

Spherical Varieties over Large Fields

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Abstract. Let k_0 be a field of characteristic 0, k its algebraic closure, G a connected reductive group defined over k . Let $H \subset G$ be a spherical subgroup. We assume that k_0 is a large field, for example, k_0 is either the field \mathbb{R} of real numbers or a p -adic field. Let G_0 be a quasi-split k_0 -form of G . We show that if H has self-normalizing normalizer, and $\Gamma = \text{Gal}(k/k_0)$ preserves the combinatorial invariants of G/H , then H is conjugate to a subgroup defined over k_0 , and hence, the G -variety G/H admits a G_0 -equivariant k_0 -form. In the case when G_0 is not assumed to be quasi-split, we give a necessary and sufficient Galois-cohomological condition for the existence of a G_0 -equivariant k_0 -form of G/H .

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1. Introduction

Let G be a connected reductive group over an algebraically closed field k of characteristic 0. Let $k_0 \subset k$ be a subfield such that k is an algebraic closure of k_0 . Let G_0 be a k_0 -form (k_0 -model) of G , i.e. an algebraic group over k_0 together with an isomorphism of algebraic k -groups $\varkappa_G: G_0 \times_{k_0} k \xrightarrow{\sim} G$.

Let Y be an irreducible G -variety. By a G_0 -equivariant k_0 -form of Y we mean a k_0 -variety Y_0 , together with an isomorphism of k -schemes $\varkappa_Y: Y_0 \times_{k_0} k \xrightarrow{\sim} Y$ and a morphism of k_0 -schemes $G_0 \times_{k_0} Y_0 \rightarrow Y_0$ defining an action of G_0 on Y_0 , such that the following diagram commutes;

$$\begin{array}{ccc} G_{0,k} \times_k Y_{0,k} & \xrightarrow{\theta_{0,k}} & Y_{0,k} \\ \varkappa_G \times_k \varkappa_Y \downarrow & & \downarrow \varkappa_Y \\ G \times_k Y & \xrightarrow{\theta} & Y \end{array}$$

Here $\theta, \theta_{0,k}$ are the G -action on Y , and the base change to k of the G_0 -action on Y_0 respectively, and $G_{0,k}, Y_{0,k}$ are the base-changes to k of G_0, Y_0 respectively.

From now on till the end of the Introduction we assume that Y is a *spherical homogeneous space of G* . This means that $Y = G/H$ (with the natural action of G) for some algebraic subgroup $H \subset G$ and that a Borel subgroup B of G has

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The proof follows ideas of Akhiezer and Cupit-Foutou [2].

Remark 1.2. Under the assumptions of Theorem 1.1, $Y = G/H$ admits a G_0 -equivariant k_0 -form Y_0 . Indeed, we may take $Y_0 := G_0/H_0$. ■

Remark 1.3. The normalizer of a spherical subgroup in general is not self-normalizing; see Avdeev [3, Example 4, Section 4]. In [8] Borovoi and Gargliardi prove that conditions (iii) and (iv) in Theorem 1.1 are in fact unnecessary, but their proof is much more involved. ■

Remark 1.4. In [20, the proof of Corollary 2.7] Moser-Jauslin and Terpereau show that condition (iv) above is satisfied if $H \subseteq G$ is *symmetric* (i.e., if there exists a nontrivial involution $\theta \in \text{Aut}_k(G)$ such that $G^\theta \subseteq H \subseteq \mathcal{N}_G(G^\theta)$). They state and prove this over \mathbb{C} , but the proof works over an arbitrary algebraically closed field of characteristic zero.

They deduce, using Theorem 1.1 (which they restate and reprove), that if $H \subseteq G$ is a symmetric subgroup of a connected reductive complex group G , and G_0 is a quasi-split \mathbb{R} -form of G (with corresponding anti-holomorphic involution σ), such that $\sigma(H)$ is conjugate to H , then there exists a k -subgroup $H' \subseteq G$ which is conjugate to H , such that $\sigma(H') = H'$. Again, they state and prove this over \mathbb{R} , but the proof generalizes easily to an arbitrary large field of characteristic zero. ■

See Section 6 for some examples where Theorem 1.1 applies.

Note that assumption (i) is a necessary condition for G/H to have a k_0 -form; see Borovoi [7, Proposition 8.12].

If assumptions (i)–(iv) of Theorem 1.1 are satisfied, but G_0 is not assumed to be quasi-split, we can give a necessary and sufficient Galois-cohomological condition for the existence of a G_0 -equivariant k_0 -form of G/H .

Theorem 1.5 (Theorem 7.1). *Let G be a reductive group over an algebraically closed field k . Let $H \subset G$ be a spherical subgroup. Let $k_0 \subset k$ be a subfield such that k is an algebraic closure of k_0 . Let G_0 be a k_0 -form of G . Assume that:*

(i) $\Gamma := \text{Gal}(k/k_0)$ preserves the combinatorial invariants of G/H when acting on $\text{BRD}(G)$ via the homomorphism $\varepsilon: \Gamma \rightarrow \text{Aut BRD}(G)$ defined by the k_0 -form G_0 of G ;

(ii) $\text{char}(k_0) = 0$; (iii) k_0 is a large field; (iv) $\mathcal{N}_G(\mathcal{N}_G(H)) = \mathcal{N}_G(H)$.

Then $Y = G/H$ admits a G_0 -equivariant k_0 -form if and only if the Tits class $t(\tilde{G}_0) \in H^2(k_0, Z(\tilde{G}_0))$ has trivial image in $H^2(k_0, A_{\text{qs}})$ (see Subsection 7.1).

We explain the notation. Let \tilde{G}_0 denote the universal covering of the commutator subgroup $[G_0, G_0]$ of G_0 . We write \tilde{Z}_0 for the center of the simply connected group \tilde{G}_0 , which is a finite abelian k_0 -group. Write $A = \mathcal{N}_G(H)/H$, which is an abelian group of multiplicative type (i.e., it is a closed subgroup of a torus); see Losev [16, Theorem 2 and Definition 4.1.1(1)]. Then we have a canonical isomorphism $A(k) \xrightarrow{\sim} \text{Aut}^G(Y)$, see e.g. Borovoi [7, Lemma 5.1], and a canonical homomorphism of abelian k -groups $\tilde{Z} \rightarrow Z(G) \hookrightarrow \mathcal{N}_G(H) \rightarrow A$, where \tilde{Z} is the center of \tilde{G} .

Using the $*$ -action $\varepsilon: \Gamma \rightarrow \text{Aut BRD}(G, T, B)$, we define a k_0 -form A_{qs} of A and a k_0 -homomorphism

$$\tilde{\varkappa}: \tilde{Z}_0 \rightarrow A_{\text{qs}}.$$

See Subsection 7.2 below. This homomorphism induces a homomorphism on the second cohomology

$$\tilde{\varkappa}_*: H^2(k_0, \tilde{Z}_0) \rightarrow H^2(k_0, A_{\text{qs}}).$$

The k_0 -form \tilde{Z}_0 of \tilde{Z} , the k_0 -form A_{qs} of A , the k_0 -homomorphism $\tilde{\varkappa}: \tilde{Z}_0 \rightarrow A_{\text{qs}}$, and the induced homomorphism on cohomology $\tilde{\varkappa}_*$, can be constructed in terms of the homomorphism ε ; see Borovoi and Gagliardi [8, Section 3]. Now, the k_0 -form \tilde{G}_0 of \tilde{G} defines the *Tits class* $t(\tilde{G}_0) \in H^2(k_0, \tilde{Z}_0)$; see Subsection 7.1 below. Theorem 1.5 says that G/H admits a G_0 -equivariant k_0 -form if and only if the class $\tilde{\varkappa}_*(t(\tilde{G}_0)) \in H^2(k_0, A_{\text{qs}})$ is trivial.

Note that if G_0 is quasi-split, then $t(\tilde{G}_0) = 1$ and hence $\tilde{\varkappa}_*(t(\tilde{G}_0)) = 1$. We see that Theorem 1.5 generalizes the assertion of Corollary 1.2. Our proof of Theorem 1.5 uses Theorem 1.1, see Theorem 7.1 below.

For examples where Theorem 1.5 applies, see Borovoi and Gagliardi [8].

The plan for the rest of the paper is as follows. In Section 2 we recall basic definitions and results related to forms and semilinear actions, and the $*$ -action on the based root datum of a reductive group induced from a k_0 -form. In Section 3 we recall basic definitions and facts related to the combinatorial invariants of spherical varieties. In Section 4 we show that if $\mathcal{N}_G(H) = H$ and ε_γ preserves the combinatorial invariants of G/H then the (unique) wonderful embedding $\iota: G/H \hookrightarrow W$ admits a G_0 -equivariant k_0 -form.

In Section 5 we show that if W_0 is a G_0 -equivariant k_0 -form of a wonderful G -variety W , and the field k_0 is large, then the open orbit $G/H \subset W$ admits a k_0 -rational point. In Section 6 we combine these results to prove Theorem 1.1, and give some examples. In Section 7 we deduce Theorem 1.5 from Theorem 1.1 and Borovoi and Gagliardi [8, Theorem 1.6].

Remark 1.6. Since the original writing of this text, this work was used by Moser-Jauslin and Terpereau in [20], and generalized by Borovoi and Gagliardi in [8]. ■

Notation and assumptions: k is an algebraically closed field. k_0 is a subfield of k such that k is a Galois extension of k_0 , hence k_0 is perfect. We denote the Galois group $\text{Gal}(k/k_0)$ by Γ .

A k -variety is a reduced separated scheme of finite type over k , not necessarily irreducible. An *algebraic k -group* is a smooth k -group scheme of finite type over k , not necessarily connected. All algebraic k -subgroups are assumed to be smooth.

All homogeneous spaces are assumed to be spherical. G is a connected reductive algebraic k -group and G_0 is a k_0 -form of G . $H \subset G$ is a spherical k -subgroup. For a field k , we write \times_k for $\times_{\text{Spec } k}$. For a field extension k_1/k_0 , we write $\times_{k_0} k_1$ for $\times_{\text{Spec } k_0} \text{Spec } k_1$.

2. Preliminaries

2.1. Let k_0 be a field of characteristic zero with a fixed algebraic closure k . By a k_0 -form of a k -scheme X we mean a k_0 -scheme X_0 together with an isomorphism of k -schemes

$$\varkappa_X: X_0 \times_{k_0} k \xrightarrow{\sim} X.$$

Two k_0 -forms $(X_0, \varkappa_X), (X'_0, \varkappa_{X'})$ of X are said to be *isomorphic* if there exists an isomorphism of k_0 -schemes $f: X_0 \xrightarrow{\sim} X'_0$.

We write $\Gamma = \text{Gal}(k/k_0)$. For $\gamma \in \Gamma$, denote by $\gamma^*: \text{Spec } k \rightarrow \text{Spec } k$ the morphism of schemes induced by γ . Notice that $(\gamma \circ \delta)^* = \delta^* \circ \gamma^*$.

Let $(X, p_X: X \rightarrow \text{Spec } k)$ be a k -scheme. For every $\gamma \in \Gamma$ we define the γ -conjugated scheme $(\gamma X, \gamma p_X)$ to be the k -scheme which is the same abstract scheme as X , but with the structural morphism $\gamma p_X := (\gamma^*)^{-1} \circ p_X: X \rightarrow \text{Spec } k$. It is an elementary fact that $(\gamma X, \gamma p_X)$ realizes the pull-back of $X \xrightarrow{p_X} \text{Spec } k$ along $\gamma^*: \text{Spec } k \rightarrow \text{Spec } k$.

A k/k_0 -semilinear automorphism of X is a pair (γ, μ) where $\gamma \in \Gamma$ and $\mu: X \rightarrow X$ is an isomorphism of schemes such that the diagram below commutes:

$$\begin{array}{ccc} X & \xrightarrow{\mu} & X \\ p_X \downarrow & & \downarrow p_X \\ \text{Spec } k & \xrightarrow{(\gamma^*)^{-1}} & \text{Spec } k \end{array}$$

In this case we say also that μ is a γ -semilinear automorphism of X . We shorten “ γ -semilinear automorphism” to “ γ -semi-automorphism”. Note that if (γ, μ) is a semi-automorphism of X , then μ uniquely determines γ ; see Borovoi [7, Lemma 1.2].

We denote by $\text{SAut}(X)$ the group of all γ -semilinear automorphisms μ of X where γ runs over $\Gamma = \text{Gal}(k/k_0)$. There is a canonical bijection (which is set-theoretically the identity) between γ -semilinear automorphisms of X , and k -isomorphisms $\gamma X \xrightarrow{\sim} X$. By a *semilinear action* of Γ on X we mean a homomorphism of groups

$$\mu: \Gamma \rightarrow \text{SAut}(X), \quad \gamma \mapsto \mu_\gamma$$

such that for each $\gamma \in \Gamma$, μ_γ is γ -semilinear.

If we have a k_0 -scheme X_0 , then the formula $\gamma \mapsto \text{id}_{X_0} \times (\gamma^*)^{-1}$ defines a semilinear action of Γ on $X := X_0 \times_{k_0} k$. Thus a k_0 -form of X induces a semilinear action of Γ on X . If X is of finite type then two forms X_0, X'_0 are isomorphic if and only if they induce the same semilinear action $\Gamma \rightarrow \text{SAut}(X)$, up to conjugation by an element of $\text{Aut}(X)$.

If $(G, p_G: G \rightarrow \text{Spec } k)$ is an algebraic k -group, we define a k_0 -form of G similarly as above, with the additional assumption that the isomorphism \varkappa_G is an isomorphism of algebraic k -groups (rather than just k -schemes). An isomorphism of k_0 -forms of G is defined as above.

For $\gamma \in \Gamma$, the γ -conjugated scheme γG has a canonical structure of an algebraic k -group (since it is the pullback of an algebraic k -group along the morphism $\gamma^*: \text{Spec } k \rightarrow \text{Spec } k$).

A γ -semi-linear automorphism of G is defined as above, with the additional assumption that the k -morphism $\mu: \gamma G \rightarrow G$ is a morphism of algebraic k -groups.

We denote by $\text{SAut}(G)$ the group of all γ -semilinear automorphisms τ of G where γ runs over $\Gamma = \text{Gal}(k/k_0)$. By a semilinear action of Γ on G we mean a homomorphism

$$\sigma: \Gamma \rightarrow \text{SAut}(G), \quad \gamma \mapsto \sigma_\gamma$$

such that for all $\gamma \in \Gamma$, σ_γ is γ -semilinear. As above, a k_0 -form G_0 of G induces a semilinear action of Γ on G . Two k_0 -forms of G are isomorphic if and only if they induce the same semilinear action of Γ on G , up to conjugation by an element of $\text{Aut}(G)$.

Let G be an algebraic group over k and let X be a G - k -variety. Let G_0 be a k_0 -form of G . It gives rise to a semilinear action $\sigma: \Gamma \rightarrow \text{SAut}(G)$, $\gamma \mapsto \sigma_\gamma$. Let X_0 be a G_0 -equivariant k_0 -form of X , i.e. a k_0 -form X_0 of X together with a morphism $G_0 \times_{k_0} X_0 \rightarrow X_0$ compatible with the action $G \times_k X \rightarrow X$. It gives rise to a semilinear action $\mu: \Gamma \rightarrow \text{SAut}(X)$ such that for all γ in Γ :

$$\mu_\gamma(g \cdot x) = \sigma_\gamma(g) \cdot \mu_\gamma(x) \quad \forall x \in X(k), g \in G(k).$$

We say then that μ_γ is σ_γ -equivariant.

2.2. Let k be an algebraically closed field, and let G be a linear algebraic group over k . We denote by $\text{Aut}(G)$ the group of k -automorphisms of G , regarded as an abstract group. For $g \in G(k)$, we denote conjugation by g by $i_g \in \text{Aut}(G)$, and we denote $\text{Inn}(G) = \{i_g \mid g \in G(k)\}$.

Let $k_0 \subset k$ be a subfield such that k is a Galois extension of k_0 . We write $\Gamma = \text{Gal}(k/k_0)$. Let G_0 be a k_0 -form of G , it defines a semilinear action

$$\sigma: \Gamma \rightarrow \text{SAut}(G).$$

This action induces an action of Γ on the abstract group $\text{Aut}(G)$ by

$$({}^\gamma a)(g) = \sigma_\gamma(a(\sigma_\gamma^{-1}(g))) \quad \text{for } g \in G(k), a \in \text{Aut}(G).$$

Recall that a map $c: \Gamma \rightarrow \text{Aut}(G)$ is called a 1-cocycle if the map c is continuous (for the profinite topology on Γ and the discrete topology on $\text{Aut}(G)$) and satisfies the following condition:

$$c_{\gamma\delta} = c_\gamma \cdot {}^\gamma c_\delta \quad \text{for all } \gamma, \delta \in \Gamma. \quad (1)$$

The set of such cocycles is denoted by $Z^1(\Gamma, \text{Aut}(G))$ or $Z^1(k_0, \text{Aut}(G))$. For $c \in Z^1(k_0, \text{Aut}(G))$, we consider the c -twisted semilinear action

$$\sigma': \Gamma \rightarrow \text{SAut}(G), \quad \gamma \mapsto c_\gamma \circ \sigma_\gamma.$$

Then, clearly, σ'_γ is a γ -semi-automorphism of G for any $\gamma \in \Gamma$. It follows from the cocycle condition (1) that

$$\sigma'_{\gamma\delta} = \sigma'_\gamma \circ \sigma'_\delta \quad \text{for all } \gamma, \delta \in \Gamma.$$

Since G is an affine variety, the semilinear action σ' comes from some k_0 -form G'_0 of G , see Serre [25, III.1.3, Proposition 5]. We write $G'_0 = {}_c(G_0)$ and say that G'_0 is the twisted form of G_0 defined by the cocycle c .

If $c \in Z^1(\Gamma, \text{Inn}(G))$ is a 1-cocycle with coefficients in $\text{Inn}(G)$, then we say that ${}_c G_0$ is an inner twist or inner form of G_0 .

2.3. From now on we assume that G is a connected reductive group over an algebraically closed field k of characteristic 0 and that $H \subset G$ is a spherical k -subgroup. Let $B \subset G$ be a Borel subgroup, and let $T \subset B$ be a maximal torus. We consider the based root datum

$$\text{BRD}(G) = \text{BRD}(G, T, B) = (X, X^\vee, R, R^\vee, S, S^\vee),$$

where

- * $X = \mathbf{X}^*(T) := \text{Hom}(T, \mathbb{G}_{m,k})$ is the character group of T ;
- * $X^\vee = \mathbf{X}_*(T) := \text{Hom}(\mathbb{G}_{m,k}, T)$ is the cocharacter group of T ;
- * $R = R(G, T) \subset X$ is the root system;
- * $R^\vee \subset X^\vee$ is the coroot system;
- * $S = S(G, T, B) \subset R$ is the system of simple roots (the basis of R) defined by B ;
- * $S^\vee \subset R^\vee$ is the system of simple coroots.

There is a canonical pairing $X \times X^\vee \rightarrow \mathbb{Z}$, $(\chi, x) \mapsto \langle \chi, x \rangle$, and a canonical bijection $\alpha \mapsto \alpha^\vee: R \rightarrow R^\vee$ such that $S^\vee = \{\alpha^\vee \mid \alpha \in S\}$. See Springer [26, Sections 1 and 2] for details.

2.4. We write $\text{SAut}_{k/k_0}(G)$ or just $\text{SAut}(G)$ for the group of k/k_0 -semi-automorphism of G . Let T and B be as in Subsection 2.3. We construct a canonical homomorphism (cf. Tits [29, Section 2.3])

$$\phi: \text{SAut}(G) \rightarrow \text{Aut BRD}(G, T, B). \tag{2}$$

Let (γ, τ) be a semi-automorphism of G . Consider $\tau(T) \subset \tau(B) \subset G$. Then $\tau(T)$ is a maximal torus and $\tau(B)$ is a Borel subgroup. By Theorem 11.1 and Theorem 10.6(4) in Borel’s book [5], there exists an element $g \in G(k)$ such that

$$g \cdot \tau(T) \cdot g^{-1} = T \quad \text{and} \quad g \cdot \tau(B) \cdot g^{-1} = B.$$

We obtain a γ -semi-automorphism $\tau' := \text{inn}(g) \circ \tau$. The pair (γ, τ') induces an automorphism of the based root datum

$$\phi(\gamma, \tau) \in \text{Aut BRD}(G, T, B)$$

as follows: Since $\tau'(T) = T$, the pair (γ, τ') acts on the character group $X = \mathbf{X}^*(T)$ by $\chi \mapsto (\gamma, \tau')(\chi)$, where

$$(\gamma, \tau')(\chi)(t) = \gamma(\chi((\tau')^{-1}(t))),$$

and similarly it acts on the dual lattice $X^\vee = \mathbf{X}_*(T)$ and preserves the pairing $X \times X^\vee \rightarrow \mathbb{Z}$. One can easily check that (γ, τ') , when acting on $\text{Lie}(G)$, takes the eigenspace corresponding to a root $\alpha \in R$ to the eigenspace corresponding to $(\gamma, \tau')(\alpha)$, hence (γ, τ') preserves R , and, similarly, it preserves, R^\vee and the bijection $\alpha \mapsto \alpha^\vee: R \rightarrow R^\vee$. Since (γ, τ') preserves B , it preserves the set of positive roots with respect to B , hence, it preserves S and S^\vee . If $g' \in G(k)$ is another element such that

$$g' \cdot \tau(T) \cdot (g')^{-1} = T \quad \text{and} \quad g' \cdot \tau(B) \cdot (g')^{-1} = B,$$

then $g' = tg$ for some $t \in T(k)$. Since $\text{inn}(t)$ acts trivially on $\text{BRD}(G, T, B)$, we see that $\phi(\gamma, \tau)$ does not depend on the choice of g . One checks easily that $\phi: \text{SAut}(G) \rightarrow \text{Aut BRD}(G, T, B)$ is a homomorphism; see [9, Proposition 3.1].

If G_0 is a k_0 -form of G with corresponding semilinear action $\sigma: \Gamma \rightarrow \text{SAut}(G)$, we denote by

$$\varepsilon = \phi \circ \sigma: \Gamma \rightarrow \text{Aut BRD}(G, T, B), \quad \gamma \mapsto \varepsilon_\gamma$$

the induced action of Γ on $\text{BRD}(G, T, B)$, and call it “the $*$ -action of Γ on $\text{BRD}(G, T, B)$ induced by the k_0 -form G_0 ” (or simply “the $*$ -action” when there is no risk of confusion).

Now let G be a connected reductive group, and let G_0 be a k_0 -form of G defining a semilinear action $\sigma: \Gamma \rightarrow \text{SAut}(G)$. Then a subgroup $H \subset G$ is “defined over k_0 ” if and only if it is Γ -invariant. Recall that a k_0 -form G_0 of G is called *split* if G_0 has a split maximal k_0 -torus. All split k_0 -forms of G are isomorphic. A k_0 -form G_0 of G is called *quasi-split* if G has a Borel subgroup defined over k_0 . Every split k_0 -form of G is quasi-split.

We shall need the following well-known result:

Proposition 2.1. *Let G_0 be a connected reductive group over a field k_0 . Then there exist a unique (up to isomorphism) quasi-split inner form G_{qs} of G_0 . Moreover, G_0 and G_{qs} induce the same action of Γ on $\text{BRD}(G, B, T)$.*

Proof. See The Book of Involutions [15, Proposition(31.5)], or Conrad [11, Proposition 7.2.12], the existence only in Springer [27, Proposition 16.4.9]. ■

3. Spherical varieties and their combinatorial invariants

From now on all varieties are assumed to be geometrically integral.

A normal G -variety X is called *spherical* if B has an open orbit in X for some (equivalently: for every) Borel subgroup $B \subseteq G$. This is equivalent to there being only finitely many B -orbits in X for some (equivalently: any) Borel subgroup $B \subseteq G$, see Perrin [22, Theorem 2.1.2].

X is called a *homogeneous spherical variety* if X is spherical and $X \cong G/H$ as G -varieties for some closed subgroup $H \subseteq G$.

If X is a spherical G -variety, then it has an open G orbit, and hence, admits an open embedding $\iota: G/H \hookrightarrow X$ where G/H is a homogeneous spherical G -variety.

Thus the classification of spherical varieties splits into two parts:

- (i) Given a homogeneous spherical G -variety $Y = G/H$, classify all possible open G -embeddings $Y \hookrightarrow X$ of Y into a normal G -variety X .
- (ii) Classify all homogeneous spherical G -varieties.

3.1. Homogeneous spherical varieties and their combinatorial invariants

Let $Y = G/H$ be a spherical homogeneous G -variety. We associate to Y the following combinatorial data:

Let $K(Y)$ denote the field of rational functions on Y (notice that $G(k)$ naturally acts on $K(Y)$). For any character $\chi \in X^*(B)$ we define

$$K(Y)_\chi^{(B)} = \{f \in K(Y) \mid b \cdot f = \chi(b) \cdot f \text{ for all } b \in B(k)\}$$

Because Y is spherical, $\dim_k K(Y)_\chi^{(B)} \leq 1$ for all $\chi \in X^*(B)$.

We define the *weight lattice* \mathcal{X} of G/H by

$$\mathcal{X} = \mathcal{X}(Y) = \{\chi \in X^*(B) \mid \dim_k K(Y)_\chi^{(B)} \neq 0\}.$$

Clearly, \mathcal{X} is a subgroup of $X^*(B)$. We define $V = V(Y) = \text{Hom}_{\mathbb{Z}}(\mathcal{X}, \mathbb{Q})$.

Let $\text{Val}(K(Y))$ denote the set of \mathbb{Q} -valued valuations on $K(Y)$ that are trivial on k . We have a natural action of $G(k)$ on $\text{Val}(K(Y))$, and we denote by $\text{Val}^G(K(Y))$ (resp. $\text{Val}^B(K(Y))$) the set of G -invariant (resp. B -invariant) valuations.

We have a natural map $\rho: \text{Val}^B(K(Y)) \rightarrow V$ given by $v \mapsto (\chi \mapsto v(f_\chi))$. It is known (see Knop [13, Corollary 1.8]) that the restriction of ρ to $\text{Val}^G(K(Y))$ is injective. Denote the image

$$\mathcal{V} = \mathcal{V}(Y) = \rho(\text{Val}^G(K(Y))) \subseteq V$$

We call \mathcal{V} the *valuation cone* of Y .

Let \mathcal{D} denote the set of B -invariant prime divisors in Y (these are called “colors”). This is a finite set (it may be empty). Each element $D \in \mathcal{D}$ defines a B -invariant valuation denoted $\text{val}(D) \in \text{Val}^B(K(Y))$ given by sending a rational function $f \in K(Y)$ to its order of vanishing at D . By abuse of notation we write ρ for $\rho \circ \text{val}: \mathcal{D} \rightarrow V$. In general, this map is not injective.

For every $D \in \mathcal{D}$ denote by $\text{Stab}_G(D)$ the stabilizer of $D \subset Y$ in G . Since D is B -invariant, we obviously have $B \subseteq \text{Stab}_G(D)$. For every $\alpha \in S$ we denote by P_α the corresponding minimal parabolic subgroup in G containing B . Denote by $\mathcal{P}(S)$ the power set of S . Define $\varsigma: \mathcal{D} \rightarrow \mathcal{P}(S)$ by letting $\varsigma(D)$ be the set of all $\alpha \in S$ such that $P_\alpha \not\subseteq \text{Stab}_G(D)$.

Lemma 3.1. *Every fiber of the map $\varsigma: \mathcal{D} \rightarrow \mathcal{P}(S)$ consists of ≤ 2 elements.*

Proof. See [7, Lemma 7.1] ■

Corollary 3.2. *Every fiber of the map $\rho \times \varsigma: \mathcal{D} \rightarrow V \times \mathcal{P}(S)$ consists of ≤ 2 elements.*

Denote by $\Omega^{(1)}$ (resp. $\Omega^{(2)}$) the subset of $V \times \mathcal{P}(S)$ consisting of those elements with precisely one (resp. two) preimages under the map $\rho \times \varsigma$.

Definition 3.3. Let G be a connected reductive group over k , Y a spherical homogeneous space for G . By the *combinatorial invariants* of Y we will mean:

$$\mathcal{X} \subseteq X^*(B), \quad \mathcal{V} \subseteq V, \quad \text{and } \Omega^{(1)}, \Omega^{(2)} \subseteq V \times \mathcal{P}(S)$$

The combinatorial invariants of a spherical homogeneous space completely determine it as a G -variety:

Theorem 3.4 (Losev’s Uniqueness Theorem [16, Theorem 1]). *Let G be a connected reductive group over k , H_1, H_2 two spherical subgroups, Y_1, Y_2 the corresponding spherical homogeneous spaces. Assume that $\text{char } k = 0$. Then H_1 and H_2 are conjugate in G if and only if*

$$\mathcal{X}(Y_1) = \mathcal{X}(Y_2), \quad \mathcal{V}(Y_1) = \mathcal{V}(Y_2), \quad \text{and } \Omega^{(1)}(Y_1) = \Omega^{(1)}(Y_2), \quad \Omega^{(2)}(Y_1) = \Omega^{(2)}(Y_2)$$

i.e., if and only if they have the same combinatorial invariants.

Now let $H \subseteq G$ be a spherical subgroup, G_0 a k_0 -form of G with corresponding semilinear action $\sigma: \Gamma \rightarrow \text{SAut}(G)$, and $\gamma \in \Gamma$. We denote by $Y = G/H$ and $\varepsilon_\gamma \in \text{Aut}(X^*(B), S)$ the automorphism arising from σ_γ .

Definition 3.5. We say that ε_γ preserves the combinatorial invariants of Y if

$$\varepsilon_\gamma(\mathcal{X}(Y)) = \mathcal{X}(Y), \quad \varepsilon_\gamma(\mathcal{V}(Y)) = \mathcal{V}(Y), \quad \text{and } \varepsilon_\gamma(\Omega^{(i)}(Y)) = \Omega^{(i)}(Y), \quad i = 1, 2$$

Here, once we require $\varepsilon_\gamma(\mathcal{X}(Y)) = \mathcal{X}(Y)$, we get induced automorphisms of $V(Y) = \text{Hom}_{\mathbb{Z}}(\mathcal{X}(Y), \mathbb{Q})$ and $V(Y) \times \mathcal{P}(S)$, which (abusing notations) we also denote by ε_γ .

Remark 3.6. This condition is automatic when G_0 is split, since in this case ε_γ acts trivially on $(X^*(B), S)$.

3.2. Spherical embeddings

Let $Y = G/H$ be a homogeneous spherical G -variety.

A *colored cone* is a pair $(\mathcal{C}, \mathcal{F})$ where $\mathcal{F} \subseteq \mathcal{D}$ is a subset and $\mathcal{C} \subseteq V$ is a strictly convex cone generated by $\rho(\mathcal{F})$ and finitely many elements of \mathcal{V} , such that $0 \notin \rho(\mathcal{F})$.

A *face of a colored cone* $(\mathcal{C}, \mathcal{F})$ is a colored cone $(\mathcal{C}_0, \mathcal{F}_0)$ such that \mathcal{C}_0 is a face of \mathcal{C} and $\mathcal{F}_0 = \mathcal{F} \cap \rho^{-1}(\mathcal{C}_0)$.

A *colored fan* is a nonempty finite set Σ of colored cones satisfying:

- (i) For every $(\mathcal{C}, \mathcal{F}) \in \Sigma$, $\mathcal{C}^\circ \cap \mathcal{V} \neq \emptyset$ (where \mathcal{C}° denotes the relative interior of \mathcal{C}).
- (ii) For every face $(\mathcal{C}_0, \mathcal{F}_0)$ of a colored cone $(\mathcal{C}, \mathcal{F}) \in \Sigma$ satisfying $\mathcal{C}_0^\circ \cap \mathcal{V} \neq \emptyset$, $(\mathcal{C}_0, \mathcal{F}_0)$ also belongs to Σ .
- (iii) For every $u \in \mathcal{V}$ there is at most one $(\mathcal{C}, \mathcal{F}) \in \Sigma$ with $u \in \mathcal{C}^\circ$.

To any spherical embedding $G/H \hookrightarrow X$ we can associate a colored fan in the following manner: Denote by \mathcal{A} the set of G -invariant prime divisors in X . Then $\Delta = \mathcal{D} \cup \mathcal{A}$ is the set of B -invariant prime divisors in X . The definition of $\rho: \mathcal{D} \rightarrow V$ extends to $\rho: \Delta \rightarrow V$. For any G -orbit $Y \subseteq X$ we denote by $I_Y \subseteq \Delta$ the set of B -invariant prime divisors containing Y , and we set

$$(\mathcal{C}_Y, \mathcal{F}_Y) := (\text{cone}(\rho(I_Y)), I_Y \cap \mathcal{D}).$$

Theorem 3.7. (Knop [13, Theorem 3.3 and the paragraph before it]) *The map*

$$(G/H \hookrightarrow X) \mapsto \Sigma(X) := \{(\mathcal{C}_Y, \mathcal{F}_Y) : Y \subseteq X \text{ is a } G\text{-orbit}\}$$

defines a bijection between isomorphism classes of spherical embeddings of G/H and colored fans. Furthermore, the assignment $Y \mapsto (\mathcal{C}_Y, \mathcal{F}_Y)$ defines a bijection

$$\{G\text{-orbits in } X\} \rightarrow \Sigma(X)$$

such that for two G -orbits $Y, Z \subseteq X$ we have $Y \subseteq \overline{Z}$ if and only if $(\mathcal{C}_Z, \mathcal{F}_Z)$ is a face of $(\mathcal{C}_Y, \mathcal{F}_Y)$.

3.3. Wonderful varieties

The following definition is due to Luna [17].

Definition 3.8. A G -variety W is called *wonderful* if:

- (i) W is complete and smooth;
- (ii) W admits an open G -orbit whose complement consists of a finite union of smooth prime divisors X_1, \dots, X_r with normal crossings.
- (iii) The G -orbit closures of W are given by partial intersections of the X_i 's, and they are all nonempty.

In particular, a wonderful G -variety W has a unique closed G -orbit, which we denote by $X := \bigcap_{i=1}^r X_i$. Every wonderful G -variety is spherical; see Luna [17]. As Luna [17] noticed, a wonderful variety is always projective (see also Avdeev [4, Proposition 3.18]).

A *wonderful embedding* of a spherical homogeneous space G/H is an open G -embedding $\iota: G/H \hookrightarrow W$ into a wonderful G -variety. Any spherical variety G/H with $\mathcal{N}_G(H) = H$ admits a wonderful embedding; see Knop [14, Corollary 7.2]. Wonderful varieties have a very nice description in terms of colored fans:

Theorem 3.9. *Under the correspondence between colored fans and spherical embeddings, a wonderful embedding $G/H \hookrightarrow W$ corresponds to the colored fan that has the following two properties:*

- (i) *It has exactly one maximal colored cone, and it equals the valuation cone $\mathcal{V}(Y)$.*
- (ii) *It has no colours.*

Proof. See Perrin [22, Corollary 3.4.2, Definition 3.4.8 and Theorem 3.4.16]. ■

3.4. Galois descent for spherical embeddings

Let $k_0 \subseteq k$ be a Galois extension of fields such that k is algebraically closed and of characteristic zero, and let $\Gamma = \text{Gal}(k/k_0)$ the Galois group. Let G be a connected reductive k -group, G_0 a k_0 -model of G . Let $Y = G/H$ a spherical homogeneous G -variety, Y_0 a G_0 -equivariant k_0 -model of Y with induced semilinear action

$$\Gamma \rightarrow \text{SAut}(Y), \gamma \mapsto \sigma_\gamma^{(Y)}$$

Theorem 3.10. *The G_0 -equivariant k_0 -form Y_0 of Y naturally induces compatible actions of Γ on $K(Y), \text{Val}^B(K(Y)), \mathcal{X}, V, \mathcal{V}, \mathcal{D}$.*

Proof. See Huruguen [12, Section 2.2]. (Huruguen assumes that Y_0 admits a k_0 -point, but he does not use this assumption.) ■

We call a fan $\Sigma = \{(\mathcal{C}_i, \mathcal{F}_i)\}_{i \in I}$ Γ -stable if for every $(\mathcal{C}_i, \mathcal{F}_i) \in \Sigma$ and for every $\gamma \in \Gamma$, the colored cone $(\gamma(\mathcal{C}_i), \gamma(\mathcal{F}_i))$ still belongs to Σ .

Let $\iota: Y \hookrightarrow X$ be a spherical embedding. We say that “the semilinear action of Γ extends to X ” if there exists a semilinear action

$$\Gamma \rightarrow \text{SAut}(X), \gamma \mapsto \sigma_\gamma^{(X)}$$

such that for every $\gamma \in \Gamma$ one has $\iota \circ \sigma_\gamma^{(Y)} = \sigma_\gamma^{(X)} \circ \iota$.

We will need the following result.

Theorem 3.11 (Huruguen [12, Theorem 2.23]). *Let $k, k_0, G, G_0, Y, Y_0, X$ be as above. Then the semilinear action of Γ extends to X if and only if the colored fan $\Sigma(X)$ corresponding to X is Γ -stable. Furthermore, in this case, for every G -orbit Y in X and every $\gamma \in \Gamma$ one has*

$$\gamma(\mathcal{C}_Y) = \mathcal{C}_{\gamma(Y)}, \quad \gamma(\mathcal{F}_Y) = \mathcal{F}_{\gamma(Y)}.$$

4. Equivariant forms of wonderful varieties

4.1. The setup for the rest of the paper is as follows: k_0 is a field of characteristic zero with a fixed algebraic closure k , and $\Gamma = \text{Gal}(k/k_0)$ is the Galois group. We fix a connected reductive k -group G , a spherical k -subgroup $H \subseteq G$, and a k_0 -form G_0 of G .

The following Lemma 4.1 and Theorem 4.2 are generalizations of results of Akhiezer [1], cf. Borovoi [7].

Lemma 4.1 ([7, Proposition 8.12]). *Let $k_0, k, \Gamma, G, H, G_0$ be as in Subsection 4.1, and let $\gamma \in \Gamma$. Then the following are equivalent:*

- (i) ε_γ preserves the combinatorial invariants of $Y = G/H$;
- (ii) there exists $a_\gamma \in G(k)$ such that $\sigma_\gamma(H) = a_\gamma H a_\gamma^{-1}$

Theorem 4.2 ([7, Theorem 9.3]). *Let $k, k_0, \Gamma, G, H, G_0$ be as in Subsection 4.1, and let $Y = G/H$. Assume that:*

- (i) For all $\gamma \in \Gamma$, ε_γ preserves the combinatorial invariants of Y ;
- (ii) $\mathcal{N}_G(H) = H$;

Then Y admits a G_0 -equivariant k_0 -form Y_0 . This k_0 -form is unique up to a unique isomorphism.

We will need the following result, which is a generalization of Theorem 4.3 of Akhiezer and Cupit-Foutou [2].

Theorem 4.3. *Let $k, k_0, \Gamma, G, H, G_0$ be as in Subsection 4.1, and let $Y = G/H$. Assume that:*

- (i) For all $\gamma \in \Gamma$, ε_γ preserves the combinatorial invariants of Y ;
- (ii) $\mathcal{N}_G(H) = H$;

Let $\iota: Y \hookrightarrow W$ be a wonderful embedding of Y . Then ι admits a unique G_0 -equivariant k_0 -form $\iota_0: Y_0 \hookrightarrow W_0$.

Proof. Since a form is uniquely determined by the semilinear action that it induces, and a semilinear action $\Gamma \rightarrow \text{SAut}(W)$ is uniquely determined by its restriction to the (dense) open G -orbit, we obtain uniqueness from Theorem 4.2.

Let Y_0 be the (unique) G_0 -equivariant k_0 -form of Y guaranteed by Theorem 4.2. Let \mathcal{E}_W be the colored fan corresponding to the spherical embedding $\iota: G/H \hookrightarrow W$. The k_0 -form Y_0 gives compatible Γ -actions on $X^*(B), \mathcal{X}(Y), \mathcal{V}(Y), \mathcal{D}(Y)$; see Huruguen [12, Section 2.2]. Since the embedding $\iota: G/H \hookrightarrow W$ is wonderful, there is a unique maximal colored cone in \mathcal{E}_W , and it consists of $\mathcal{V}(Y)$ *without colors* (see Theorem 3.9). Since by assumption Γ preserves $\mathcal{V}(Y)$, we see that the colored fan \mathcal{E}_W corresponding to $\iota: G/H \hookrightarrow W$ is Γ -stable. By Theorem 3.11, the semilinear

action of Γ extends to W . Since W is projective, Theorem 2.26 in Huruguen [12] concludes the proof. ■

5. Rational points in wonderful varieties over large fields

Lemma 5.1 (well-known). *Let G be an algebraic group over k and let $Y = G/H$ be a homogeneous space. Let G_0 be a k_0 -form of G with corresponding semilinear action*

$$\sigma: \Gamma \rightarrow \text{SAut}(G).$$

Assume that Y admits a G_0 -equivariant k_0 -form Y_0 . Then $\sigma_\gamma(H)$ is conjugate to H for every $\gamma \in \Gamma$.

Proof. Denote by $\sigma: \Gamma \rightarrow \text{SAut}(G)$ the semilinear action of Γ on G induced by the k_0 -form G_0 , and by $\mu: \Gamma \rightarrow \text{SAut}(Y)$ the semilinear action of Γ on Y induced by the k_0 -form Y_0 .

Let $\gamma \in \Gamma$. Since the form Y_0 is G_0 -equivariant, we see that μ_γ is σ_γ -equivariant. Thus we see that $\text{Stab}_G(\mu_\gamma(eH)) = \sigma_\gamma(H)$. Choose $g_0 \in G(k)$ such that $\mu_\gamma(eH) = g_0H$. An elementary calculation shows that $\text{Stab}_G(g_0H) = g_0Hg_0^{-1}$. We see that $\sigma_\gamma(H) = g_0Hg_0^{-1}$ for some $g_0 \in G(k)$. ■

For a semilinear action $\mu: \Gamma \rightarrow \text{SAut}(X)$ of Γ on X we will call a “ Γ -fixed k -point in X ” a point $x \in X(k)$ such that for all $\gamma \in \Gamma$, $\mu_\gamma(x) = x$. If μ arises from a k_0 -form X_0 of X , then the Γ -fixed k -points in X are in canonical bijection with k_0 -rational points in X_0 , given by sending a k_0 -rational point $x: \text{Spec } k_0 \rightarrow X_0$ in X_0 to the Γ -fixed k -point $\text{Spec } k \rightarrow \text{Spec } k_0 \xrightarrow{x} X_0$.

Lemma 5.2. *Let k_0, k, Γ, G, G_0 be as in Subsection 4.1. Let $P \subseteq G$ be a parabolic subgroup, and let $X = G/P$ denote the corresponding generalized flag variety. Assume that G_0 is **quasi-split** and that X admits a G_0 -equivariant k_0 -form X_0 . Then X_0 has a k_0 -point.*

See Moser-Jauslin and Terpereau [19, Remark 3.13] for a generalization to homogeneous *horospherical* varieties (again, the authors only consider the case $k_0 = \mathbb{R}$, but this does not change anything).

Proof. Let $B_0 \subseteq G_0$ be a Borel subgroup defined over k_0 , and let $T_0 \subseteq B_0$ be a maximal torus. Set $B = (B_0)_k$, $T = (T_0)_k$. Consider the set of simple roots $S = S(G, T, B)$. For $I \subseteq S$, we denote by P_I the standard parabolic subgroup generated by B and the unipotent subgroups of G associated with the simple roots $\alpha \in I$; see Malle and Testerman [18, Definition 12.3]. Any parabolic subgroup of G is conjugate to P_I for some $I \subseteq S$. Moreover, two standard parabolic subgroups P_I and P_J for $I \neq J$ are not conjugate; see Malle and Testerman [18, Proposition 12.2].

Our parabolic subgroup P is conjugate to P_I for some $I \subseteq S$. Consider the semilinear action $\sigma: \Gamma \rightarrow \text{SAut}(G)$ defined by G_0 . Since G_0 is quasi-split, it follows from the definition of the $*$ -action

$$\varepsilon: \Gamma \rightarrow \text{Aut BRD}(G, T, B)$$

in Subsection 2.4 that for all $\gamma \in \Gamma$, we have $\sigma_\gamma(P_I) = P_{\varepsilon_\gamma(I)}$. Since X admits a G_0 -equivariant k_0 -form, and P_I is the stabilizer of some point in X , by Lemma 5.1 the subgroup $\sigma_\gamma(P_I)$ is conjugate to P_I . It follows that $\varepsilon_\gamma(I) = I$, and therefore,

$\sigma_\gamma(P_I) = P_I$ (for all $\gamma \in \Gamma$). This Γ -invariant subgroup P_I corresponds to a Γ -fixed k -point $x_0 \in X(k)$. Clearly, the point $x_0 \in X_0(k) = X(k)$ is defined over k_0 . ■

5.1. Assume now that W is a wonderful G -variety with a given G_0 -equivariant k_0 -form W_0 , given by a semilinear action $\mu: \Gamma \rightarrow \text{SAut}(W)$. Denote by $X = \bigcap_{i=1}^r X_i$ the unique closed G -orbit in W . Note that since μ_γ is σ_γ -equivariant for all $\gamma \in \Gamma$, $\mu_\gamma(X)$ is a closed G -orbit for every $\gamma \in \Gamma$. This implies that X has a canonical G_0 -equivariant k_0 -form. Since W has a *unique* closed G -orbit, X is Γ -invariant. Since X is a complete homogeneous G -variety, it is a generalized flag variety, i.e. $X = G/P$ for some parabolic subgroup $P \subseteq G$. By Lemma 5.2, if G_0 is quasi-split then the variety X contains a Γ -fixed k -point.

Theorem 5.3. *Let k_0, k, Γ, G, G_0 be as in Subsection 4.1. Let W be a wonderful G -variety with a G_0 -equivariant k_0 -form W_0 , given by a semilinear action $\mu: \Gamma \rightarrow \text{SAut}(W)$. Assume that k_0 is **large** and that the k_0 -form G_0 is **quasi-split**. Then:*

- (i) *Every Γ -invariant G -orbit in W has a Γ -fixed k -point.*
- (ii) *The open G -orbit in W has a Γ -fixed k -point.*
- (iii) *If moreover the k_0 -form G_0 is split, then every G -orbit in W is Γ -invariant.*

This theorem is a generalization of Theorem 3.6(ii) of Akhiezer and Cupit-Foutou [2], where the authors considered the case when $k = \mathbb{R}$ and G_0 is split.

Proof. (i) Let X' be a Γ -invariant G -orbit in W . Since X' is Γ -invariant, so is its closure $Z := \overline{X'}$. Denote by X the (unique) closed G -orbit in W . Notice that by condition (iii) in Definition 3.8, we have $X \subseteq Z$. By Subsection 5.1, X contains a Γ -fixed k -point, and hence, Z contains a Γ -fixed k -point. Since the divisors X_i have normal crossings, and Z is the intersection of some of these divisors, the variety Z is smooth, and hence also reduced (see Definition 3.8). Since Z is reduced, and $Z(k) \subseteq W(k)$ is Γ -invariant, we see that the semilinear action of Γ on W restricts to a semilinear action of Γ on Z , which we denote by $\nu: \Gamma \rightarrow \text{SAut}(Z)$. Denote by ι the embedding of Z into W . Since everything is of finite type, there exists a finite extension k_1/k_0 in k , a k_1 -scheme Z_1 and a closed k_1 -embedding $\iota_1: Z_1 \hookrightarrow W_1 := W_0 \times_{k_0} k_1$ such that $\iota_1 \times_{k_1} k = \iota$. The semilinear action of $\Gamma_1 := \text{Gal}(k_1/k_0)$ on Z induced from Z_1 is the same as $\nu|_{\Gamma_1}$, since Z is reduced and on closed points both are given by μ . Since Z can be covered by Γ -invariant affine open subsets (take such a covering of W and intersect with Z), by Borel and Serre [6, Lemma 6.12], we see that ν is induced from a k_0 -form Z_0 of Z . Since k_0 is large, and Z is smooth and has a Γ -fixed k -point, we obtain that Γ -fixed k -points (or equivalently: k_0 -rational points in Z_0) are Zariski dense in Z . In particular, the open subset $X' \subseteq Z$ contains a Γ -fixed k -point.

(ii) Let Y be the (unique) open G -orbit in W . Since $\mu_\gamma(Y)$ is an open G -orbit for every $\gamma \in \Gamma$, we get that Y is Γ -invariant, hence Y contains a Γ -fixed k -point by (i).

(iii) Denote by $Y = G/H$ the open G -orbit in W . Assume now that G_0 is split, and denote by \mathcal{E}_W the colored fan corresponding to the embedding $G/H \hookrightarrow W$. Since G_0 is split, Γ acts trivially on $X^*(B)$, and hence preserves $\mathcal{X}(Y)$. We see that it also acts trivially on $V = \text{Hom}(\mathcal{X}(Y), \mathbb{Q})$. Since W is wonderful, by Theorem 3.9 there are no colors in the colored fan \mathcal{E}_W . Hence Γ preserves each colored cone in \mathcal{E}_W . From the second assertion of Theorem 3.11, we see that every G -orbit in W is Γ -invariant. ■

6. Spherical subgroup defined over the base field

The following theorem is a generalization of Theorem 4.4 of Akhiezer and Cupit-Foutou [2], where the authors consider the case when $k_0 = \mathbb{R}$ and G_0 is split.

Theorem 6.1. *Let k be an algebraically closed field of characteristic 0, and let $k_0 \subseteq k$ be a subfield such that k/k_0 is Galois. Denote by $\Gamma = \text{Gal}(k/k_0)$ the Galois group. Let G be a connected reductive k -group, G_0 a k_0 -form of G . Let $H \subseteq G$ be a spherical subgroup. Assume that k_0 is **large** and that the k_0 form G_0 is **quasi-split**. Assume moreover that $\text{char } k_0 = 0$, and that:*

- (i) Γ preserves the combinatorial invariants of G/H when acting on $\text{BRD}(G)$ via the homomorphism $\varepsilon: \Gamma \rightarrow \text{Aut BRD}(G)$ defined by the k_0 -form G_0 of G ;
- (ii) $\mathcal{N}_G(\mathcal{N}_G(H)) = \mathcal{N}_G(H)$.

Then H is conjugate to a subgroup defined over k_0 , i.e., there exist $g \in G(k)$ and an algebraic subgroup $H_0 \subset G_0$ such that $gHg^{-1} = H_0 \times_{k_0} k$.

Remark 6.2. When G_0 is split, condition (i) is automatically satisfied. ■

Proof. Denote $N = \mathcal{N}_G(H)$. Since ε_γ preserves the combinatorial invariants of G/H , by Lemma 4.1 for every $\gamma \in \Gamma$ there exists $a_\gamma \in G(k)$ such that $\sigma_\gamma(H) = a_\gamma H a_\gamma^{-1}$. Then, of course, we have $\sigma_\gamma(N) = a_\gamma N a_\gamma^{-1}$. By assumption (ii) we have $\mathcal{N}_G(N) = N$, hence G/N admits a wonderful embedding, say, $G/H \hookrightarrow X$.

Since $\sigma_\gamma(N)$ is conjugate to N , we see from Lemma 4.1 that ε_γ preserves the combinatorial invariants of G/N for all $\gamma \in \Gamma$. By Theorem 4.3, X admits a G_0 -equivariant k_0 -form X_0 , given by a semilinear action $\mu: \Gamma \rightarrow \text{SAut}(X)$. By Theorem 5.3, there exists a Γ -fixed k -point, say gN , in the open orbit $G/N \subseteq X$. Then $\mu_\gamma(gN) = gN$ for all $\gamma \in \Gamma$. Define $N' = gNg^{-1}$ and $H' = gHg^{-1}$. Then $N' = \text{Stab}_G(gN)$ and $N' = \mathcal{N}_G(H')$. For every $\gamma \in \Gamma$, since ε_γ preserves the combinatorial invariants of $G/H' = G/H$, there exists $a'_\gamma \in G(k)$ such that $\sigma_\gamma(H') = a'_\gamma H' (a'_\gamma)^{-1}$ (we may take $a'_\gamma = \sigma_\gamma(g)a_\gamma g^{-1}$). Clearly we have $\sigma_\gamma(N') = a'_\gamma N' (a'_\gamma)^{-1}$. We shall show that $\sigma_\gamma(N') = N'$ and $\sigma_\gamma(H') = H'$ for all $\gamma \in \Gamma$.

Let $\gamma \in \Gamma$. Since μ_γ is σ_γ -equivariant, we see that

$$\text{Stab}_G(\mu_\gamma(gN)) = \sigma_\gamma(\text{Stab}_G(gN)) = \sigma_\gamma(N').$$

Since $\mu_\gamma(gN) = gN$, we see that $\sigma_\gamma(N') = N'$. Since $\sigma_\gamma(N') = a'_\gamma N' (a'_\gamma)^{-1}$, we obtain that $a'_\gamma N' (a'_\gamma)^{-1} = N'$. This means that a'_γ normalizes N' . By assumption $\mathcal{N}_G(N) = N$, hence $\mathcal{N}_G(N') = N'$, and therefore, $a'_\gamma \in N'(k) = \mathcal{N}_G(H')(k)$. We see that a'_γ normalizes H' , i.e. $a'_\gamma H' (a'_\gamma)^{-1} = H'$.

Since $a'_\gamma H' (a'_\gamma)^{-1} = \sigma_\gamma(H')$, we obtain

$$\sigma_\gamma(H') = H'. \tag{3}$$

Since (3) holds for all $\gamma \in \Gamma$, by Galois descent there exists a k_0 -subgroup $H_0 \subseteq G_0$ such that $H_0 \times_{k_0} k = H'$. Thus $gHg^{-1} = H_0 \times_{k_0} k$, as required. ■

Example 6.3. Let k, k_0, G, G_0 be as in Theorem 6.1. Let $H \subseteq G$ be a horospherical subgroup, that is, $H \supseteq U$, where U is the unipotent radical of a Borel subgroup of G . Then $\mathcal{N}_G(H)$ is parabolic; see Pasquier [21, Proposition 2.2].

It follows that $\mathcal{N}_G(H)$ is self-normalizing, and hence, hypothesis (ii) of Theorem 6.1 is satisfied. Thus, for such H , and for quasi-split G_0 , G/H admits a G_0 -equivariant k_0 -model if and only if the combinatorial invariants of G/H (or, equivalently in this case, the *horospherical datum* of G/H ; see [19, Definition 3.6]) are preserved under the $*$ -action (see Moser-Jauslin and Terpereau [19, Corollary 3.12] for this result in the case where $k_0 = \mathbb{R}$). This happens for example for $H = U$ (This can be seen easily: we simply take $U_0 = [B_0, B_0]$). See Example 6.9 below.

Example 6.4. Let k, k_0, G, G_0 be as in Theorem 6.1. In Moser-Jauslin and Terpereau [20, The proof of Corollary 2.7] the authors showed that if $H \subseteq G$ is *symmetric* (i.e., there exists an involutive k -automorphism θ of G such that $G^\theta \subseteq H \subseteq \mathcal{N}_G(G^\theta)$), then condition (ii) in the theorem above is satisfied. Thus for such an H , the homogeneous variety G/H will admit a G_0 -equivariant k_0 -form if and only if for every $\gamma \in \Gamma$, $\sigma_\gamma(H)$ will be conjugate in G to H . This happens for example if G_0 is split. See Example 6.6 below. ■

Example 6.5. Let k, k_0, G be as in Theorem 6.1. Let G_0 be a split k_0 -form of G , then hypothesis (i) is satisfied. Thus, if H has self-normalizing normalizer, which happens e.g., if $H \subseteq G$ is horospherical (see Example 6.3), or if $H \subseteq G$ is symmetric (see Example 6.4), then the subgroup H is conjugate to a subgroup of G_0 defined over k_0 . Hence, in this case, G/H admits a G_0 -equivariant k_0 -form. ■

Example 6.6. Let k_0 be a p -adic field, that is, a finite extension of the field of p -adic numbers \mathbb{Q}_p . Let $G = \text{Spin}_8$ over $k = \overline{k_0}$, and let G_0 be a *trialitary* k_0 -form of G , i.e., a k_0 -form of G where the image of the Galois group in the group of automorphisms of the Dynkin diagram is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ or S_3 . Then G_0 is quasi-split. Indeed, G_0 is an inner twisted form of a quasi-split trialitary k_0 -form G_{qs} of G : we have $G_0 = {}_c G_{\text{qs}}$, where $c \in Z^1(k_0, G_{\text{qs}}/Z(G_{\text{qs}}))$. By Lemma 6.7 below we have $H^1(k_0, G_{\text{qs}}/Z(G_{\text{qs}})) = 1$. It follows that $[c] = 1 \in H^1(k_0, G_{\text{qs}}) = 1$, and hence, $G_0 \simeq G_{\text{qs}}$.

Let $H = \text{Spin}(4) \cdot \text{Spin}(4) \subset \text{Spin}(8)$. Then H is a semisimple subgroup of type A_1^4 , that is, H is isogenous to $\text{SL}(2)^4$. Let $\gamma \in \Gamma = \text{Gal}(k/k_0)$, and let $\sigma_\gamma: G \rightarrow G$ denote the corresponding γ -semilinear automorphism of G . Then $\sigma_\gamma(H)$ is again a semisimple subgroup of type A_1^4 . An easy argument (see Lemma 6.8 below) shows that any two subgroups of type A_1^4 in a group of type D_4 are conjugate. Thus $\sigma_\gamma(H)$ is conjugate to H for all $\gamma \in \Gamma$. Since $H \subset G$ is symmetric, by Remark 1.4, H has self-normalizing normalizer. By Theorem 6.1, G/H admits a G_0 -equivariant k_0 -form. ■

I thank Mikhail Borovoi for help with proving Lemmas 6.7 and 6.8 below.

Lemma 6.7 (well-known). *Let G_{qs} be a quasi-split trialitary k_0 -form of a simply connected algebraic k -group G of type D_4 , where k_0 is a p -adic field. Write $G_{\text{qs}}^{\text{ad}} = G_{\text{qs}}/Z_{\text{qs}}$, where $Z_{\text{qs}} = Z(G_{\text{qs}})$. Then $H^1(G_{\text{qs}}^{\text{ad}}) = 1$.*

Proof. We compute $H^2(k_0, Z_{\text{qs}})$. By the Tate-Nakayama duality (see Serre [25, Section II.5.2, Theorem 2]), $H^2(k_0, Z_{\text{qs}})$ is dual to the group of fixed points $X^*(Z)^\Gamma$, where $Z = Z(G)$ and the Galois action on $X^*(Z)$ is induced from the k_0 -form Z_{qs}

of Z . It is clear that $X^*(Z)^\Gamma = 1$ (notice that $Z(k) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and the Galois action is transitive on $Z(k) \setminus \{e\}$), hence $H^2(k_0, Z_{\text{qs}}) = 1$.

Now we compute $H^1(k_0, G_{\text{qs}}^{\text{ad}})$. The short exact sequence

$$1 \rightarrow Z_{\text{qs}} \rightarrow G_{\text{qs}} \rightarrow G_{\text{qs}}^{\text{ad}} \rightarrow 1$$

gives rise to a cohomology exact sequence

$$1 = H^1(k_0, G_{\text{qs}}) \rightarrow H^1(k_0, G_{\text{qs}}^{\text{ad}}) \rightarrow H^2(k_0, Z_{\text{qs}}) = 1,$$

in which $H^1(k_0, G_{\text{qs}}) = 1$ by Kneser's theorem; see Platonov and Rapinchuk [23, Theorem 6.4]. We conclude that $H^1(k_0, G_{\text{qs}}^{\text{ad}}) = 1$, as required. ■

Lemma 6.8 (well-known). *Any two connected k -subgroups H, H' of type A_1^4 in a simple k -group of type D_4 are conjugate.*

Proof. Let T, T' be maximal tori in H, H' respectively; then T and T' are maximal tori in G . Up to conjugacy, we may assume that $T' = T$. Thus $H, H' \supset T$. We see that H defines a subset $\Theta = R(H, T) \subset R(G, T) = R$ with the following properties: (a) if $\beta \in \theta$, then $-\beta \in \Theta$, and (b) there are four pairwise orthogonal roots $\beta_0, \beta_1, \beta_3, \beta_4$ in Θ . Similarly, H' defines a subset $\Theta' = R(H', T) \subset R$ with the same properties. It suffices to show that there exists an element w of the Weyl group $W(R) = W(G, T)$ sending Θ to Θ' .

We write R as in Bourbaki [10, Plate IV]: $R = \{\pm \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 4\}$.

We set $\Phi = \{\alpha_0 = \varepsilon_1 + \varepsilon_2, \alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_3 = \varepsilon_3 - \varepsilon_4, \alpha_4 = \varepsilon_3 + \varepsilon_4\}$.

Since W acts on R transitively, there exists an element $w \in W$ sending β_0 to α_0 . Up to W -conjugacy we may assume that $\beta_0 = \alpha_0$. Up to sign, we may assume that $\beta_1, \beta_3, \beta_4$ are positive roots. Thus $\beta_1, \beta_3, \beta_4$ are three positive roots in R which are orthogonal to α_0 . Now it is clear that the set $\{\beta_1, \beta_3, \beta_4\}$ coincides with the set $\{\alpha_1, \alpha_3, \alpha_4\}$. It follows that $\Theta = \Phi$.

We have shown that Θ is W -conjugate to Φ . Similarly, Θ' is W -conjugate to Φ . Thus Θ and Θ' are W -conjugate, and hence, H and H' are conjugate in G . ■

Example 6.9. Let k, G, k_0, G_0 be as in Example 6.6. Let $H \subseteq G$ be a horospherical subgroup with horospherical datum (I, M) ; see Moser-Jauslin and Terpereau [19, Definition 3.6]. Here $I \subseteq S$ is a subset of the set of simple roots, and $M \subseteq X^*(B)$ is a sublattice whose elements satisfy $\langle \chi, \alpha \rangle = 0$ for all $\alpha \in I$ (i.e., characters admitting an extension to P_I). Then, since $\mathcal{N}_G(\mathcal{N}_G(H)) = \mathcal{N}_G(H)$, by Theorem 6.1 and Pasquier [21, Proposition 2.4], G/H admits a G_0 -equivariant k_0 -form if and only if I, M are invariant under the $*$ -action. Here are some examples of such (I, M) :

In the notation of Lemma 6.8 (with $\alpha_2 = \varepsilon_2 - \varepsilon_3$), we can take e.g. $I = \{\alpha_2\}$ or $I = \{\alpha_1, \alpha_3, \alpha_4\}$. In the second case, any

$$M \subseteq \text{Span}_{\mathbb{Z}}\{\alpha_1, \alpha_3, \alpha_4\}^\perp = \text{Span}_{\mathbb{Z}}\left\{\frac{\varepsilon_1 + \varepsilon_2}{2}\right\} = \text{Span}_{\mathbb{Z}}\left\{\frac{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4}{2}\right\}$$

would be Γ -invariant. Here are some examples of Γ -invariant M for the case $I = \{\alpha_2\}$, where n is an integer:

- (i) $M_1 = n \cdot \text{Span}_{\mathbb{Z}}\left\{\varepsilon_1 + \frac{\varepsilon_2 + \varepsilon_3}{2}\right\} = n \cdot \text{Span}_{\mathbb{Z}}\left\{\frac{3\alpha_2}{2} + \alpha_1 + \alpha_3 + \alpha_4\right\}$
- (ii) $M_2 = n \cdot \text{Span}_{\mathbb{Z}}\left\{\frac{\varepsilon_2 + \varepsilon_3 - \varepsilon_1 - \varepsilon_4}{2}, \varepsilon_4\right\} = n \cdot \text{Span}_{\mathbb{Z}}\left\{\frac{\alpha_1 - \alpha_3}{2}, \frac{\alpha_3 - \alpha_4}{2}\right\}$
- (iii) $M_3 = M_1 \oplus M_2$, (iv) $M_4 = n \cdot \text{Ker}(\alpha_2^\vee)$.

7. Existence of a form over a large field

Let k, k_0, G, G_0 be as in Subsection 4.1. Write $\overline{G} = G/Z(G)$ for the corresponding adjoint group, and \tilde{G} for the universal covering of $G' = [G, G]$. By Proposition 2.1 we may write $G_0 = {}_c(G_{\text{qs}})$, where G_{qs} is a quasi-split k_0 -form of G and $c \in Z^1(k_0, \overline{G}_{\text{qs}})$. Here we write $\overline{G}_{\text{qs}} = G_{\text{qs}}/Z(G_{\text{qs}})$.

7.1. We write \tilde{Z}_{qs} for the center $Z(\tilde{G}_{\text{qs}})$ of the universal cover \tilde{G}_{qs} of the connected semisimple group $[G_{\text{qs}}, G_{\text{qs}}]$. Note that $\tilde{Z}_{\text{qs}} = \tilde{Z}_0$. The short exact sequence

$$1 \rightarrow \tilde{Z}_{\text{qs}} \rightarrow \tilde{G}_{\text{qs}} \rightarrow \overline{G}_{\text{qs}} \rightarrow 1$$

induces a cohomology exact sequence

$$H^1(k_0, \tilde{Z}_{\text{qs}}) \rightarrow H^1(k_0, \tilde{G}_{\text{qs}}) \rightarrow H^1(k_0, \overline{G}_{\text{qs}}) \xrightarrow{\Delta} H^2(k_0, \tilde{Z}_{\text{qs}}).$$

By definition, the *Tits class* of \tilde{G}_0 is the image of $[c] \in Z^1(k_0, \overline{G}_{\text{qs}})$ in $H^2(k, \tilde{Z}_{\text{qs}})$ under the connecting map $\Delta: H^1(k_0, \overline{G}_{\text{qs}}) \rightarrow \Delta H^2(k_0, \tilde{Z}_{\text{qs}})$; see [15], Section 31, before Proposition 31.7.

7.2. Let $H \subset G$ be a spherical subgroup. Set $A = \mathcal{N}_G(H)/H$, which is a group multiplicative type (i.e. a closed subgroup of a torus); see Losev [16, Theorem 2 and Definition 4.1.1(1)]. The character group $X^*(A)$ of A is a quotient group of the weight lattice \mathcal{X} of $Y = G/H$. The group $X^*(A)$ and the surjective homomorphism $\mathcal{X} \rightarrow X^*(A)$ can be computed from the combinatorial invariants of G/H ; see Losev [16, Theorem 2 and Definition 4.1.1(1)] again.

We use the notation of Subsection 2.4. In particular, for any $\gamma \in \Gamma$ we have an automorphism $\varepsilon_\gamma \in \text{Aut BRD}(G, T, B)$. Assume that ε_γ preserves the combinatorial invariants of G/H for all $\gamma \in \Gamma$. Then ε_γ naturally acts on \mathcal{X} , on $X^*(A)$, and on $X^*(\tilde{Z})$, and the homomorphisms

$$\mathcal{X} \rightarrow X^*(A) \quad \text{and} \quad X^*(A) \rightarrow X^*(\tilde{Z})$$

are Γ -equivariant. Thus we obtain a k_0 -form of A , which we denote A_{qs} , and a k_0 -homomorphism $\tilde{Z}_{\text{qs}} \rightarrow A_{\text{qs}}$.

Theorem 7.1. *Let k be an algebraically closed field of characteristic 0, and let $k_0 \subseteq k$ be a subfield such that k/k_0 is Galois. Denote by $\Gamma = \text{Gal}(k/k_0)$ the Galois group. Let G be a connected reductive k -group, G_0 a k_0 -form of G . Let $H \subseteq G$ be a spherical subgroup. Assume that:*

- (i) Γ preserves the combinatorial invariants of G/H when acting on $\text{BRD}(G)$ via the homomorphism $\varepsilon: \Gamma \rightarrow \text{Aut BRD}(G)$ defined by the k_0 -form G_0 of G ;
- (ii) k_0 is a large field; (iii) $\text{char } k_0 = 0$; (iv) $\mathcal{N}_G(\mathcal{N}_G(H)) = \mathcal{N}_G(H)$.

Then $Y = G/H$ admits a G_0 -equivariant k_0 -form if and only if the image in $H^2(k_0, A_{\text{qs}})$ of the Tits class $t(\tilde{G}_0) \in H^2(k_0, Z(\tilde{G}_0))$ is 1.

Note that assumption (i) is a necessary condition for G/H to have a k_0 -form; see Borovoi [7, Proposition 8.12].

Proof. The k_0 -forms G_0 and G_{qs} of G define the same homomorphism

$$\varepsilon^0 = \varepsilon^{\text{qs}}: \Gamma \rightarrow \text{Aut BRD}(G).$$

Since ε^0 preserves the combinatorial invariants of G/H , so does ε^{qs} . Since G_{qs} is quasi-split, by Theorem 6.1 the subgroup H is conjugate to some subgroup defined over k_0 in G_{qs} , say $H_{\text{qs}} \times_{k_0} k$, where $H_{\text{qs}} \subset G_{\text{qs}}$. Then clearly $G_{\text{qs}}/H_{\text{qs}}$ is a G_{qs} -equivariant k_0 -form of G/H . Recall that $G_0 = {}_c(G_{\text{qs}})$, where $c \in Z^1(k_0, \overline{G_{\text{qs}}})$. By Borovoi and Gagliardi [8, Theorem 3.8] the homogeneous variety G/H admits a G_0 -equivariant k_0 -form if and only if the image of $t(\widetilde{G}_0)$ in $H^2(k_0, A_{\text{qs}})$ is 1, as required. ■

Example 7.2. Let $G = \text{SL}_{2,k}$, and let $H = T$ or $H = \mathcal{N}_G(T)$. Then G/H admits a G_0 -equivariant k_0 -form (for any k_0 and any G_0). Indeed, for any k_0 -form G_0 of G (and any field k_0), G_0 has a maximal torus T_0 defined over k_0 , and thus we obtain a k_0 -form G_0/T_0 of G/T .

See [8] for many more interesting non-quasi-split examples, as well as a generalization of the main theorems presented in this paper to the case where k_0 is an arbitrary field of characteristic 0, and $H \subseteq G$ is an arbitrary spherical subgroup.

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