of these three possibilities using Theorems 2.11, 2.13 and Proposition 2.15 since $l_{\mathbb{C}}\big(A_5(\mathbb{C})\big) = 25 < 29 = l\big(\mathrm{Sp}(10)\big)$, $l_{\mathbb{C}}\big(D_6(\mathbb{C})\big) = 42 < 47 = l\big(\mathrm{Sp}(16)\big)$ and $l_{\mathbb{C}}\big(E_7(\mathbb{C})\big) = 77 < 83 = l\big(\mathrm{Sp}(28)\big)$. If $(M(\mathbb{C}), d)$ is $(B_3, 8)$, $(B_4, 16)$, $(G_2, 7)$ or $(F_4, 26)$ then $G(\mathbb{C}) = \mathrm{SO}_d(\mathbb{C})$ by Lemma 79 of [17]. Since G is not compact, G is isomorphic to either $\mathrm{SO}^*(d)$ (if d is even) or to $\mathrm{SO}(p, q)$ for some $p \ge q > 0$ satisfying $p + q = d$. Observe that

$$l(G) \ge \min\big\{(d/2)^2 + \lfloor d/4 \rfloor, pq + p - 1 + \lfloor (p-q)/4 \rfloor\big\} \ge 2d - 3$$

by Lemma 2.14. If $(M(\mathbb{C}), d)$ is $(B_4, 16)$, $(G_2, 7)$ or $(F_4, 26)$ then we use Theorem 2.11 to check that $l_{\mathbb{C}}\big(M(\mathbb{C})\big) < 2d - 3$. Hence $l(M) \le l_{\mathbb{C}}\big(M(\mathbb{C})\big) < l(G)$ for all possible $M < G$ by Proposition 2.15, as required. It remains to consider the case where $(M(\mathbb{C}), d) = (B_3, 8)$. If $M = \mathrm{SO}(6,1)$ or $\mathrm{SO}(7)$ then we check that $l(M) < 2d - 3 = 13$ using the inductive hypothesis and Table 1. If $M = \mathrm{SO}(4,3)$ or $\mathrm{SO}(5,2)$ then $2 \le r_{\mathbb{R}}(M) \le r_{\mathbb{R}}(G)$ and so G must be either $\mathrm{SO}(6,2)$, $\mathrm{SO}(5,3)$ or $\mathrm{SO}(4,4)$, but then $l(M) \le l_{\mathbb{C}}\big(M(\mathbb{C})\big) = 15 < l(G)$ by Theorem 2.11 and Lemma 2.14.

We now assume that $G(\mathbb{C}) = \mathrm{SL}_d(\mathbb{C})$ and that $M(\mathbb{C})$ satisfies the tightened inequality $l_{\mathbb{C}}\big(M(\mathbb{C})\big) \ge 2d - 2 = l\big(\mathrm{SU}(d)\big)$. By Tables $6.6 - 6.53$ of [13], the only possibility for $(M(\mathbb{C}), d)$ is $(A_4, 10)$. We use Theorems 2.11, 2.13 and Proposition 2.15 (since G is not compact) to check that $l_{\mathbb{C}}\big(A_4(\mathbb{C})\big) = 18 = l\big(\mathrm{SU}(10)\big) < l(G)$. We have our contradiction.

Case D: In (i) to (v) we consider the remaining non-parabolic maximal connected real subgroups M of classical real G (which are listed in Table 3) and check that $l(M) < l(G)$. Recall that Lemma 2.14 gives a lower bound for $l(G)$ and that we can compute $l(M)$ using Lemma 2.12, the inductive hypothesis and Tables 1 and 2.

(i): $G = \mathrm{SU}(p, q)$ $(p \ge q,\ p + q \ge 2)$, where $l(G) \ge 2(pq + p - 1) + \delta_{pq}$ by Lemma 2.14 and Table 1.

Let $M = \mathrm{SU}(p_1, q_1) \times \mathrm{SU}(p_2, q_2) \times \mathbb{T}$, where we have $p = p_1 + p_2$, $q = q_1 + q_2$ and $(p_1 + q_1)(p_2 + q_2) \ne 0$. We compute

$$\begin{aligned} l(M) &= l\big(\mathrm{SU}(p_1, q_1)\big) + l\big(\mathrm{SU}(p_2, q_2)\big) + l(\mathbb{T}) && \text{(Lemma 2.12, } \textit{additivity}\text{)} \\ &= 2(p_1q_1 + p_1 - 1) + \delta_{p_1q_1} + 2(p_2q_2 + p_2 - 1) + \delta_{p_2q_2} + 1 && \text{(ind. hyp., Table 1)} \\ &< 2(pq + p - 1) + \delta_{pq} \le l(G)\,. && \text{(Lemma 2.14, Table 1)} \end{aligned}$$

Similarly, let $M = \mathrm{SU}(p_1, q_1) \otimes \mathrm{SU}(p_2, q_2)$, where $p = p_1q_2 + p_2q_1$ and $q = p_1p_2 + q_1q_2$ for non-negative integers $p_1 \geq q_1$ and $q_2 \geq p_2$ that satisfy $p_1+q_1 \geq 2$ and $p_2+q_2 \geq 2$.

Then
$$\begin{aligned} l(M) &= 2(p_1q_1 + p_1 - 1) + \delta_{p_1q_1} + 2(p_2q_2 + q_2 - 1) + \delta_{p_2q_2} \\ &< 2(pq + p - 1) + \delta_{pq}\,. \end{aligned}$$

Finally, let $M = \mathrm{SL}_n(\mathbb{C})_{\mathbb{R}}$ where $p = n(n+1)/2$, $q = n(n-1)/2$ and $n > 1$. Then

$$l(M) = (n+3)(n-1) < 2(pq + p - 1)\,.$$

(ii): $G = \mathrm{SL}_n(\mathbb{H})$ $(n \geq 2)$, where $l(G) \geq 2(n^2+n-1)$ by Lemma 2.14 and Table 1. If $M = \mathrm{SL}_n(\mathbb{C})_{\mathbb{R}} \times \mathbb{T}$ then $l(M) = (n+3)(n-1) + 1 < 2(n^2 + n - 1)$.
Let $M = \mathrm{SL}_{n_1}(\mathbb{R}) \otimes \mathrm{SL}_{n_2}(\mathbb{H})$ where $n = n_1n_2$ and $n_1 > 1$. Then

$$l(M) = (n_1 - 1)(n_1 + 4)/2 + 2(n_2^2 + n_2 - 1) < 2(n^2 + n - 1)\,.$$

(iii): $G = \mathrm{SO}(p, q)$ $(p > q,\ p + q \geq 5)$, where $l(G) \geq pq + p - 1 + \lfloor (p - q)/4 \rfloor$ by Lemma 2.14 and Table 1.
First we let $M = \mathrm{SO}(p_1, q_1) \times \mathrm{SO}(p_2, q_2)$ where $p = p_1 + p_2$, $q = q_1 + q_2$, $p_1 > q_1$ and $p_2 + q_2 \neq 0$. We compute

$$\begin{aligned} l(M) &= p_1q_1 + p_1 - 1 + \lfloor (p_1 - q_1)/4 \rfloor + p_2q_2 + \max\{p_2, q_2\} - 1 + \delta_{p_2q_2} + \lfloor |p_2 - q_2|/4 \rfloor \\ &< pq + p - 1 + \lfloor (p - q)/4 \rfloor \end{aligned}$$

which holds even if $p_2 < q_2$. If $M = \mathrm{SU}(p/2, q/2) \times \mathbb{T}$ where p and q are both even and $p + q \geq 4$ then

$$l(M) = (pq)/2 + p - 2 + 1 < pq + p - 1 + \lfloor (p - q)/4 \rfloor\,.$$

Now consider $M = \mathrm{SO}(p_1, q_1) \otimes \mathrm{SO}(p_2, q_2)$ where $p = p_1q_2 + p_2q_1$ and $q = p_1p_2 + q_1q_2$ for non-negative integers $p_1 > q_1$ and $q_2 > p_2$ satisfying $p_1 \geq 2$ and $q_2 \geq 2$. Then

$$\begin{aligned} l(M) &= p_1q_1 + p_1 - 1 + \lfloor (p_1 - q_1)/4 \rfloor + p_2q_2 + q_2 - 1 + \lfloor (q_2 - p_2)/4 \rfloor \\ &< pq + p - 1 + \lfloor (p_1 - q_1)(q_2 - p_2)/4 \rfloor \\ &= pq + p - 1 + \lfloor (p - q)/4 \rfloor\,. \end{aligned}$$

Next, let $M = \mathrm{Sp}(p_1, q_1) \otimes \mathrm{Sp}(p_2, q_2)$ where $p = 4(p_1q_2 + p_2q_1)$ and $q = 4(p_1p_2 + q_1q_2)$ for non-negative integers $p_1 > q_1$ and $q_2 > p_2$. Then

$$\begin{aligned} l(M) &= (4p_1q_1 + 3p_1 - 1) + (4p_2q_2 + 3q_2 - 1) \\ &\leq 4\big(p_1q_1(p_2^2 + q_2^2) + p_2q_2(p_1^2 + q_1^2)\big) + (4p_1q_2 + 2) - 2 \\ &< pq + p + \lfloor (p - q)/4 \rfloor - 1\,. \end{aligned}$$

Let $M = \mathrm{SO}_n(\mathbb{C})_{\mathbb{R}}$ where $p = n(n+1)/2$, $q = n(n-1)/2$ and $n > 1$. Then

$$l(M) \leq n(n+1)/2 < (n^4 + n^2 + 2n - 4)/4 + \lfloor n/4 \rfloor = pq + p - 1 + \lfloor (p - q)/4 \rfloor\,.$$

Finally, let $M = \mathrm{Sp}_{2n}(\mathbb{C})_{\mathbb{R}}$ where $p = n(2n+1)$ and $q = n(2n-1)$. Then

$$l(M) = n(2n+3) < 4n^4 + 3n^2 + n - 1 + \lfloor n/2 \rfloor = pq + p - 1 + \lfloor (p - q)/4 \rfloor\,.$$

(iv): $G = \mathrm{SO}^*(2n)$ $(n \geq 3)$, where $l(G) \geq n^2 + \lfloor n/2 \rfloor$ by Lemma 2.14 and Table 1.

First we take $M = \mathrm{SO}^*(2n_1) \times \mathrm{SO}^*(2n_2)$ where $n = n_1 + n_2$ for positive integers n_1 and n_2. We compute $l(M) = n_1^2 + \lfloor n_1/2 \rfloor + n_2^2 + \lfloor n_2/2 \rfloor < n^2 + \lfloor n/2 \rfloor$.

If $M = \mathrm{SO}_n(\mathbb{C})_\mathbb{R}$ for $n > 1$ then $l(M) \leq n(n+1)/2 < n^2 + \lfloor n/2 \rfloor$.

Now if $M = \mathrm{SU}(p,q) \times \mathbb{T}$ where $p + q = n$ then $l(M) \leq 2(pq + p) < n^2 + \lfloor n/2 \rfloor$.

Let $M = \mathrm{Sp}_{2m}(\mathbb{R}) \otimes \mathrm{Sp}(p,q)$ where $n = 2m(p+q)$ and $p \geq q$. Then

$$\begin{aligned} l(M) &= m(m+2) + 4pq + 3p - 1 + \delta_{pq} \leq (2m^2+1) + 4pq + 2p^2 + \delta_{pq} \\ &\leq 2\big(m^2 + (p+q)^2\big) + 1 + \delta_{pq} < n^2 + \lfloor n/2 \rfloor . \end{aligned}$$

Finally, let $M = \mathrm{SO}^*(2m) \otimes \mathrm{SO}(p,q)$ where $n = m(p+q)$ and $p + q \geq 2$. Then

$$\begin{aligned} l(M) &= m^2 + \lfloor m/2 \rfloor + pq + p - 1 + \delta_{pq} + \lfloor (p-q)/4 \rfloor \\ &< (m^2 + (p+q)^2 - 1) + \lfloor n/2 \rfloor \leq n^2 + \lfloor n/2 \rfloor . \end{aligned}$$

(v): $G = \mathrm{Sp}(p,q)$ $(p \geq q)$, where $l(G) \geq 4pq + 3p - 1 + \delta_{pq}$ by Lemma 2.14 and Table 1.

First let $M = \mathrm{Sp}(p_1,q_1) \times \mathrm{Sp}(p_2,q_2)$, where we have $p = p_1 + p_2$, $q = q_1 + q_2$ and $(p_1+q_1)(p_2+q_2) \neq 0$. We compute

$$l(M) = (4p_1q_1 + 3p_1 - 1 + \delta_{p_1q_1}) + (4p_2q_2 + 3p_2 - 1 + \delta_{p_2q_2}) < 4pq + 3p - 1 + \delta_{pq} .$$

If $M = \mathrm{SU}(p,q) \times \mathbb{T}$ then $l(M) = 2(pq + p - 1) + \delta_{pq} + 1 < 4pq + 3p - 1 + \delta_{pq}$.

Now let $M = \mathrm{Sp}(p_1,q_1) \otimes \mathrm{SO}(p_2,q_2)$ where $p = p_1q_2 + p_2q_1$ and $q = p_1p_2 + q_1q_2$ for non-negative integers $p_1 \geq q_1$ and $q_2 \geq p_2$ satisfying $q_2 \geq 2$. Then

$$\begin{aligned} l(M) &= 4p_1q_1 + 3p_1 - 1 + \delta_{p_1q_1} + p_2q_2 + q_2 - 1 + \delta_{p_2q_2} + \lfloor (q_2 - p_2)/4 \rfloor \\ &< 4p_1q_1 + p_2q_2 + 3p_1 - 2 + \delta_{p_1q_1} + \delta_{p_2q_2} + \big(3q_2 - 2\big) \\ &\leq 4(p_1q_1 + p_2q_2) + 3(p_1 + q_2 - 1) - 1 + \delta_{p_1q_1} \leq 4pq + 3p - 1 + \delta_{pq} . \end{aligned}$$

Next, let $M = \mathrm{Sp}_{2n_1}(\mathbb{R}) \otimes \mathrm{SO}^*(2n_2)$ for $p = q = n_1n_2$. Then

$$l(M) = n_1(n_1+2) + n_2^2 + \lfloor n_2/2 \rfloor < 4n_1^2n_2^2 + 3n_1n_2 = 4p^2 + 3p .$$

Finally, let $M = \mathrm{Sp}_{2n}(\mathbb{C})_\mathbb{R}$ where $p = q = n$. Then $l(M) = n(2n+3) < 4p^2 + 3p$.

Case E: In Table 5 we consider the remaining non-parabolic maximal connected real subgroups M of exceptional real G (which are taken from Tables $4-62$ of [12]) and check that $l(M) < l(G)$. Recall that Lemma 2.14 gives a lower bound for $l(G)$ and that we can compute $l(M)$ using Lemma 2.12, the inductive hypothesis and Tables 1 and 2.

This completes the proof of Theorem 1.1. ■

Table 5: Reductive maximal connected subgroups of non-split, non-compact exceptional real groups

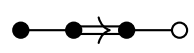
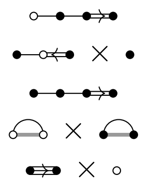
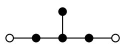
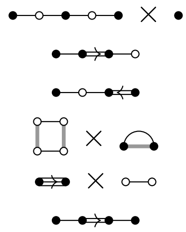
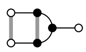
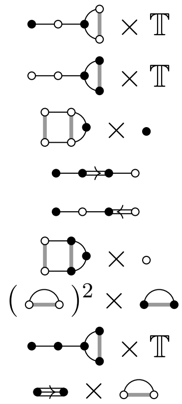
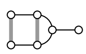
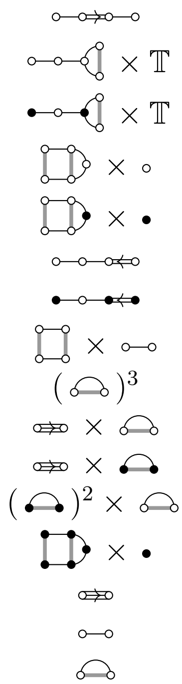
G	$\mathcal{S}(G)$	$l(G) \geq$	$\mathcal{S}(M)$	$l(M)$
FII		24		16 15 10 10 8
EIV		37		24 24 20 16 12 11
$EIII$		41		28 25 24 24 22 21 16 12 11
EII		46		32 30 28 26 24 24 20 19 18 16 14 14 12 10 7 6

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Table 5 – continued from previous page

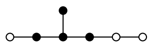
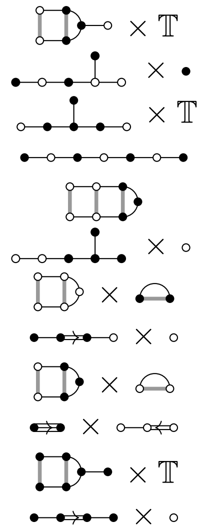
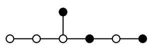
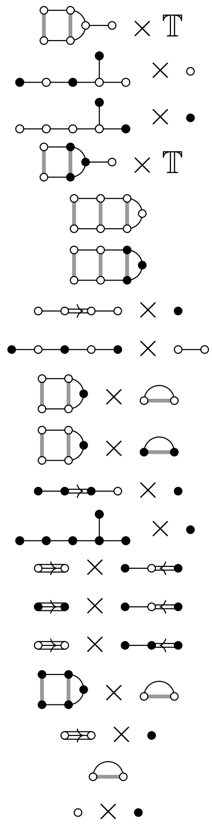
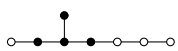
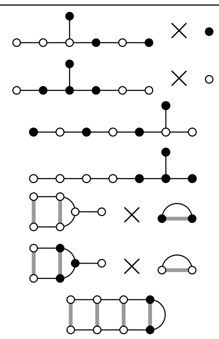
G	$\mathcal{S}(G)$	$l(G)\geq$	$\mathcal{S}(M)$	$l(M)$
$EVII$		66		42 41 38 38 34 34 27 27 24 20 14 14
EVI		74		47 42 42 42 39 34 34 29 28 26 26 24 23 18 18 16 12 6 5
EIX		125		76 69 68 61 50 47 46

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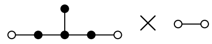
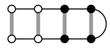
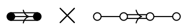
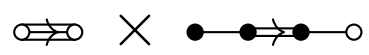
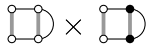
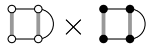
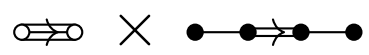
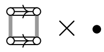
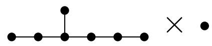
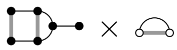
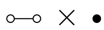
Table 5 – continued from previous page

G	$\mathcal{S}(G)$	$l(G) \geq$	$\mathcal{S}(M)$	$l(M)$
				44
				40
				37
				34
				30
				24
				21
				20
				19
				19
				9

4. Proof of Theorem 1.2

Let G be a simple real algebraic group. If G is compact then the depth $\lambda(G)$ has already been computed in [4]. An outline of the proof of Theorem 1.2 is as follows.

Throughout the proof we use the results in §2.1.2 which classify the maximal connected subgroups of a real algebraic group. In Lemma 4.3 we find all real algebraic groups of depth at most 2 and in Lemma 4.4 we find all semisimple real algebraic groups of depth 3. We then show that various simple real algebraic groups have depth 4 and 5 respectively in Lemmas 4.5 and 4.6.

In Lemma 4.7 we consider the case where $G(\mathbb{C})$ is not simple and show that $\lambda(G) = \lambda_{\mathbb{C}}\big(G(\mathbb{C})\big) - 1$. At this point we have proved parts (i) and (ii) of Theorem 1.2. We then use Theorem 2.6 and Proposition 2.10 to construct unrefinable chains for the classical groups G given in part (iii) of Theorem 1.2. This gives us an upper bound for $\lambda(G)$ and proves part (iii).

Finally, we check that $\lambda_{\mathbb{C}}\big(G(\mathbb{C})\big) - 1 \leq \lambda(G) \leq 9$ holds for any simple real algebraic group G.

Proof. Let X be a simple complex algebraic group and let

$$\mathcal{X} := \big\{X = X_0 > X_1 > ... > X_k = H\big\}$$

be an unrefinable chain of connected reductive complex subgroups of X (where H is not necessarily the trivial group). For every i let $(X_i)_s$ be a split form of X_i. Recall from Proposition 2.4 that such an $(X_i)_s$ always exists and is unique up to conjugacy in X_i. If $(X_i)_s$ is contained in some conjugate of $(X_{i+1})_s$ for every i then we say that the chain $\mathcal{X}$ *splits.* That is, after adjusting by an appropriate set of conjugates, there exists a chain

$$\mathcal{X}_s := \big\{(X_0)_s > (X_1)_s > (X_2)_s > ... > (X_k)_s\big\}$$

of split real subgroups. Observe that $\mathcal{X}_s$ is unrefinable by Lemma 2.8.

By Dynkin's classification [7, 8], the following unrefinable complex chains exist

$$\begin{cases} A_{2k-1} > C_k > A_1 & (k \geq 2) \\ A_{2k} > B_k > A_1 & (k \geq 4 \text{ or } k = 2) \\ A_6 > B_3 > G_2 > A_1 & \\ D_4 > A_2 > A_1 & \\ D_k > B_{k-1} > A_1 & (k \geq 5) \\ E_6 > F_4 > A_1 & \\ E_7 > A_1 & \\ E_8 > A_1 & \end{cases}.$$

In particular, the penultimate group in each chain contains a maximal connected A_1. Consider the subchain $\mathcal{X}(X) = X > ... > A_1$ of one of the listed chains. Observe that $\mathcal{X}(X)$ has length $\lambda_{\mathbb{C}}(X) - 3$ by Theorem 2.19.

Lemma 4.1. *Let X be a simple complex group other than D_4. Then the chain $\mathcal{X}(X)$ splits.*

Proof. If $X = G_2$, E_7 or E_8 then $\mathcal{X}(X)$ splits by Tables 6, $38-39$ and $55-57$ of [12] respectively. The chain $\mathcal{X}(E_6) = E_6 > F_4 > A_1$ splits by Tables 12 and 29 of [12]. The embedding $\mathrm{SO}_{2k}(\mathbb{C}) > \mathrm{SO}_{2k-1}(\mathbb{C})$ splits into $\mathrm{SO}(k,k) > \mathrm{SO}(k,k-1)$ by definition of the indefinite orthogonal group. That is, the chain $D_k > B_{k-1}$ splits.

Now let $R = (r_{ij})$ be the $n \times n$ matrix with entries given by $r_{ij} = i$ if $j = i+1$ and $r_{ij} = 0$ otherwise. Similarly, let $S = (s_{ij})$ be the $n \times n$ matrix with entries given by $s_{ij} = n-j$ if $j = i-1$ and $s_{ij} = 0$ otherwise. In addition, let $P = (p_{ij})$ be a $n \times n$ antidiagonal matrix with non-zero entries that satisfy $ip_{i(n-i+1)} + (n-i)p_{(i+1)(n-i)} = 0$ for all $1 \leq i \leq n-1$. For example, if $n = 3$ then P is a scalar multiple of $\left(\begin{smallmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{smallmatrix}\right)$. Note that P is symmetric if n is odd and skew-symmetric if n is even.

Let $V = \mathbb{C}^n$ be equipped with a non-degenerate bilinear form $\mathcal{P}$ that corresponds to the matrix P. Henceforth let $X = \mathrm{SL}_n(\mathbb{C})$ and let $Y = \{A \in X \,|\, A^\top P A = P\}$ be the subgroup of X that preserves $\mathcal{P}$. Let $H = A_1$ be the irreducibly embedded complex subgroup of X that is generated by the matrices $\exp(tR)$ and $\exp(tS)$ for all $t \in \mathbb{C}$. We check that $R^\top P + PR = 0 = S^\top P + PS$ and hence $H < Y$ by Lemma 11.2.2 of [6]. Let σ be the antiholomorphic involution of X that sends $A \mapsto \overline{A}$. Then σ stabilises the chain $X > Y > H$ with fixed points

$$\begin{cases} \mathrm{SL}_{2k+1}(\mathbb{R}) > \mathrm{SO}(k+1,k) > \mathrm{PGL}_2(\mathbb{R}) & \text{if } n = 2k+1 \geq 3 \\ \mathrm{SL}_{2k}(\mathbb{R}) > \mathrm{Sp}_{2k}(\mathbb{R}) > \mathrm{SL}_2(\mathbb{R}) & \text{if } n = 2k \geq 4 \end{cases}.$$

That is, the chains $A_{2k} > B_k > A_1$ and $A_{2k-1} > C_k > A_1$ both split. If $n = 7$ then there exists a σ-stable copy of G_2 in Y that contains H. Hence the chain $A_6 > B_3 > G_2 > A_1$ splits since the only non-compact real form of G_2 is the split form. ■

Lemma 4.2. *Let G be a quasisplit simple real algebraic group. Then $\lambda(G) \leq \lambda_{\mathbb{C}}\big(G(\mathbb{C})\big) - 1$.*

Proof. We first consider the case where $G = X_{\mathbb{R}}$ for some simple complex group X. Recall that $G(\mathbb{C}) \cong X^2$ and so $\lambda_{\mathbb{C}}\big(G(\mathbb{C})\big) = \lambda_{\mathbb{C}}(X) + 1$ by Corollary 2.18. The maximal compact subgroup of G is isomorphic to the compact form X_c of X. So $\lambda(G) \leq \lambda(X_c) + 1 = \lambda_{\mathbb{C}}(X) = \lambda_{\mathbb{C}}\big(G(\mathbb{C})\big) - 1$ by Proposition 2.10 and Theorem 2.20.

Henceforth we can assume that $G(\mathbb{C})$ is a simple complex group. The maximal compact subgroup of $(A_1)_s$ is a compact torus $\mathbb{T}$. So the real chain $(A_1)_s > \mathbb{T} > 1$ is unrefinable by Proposition 2.10. If G is split and not of type D_4 then there exists an unrefinable chain

$$G > ... > (A_1)_s > \mathbb{T} > 1$$

of length $\lambda_{\mathbb{C}}\big(G(\mathbb{C})\big) - 1$ by Lemma 4.1. If G is split and of type D_4 then the chain

$$\mathrm{SO}(4,4) > \mathrm{SU}(2,1) > \mathrm{SO}(2,1) > \mathbb{T} > 1$$

is unrefinable by Table 19 of [12] and Lemma 2.8.

It remains to consider the cases where G is quasisplit but not split. If G is of type A_{2k} for $k \geq 1$ (resp. A_{2k-1} for $k \geq 2$, D_k for $k \geq 5$, D_4 or E_6) then by Lemmas 2.8 and 4.1 it suffices to check that G contains a split form of B_k (resp. C_k, B_{k-1}, A_2 or F_4).

For any k the embeddings $\mathrm{SU}(k+1,k) > \mathrm{SO}(k+1,k)$ and $\mathrm{SO}(k+1,k-1) > \mathrm{SO}(k,k-1)$ follow immediately from their definitions. The embeddings $\mathrm{SO}(5,3) > \mathrm{SL}_3(\mathbb{R})$ and $EII > FI$ are given in Tables 12 and 19 of [12] respectively.

Finally, let $Y = \big\{A \in \mathrm{SL}_{2k}(\mathbb{C}) \,\big|\, A^{\top}QA = Q\big\} \cong \mathrm{Sp}_{2k}(\mathbb{C})$ for $k \geq 2$ where $Q := \left(\begin{smallmatrix} 0 & I_k \\ -I_k & 0 \end{smallmatrix}\right)$ and I_k denotes the k by k identity matrix. Let $s := \mathrm{diag}\big(i^{(k)}, (-i)^{(k)}\big) \in Y$. The antiholomorphic involution $A \mapsto s\big(\overline{A}^{\top}\big)^{-1}s^{-1}$ of $\mathrm{SL}_{2k}(\mathbb{C})$ stabilises Y with fixed points $\mathrm{SU}(k,k) > \mathrm{Sp}_{2k}(\mathbb{R})$. ■

The following Lemma is similar to Lemma 2.3 of [3].

Lemma 4.3. *Let G be a real algebraic group. Then $\lambda(G) = 1$ if and only if $\dim(G) = 1$ and $\lambda(G) = 2$ if and only if $\dim(G) = 2$ or $G(\mathbb{C}) = A_1$.*

Proof. The case where $\lambda(G) = 1$ is obvious. So assume that $\lambda(G) = 2$. If G is soluble then $\dim(G) = 2$ by Lemma 2.2 of [3]. If G is insoluble then $G' \neq 1$ and $\lambda(G') \geq 2$. Applying Lemma 2.16 to the decomposition $G = R(G) \cdot G'$ implies that $\lambda(G') = 2$ and $R(G) = 1$.

If G is compact then $r(G) = 1$ by Theorems 2.19, 2.20 and Corollary 2.17. So we can assume that G is not compact. Let M be a maximal connected subgroup of G of dimension 1. If M is unipotent then M is strictly contained in a parabolic subgroup of G, a contradiction. If M is a torus then M is contained in a maximal torus T of G with $\dim(T) = r(G)$ and hence $r(G) = 1$.

There are two real forms of A_1 up to isogeny, a split form $(A_1)_s$ and a compact form $(A_1)_c$. The chain $(A_1)_s > \mathbb{T} > 1$ is unrefinable by Proposition 2.10 and $\lambda\big((A_1)_c\big) = 2$ by Theorem 2.20. ■

Now let U_1 denote a unipotent group of dimension 1 and let T^k denote a k-dimensional torus.

Lemma 4.4. *Let G be a non-compact semisimple real algebraic group. Then $\lambda(G) = 3$ if and only if G is quasisplit and is of type $(A_1)^2$, A_2, C_r ($r \geq 2$), B_r ($r \geq 4$), G_2, F_4, E_7 or E_8.*

Proof. By Lemma 4.3 we can assume that $r(G) > 1$. Let $\lambda(G) = 3$ and let M be a maximal connected subgroup of G such that $\lambda(M) = 2$. Assume (for a contradiction) that $\dim(M) = 2$. If $M = \mathbb{C}^\times$, $(\mathbb{R}^\times)^2$ or if M is unipotent then M is strictly contained in a minimal parabolic subgroup of G. If $M = \mathbb{T}^2$ then M is strictly contained in a maximal compact subgroup of G. If $M = U_1T^1$ then $M < N_G(U_1) \leq P$ for some parabolic subgroup P of G (Borel-Tits, Theorem 2.5 of [2]). All of these cases contradict the maximality of M. Hence M is of type A_1 by Lemma 4.3.

If G is not simple then G must be of type $(A_1)^2$ where M is diagonally embedded in G by Corollary 2.17 and Lemma 4.3. Otherwise, G is simple and so either $G = X_\mathbb{R}$ for some simple complex group X or $G(\mathbb{C})$ is simple. If $G = X_\mathbb{R}$ then $X = A_1$ by Proposition 2.9 and so again G is of type $(A_1)^2$. If $G(\mathbb{C})$ is simple then $M(\mathbb{C})$ is maximal connected in $G(\mathbb{C})$ by Lemmas 2.4 and 2.7 of [12] (since M is of type A_1). Hence either $G(\mathbb{C}) = (A_1)^2$ or $G(\mathbb{C})$ is simple and contains a maximal connected copy of A_1. That is, $G(\mathbb{C})$ is one of $(A_1)^2$, A_2, C_r ($r \geq 2$), B_r ($r \geq 4$), G_2, F_4, E_7 or E_8.

Without loss of generality we may assume that $G(\mathbb{C})$ is adjoint. Observe that $C_{G(\mathbb{C})}\big(M(\mathbb{C})\big)$ is finite since $M(\mathbb{C})$ is maximal connected and $Z\big(M(\mathbb{C})\big)$ is finite. Hence all elements of $C_{G(\mathbb{C})}\big(M(\mathbb{C})\big)$ are semisimple. If $z \in C_{G(\mathbb{C})}\big(M(\mathbb{C})\big)$ is non-trivial then $M(\mathbb{C}) = C_{G(\mathbb{C})}(z)^\circ < G(\mathbb{C})$ by maximality. But $C_{G(\mathbb{C})}(z)^\circ$ contains some maximal torus of $G(\mathbb{C})$ by Theorem 14.2 of [14], which is a contradiction. Hence $C_{G(\mathbb{C})}\big(M(\mathbb{C})\big) = 1$ and in particular $M(\mathbb{C}) \cong \mathrm{PGL}_2(\mathbb{C})$.

Now let G be an inner real form of $(A_1)^2$, A_2, C_r ($r \geq 2$), B_r ($r \geq 4$), G_2, F_4, E_7 or E_8. For a given $M(\mathbb{C})$ and $G(\mathbb{C})$, it follows from Theorem 2.3 that conjugacy classes of embeddings of real forms $M < G$ are in bijection with conjugacy classes of holomorphic inner involutions θ of $G(\mathbb{C})$ that stabilise $M(\mathbb{C})$. We can ignore the case where θ is trivial as this corresponds to the case where M and G are both compact. Since $C_{G(\mathbb{C})}\big(M(\mathbb{C})\big) = 1$, the only remaining possibility for θ is conjugation by an element $s \in M(\mathbb{C})$ which has order 2. But all elements of order 2 in $M(\mathbb{C}) \cong \mathrm{PGL}_2(\mathbb{C})$ are conjugate. So θ corresponds to the case where M and G are both split by Lemma 4.1.

It remains to consider the cases where G is an outer real form. Then $\mathrm{Out}\big(G(\mathbb{C})\big)$ is non-trivial and so G can only be of type A_2 or $(A_1)^2$. The split form $(A_2)_s$ is the unique outer real form of A_2 and the unique outer real form $(A_1)_\mathbb{R}$ of $(A_1)^2$ is quasisplit. Both $(A_2)_s$ and $(A_1)_\mathbb{R}$ contain a maximal connected subgroup of type A_1 by Lemmas 4.1 and 4.2 respectively.

The converse follows from Theorem 2.19 and Lemmas 4.2 and 4.3. ∎

Lemma 4.5. *Let G be quasisplit and of type A_r ($r \geq 3, r \neq 6$), D_r ($r \geq 4$), B_3 or E_6, or let $G = \mathrm{SL}_k(\mathbb{H})$ ($k > 1$), $\mathrm{SO}(2k+1, 1)$ ($k > 3$), FII, EIV or EVI. Then $\lambda(G) = 4$.*

Proof. It suffices to show that $\lambda(G) \leq 4$ since G is a non-compact simple real group that is not of type A_1 and is not one of the groups listed in Lemma 4.4.

If G is quasisplit then $\lambda(G) \leq 4$ by Lemma 4.2 and Theorem 2.19.

If $G = EVI$ then by Table 5 there exists a maximal connected $M \cong \mathrm{PSU}(2,1)$ in G. Then $\lambda(M) = 3$ by Lemma 4.4 since M is quasisplit and of type A_2.

For all remaining cases, let K be a maximal compact subgroup of G. If $G = \mathrm{SL}_k(\mathbb{H})$ for $k > 1$ (resp. $\mathrm{SO}(2k+1,1)$ for $k > 3$, FII, EIV) then $K^\circ = (C_k)_c$ (resp. $(B_k)_c$, $(B_4)_c$, $(F_4)_c$). For each case $\lambda(K^\circ) = 3$ by Theorems 2.19 and 2.20 and hence $\lambda(G) \leq 4$ by Proposition 2.10. ■

Lemma 4.6. *If $G = EIII$, $EVII$ or EIX then $\lambda(G) = 5$. If G is non-compact of type A_6 then $\lambda(G) \geq 5$ with equality if G is quasisplit.*

Proof. We have $\lambda(G) \geq 4$ by Lemmas 4.3 and 4.4. Let M be a maximal connected subgroup of G.

If G is of type A_6 then by Table 3 of [12] either M is parabolic, M is of type $A_k A_{6-k-1}\mathbb{T}$ or M is simple and irreducibly embedded in G (in particular, M is of type A_1, G_2 or B_3). If M is simple and irreducibly embedded in G then $M(\mathbb{C})$ is maximal connected in $G(\mathbb{C})$ by Lemma 2.4 of [12], and so M must be of type B_3.

We first consider the case where M is a maximal parabolic subgroup of G. Let $M = R_u(M) \rtimes L$ where L is a Levi subgroup of M. Then $\lambda(M) \geq \lambda(L) + 1 \geq 4$ by Corollary 2.17 and Lemma 4.3 since $\dim(L) > 2$ and L is not of type A_1.

Now assume that M is reductive and $\lambda(M) = 3$. Then one of the following possibilities occurs. If M is not semisimple then $\lambda(M) \geq \lambda(M') + 1$ by Lemma 2.16 (since $R(M) \lhd M$) and so either M' is trivial or M' is of type A_1 by Lemma 4.3. If M is semisimple but not simple then $M = M_1 M_2$ where M_1 and M_2 are both simple groups of type A_1 (by Corollary 2.17 and Lemma 4.3). If M is simple then M is compact or quasisplit by Lemma 4.4. If $G = EIII$, $EVII$ or EIX then it is easy to see from Table 5 that none of these possibilities can occur. If G is of type A_6 then M must be of type B_3 but then $\lambda(M) \neq 3$ by Lemma 4.4. We have a contradiction.

It remains to show that $\lambda(G) \leq 5$ if $G = EIII$, $EVII$, EIX, $\mathrm{SL}_7(\mathbb{R})$ or $\mathrm{SU}(4,3)$. If $G = \mathrm{SL}_7(\mathbb{R})$ or $\mathrm{SU}(4,3)$ this follows from Theorem 2.19 and Lemma 4.2. For each remaining case we use Table 5 and Lemma 4.5 to find a maximal connected subgroup M of G with $\lambda(M) = 4$. If $G = EIII$ then $M = FII$ is such a subgroup and if $G = EVII$ then we take $M = \mathrm{SL}_4(\mathbb{H})$. If $G = EIX$ then let $M = \mathrm{PGL}_3(\mathbb{R}) \times \mathrm{PSU}(2)$ and observe that the chain

$$G > M = \mathrm{PGL}_3(\mathbb{R}) \times \mathrm{PSU}(2) > \big(\mathrm{PSU}(2)\big)^2 > \mathrm{PSU}(2) > \mathbb{T} > 1$$

is unrefinable by Proposition 2.10 and Corollary 2.18. ■

By combining Theorem 2.19 with Lemmas 4.3, 4.4, 4.5 and 4.6, for $G(\mathbb{C})$ a simple complex group, we have shown that $\lambda(G) \geq \lambda_{\mathbb{C}}\big(G(\mathbb{C})\big) - 1$ with equality if G is quasisplit and proved parts (i) and (ii) of Theorem 1.2. The following lemma takes care of the case where $G(\mathbb{C})$ is not simple.

Lemma 4.7. *Let $G = X_{\mathbb{R}}$ for X a simple complex group. Then $\lambda(G) = \lambda_{\mathbb{C}}\big(G(\mathbb{C})\big) - 1 = \lambda_{\mathbb{C}}(X)$.*

Proof. The case where $X = A_1$ has been done in Lemma 4.4. Recall that $G(\mathbb{C}) \cong X^2$ and so $\lambda_{\mathbb{C}}\big(G(\mathbb{C})\big) = \lambda_{\mathbb{C}}(X) + 1$ by Corollary 2.18. The upper bound $\lambda(G) \leq \lambda_{\mathbb{C}}(X)$ was shown in Lemma 4.2 since G is quasisplit. It remains to show that $\lambda(G) \geq \lambda_{\mathbb{C}}(X)$ for X a simple complex group with $r(X) > 1$. Let M be a maximal connected subgroup of G. Then one of the following three possibilities occurs by Proposition 2.9. Either M is parabolic, M is a real form of X or $M = H_{\mathbb{R}}$ for some reductive maximal connected complex subgroup H of X.

Let M be a maximal parabolic subgroup of G and recall that $r(X) > 1$. Then $M/R_u(M) =: L \cong L'\mathbb{C}^{\times}$ where $\mathbb{C}^{\times}$ is a real torus of dimension 2 and $r(L') > 1$. Hence $\lambda(M) \geq \lambda(L)+1 \geq \lambda(L')+3 \geq 6$ by Lemma 2.16, Corollary 2.17 and Lemma 4.3 since L' is not of type A_1. So $\lambda(M) \geq 6 \geq \lambda_{\mathbb{C}}(X)$ by Theorem 2.19.

If M is a real form of X then $\lambda(M) \geq \lambda_{\mathbb{C}}(X) - 1$ by combining Lemmas 4.3, 4.4 and 4.6.

Finally, let $M = H_{\mathbb{R}}$ for $H < X$ as above. Observe that H' is non-trivial since $r(X) > 1$. Hence $\lambda(M) \geq 3$ with equality if and only if $H = A_1$ by Lemmas 2.16, 4.3 and 4.4. So it suffices to consider the cases where $\lambda_{\mathbb{C}}(X) \geq 5$. If $X = A_r$ ($r \geq 3$, $r \neq 6$), D_r ($r \geq 4$), B_3 or E_6 then $\lambda_{\mathbb{C}}(X) = 5$ by Theorem 2.19 and $\lambda(M) \geq 4$ since X does not contain a maximal connected copy of A_1. By Theorem 2.19 the only remaining case is $X = A_6$, which satisfies $\lambda_{\mathbb{C}}(X) = 6$. Then $H = B_3$ by §18 of [14], and $\lambda\big((B_3)_{\mathbb{R}}\big) \geq 5$ by the preceding arguments. ■

Completion of the proof of Theorem 1.2: It remains to show the upper bounds for classical G given in part (iii) of Theorem 1.2. We use Theorem 2.6 and Proposition 2.10 to construct unrefinable chains for classical G.

Firstly, let $G = \mathrm{SO}^*(2k)$ for $k \geq 4$. By Theorem 2.6 there exists a maximal connected $M \cong \mathrm{SO}_k(\mathbb{C})_{\mathbb{R}}$ in G. The maximal compact subgroup of M is isomorphic to $\mathrm{SO}(k)$ and so $\lambda(G) \leq \lambda(M) + 1 \leq \lambda\big(\mathrm{SO}(k)\big) + 2 = 6 - \zeta_k$ by Theorem 2.20.

Next let $G = \mathrm{Sp}(p,q)$ for $p \geq q > 0$. If $q > 1$ then the chain

$$G > \mathrm{Sp}(p) \times \mathrm{Sp}(q) > \mathrm{Sp}(p) \times \mathrm{Sp}(1) > \big(\mathrm{Sp}(1)\big)^2 > \mathrm{Sp}(1) > \mathbb{T} > 1$$

is unrefinable and so $\lambda(G) \leq 6$. Now if $p > q = 1$ then

$$G > \mathrm{Sp}(p) \times \mathrm{Sp}(1) > \big(\mathrm{Sp}(1)\big)^2 > \mathrm{Sp}(1) > \mathbb{T} > 1$$

is unrefinable and so $\lambda(G) \leq 5$. If $p = q > 1$ then

$$G > \big(\mathrm{Sp}(p)\big)^2 > \mathrm{Sp}(p) > \mathrm{SU}(2) > \mathbb{T} > 1$$

is unrefinable and so again $\lambda(G) \leq 5$. Finally, if $p = q = 1$ then

$$G > \big(\mathrm{Sp}(1)\big)^2 > \mathrm{Sp}(1) > \mathbb{T} > 1$$

is unrefinable and hence $\lambda(G) = 4$ by Lemmas 4.3 and 4.4.

Now let $G = \mathrm{SU}(p,q)$ for $p \geq q > 0$. Then G contains a copy of $\mathrm{SO}(p,q)$ which is maximal connected by Lemma 2.8. Hence $\lambda(G) \leq \lambda\big(\mathrm{SO}(p,q)\big) + 1$.

It remains to consider the most complicated case, let $G = \mathrm{SO}(p,q)$ for $p \geq q > 0$. For any choice of integers satisfying $p_1 + p_2 = p$ and $q_1 + q_2 = q$, recall from Table 3 that (the connected component of) $\mathrm{SO}(p_1,q_1)\times\mathrm{SO}(p_2,q_2)$ is a maximal connected subgroup of G.

If $p-q\le 2$ then G is quasisplit and so $\lambda(G)\le 4$ by Lemma 4.2. If $p-q=3$ then $M=\mathrm{SO}(p-1,q)$ is a quasisplit maximal connected subgroup of G and so $\lambda(G)\le 5$. Similarly, if $p-q=4$ then $M=\mathrm{SO}(p-1,q)$ is a maximal connected subgroup of G with $\lambda(M)\le 5$ and so $\lambda(G)\le 6$. Note that if $p=12$ and $q=7$ then $\lambda(G)\le\lambda\big(\mathrm{SO}(11,7)\big)+1\le 7$.

If $q=0$ then G is compact and so $\lambda(G)\le 4$ by Theorem 2.20. If $q=1$ then $M=\mathrm{SO}(p)$ is a compact maximal connected subgroup of G and so $\lambda(G)\le 5$. Similarly, if $q=2$ then $M=\mathrm{SO}(p,1)$ is a maximal connected subgroup of G with $\lambda(G)\le 5$ and so $\lambda(G)\le 6$.

So henceforth we can assume that $p-q>4$ and $q>2$. In particular, $p\ne 7$.

If p is odd and $q=7$ then

$$G>\mathrm{SO}(p)\times\mathrm{SO}(7)>(A_1)_c\times\mathrm{SO}(7)>(A_1)_c\times(G_2)_c>(A_1)_c^2>(A_1)_c>\mathbb{T}>1$$

is unrefinable and so $\lambda(G)\le 7$. If p and q are odd and $q\ne 7$ then the chain

$$G>\mathrm{SO}(p)\times\mathrm{SO}(q)>(A_1)_c\times\mathrm{SO}(q)>(A_1)_c^2>(A_1)_c>\mathbb{T}>1$$

is unrefinable and so $\lambda(G)\le 6$. So if p is even and $q\ne 7$ is odd then we have $\lambda(G)\le\lambda\big(\mathrm{SO}(p-1,q)\big)+1\le 7$. Similarly, if p is odd and $q\ne 8$ is even then $\lambda(G)\le\lambda\big(\mathrm{SO}(p,q-1)\big)+1\le 7$. If p is odd and $q=8$ then

$$G>\mathrm{SO}(p)\times\mathrm{SO}(8)>(A_1)_c\times\mathrm{SO}(8)>(A_1)_c\times(A_2)_c>(A_1)_c^2>(A_1)_c>\mathbb{T}>1$$

is unrefinable and so again $\lambda(G)\le 7$.

If p and q are even and $p-q\ne 8$ then the chain $G>\mathrm{SO}(p-q-1)\times\mathrm{SO}(q+1,q)>\ldots$

$$\ldots(A_1)_c\times\mathrm{SO}(q+1,q)>(A_1)_c\times(A_1)_s>(A_1)_c\times\mathbb{T}>\mathbb{T}^2>\mathbb{T}>1$$

is unrefinable and so $\lambda(G)\le 7$. Similarly, if p and q are even and $p-q=8$ but $q\ne 4$ then the chain $G>\mathrm{SO}(p-q+1)\times\mathrm{SO}(q-1,q)>\ldots$

$$\ldots(A_1)_c\times\mathrm{SO}(q-1,q)>(A_1)_c\times(A_1)_s>(A_1)_c\times\mathbb{T}>\mathbb{T}^2>\mathbb{T}>1$$

is unrefinable and so again $\lambda(G)\le 7$. Finally, $p=12$ and $q=4$ then the chain

$$G>\mathrm{SO}(11)\times\mathrm{SO}(1,4)>\mathrm{SO}(11)\times(A_1)_c^2>(A_1)_c^3>(A_1)_c^2>(A_1)_c>\mathbb{T}>1$$

is unrefinable and so once more $\lambda(G)\le 7$. This completes the proof of part (iii) of Theorem 1.2.

Finally, we justify the first assertion of Theorem 1.2. Consider any simple real algebraic group G. By the classification of simple real algebraic groups, G is listed in at least one of parts (i), (ii) or (iii) of Theorem 1.2. If $G(\mathbb{C})$ is a simple complex group then $\lambda_{\mathbb{C}}\big(G(\mathbb{C})\big)$ has been computed in [3]. Otherwise, $\lambda(G)=\lambda_{\mathbb{C}}\big(G(\mathbb{C})\big)-1$ by Lemma 4.7. It is then easy to see that $\lambda_{\mathbb{C}}\big(G(\mathbb{C})\big)-1\le\lambda(G)\le 9$. ∎

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