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Some Harmonic Analysis on Commutative Nilmanifolds

Andrea L. Gallo and Linda V. Saal*

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Abstract. We consider a family of Gelfand pairs $(K \ltimes N, N)$ (in short (K, N)) where N is a two step nilpotent Lie group, and K is the group of orthogonal automorphisms of N. This family has a nice analytic property: almost all these 2-step nilpotent Lie group have square integrable representations. In these cases, following Moore-Wolf's theory, we find an explicit expression for the inversion formula of N, and as a consequence, we decompose the regular action of $K \ltimes N$ on $L^2(N)$. This explicit expression for the Fourier inversion formula of N, specialized to a class of commutative nilmanifolds described by J. Lauret, sharpens the analysis of J. A. Wolf in Section 14.5 in "Harmonic Analysis on Commutative Spaces" [Mathematical Surveys and Monographs 142, American Mathematical Society, Providence (2007)], and in "On the analytic structure of commutative nilmanifolds" [J. Geometric Analysis 26 (2016) 1011–1022], concerning the regular action of $K \ltimes N$ on $L^2(N)$. When N is the Heisenberg group, we obtain the decomposition of $L^2(N)$ under the action of $K \ltimes N$ for all K such that (K, N) is a Gelfand pair. Finally, we also give a parametrization for the generic spherical functions associated to the pair (K, N), and we give an explicit expression for these functions in some cases.

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1. Introduction

Let G be a connected Lie group, and K a compact subgroup of G. It is well known that the following conditions are equivalent:

1. The convolution algebra $L^1(K\backslash G/K)$ is commutative.
2. The algebra $\mathcal{U}(\mathfrak{g})^K$ of K-invariant and left invariant differential operators on G/K is commutative.
3. The regular representation of G on G/K is multiplicity free.
4. For any irreducible unitary representation $(\rho, \mathcal{H})$ of $K \ltimes N$, the space $\mathcal{H}_K := \{v \in \mathcal{H} : \rho(k)\, v = v \text{ for all } k \in K\}$ is at most one dimensional.

When any of the above holds, we say that (G, K) is a Gelfand pair.

Also, G/K is called a nilmanifold if a nilpotent subgroup N of G acts transitively, and we say that G/K is a commutative nilmanifold if (G, K) is a Gelfand pair. In this work, G/K is connected and simply connected and then, N acts simply transitively on G/K and G is the semidirect product $K \ltimes N$. We denote the Gelfand pair $(K \ltimes N, K)$ by (K, N).

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In Vinberg's classification theorem of commutative nilmanifolds, there is a large family that was defined by J. Lauret in [9]. Lauret's construction corresponds to pairs where K is the maximal orthogonal automorphism group, with exception of four cases. As it is proved in [11](pages 339-341), in almost all cases N has a very surprising property: it has square integrable representations, with exception of three cases (two of which are in Lauret's list). For these N, we develop the corresponding harmonic analysis, finding explicitly the inversion formula, and as a consequence we obtain the decomposition of the regular action of G on $L^2(N)$. The remaining inversion formulas of Lauret's list can be found in [12].

We now give a brief summary of the results in this paper. In Section 2, we introduce some preliminaries about the family described by Lauret, and some results concerning nilpotent Lie groups. We present our main result in Section 3. In Section 4, we develop the harmonic analysis in the case (K, H_n), where H_n is the $(2n+1)$-dimensional Heisenberg group. In Section 5, we describe the set of generic spherical functions associated to the Gelfand pair (K, N) of Lauret's list such that N has square integrable representations.

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2. Preliminaries

Let $\mathfrak{n}$ be a two step nilpotent Lie algebra with Lie bracket $[\cdot,\cdot]$ and equipped with an inner product $\langle\cdot,\cdot\rangle$. Then we write $\mathfrak{n} = \mathfrak{z} \oplus V$ where $\mathfrak{z}$ is its center and V is the orthogonal complement of $\mathfrak{z}$. Let N be the connected simply connected Lie group with Lie algebra $\mathfrak{n}$, equipped with the left invariant Riemannian metric determined by $\langle\cdot,\cdot\rangle$.

The group N acts on $\mathfrak{n}$ by the adjoint action Ad, and N acts on $\mathfrak{n}^*$, the dual space of $\mathfrak{n}$, by the dual representation $Ad^*(n)\lambda = \lambda \circ Ad(n^{-1})$. Fixed a non trivial $\lambda \in \mathfrak{n}^*$, let $O_\lambda := \{Ad^*(n)\lambda : n \in N\}$ be its coadjoint orbit.

We denote by $\widehat{N}$ the set of equivalence classes of irreducible unitary representations of N. From Kirillov's theory there is a correspondence between $\widehat{N}$ and the set of coadjoint orbits in $\mathfrak{n}^*$. Indeed, let

$$B_\lambda(x,y) := \lambda([x,y]), \; x,y \in \mathfrak{n}. \tag{1}$$

Let $\mathfrak{m}$ be a maximal isotropic subspace of $\mathfrak{n}$, and set $M = \exp(\mathfrak{m})$. Defining on M the character $\chi_\lambda(\exp y) = e^{i\lambda(y)}$, the irreducible representation corresponding to O_λ is the induced representation $\rho_\lambda := Ind_M^N(\chi_\lambda)$.

Let Z be the center of N. Recall that an irreducible unitary representation is called *square integrable* if its matrix entries are in $L^2(N/Z)$. We denote by $\widehat{N}_{sq}$ the subset of $\widehat{N}$ of square integrable classes. It follows from Moore-Wolf's theory that

(i) If $\rho_\lambda \in \widehat{N}$ has a matrix entry in $L^2(N/Z)$, then $\rho_\lambda \in \widehat{N}_{sq}$.

(ii) If N has a square integrable representation then its Plancherel measure is concentrated on $\widehat{N}_{sq}$.

We have that if ρ_λ is a square integrable representation then B_λ is non degenerate on V and the orbit is maximal, that is $O_\lambda = \lambda\mid_{\mathfrak{z}} + V^*$, where we identify V^* with

$\mathfrak{z}^{\perp} := \{f \in \mathfrak{n}^* : f\,|_{\mathfrak{z}} = 0\}$. Indeed, let $x_\lambda \in \mathfrak{z}$ be the representative of $\lambda|_{\mathfrak{z}}$, that is $\lambda(y) = \langle y, x_\lambda\rangle$ for all $y \in \mathfrak{z}$, and denote by $\mathfrak{z}_\lambda$ the kernel of $\lambda|_{\mathfrak{z}}$. Let $\mathfrak{a}_\lambda$ be the subspace of V where B_λ is degenerate and let $\mathfrak{b}_\lambda$ be the complement of $\mathfrak{a}_\lambda$ in V. Consider $\mathfrak{n}_\lambda = \mathfrak{a}_\lambda \oplus \mathfrak{b}_\lambda \oplus \mathbb{R}x_\lambda$ and $N_\lambda := \exp(\mathfrak{n}_\lambda)$. We equip $\mathfrak{a}_\lambda$ with the trivial Lie bracket and $\mathfrak{h}_\lambda := \mathfrak{b}_\lambda \oplus \mathbb{R}x_\lambda$ with Lie bracket

$$[u,v]_{\mathfrak{h}_\lambda} = B_\lambda(u,v)\,y_\lambda,\ u,v \in \mathfrak{b}_\lambda,\ y_\lambda := \frac{x_\lambda}{|x_{\lambda|}}.$$

It is clear that $\mathfrak{h}_\lambda$ is a Heisenberg algebra and we denote by H_λ the corresponding Heisenberg group and we set $A_\lambda := \exp(\mathfrak{a}_\lambda)$. Since the representation ρ_λ is trivial on $\exp(\mathfrak{z}_\lambda)$, it factors through N_λ. Identifying N_λ with $A_\lambda \times H_\lambda$, we can write $\rho_\lambda(a,n) = \chi(a)\,\rho'_\lambda(n)$ where χ is a unitary character of A_λ and ρ'_λ is an irreducible representation of H_λ. Thus ρ_λ cannot be square integrable unless $\mathfrak{a}_\lambda = 0$.

The reciprocal assertion is also true: if B_λ is non degenerate on V (and thus O_λ is maximal), then ρ_λ gives rise to an irreducible representation of N_λ because λ restricted to $\mathfrak{z}_\lambda$ is trivial. In this case N_λ is a Heisenberg group and since every irreducible representation of infinite dimension of N_λ is square integrable, so is ρ_λ.

This is a particular case of the following general result in the Moore-Wolf's theory. If N is a connected simply connected nilpotent Lie group, the following are equivalent:

(i) ρ_λ is square integrable.

(ii) The orbit O_λ is determined by $\lambda_{|\mathfrak{z}}$.

(iii) B_λ is non degenerate over $\mathfrak{n}/\mathfrak{z}$.

The family to be considered in this work was introduced by J. Lauret (see [9]). Starting from a real representation (π, V) of a compact Lie algebra $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}'$, where $\mathfrak{c}$ is the center of $\mathfrak{g}$ and $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$, let $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and $\langle\cdot,\cdot\rangle_V$ be inner products on $\mathfrak{g}$ and V respectively, such that $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is $ad(\mathfrak{g})$-invariant and $\langle\cdot,\cdot\rangle_V$ is π-invariant. Let $\mathfrak{n} = \mathfrak{g} \oplus V$ and let $\langle\cdot,\cdot\rangle$ be the inner product in $\mathfrak{n}$ such that $\langle\cdot,\cdot\rangle_{\mathfrak{g}\times\mathfrak{g}} = \langle\cdot,\cdot\rangle_{\mathfrak{g}}$, $\langle\cdot,\cdot\rangle_{V\times V} = \langle\cdot,\cdot\rangle_V$ and $\langle\mathfrak{g}, V\rangle = 0$. Such inner product $\langle\cdot,\cdot\rangle$ is called $\mathfrak{g}$-invariant. The Lie algebra structure on $\mathfrak{n}$ is defined by assuming that $\mathfrak{g}$ is the center of $\mathfrak{n}$ and the Lie bracket on V is given by

$$\langle [u,v], x\rangle = \langle \pi(x)\,u, v\rangle \text{ for all } u,v \in V, x \in \mathfrak{g}. \tag{2}$$

We denote by $N(\mathfrak{g},V)$ the connected simply connected Lie group with Lie algebra $\mathfrak{n}$. It is remarked that this construction does not depend on the $\mathfrak{g}$-invariant inner product $\langle\cdot,\cdot\rangle$ (up to Lie group isomorphism). Moreover, if (π, V) and (π', V') are two representations of $\mathfrak{g}$ and there exists an automorphism φ of $\mathfrak{g}$ and an isomorphism $T : V \to V'$ such that $T\pi(x)T^{-1} = \pi'(\varphi(x))$ for all $x \in \mathfrak{g}$, then $N(\mathfrak{g},V)$ and $N(\mathfrak{g},V')$ are isomorphic Lie groups, see [9].

The group of orthogonal automorphisms of $N(\mathfrak{g},V)$ is $K = G' \times U$, where G' is the connected simply connected Lie group with Lie algebra $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ and U is the connected component of the identity of the orthogonal group of intertwining operators of (π, V). The component U acts trivially on the center of $\mathfrak{n}$ and each $g \in G'$ acts on $\mathfrak{n}$ by $(Ad(g), \pi(g))$ where we also denote by π the corresponding representation of G' (see [8], Theorem 3.12).

The group $N(\mathfrak{g}, V)$ is said to be *decomposable* if it is a direct product of Lie groups of the form

$$N(\mathfrak{g}, V) = N(\mathfrak{h}_1, V_1) \times N(\mathfrak{h}_2, V_2).$$

Otherwise we will say that $N(\mathfrak{g}, V)$ is *indecomposable*. The list of Gelfand pairs of the form $(G' \times U, N(\mathfrak{g}, V))$ where $N(\mathfrak{g}, V)$ is indecomposable is the following:

(I) $(SU(2)\times Sp(n), N(\mathfrak{su}(2), (\mathbb{C}^2)^n))$, $n \geq 1$, where $\mathfrak{su}(2)$ acts on $(\mathbb{C}^2)^n$ as $\mathrm{Im}(\mathbb{H})$ acts component-wise on $\mathbb{H}^n$ by quaternion product on the left side, where $\mathbb{H}$ denotes the quaternions and $\mathrm{Im}(\mathbb{H})$ the imaginary quaternions. (Heisenberg type)

(II) $(SU(2) \times Sp(n), N(\mathfrak{su}(2), \mathbb{R}^3 \oplus (\mathbb{C}^2)^n))$, $n \geq 0$, where $\mathfrak{su}(2)$ acts as $\mathfrak{so}(3)$ by rotations on $\mathbb{R}^3$, and $\mathfrak{su}(2)$ acts component-wise on $(\mathbb{C}^2)^n$ in the standard way.

(III) $(Spin(4) \times Sp(k_1) \times Sp(k_2), N(\mathfrak{su}(2) \oplus \mathfrak{su}(2), (\mathbb{C}^2)^{k_1} \oplus \mathbb{R}^4 \oplus (\mathbb{C}^2)^{k_2}))$ with $k_1 + k_2 \geq 1$, where the real vector space $\mathbb{R}^4 = (\mathbb{C}^2 \otimes \mathbb{C}^2)_{\mathbb{R}}$ denotes the standard representation of $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and the first copy of $\mathfrak{su}(2)$ acts only on $(\mathbb{C}^2)^{k_1}$ and the second one only on $(\mathbb{C}^2)^{k_2}$.

(IV) $(Sp(2) \times Sp(n), N(\mathfrak{sp}(2), (\mathbb{C}^4)^n)$, $n \geq 1$, where $\mathfrak{sp}(2)$ acts component-wise on $(\mathbb{H}^2)^n$ in the standard way (identifying $\mathbb{H}^2$ with $\mathbb{C}^4$).

(V) $(SU(n) \times \mathbb{S}^1, N(\mathfrak{su}(n), \mathbb{C}^n))$, $n \geq 3$, where $\mathbb{C}^n$ denotes the standard representation of $\mathfrak{su}(n)$ regarded as a real representation.

(VI) $(SO(n), N(\mathfrak{so}(n), \mathbb{R}^n))$, $n \geq 2$ (free two-step nilpotent Lie group), where $\mathbb{R}^n$ denotes the standard representation of $\mathfrak{so}(n)$.

(VII) $(U(n), N(\mathbb{R}, \mathbb{C}^n))$, $n \geq 1$ (Heisenberg group).

(VIII) $(SU(2) \times U(k) \times Sp(n), N(\mathfrak{u}(2), (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n))$, $k \geq 1, n \geq 0$, where the center of $\mathfrak{u}(2)$ acts non-trivially only on $(\mathbb{C}^2)^k$, in fact, $(\mathbb{C}^2)^n$ denotes the representation of $\mathfrak{su}(2)$ described in the item (I) and $\mathfrak{u}(2)$ acts component-wise on $(\mathbb{C}^2)^k$ in the standard way.

(IX) $(SU(n) \times \mathbb{S}^1, N(\mathfrak{u}(n), \mathbb{C}^n))$, $n \geq 3$, where $\mathbb{C}^n$ denotes the standard representation of $\mathfrak{u}(n)$ regarded as a real representation.

(X) $(G' \times U, N(\mathfrak{g}, V))$ where:

- $\mathfrak{g} := \mathfrak{su}(m_1) \oplus \cdots \oplus \mathfrak{su}(m_\beta) \oplus \mathfrak{su}(2) \oplus \cdots \oplus \mathfrak{su}(2) \oplus \mathfrak{c}$, with α copies of $\mathfrak{su}(2)$, $m_i \geq 3$ for all $1 \leq i \leq \beta$ and $\mathfrak{c}$ is an abelian component.
- $V := \mathbb{C}^{m_1} \oplus \cdots \oplus \mathbb{C}^{m_\beta} \oplus \mathbb{C}^{2k_1+2n_1} \oplus \cdots \oplus \mathbb{C}^{2k_\alpha+2n_\alpha}$, where $k_j \geq 1$ and $n_j \geq 0$ for all $1 \leq j \leq \alpha$.
- $\mathfrak{g}$ acts on V as follows: for each $1 \leq i \leq \beta + \alpha$, $\mathfrak{c}$ has a maximal subespace, denoted by $\mathfrak{c}_i$, and $\dim(\mathfrak{c}_i)$=1, acting non-trivially only on $\mathbb{C}^{m_i}$ (as the representation described in item (V)) and for $\beta + 1 \leq i \leq \beta+\alpha$, $\mathfrak{su}(2)\oplus\mathfrak{c}_i$ acts non-trivially only on $\mathbb{C}^{2k_i+2n_i}$ (as the representation in item (VIII)).
- $U := \mathbb{S}^1 \times \cdots \mathbb{S}^1 \times U(k_1) \times sp(n_1) \times \cdots \times U(k_\alpha) \times Sp(n_\alpha)$, with β copies of $\mathbb{S}^1$.

Here we only consider the groups $N(\mathfrak{g}, V)$ which do have *square integrable representations*; this condition holds for all $N(\mathfrak{g}, V)$ in the family, with the exception of two cases: Case II and Case VI with n odd as is shown in [11], pages 339–341. From Moore-Wolf's theory it follows that the Plancherel measure for $N(\mathfrak{g}, V)$ is concentrated on the equivalent classes of square integrable representations, which are parametrized by the elements of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$.

We denote by N the group $N(\mathfrak{g}, V)$, and we let $(\rho_\lambda, \mathcal{H}_\lambda)$ be the irreducible representation of N corresponding to λ, with $\lambda \in \mathfrak{n}^*$. For $k \in K$, let $\rho_\lambda^k(n) := \rho_\lambda(k \cdot n)$. So ρ_λ^k is another irreducible representation of N acting on $\mathcal{H}_\lambda$, and the stabilizer of ρ_λ is

$$K_{\rho_\lambda} := \{k \in K \; : \; \rho_\lambda^k \text{ is equivalent to } \rho_\lambda\}.$$

Thus, for $k \in K_{\rho_\lambda}$ there exists a unitary operator $\varpi(k)$ which intertwines ρ_λ and ρ_λ^k. This gives rise to a non projective representation ϖ_λ (see Theorem 2.3 in [3]) of K_{ρ_λ} called the *metaplectic* representation.

There is an action of K on $\mathfrak{n}^*$ defined by $(k \cdot \lambda)(x) = \lambda(k^{-1} \cdot x)$, so that we have $K_{\rho_\lambda} = \{k \in K \mid k \cdot \lambda \in O_\lambda\}$. Since the equivalent class of ρ_λ depends only of $\lambda\mid_{\mathfrak{g}^*}$, if x_λ is the vector in $\mathfrak{g}$ such that $\lambda(x) = \langle x, x_\lambda \rangle$ for all $x \in \mathfrak{g}$, clearly K_{ρ_λ} coincides with the stabilizer of x_λ, $K_\lambda := \{k \in K : k \cdot x_\lambda = x_\lambda\}$. If $y_\lambda = \frac{x_\lambda}{|x_\lambda|}$, let N_λ be the Heisenberg group with Lie algebra $\mathfrak{n}_\lambda = \mathbb{R} y_\lambda \oplus V$ and Lie bracket

$$[u, v]_\lambda = B_\lambda(u, v)\, y_\lambda, \; u, v \in V.$$

Notice that for $k \in K_\lambda$ and $u, v \in V$, we have $B_\lambda(k \cdot u, k \cdot v) = \langle x_\lambda, [k \cdot u, k \cdot v] \rangle = \langle x_\lambda, k[u, v] \rangle = \langle k^{-1} x_\lambda, [u, v] \rangle = B_\lambda(u, v)$. Thus K_λ is contained in the symplectic group $Sp(B_\lambda)$.

Let $\mathfrak{z}_\lambda = Ker(\lambda\mid_{\mathfrak{g}})$. Since λ restricted to $\mathfrak{z}_\lambda$ is trivial, ρ_λ is an irreducible representation of N_λ, and the metaplectic action of K_{ρ_λ} coincides with the metaplectic action of K_λ. Moreover, if π_s denotes the irreducible representation of N_λ such that $\pi_s(t, 0) = e^{ist}$ realized on the Fock space of holomorphic (resp. antiholomorphic) functions on $\mathbb{C}^n$ which are square integrable with respect to the measure $e^{-|\lambda|(|z|^2/2)}$, we have

$$\rho_\lambda(z, 0) = \rho_\lambda(\langle z, y_\lambda \rangle y_\lambda, 0) = e^{i\langle z, y_\lambda \rangle \lambda(y_\lambda)} = e^{i|\lambda|\langle z, y_\lambda \rangle},$$

that is $\rho_\lambda(z, 0) = \pi_{|\lambda|}(\langle z, y_\lambda \rangle, 0)$. Therefore $\rho_\lambda(z, v) = \pi_{|\lambda|}(\langle z, y_\lambda \rangle, v)$.

For fixed $\lambda \in \mathfrak{n}^*$, let $\mathcal{H}_\lambda = \oplus_{j \in \Lambda} W_{\lambda, j}$ be the decomposition of the metaplectic representation ϖ_λ into irreducible K_λ-modules. For $j \in \Lambda$, let $d_j = dim(W_{\lambda, j})$ and let $\{v_l^j\}_{l=1}^{d_j}$ be an orthonormal basis of $W_{\lambda, j}$. We define

$$\psi_{x_\lambda, j}(n) = \frac{1}{d_j} \sum_{l=1}^{d_j} \langle \rho_\lambda(n) v_l^j, v_l^j \rangle. \tag{3}$$

3. The main result

We begin this section by recalling some results of the Moore-Wolf theory developped in [10], which will be applied to our groups $N(\mathfrak{g}, V)$.

Since the product in N is given by $(x, v)(x_0, v_0) = (x + x_0 + \frac{1}{2}[v, v_0], v + v_0)$ for $v, v_0 \in V$ and $x, x_0 \in \mathfrak{g}$, the Haar measure dn on N is the Lebesgue measure, and we denote by dx (resp dv) the Lebesgue measure on $\mathfrak{g}$ (resp. on V) so that $dn = dxdv$.

We select Lebesgue measures dn^*, dx^* and dv^* on $\mathfrak{n}^*, \mathfrak{g}^*$ and V^* respectively, such that the Fourier transform of functions on $\mathfrak{n}, \mathfrak{g}$ or V into functions on $\mathfrak{n}^*, \mathfrak{g}^*$, V^* be an isometry.

For $\lambda \in \mathfrak{n}^*, u, v \in V$, let $B_\lambda(u,v) := \lambda([u,v])$. Since B_λ is a non degenerate skew-symmetric form on $V \times V$, the corresponding two form ϖ_λ is non degenerate and ϖ_λ^m is a multiple of dv. The Pfaffian $Pf(B_\lambda)$ of B_λ is by definition this multiple, that is, $\varpi_\lambda^m = Pf(B_\lambda)\,dv$. One then has that $\det\left(B_{\lambda|V\times V}\right) = Pf(B_\lambda)^2$.

Let $P(\lambda) := Pf(B_\lambda)$. Then $P(\lambda)$ is a homogeneous polynomial function on $\mathfrak{n}^*$. Moreover, it follows from Lemma 3.2 in [10] that $P(\lambda)$ depends only on the restriction of λ to $\mathfrak{g}$. Hence there is a homogeneous polynomial on $\mathfrak{g}^*$, also denoted by P, such that $P(\lambda) = P(\lambda \mid \mathfrak{g})$.

Let $\mathcal{V} = \{\lambda : P(\lambda) \neq 0\}$. By Theorem 2 in [10] we know that there is a correspondence between the coadjoint orbits O_λ such that $P(\lambda) \neq 0$ and the set of square integrable representations. Moreover if ϕ is the map which sends $\lambda\,|_{\mathfrak{g}} \in \mathfrak{g}^* \setminus \mathcal{V}$ to $[\rho_\lambda] \in \widehat{N}_{sq}$, then ϕ is a homeomorphism from $\mathfrak{g}^* \setminus \mathcal{V}$ with the natural topology to the Fell topology on representations.

Assume that N has square integrable representations. Then in Theorem 6 it is proved that the Plancherel measure is concentrated on $\widehat{N}_{sq}$ and its image under ϕ^{-1} is $m!\, 2^m P(\lambda)\, dx^*(\lambda)$ where dx^* is the Lebesgue measure chosen as above.

Since we are interested in the inversion formula for a Schwartz function on N, we recall some lines of the proof.

The form B_λ determines an isomorphism T_B of V into V^* by $T_B(u)(v) = B_\lambda(u,v)$, for $u, v \in V$. For $\lambda\,|_{\mathfrak{g}} \in \mathfrak{g}^* \setminus \mathcal{V}$ the orbit O_λ is the hyperplane $(\lambda \mid \mathfrak{g}) + V^*$ of $\mathfrak{n}^*$ (once we have identified $\mathfrak{g}^\perp$ naturally with V^*). Then $B_{\lambda|V\times V}$ is transported via T_B to a bilinear form on the tangent space of O_λ at the point $\lambda\,|_{\mathfrak{g}}$ which, in turn, defines a two form ϖ_λ^* so that $(\varpi_\lambda^*)^m$ determines a measure on O_λ (or a measure on $\mathfrak{n}^*$ supported on O_λ,) called the *canonical* measure on O_λ and which we denote by μ_λ. It follows from Lemma 3.1 in [10] that if dv^* is the Lebesgue measure on V^* and dv_λ is the translated measure on O_λ, then $\mu_\lambda = |P(\lambda)|^{-1}\, dv_\lambda$.

Let f be a Schwartz function on N and let $f_0(y) := f(\exp y)$ be the corresponding function on $\mathfrak{n}$. Then by general theory

$$tr(\rho_\lambda(f)) := \Theta(f) = c^{-1} \int_{O_\lambda} \widehat{f_0}(y)\, d\mu_\lambda(y), \quad \text{where } \widehat{f_0}(y) := \int_n f(x)\, e^{2\pi i y(x)} dx$$

and
$$f(e) = \int_{\mathfrak{g}^*\setminus\mathcal{V}} tr(\rho_\lambda(f))\, d\mu(\lambda),$$

where $d\mu$ is the Plancherel measure and $c = m!\; 2^m$. Then, on the one hand, we have $f(e) = c^{-1} \int_{\mathfrak{g}^*\setminus\mathcal{V}} \left(\int \widehat{f_0}(y)\, |P(\lambda)|^{-1}\, dv_\lambda(y)\right) d\mu(\lambda)$, and by the inversion formula on the vector group $\mathfrak{n}^*$, $f(e) = \int_{\mathfrak{g}^*\setminus\mathcal{V}} \left(\int \widehat{f_0}(y)\, dv_\lambda(y)\right) dx^*(\lambda)$. Hence we have $d\mu = m!\; 2^m\, |P(\lambda)|\, dx^*(\lambda)$. Then, the inversion formula for a Schwartz function f is

$$f(n) = m!\; 2^m \int_{\mathfrak{g}} tr(\rho_\lambda(f)\rho_\lambda(n))|P(\lambda)|dx^*(\lambda) \quad \text{for } n \in N.$$

Decomposing the metaplectic action of K_λ on the Fock space as

$$\mathcal{H}_\lambda = \bigoplus_{j\in\Lambda} W_{\lambda,j},$$

we obtain by a straightforward computation that

$$f(n) = m!\, 2^m \sum_{j\in\Lambda} d_j \int_{\mathfrak{g}} f * \psi_{x_\lambda,j}(n)\, |P(\lambda)|\, dx^*(\lambda) \tag{4}$$

where $\psi_{x_\lambda,j}$ is defined in (3).

From now on, we will use the notation $P(\lambda)$ (resp. $\rho(\lambda)$, $\psi_{\lambda,j}$, etc) or $P(x_\lambda)$ (resp. $\rho(x_\lambda)$, $\psi_{x_\lambda,j}$, etc) interchangeably.

Lemma 3.1. *If $g \in G$ and $x_\lambda \in \mathfrak{g}$ then $|P(Ad(g)\,x_\lambda)| = |P(x_\lambda)|$.*

Proof. Since for $u, v \in V$, $B_\lambda(u,v) = \lambda([u,v]) = \langle [u,v], x_\lambda\rangle$, we have that

$$\begin{aligned} B_{Ad(g)x_\lambda}(u,v) &= \langle [u,v], Ad(g)\,x_\lambda\rangle = \left\langle Ad\left(g^{-1}\right)[u,v], x_\lambda\right\rangle \\ &= \left\langle \left[\pi\left(g^{-1}\right)u, \pi\left(g^{-1}\right)v\right], x_\lambda\right\rangle = \lambda\left(\left[\pi\left(g^{-1}\right)u, \pi\left(g^{-1}\right)v\right]\right), \end{aligned}$$

where in the third equality we have used that $(Ad(g), \pi(g))$ is an automorphism of N. Therefore $|P(Ad(g)\,x_\lambda)| = |P(x_\lambda)|$, as desired. ■

Recall that $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{c}$ where $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ and $\mathfrak{c}$ is the center of $\mathfrak{g}$. Thus if we denote by $\mathfrak{g}'_r$ the set of regular elements of $\mathfrak{g}'$, we can consider that the Plancherel measure is defined on $\mathfrak{g}'_r \oplus \mathfrak{c}$, since the complement of $\mathfrak{g}'_r$ in $\mathfrak{g}'$ has Lebesgue measure zero.

Let T be a maximal torus of G' with Lie algebra $\mathfrak{h}$. Denote by $\mathfrak{g}'_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ the complexified Lie algebras of $\mathfrak{g}'$ and $\mathfrak{h}$ respectively, and by Δ the root system corresponding to $(\mathfrak{g}'_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Let $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{h}$ and let $\mathfrak{C}$ be a fixed Weyl chamber of $\mathfrak{h}_{\mathbb{R}}$.

Let $\Phi : G'/T \times \mathfrak{C} \to \mathfrak{g}'_r$ be the function defined by $\Phi(gT, x) = Ad(g)x$, with $g \in G, x \in \mathfrak{C}$. It follows from basic Lie theory that the map Φ is a diffeomorphism, and a computation shows that $\det(d\Phi_{(g,ix)}) = (-1)^{\#\Delta}\prod_{\alpha\in\Delta}\alpha(x)$.

Set $\theta(x) := |\det(d\Phi_{(g,x)})|$. Integrating on $\mathfrak{g}$ in place of $\mathfrak{g}^*$ and using change of variables we obtain

$$f(n) = m!\, 2^m \sum_{j\in\Lambda} d_j \int_{\mathfrak{c}}\int_{G'/T}\int_{\mathfrak{C}} f * \psi_{(Ad(g)x+z),j}(n)\, |P(Ad(g)\,x+z)|\, \theta(x)\; dx\; d\dot{g}\; dz.$$

where $d\dot{g}$ denotes the G'-invariant measure on G'/T, and dx, dz are the Lebesgue measures on $\mathfrak{C}$ and $\mathfrak{c}$ respectively. By Lemma 3.1

$$f(n) = m!\, 2^m \sum_{j\in\Lambda} d_j \int_{\mathfrak{c}}\int_{\mathfrak{C}} f * \left(\int_{G'/T} \psi_{Ad(g)(x+z),j}(n)\, d\dot{g}\right) |P(x+z)|\, \theta(x)\; dxdz. \tag{5}$$

Recall that for $k \in K$, ρ_λ^k is the irreducible representation of N corresponding to $k\cdot\lambda$, and thus if $\lambda|_{\mathfrak{g}}$ is represented by the vector $x+z$ then $k\cdot\lambda$ corresponds to $k\cdot(x+z)$. Since $(Ad(g), \pi(g))$ is an automorphism of N, we have that $\rho_{Ad(g)(x+z)}(n) = \rho_{x+z}^{Ad(g)}(n) = \rho_{x+z}(Ad(g)\cdot n)$, so $\psi_{Ad(g)(x+z),j}(n) = \psi_{x+z,j}(Ad(g)\cdot n)$.

We will now express (5) in terms of the set of spherical functions associated to the pair (K,N).
Let $C_c^\sharp(N)$ be the algebra of K-invariant continuous functions on N with compact support. We say that a K-invariant continuous function ϕ on N is a *spherical function* if the linear functional $\chi(f) := \int f(n)\phi(n^{-1})\,dn$ is a non trivial character of $C_c^\sharp(N)$. It is well known that the set of bounded spherical functions can be identified with the homomorphisms of the space of the K-invariant integrable functions on N via the map

$$\phi \longrightarrow \chi(f) = \int f(n)\phi(n^{-1})\,dn.$$

Lemma 3.2. (i) *If $x_\lambda = x' + z$, with $x' \in \mathfrak{g}'_r$, $x' \neq 0$ and $z \in \mathfrak{c}$, then we have $K/K_\lambda = G'/T$. Moreover,*

$$\phi_{\lambda,j}(n) := \int_{G'/T} \psi_{Ad(g)(x+z),j}(n)\,d\dot{g}$$

is a spherical function of (K,N).

(ii) *If $x_\lambda \in \mathfrak{c}$, then $K_\lambda = K$. In particular, if $\lambda \in \mathfrak{c}^*$, $\phi_{\lambda,j} = \psi_{\lambda,j}$.*

Proof. (i) Let $C_{G'}(x_\lambda) = \{g \in G' : Ad(g)\,x_\lambda = x_\lambda\}$ be the centralizer of x_λ in G'. Since U acts on $\mathfrak{g}$ by the identity, $K_\lambda = C_{G'}(x_\lambda) \times U$ and since x' is a regular element, $C_{G'}(x_\lambda) = C_{G'}(x')$ is a maximal torus of G'.

The description of the bounded spherical functions of a Gelfand pair (K,N) is given in Theorem 8.7 in [1]. Indeed, let $(\rho,\mathcal{H}_\lambda) \in \widehat{N}$, let $\mathcal{H}_\lambda = \oplus_{j\in\Lambda} W_{j,\lambda}$ be the decomposition of the metaplectic representation of K_{ρ_λ} into irreducible components and let $\{v_1^j,\cdots,v_{d_j}^j\}$ be an orthonormal basis of $W_{\lambda,j}$. Then the proof of Theorem 8.7 shows that the spherical functions are given by

$$\phi_{\lambda,j}(n) = \int_{K/K_{\rho_\lambda}} \frac{1}{d_j}\sum_{l=1}^{d_j}\langle \rho_\lambda(\dot{k}\cdot n)v_l^j, v_l^j\rangle\,d\dot{k}, \tag{6}$$

where $\dot{k}$ denotes the K-invariant measure on K/K_{ρ_λ}. In our case $K_{\rho_\lambda} = K_\lambda$, $K/K_\lambda = G'/T$ and

$$\int_{G'/T} \psi_{Ad(g)(x'+z),j}(n)\,d\dot{g} = \int_{G'/T} \psi_{x'+z,j}(Ad(g)\cdot n)\,d\dot{g} = \int_{K/K_\lambda} \psi_{x'+z,j}(k\cdot n)\,d\dot{k},$$

which implies assertion (i).

(ii) As above, $K_\lambda = C_{G'}(x_\lambda) \times U$, and since $x_\lambda \in \mathfrak{c}$, $C_{G'}(x_\lambda) = G'$ and $K_\lambda = K$. ■

Remark 3.3. For $x \in \mathfrak{C}$, $z \in \mathfrak{c}$, it would be more proper to denote by $\phi_{x,z,j}$ the spherical function corresponding to $x_\lambda = x + z$. But this notation seems to be cumbersome and we prefer to set $\phi_{x,z,j} := \phi_{\lambda,j}$. ■

We denote by $V_{\mathbb{C}}$ the complexification of V and by $(\pi_{\mathbb{C}}, V_{\mathbb{C}})$ the extension of π to $\mathfrak{g}_{\mathbb{C}}$. Let $V_{\mathbb{C}} = \oplus_r W_r$ be the decomposition into irreducible subespaces and we let $W_r = \oplus_j W^{\nu_r^j}$ be the decomposition into weight spaces, that is

$$W^{\nu_r^j} := \{v \in W_r \mid \pi_{\mathbb{C}}(x)(v) = \nu_r^j(x)v \text{ for all } x \in \mathfrak{h}_{\mathbb{C}}\}.$$

Then we have $$P(x)=\prod_{r,j}|\nu_r^j(x)|^{m_r^j/2},$$

for $x\in\mathfrak{C}$, where m_r^j is the dimension of W_r^j. Let ζ_r be the central character of $\pi_C|_{W_r}$. Also, for $z\in\mathfrak{c}$ and $x\in\mathfrak{C}$, we have

$$P(x+z)=\prod_{r,j}|\nu_r^j(x)+\zeta_r(z)|^{m_r^j/2} \tag{7}$$

Hence, we have proved our main result:

Theorem 3.4. *Let f be a Schwartz function on N. Then*

$$f(n)=m!\;2^m\sum_{j\in\Lambda}d_j\int_{\mathfrak{c}}\int_{\mathfrak{C}}f*\phi_{\lambda,j}(n)\,|P(x+z)|\;\theta(x)\,dx\,dz,$$

where $\phi_{\lambda,j}$ is the spherical function defined as in (6) *and the function P is as* (7)*. The support of the Plancherel measure is $\Lambda\times\mathfrak{C}\times\mathfrak{c}$, and the measure is given by the product of the weighted counting measure and $d\mu(\lambda)=|P(x+z)|\,\theta(x)\,dx\,dz$.*

We write $\lambda=\lambda'+\lambda_0$, with $\lambda'\in[\mathfrak{g},\mathfrak{g}]^*$, and $\lambda_0\in\mathfrak{c}^*$. As a consequence of the previous result we obtain the decomposition of the regular action on $L^2(N)$.

Theorem 3.5. *Let $\mathfrak{g}$ be any compact Lie algebra that appears in Lauret's list and such that the corresponding $N(\mathfrak{g},V)$ has square integrable representations. Then the regular action of $K\ltimes N$ on $L^2(N)$ decomposes as a direct integral of irreducible components by*

$$L^2(N)=\sum_{j\in\Lambda}\int_{\mathfrak{c}}\int_{\mathfrak{C}}\mathcal{H}_{\lambda,j}\;\;d\mu(\lambda),$$

where μ is the measure $\mu(\lambda)=|P(\lambda)|\,\theta(\lambda')\,d\lambda$ and $d\lambda$ is the Lebesgue measure on $\mathfrak{c}\times\mathfrak{C}$. Moreover, the projection on $\mathcal{H}_{\lambda,j}$ is $Q_{\lambda,j}(f)=f*\phi_{\lambda,j}$, where $\phi_{\lambda,j}$ is the spherical function given by the following:

(i) *If $\lambda'\neq 0$,*
$$\phi_{\lambda,j}(n)=\int_{G'/T}\psi_{\lambda,j}(\dot g\cdot n)\,d\dot g, \tag{8}$$
where $g\cdot n$ denotes the action of G' by automorphism on N, $d\dot g$ is the G'-invariant measure on G'/T and $\psi_{\lambda.j}$ is as in (3)*.*

(ii) *Assume that $\lambda'=0$. If $\mathfrak{g}$ is as in Case* VIII *with $k\geq 1$ and $n=0$, Case* IX *and Case* X *with $k_j\geq 1$ and $n_j=0$ for all $1\leq j\leq\alpha$, then $\phi_{\lambda,j}=\psi_{\lambda,j}$ with $\psi_{\lambda,j}$ as in* (3)*.*

In Case VII*, $\mathfrak{g}=\mathbb{R}$, and $\phi_{\lambda,j}=\psi_{\lambda,j}$ with $\psi_{\lambda,j}$ as in* (3)*.*

In the other cases, the Plancherel measure vanishes on $\mathfrak{c}$.

Proof. (i) Follows from Theorem (3.4).

(ii) In Cases I, III, IV ,V and VI with n even $\mathfrak{g}$ is semisimple and has trivial center. In Case IX, $\mathfrak{g}=\mathfrak{u}(n)=\mathfrak{su}(n)\oplus i\mathbb{R}, V=\mathbb{C}^n$, $n\geq 3$, where $\mathbb{C}^n$ denotes the standard representation of $\mathfrak{u}(n)$. Since $Ker(\pi(x))$ is trivial for all $x\in\mathfrak{u}(n)$, it follows that B_λ is non degenerate. Then, the Plancherel measure is concentrated in $\mathfrak{g}=i\mathbb{R}\oplus[\mathfrak{g},\mathfrak{g}]$. The expression of the spherical functions follows from Lemma 3.2(ii).

In Case VIII with $k \geq 1,\ n = 0$, and case X with $k_j \geq 1,\ n_j = 0$ for all $1 \leq j \leq \alpha$ the analysis is similar to that in case IX since π has trivial kernel.

In Case VIII with $k \geq 1, n > 0,\ \mathfrak{g} = \mathfrak{u}(2) = \mathfrak{su}(2) \oplus i\mathbb{R},\ V = (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n$. The center of $\mathfrak{u}(2)$ acts non-trivially only on $(\mathbb{C}^2)^k$, in fact, $\mathfrak{su}(2)$ acts on $(\mathbb{C}^2)^n$ as $\mathrm{Im}(\mathbb{H})$ acts component-wise on $\mathbb{H}^n$ by quaternion product on the left hand side. Thus, if $t \in \mathbb{R},\ \pi(it)(0,v) = (0,0)$ for all $v \in (\mathbb{C}^2)^n$, that is, $(0,v) \in Ker(\pi(it))$. For (2) and (1) it follows that B_{it} is degenerate for all $it \in i\mathbb{R}$. Then, by Theorem 6 in [10], the Plancherel measure is concentrated in $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.

In Case X with $k_j \geq 1$ for all $1 \leq j \leq \alpha$ and $n_{j_0} > 0$ for some $1 \leq j_0 \leq \alpha$ the analysis is similar to that in case VIII with $n > 0$.

Case VII corresponds to the Heisenberg group, and it is proved in Section 4, Theorem 4.1. ■

4. The Heisenberg case

We take $\mathfrak{g} = \mathbb{R}$ and $V = \mathbb{C}^n$ with the standard Hermitian form $(u,v) = Re(\sum_{i=1}^n u_i \overline{v}_i)$, where u_i, v_i are the coordinates of $u, v \in \mathbb{C}^n$ respectively and let π defined by $\pi\,(t)\,v = itv$, for $t \in \mathbb{R}$. In this case we have that

$$\langle t, [u,v]\rangle = (\pi\,(t)\,u, v) = t\,(iu, v) = -t\,\mathrm{Im}(u.\overline{v}).$$

Thus, the bracket is given by the standard simplectic form and the corresponding group $N(\mathfrak{g},V)$ is the $(2n+1)$-dimensional Heisenberg group.

It is known that the unitary irreducible representations of H_n are of two types: those of infinite dimension acting non trivially on the center and the characters $\chi_w(t,v) = e^{i\mathrm{Re}(v.\bar{w})}$. The unitary irreducible representations of infinite dimension $(\pi_\lambda, \mathcal{H}_\lambda)$ of H_n are parametrized by $0 \neq \lambda \in \mathbb{R}$. More explicitly, for $\lambda > 0$ (resp. $\lambda < 0$), they are realized on the Fock space of holomorphic (resp. antiholomorphic) functions on $\mathbb{C}^n$ which are square integrable with respect to the measure $e^{-|\lambda|(|z|^2/2)}$ and they are determined by the central action. We denote by $\mathcal{P}(V)$ the polynomial algebra which is dense in $\mathcal{H}_\lambda$.

Let $K \subseteq U\,(n)$ such that (K, H_n) is a Gelfand pair. For $k \in K$ and $(t,v) \in H_n$, we can define $\pi_\lambda^k\,(t,v) := \pi_\lambda\,((t,kv))$. Since $\pi_\lambda\,(t,0) = e^{i\lambda t}$, we have that $K = \left\{k \in K : \pi_\lambda^k \sim \pi_\lambda\right\}$. For $p \in \mathcal{P}\,(V)$ and $k \in K$ we define

$$\varpi\,(k)\,p\,(v) = p\left(k^{-1}v\right). \tag{9}$$

Then ϖ extends to a unitary representation of K, called the metaplectic representation, which intertwines π_λ^k and π_λ. According to Mackey's theory, the irreducible unitary representations of $K \ltimes H_n$ are induced by those of H_n.

For $\sigma \in \widehat{K}$, the irreducible representations of $K \ltimes H_n$ induced by $\mathcal{H}_\lambda$ are defined by

$$\rho_{\lambda,\sigma}\,(k,t,v) = \sigma\,(k) \otimes \varpi\,(k)\,\pi_\lambda\,(t,v)\,, \quad k \in K,\ (t,v) \in H_n.$$

Thus $\rho_{\lambda,\sigma}$ has a vector fixed by K if and only if σ is the dual representation of some irreducible component of ϖ.

Since the other elements of $\widehat{K \ltimes H_n}$ are induced by the characters of $\mathbb{R}^{2n}$, and $(K \ltimes \mathbb{R}^{2n}, K)$ is always a Gelfand pair, we have that (K, H_n) is a Gelfand pair if and only if ϖ is multiplicity free.

Let
$$\varpi \downarrow \mathcal{H}_\lambda = \bigoplus_{j\in\Lambda} \varpi_j$$
be the decomposition of ϖ into irreducible components. We denote by W_j the representation space of ϖ_j and by ϖ_j' its dual representation.

We select an orthonomal basis $\left\{h_1, ..., h_{d_j}\right\}$ of W_j, and let $\left\{h_1^*, ..., h_{d_j}^*\right\}$ be its dual basis. It follows immediately that $s_j = \sum_{l=1}^{d_j} h_l \otimes h_l^*$ is a vector of $\rho_{\lambda,\varpi_j'}$ fixed by K. In order to simplify the notation, we set $\rho_{\lambda,j} := \rho_{\lambda,\varpi_j'}$. The spherical function corresponding to s_j is $\phi_{\lambda,j}(t,v) = \langle \rho_{\lambda,j}(k,t,v)\, s_j, s_j\rangle$ and an easy computation gives

$$\phi_{\lambda,j}(t,v) = \frac{1}{d_j}\sum_{i=1}^{d_j} \langle \pi_\lambda(t,v)\, h_i, h_i\rangle. \tag{10}$$

For $h, h' \in \mathcal{H}_\lambda$, let $e_\lambda(h,h')(t,v) := \langle \pi_\lambda(t,v)\, h, h'\rangle$ the entry matrix of π_λ associated to h, h'. It is well known that the functions $v \longmapsto e_\lambda(h,h')(0,v) \in L^2(\mathbb{C}^n)$,

$$\int_{\mathbb{C}^n} e_\lambda(h,h')(0,v)\, \overline{e_\lambda(h_1,h_1')(0,v)}dv = d_\lambda^{-1}\langle h, h_1\rangle \overline{\langle h', h_1'\rangle}$$

and if $\lambda \neq \lambda_1$
$$\int_{\mathbb{C}^n} e_\lambda(h,h')(0,v)\, \overline{e_{\lambda_1}(h_1,h_1')(0,v)}dv = 0$$

for all $h, h' \in \mathcal{H}_\lambda$, $h_1, h_1' \in \mathcal{H}_{\lambda_1}$ (d_λ is called the *formal degree* of π_λ). For $j \in \Lambda$, we select a basis $\mathcal{B}_j$ of W_j conveniently normalized such that

$$\int_{\mathbb{C}^n} |e_\lambda(h_\alpha, h_\beta)|^2 (0,v)\, dv = 1$$

for all $h_\alpha, h_\beta \in \mathcal{B}_j$. Let $\mathcal{B} := \cup_j \mathcal{B}_j$, then $\mathcal{B}$ is a basis of $\mathcal{H}_\lambda$. Recall that the convolution is defined for integrable functions for $x \in H_n$ by

$$(f * g)(x) = \int_{H_n} f(y) g(y^{-1}x)\ dy\,.$$

Theorem 4.1. *Let $H_{\lambda,j}$ be the Hilbert space generated by $\{e_\lambda(h_\alpha, h_\beta)\}$ where $h_\alpha \in \mathcal{B}_j$ and $h_\beta \in \mathcal{B}$, with inner product $\langle \varphi, \psi\rangle_{\lambda,j} := \int_{\mathbb{C}^n} \varphi(0,v)\, \overline{\psi(0,v)}dv$. Then, the regular action of $K \ltimes N$ decomposes as a direct integral of irreducible components:*

$$L^2(H_n) = \sum_{j\in\Lambda} \int_{-\infty}^{\infty} H_{\lambda,j}\, |\lambda|^n\, d\lambda\,.$$

Moreover, the Hilbert space $H_{\lambda,j}$ is primary and equivalent to $(\dim W_j)\, \mathcal{F}_\lambda$ as H_n-module.

Proof. The inversion formula for a Schwartz function f on H_n is given by

$$f(t,v) = \int_{-\infty}^{\infty} tr\,(\pi_\lambda(t,v)\, \pi_\lambda(f))\, |\lambda|^n\, d\lambda\,.$$

Moreover,
$$\|f\|^2 = \int_{-\infty}^{\infty} \|\pi_\lambda(f)\|_{HS}^2\, |\lambda|^n\, d\lambda = \sum \int_{-\infty}^{\infty} |\langle f, e_\lambda(h_\alpha, h_\beta)\rangle|^2\, |\lambda|^n\, d\lambda,$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm, and the sum runs on $h_\alpha, h_\beta \in \mathcal{B}$. Notice that $\langle \pi_\lambda(t,v)\pi_\lambda(f)h, h'\rangle = (f * e_\lambda(h,h'))(t,v)$, then

$$\begin{aligned} f(t,v) &= \sum_{j\in\Lambda}\int_{-\infty}^{\infty}\sum_{h_\alpha\in\mathcal{B}_j}(f*e_\lambda(h_\alpha,h_\alpha))(t,v)\,|\lambda|^n\,d\lambda \\ &= \sum_{j\in\Lambda}\int_{-\infty}^{\infty} d_j\,(f*\phi_{\lambda,j})(t,v)\,|\lambda|^n\,d\lambda \end{aligned}$$

where the last equality follows from (10). By straightforward computation we obtain

$$f * e_\lambda(h_\alpha,h_\alpha) = \sum_{h_\beta\in\mathcal{B}}\langle f, e_\lambda(h_\alpha,h_\beta)\rangle_{L^2(H_n)}\, e_\lambda(h_\alpha,h_\beta). \tag{11}$$

By (11) it follows that $f * e_\lambda(h_\alpha,h_\alpha)\in H_{\lambda,j}$ and

$$||f * e_\lambda(h_\alpha,h_\alpha)||^2_{\lambda,j} = \sum_{h_\beta\in\mathcal{B}} |\langle f, e_\lambda(h_\alpha,h_\beta)\rangle|^2, \quad \text{for all } h_\alpha\in\mathcal{B}_j.$$

Hence, we obtain that the orthogonal projection $Q_{\lambda,j}(f) = d_j\; f*\phi_{\lambda,j}$ maps $L^2(H_n)$ onto $H_{\lambda,j}$, $H_{\lambda,j}$ is irreducible, and by (11) $\|f\|^2 = \sum_{j\in\Lambda}\int_{-\infty}^{\infty}\|Q_{\lambda,j}f\|^2_{\lambda,j}|\lambda|^n\,d\lambda$. This concludes the proof of the theorem. ■

5. Description of $\phi_{\lambda,j}$

In this section we describe the set $\mathfrak{B}$ of spherical functions corresponding to the set of generic (or with full Plancherel measure) representations of $N(\mathfrak{g},V)$. These computations involve integration on G/T, which is difficult to carry out with exception of a few cases. Nevertheless, we obtain a parametrization of $\mathfrak{B}$.

As we saw before the metaplectic representation ϖ_λ of K_λ is given by (9). We assume that it decomposes into irreducible components as $\mathcal{P}(V) = \oplus_{j\in\Lambda}W_{\lambda,j}$. We also saw that

$$\rho_\lambda(z,v) = e^{i|\lambda|\langle z,y_\lambda\rangle}\pi_{|\lambda|}(0,v).$$

The set of spherical functions of (K_λ, H_n) corresponding to the Fock representation $\pi_{|\lambda|}$ is given by $\{\psi_{\lambda,j}\}_{j\in\mathbb{N}\cup\{0\}}$, where $\psi_{\lambda,j}$ is defined in (3). As

$$\begin{aligned} \rho_\lambda(Ad(g)z,\pi(g)v) &= \pi_{|\lambda|}(\langle Ad(g)z,y_\lambda\rangle,0)\,\pi_{|\lambda|}(0,\pi(g)v) \\ &= e^{i|\lambda|\left\langle z,Ad\left(g^{-1}\right)y_\lambda\right\rangle}\pi_{|\lambda|}(0,\pi(g)v), \end{aligned}$$

and $\psi_{\lambda,j}$ is a K_λ-invariant function, we obtain that

$$\int_{G/T}\psi_{\lambda,j}(g\cdot(z,v))\,dg = \int_{G/T} e^{i|\lambda|\left\langle z,Ad\left(g^{-1}\right)y_\lambda\right\rangle}\psi_{\lambda,j}(0,\pi(g)v)\,d\dot{g}.$$

By the description in [2] of the set of bounded spherical functions of a Gelfand pair (K,H_n) we know that for $(t,v)\in H_n$ and $\lambda>0$

$$\psi_{\lambda,j}(t,v) = e^{it\lambda}q_j\left(\lambda^{\frac{1}{2}}v\right)e^{-\frac{\lambda}{4}|v|^2},$$

where q_j is a *real* K-invariant polynomial. Indeed, assume $\lambda=1$ and let $\mathcal{P}(V)^{\mathbb{R}}$ denote the algebra of real K-invariant polynomials. Then it is proved in [2] that there is a canonical basis $\{p_j\}_{j\in\Lambda}$ of the vector space $\mathcal{P}(V)^{\mathbb{R}}$, $p_j \in W_j := W_{j,1}$, such that the sequence $\{q_j\}_{j\in\Lambda}$ is obtained from $\{p_j\}_{j\in\Lambda}$ by applying the Gram-Schmidt process with respect to the measure $e^{-\frac{1}{4}|v|^2}dv$. Thus

$$\phi_{\lambda,j}(z,v) = e^{-\frac{|\lambda|}{4}|v|^2}\left(\int_{G/T} e^{i|\lambda|\langle z, Ad(g^{-1})y_\lambda\rangle} q_j\left(|\lambda|^{\frac{1}{2}}\pi(g)v\right)d\dot{g}\right).$$

Remark 5.1. In case that $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}'$, and $x_\lambda = x_{\lambda'} + z_\lambda$, $z_\lambda \in \mathfrak{c}$, $x_{\lambda'} \neq 0$, we have $\phi_{\lambda,j}(z,v) = e^{i|\lambda|\langle z, z_\lambda\rangle}\phi_{\lambda',j}(z,v)$, where $|\lambda'| = |\lambda||x_\lambda|$. ■

In the following, we analyse the set $\mathfrak{B}$ case by case. We denote by T_n the n-dimensional torus.

• **Case 1.** In this case $\mathfrak{g} = \mathfrak{su}(2)$, $V = \mathbb{H}^n$ and $\mathfrak{n} = \mathfrak{su}(2)\oplus\mathbb{H}^n$. $\mathfrak{su}(2)$ is isomorphic to Im($\mathbb{H}$) and the action is given by $q.(v_1,...,v_n) = (qv_1,...,qv_n)$, for $q \in$ Im($\mathbb{H}$), $v = (v_1,...,v_n) \in \mathbb{H}^n$. Thus $\mathfrak{n} =$ Im($\mathbb{H}$) $\oplus\, \mathbb{H}^n$ is a Lie algebra of Heisenberg type, $K = SU(2)\times Sp(n)$ and $K_\lambda = T_1 \times Sp(n)$ where

$$T_1 := \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} : \theta \in \mathbb{R}\right\} \tag{12}$$

is a maximal torus of $SU(2)$. It is well known that the natural action of $Sp(n)$ on the space $\mathcal{P}_j(\mathbb{C}^{2n})$ of homogeneous polynomial of degree j is irreducible and we denote it by η_j. Then the metaplectic representation of K_λ acting on $\mathcal{P}(\mathbb{C}^{2n})$ decomposes as

$$\varpi \downarrow K_\lambda = \oplus_{j=0}\chi_j \otimes \eta_j,$$

where $\chi_j(\theta) = e^{-ij\theta}$. It is also well known that $\psi_{\lambda,j}(t,v) = e^{it|\lambda|}L_j^{2n-1}\left(\frac{|\lambda|}{2}|v|^2\right)e^{-\frac{|\lambda|}{4}|v|^2}$ where L_j^{2n-1} is a Laguerre polynomial of degree j, (see for example [4] p, 64). Thus,

$$\phi_{\lambda,j}(z,v) = \int_{SU(2)/T_1}\psi_{\lambda,j}\left(\left\langle Ad\left(g^{-1}\right)y_\lambda, z\right\rangle, g.v\right)d\dot{g}.$$

Since $Ad\colon SU(2) \to SO(3)$ is a surjective morphism with kernel ± 1 and $SO(3)/SO(2)$ is homeomorphic to the two dimensional sphere S^2, we have that

$$\phi_{\lambda,j}(z,v) = \int_{SO(3)/SO(2)}\psi_{\lambda,j}\left(\left\langle g^{-1}y_\lambda, z\right\rangle, g.v\right)d\dot{g}$$

$$= \left(\int_{S^2} e^{i|\lambda|\xi.z}d\xi\right)L_j^{2n-1}\left(\frac{|\lambda|}{2}|v|^2\right)e^{-\frac{|\lambda|}{4}|v|^2} = J_{\frac{1}{2}}(|\lambda|z)\,L_j^{2n-1}\left(\frac{|\lambda|}{2}|v|^2\right)e^{-\frac{|\lambda|}{4}|v|^2},$$

where $d\xi$ denotes the $SO(3)$-invariant measure on S^2, and $J_{\frac{1}{2}}$ is the Bessel function of order $\frac{1}{2}$ of the first kind.

• **Case 2.** In this case N does not have square integrable representations.

• **Case 3.** In this case we have $\mathfrak{g} = \mathfrak{su}(2)\oplus\mathfrak{su}(2)$, $V = \mathbb{H}^{k_1}\oplus\mathbb{R}^4\oplus\mathbb{H}^{k_2}$ and $\mathfrak{n} = \mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathbb{H}^{k_1}\oplus\mathbb{R}^4\oplus\mathbb{H}^{k_2}$. The first copy (resp. the second) of $\mathfrak{su}(2)$ acts as $\mathfrak{sp}(1)$ on $\mathbb{H}^{k_1}$ and trivially on $\mathbb{H}^{k_2}$ (resp. on $\mathbb{H}^{k_2}$ and trivially on $\mathbb{H}^{k_1}$), and as $\mathfrak{so}(4)$ on $\mathbb{R}^4$. Thus $U = Sp(k_1)\times Sp(k_2)$, $K = Spin(4)\times U$ and $K_\lambda = T_2 \times U$,

where
$$T_2 := \left\{\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2}\end{pmatrix} : \theta_1,\theta_2 \in \mathbb{R}\right\} \tag{13}$$

is a maximal torus of $Spin\,(4)$. Since $\mathcal{P}\,(V)=\mathcal{P}\left(\mathbb{C}^{2k_1}\right)\otimes\mathcal{P}\,(\mathbb{C}^2)\otimes\mathcal{P}\left(\mathbb{C}^{2k_2}\right)$, we can decompose the metaplectic representation as

$$\varpi\downarrow K_\lambda=\left(\oplus_{j\geq 0}\,\chi_j\,(\theta_1)\,\eta_j^{k_1}\right)\otimes\left(\oplus_{l_1,l_2\geq 0}\,\chi_{l_1,l_2}\,(\theta_1,\theta_2)\right)\otimes\left(\oplus_{s\geq 0}\,\chi_s\,(\theta_2)\,\eta_s^{k_2}\right),$$

where $\eta_j^{k_i}$ is the natural action on the space $\mathcal{P}_j(\mathbb{C}^{2k_i})$.

Writing $v=(v_1,u,v_2)$, with $v_1\in\mathbb{C}^{2k_1},v_2\in\mathbb{C}^{2k_2}$ and $u=(u_1,u_2)\in\mathbb{C}^2$, an easy computation shows that

$$\pi_{|\lambda|}\,(t,v)=\pi_{|\lambda|}\left(\frac{t}{3},(v_1,0,0)\right)\otimes\pi_{|\lambda|}\left(\frac{t}{3},(0,u,0)\right)\otimes\pi_{|\lambda|}\left(\frac{t}{3},(0,0,v_2)\right),$$

and by applying an elementary property of the trace of a linear map defined on a tensor product, we obtain

$$\psi_{\lambda,j,l_1,l_2,s}\,(t,v)=e^{i|\lambda|t}\,L(v),$$

$$L(v)=L_j^{2k_1-1}\left(\frac{|\lambda|}{2}\,|v_1|^2\right)L_{l_1}^0\left(\frac{|\lambda|}{2}\,|u_1|^2\right)L_{l_2}^0\left(\frac{|\lambda|}{2}\,|u_2|^2\right)L_s^{2k_2-1}\left(\frac{|\lambda|}{2}\,|v_2|^2\right)e^{-\frac{|\lambda|}{4}|v|^2},$$

where L_k^n is a Laguerre polynomial of degree k.

On the one hand, $SU\,(2)$ is acting as $Im(\mathbb{H})$ (or $Sp\,(1)$) on each component of $\mathbb{C}^{2k_i}$, $i=1,2$. On the other hand, $G\simeq Sp\,(1)\times Sp\,(1)$ acts on $\mathbb{C}^2\simeq\mathbb{H}$ by the rule: $v\longmapsto g.v=q_1vq_2$, for $g=(q_1,q_2)\in G,v\in\mathbb{H}$. Then we have that

$$L_j^{2k_1-1}\left(\frac{|\lambda|}{2}\,|gv_1|^2\right)=L_j^{2k_1-1}\left(\frac{|\lambda|}{2}\,|v_1|^2\right),L_s^{2k_2-1}\left(\frac{|\lambda|}{2}\,|gv_2|^2\right)=L_s^{2k_2-1}\left(\frac{|\lambda|}{2}\,|v_2|^2\right),$$

and the spherical functions are given by

$$\phi_{\lambda,j,l_1,l_2,s}\,(z,v)=\left(\int_{G/T}e^{i|\lambda|\left\langle Ad\left(g^{-1}\right)Y_\lambda,z\right\rangle}L_{l_1}^0\left(\frac{|\lambda|}{2}\,|(gu)_1|^2\right)L_{l_2}^0\left(\frac{|\lambda|}{2}\,|(gu)_2|^2\right)dg\right)$$
$$\times\,e^{-\frac{|\lambda|}{4}|v|^2}L_j^{2k_1-1}\left(\frac{|\lambda|}{2}\,|v_1|^2\right)L_s^{2k_2-1}\left(\frac{|\lambda|}{2}\,|v_2|^2\right)$$

where $g\,(u_1,u_2)=((gu)_1\,,(gu)_2)$.

• **Case 4.** Here $\mathfrak{g}=\mathfrak{sp}\,(2)\,,V=(\mathbb{H}^2)^n$ and $\mathfrak{n}=\mathfrak{sp}\,(2)\oplus(\mathbb{H}^2)^n$. The real action is given by $\pi\,(g)\,(v_1,...,v_n)=(gv_1,...,gv_n)\,,v_j\in\mathbb{H}^2$ for $j=1,...,n$. By Schur's Lemma, the group of orthogonal intertwining operator is isomorphic to $Sp(n)$ with the action on $(\mathbb{H}^2)^n$ given by the $2n\times 2n$ matrix $a_{ij}I$, $a_{ij}\in\mathbb{H}$, and I the 2×2 identity. Thus $K=Sp\,(2)\times Sp\,(n)$, and $K_\lambda=T_2\times Sp\,(n)$ where T_2 is a maximal torus of $Sp\,(2)$ as in (13).

Writing $(v_1,...,v_n)=((u_1,w_1)\,,...,(u_n,w_n))$, with $(u_j,w_j)\in\mathbb{H}^2$ for $j=1,...n$, we have that the action of $Sp\,(n)$ is given by

$$g\,((u_1,w_1)\,,...,(u_n,w_n))=(g\,(u_1,...,u_n)\,,g\,(w_1,...,w_n))\,.$$

The action of $Sp\,(n)$ on $\mathcal{P}\,(\mathbb{C}^{4n})$ splits as $\mathcal{P}\,(\mathbb{C}^{2n})\otimes\mathcal{P}\,(\mathbb{C}^{2n})=\oplus_{r,s}\mathcal{P}_r\,(\mathbb{C}^{2n})\otimes\mathcal{P}_s\,(\mathbb{C}^{2n})$. On the other hand, T_2 acts naturally on each $\mathbb{H}^2$ by

$$\begin{pmatrix}e^{i\theta_1}&0\\0&e^{i\theta_2}\end{pmatrix}(u_j,v_j)=(e^{-i\theta_1}u_j,e^{-i\theta_2}v_j).$$

As above, we denote by η_j the irreducible representation of $Sp\,(n)$ on $\mathcal{P}_j\,(\mathbb{C}^{2n})$. In [7] it is proved that $\eta_s \otimes \eta_r = \oplus_{j=0}^{s} \oplus_{i=0}^{j} \eta_{(r+s-j-i,j-i)}$ where $\eta_{(r+s-j-i,j-i)}$ is the irreducible representation of $Sp(n)$ with highest weight $(r+s-j-i, j-i, 0, \cdots, 0)$, for $r \geq s$. Then

$$\varpi \downarrow K_\lambda = \oplus_{r,s}\chi_{r,s}\,(\theta_1,\theta_2) \otimes (\oplus_{j=0}^{s} \oplus_{i=0}^{j} \eta_{(r+s-j-i,j-i)}),$$

where $\chi_{r,s}\,(\theta_1,\theta_2) = e^{-i(r\theta_1+s\theta_2)}$.

The polynomial $q_{r,s,j,i}$ in $\mathcal{P}\,(\mathbb{C}^{4n})$ corresponding to the spherical function $\psi_{\lambda,r,s,j,i}$ is $K_\lambda-$invariant, thus $q_{r,s,j,i}\,(t,v) = q_{r,s,j,i}\left(t, |u|^2, |w|^2\right)$, but $Sp\,(2)$ preserves the norm of (u_j, w_j) for $j = 1, \ldots n$. Thus

$$\phi_{\lambda,r,s,j,i}\,(z,v) = e^{-\frac{|\lambda|}{4}|v|^2}\left(\int_{G/T} e^{i|\lambda|\langle Ad(g^{-1})y_\lambda, z\rangle} q_{r,s,j,i}(t, g\cdot v)\, d\dot{g}\right).$$

• **Case 5.** In this case $\mathfrak{g} = \mathfrak{su}\,(n)$, $V = \mathbb{C}^n$, $\mathfrak{n} = \mathfrak{su}\,(n) \oplus \mathbb{C}^n$ and π is the canonical action of $\mathfrak{su}\,(n)$ on $\mathbb{C}^n$. Since it is irreducible, the group of the orthogonal intertwining operators is a one dimensional torus which we denote by T_1. So $K = SU\,(n) \times T_1$, $K_\lambda = T_{n-1} \times T_1$ where T_{n-1} is a maximal torus of $SU\,(n)$, and

$$\varpi \downarrow K_\lambda = \oplus_{m_1,\ldots,m_n \in \mathbb{Z}\geq 0}\ \chi_{m_1,\ldots,m_n},$$

where $\chi_{m_1,\ldots,m_n}\ (\theta_1, \ldots, \theta_n) = e^{-i(m_1\theta_1+\ldots+m_n\theta_n)}$. The spherical functions corresponding to the pair (T_n, H_n) are given by (see for example [4])

$$\psi_{\lambda,m_1,\ldots,m_n}\,(t,v) = e^{i|\lambda|t}\prod_{j=1}^{n} L_{m_j}^{0}\left(\frac{|\lambda|}{2}\,|v_j|^2\right) e^{-\frac{|\lambda|}{4}|v|^2}. \tag{14}$$

Setting $gv = \left((gv)_1, \ldots, (gv)_n\right)$ for $g \in SU\,(n)$, we obtain the following expression for the set of generic spherical functions

$$\phi_{\lambda,m_1,\ldots,m_n}\,(z,v) = e^{-\frac{|\lambda|}{4}|v|^2}\left(\int_{SU(n)/T_n} e^{i|\lambda|\left\langle Ad\left(g^{-1}\right)y_\lambda, z\right\rangle}\prod_{j=1}^{n} L_{m_j}^{0}\left(\frac{|\lambda|}{2}\left|(gv)_j\right|^2\right) d\dot{g}\right),$$

with $L_{m_j}^{0}$ is a Laguerre polynomial of degree m_j.

• **Case 6.** In this case $\mathfrak{g} = \mathfrak{so}\,(2n)$, $V = \mathbb{R}^{2n}$, $\mathfrak{n} = \mathfrak{so}\,(2n) \oplus \mathbb{R}^{2n}$ and π is the canonical action of $\mathfrak{so}\,(2n)$ on $\mathbb{R}^{2n}$. Since it is irreducible, the group of the orthogonal intertwining operators is trivial. Thus $K = SO\,(2n)$ and K_λ is an n dimensional torus. As in Case 5 the metaplectic representation is decomposed into a direct sum of characters without multiplicity as

$$\varpi \downarrow_{T_n} = \bigoplus_{(m_1,\cdots,m_n)\in\mathbb{Z}\geq 0} \chi_{(m_1,\cdots,m_n)},$$

where $(\chi_{(m_1,\cdots,m_n)}(\theta_1,\cdots,\theta_n))(z_1^{m_1}\cdots z_n^{m_n}) = e^{-im_1\theta_1}z_1^{m_1}\cdots e^{-im_n\theta_n}z_n^{m_n}$.

Then $\psi_{\lambda,m_1,\ldots,m_n}$ is given by (14), but here we have to integrate on $SO\,(2n)\,/T_n$, i.e.,

$$\phi_{\lambda,m_1,\ldots,m_n}\,(z,v) = e^{-\frac{|\lambda|}{4}|v|^2}\left(\int_{SO(2n)/T_n} e^{i|\lambda|\left\langle Ad\left(g^{-1}\right)y_\lambda, z\right\rangle}\prod_{j=1}^{n} L_{m_j}^{0}\left(\frac{|\lambda|}{2}\left|(gv)_j\right|^2\right) d\dot{g}\right),$$

where $L_{m_j}^{0}$ is a Laguerre polynomial of degree m_j.

• **Case 7.** In this case $N(\mathfrak{g}, V)$ is the $(2n+1)$-dimensional Heisenberg group, $K = U(n)$ and the set of bounded spherical functions was described by many authors, see for example [2] and [4].

• **Case 8.** In this case $\mathfrak{g} = \mathfrak{u}(2)$, $V = (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n$ and $\mathfrak{n} = \mathfrak{u}(2) \oplus (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n$. $\mathfrak{u}(2)$ acts in the following way:

– on each of the k components of $(\mathbb{C}^2)^k$ it acts in the natural way, and
– in $(\mathbb{C}^2)^n$ the center of $\mathfrak{u}(2)$ acts trivially and the semisimple part acts as $\mathfrak{sp}(1)$ (or $\mathrm{Im}(\mathbb{H})$) on the left side on each of the n components of $(\mathbb{C}^2)^n$.

For n positive, $K = SU(2) \times U(k) \times Sp(n)$ and $K_\lambda = T_1 \times U(k) \times Sp(n)$ where T_1 is a maximal torus of $SU(2)$. The action of T_1 on $\mathcal{P}\left((\mathbb{C}^2)^k\right)$ is given by

$$p(u_1, w_1, ..., u_k, w_k) \to p\left(e^{i\theta}u_1, e^{-i\theta}w_1, ..., e^{i\theta}u_k, e^{-i\theta}w_k\right)$$

for $p \in \mathcal{P}(\mathbb{C}^{2k}), e^{i\theta} \in T_1$, $(u_i, w_i) \in \mathbb{C}^2$. Also $U(k)$ acts by a multiple of the 2×2 identity on each of the k components of $(\mathbb{C}^2)^k$.

We denote by ν_r (resp. η_r) the irreducible action of $U(k)$ (resp. $Sp(k)$) on $\mathcal{P}_r(\mathbb{C}^{2k})$. As $U(k)$-module $\mathcal{P}(\mathbb{C}^{2k}) = \mathcal{P}(\mathbb{C}^k) \otimes \mathcal{P}(\mathbb{C}^k) = \oplus_{r,s}\nu_r \otimes \nu_s$. Moreover,

$$\nu_r \otimes \nu_s = \oplus_{j=1}^{min(r,s)} \nu_{(r+s-2j,j)},$$

where $\nu_{(r+s-2j,j)}$ denotes the irreducible representation of the highest weight $(r+s-2j, j, 0, \cdots, 0)$ ([5], p. 225). Thus $\mathcal{P}(V) = \mathcal{P}\left(\mathbb{C}^{2k}\right) \otimes \mathcal{P}\left(\mathbb{C}^{2n}\right)$ and the decomposition of the metaplectic representation into irreducible components is

$$\varpi \downarrow_{K_\lambda} = \left(\oplus_{r,s,j \in \mathbb{Z}_{\geq 0}} \chi_{r-s}(\theta) \oplus_{j=1}^{min(r,s)} \nu_{(r+s-2j,j)}\right) \otimes \left(\oplus_{l \in \mathbb{Z}_{\geq 0}} \chi_l(\theta)\, \eta_l\right).$$

We set $v = (\mathbf{v}^1, \mathbf{v}^2)$, $\mathbf{v}^1 \in \mathbb{C}^{2k}, \mathbf{v}^2 \in \mathbb{C}^{2n}$. As in Case 3 by applying an elementary property of the trace of a linear map defined on a tensor product, we obtain the expression for the spherical function

$$\psi_{\lambda,r,s,j,l}(t,v) = e^{i|\lambda|t} e^{-\frac{|\lambda|}{4}|v|^2} q_{j,r,s}\left(\frac{|\lambda|}{2}\mathbf{v}^1\right) L_l^{2n-1}\left(\frac{|\lambda|}{2}\left|\mathbf{v}^2\right|^2\right), j = 1, ..., \min(r,s),\ l \geq 0,$$

where L_l^{2n-1} is a Laguerre polynomial of degree l. For $\mathbf{v}^1 = (u_1, w_1, ..., u_k, w_k) \in \mathbb{C}^{2k}$, we have that $q_{j,r,s}$ is a polynomial in $|u|^2, |w|^2$ since $q_{j,r,s}$ is $U(k)$-invariant, but the action of $G = SU(2)$ is componentwise on each (u_j, w_j). Thus

$$\phi_{\lambda,r,s,j,l}(t,v) = \left(\int_{G/T} e^{i|\lambda|\langle Ad(g^{-1})y_\lambda, z\rangle} q_{j,r,s}\left(\frac{|\lambda|}{2} g \cdot \mathbf{v}^1\right) d\dot{g}\right) e^{-\left(\frac{|\lambda|}{4}|v|^2\right)} L_l^{2n-1}\left(\frac{|\lambda|}{2}|\mathbf{v}^2|^2\right).$$

As we observe in the proof of Theorem (3.5), in this case there is no generic spherical functions associated to the center of $\mathfrak{g}$.

Case $n = 0$. To the set of spherical functions described above (with the obvious changes since $n = 0$) we add the set of spherical functions corresponding to the elements of the center of $\mathfrak{g}$. For this purpose, we need to decompose the metaplectic action of $K = SU(2) \times U(k)$ on $\mathcal{P}\left(\mathbb{C}^{2k}\right)$. We assume $k \geq 2$.

It is easy to see that $\mathbb{C}^{2k}$ is equivalent to $\mathbb{C}^2 \otimes \mathbb{C}^k$ with the standard action as $(SU(2) \times U(k))$-module. So we can apply the Corollary 5.2.8 in [6] to obtain the desired decomposition. Indeed, following the notation there, let $\mathcal{F}_2^{\mu}$ be the irreducible representation of $Gl\,(2,\mathbb{C})$ with highest weight $\mu=\mu_1\varepsilon_1+\mu_2\varepsilon_2$ where $\varepsilon_1,\varepsilon_2$ are the coordinate weights and $\mu_1\geq\mu_2\geq 0$. Let $\mathcal{F}_k^{\mu}$ be the irreducible representation of $Gl\,(k,\mathbb{C})$ with highest weight the k-tuple $(\mu_1,\mu_2,0,...,0)$. Then we have that the space of homogeneous polynomials of degree d over $\mathbb{C}^2\otimes\mathbb{C}^k$ decomposes as

$$\mathcal{P}_d\left(\mathbb{C}^2\otimes\mathbb{C}^k\right)=\bigoplus_{\mu}\mathcal{F}_2^{\mu}\otimes\mathcal{F}_k^{\mu},\qquad \mu_1+\mu_2=d.$$

Now in order to restrict to $Sl\,(2,\mathbb{C})\times Gl\,(k,\mathbb{C})$, let $l=\mu_1-\mu_2$ and let $\mathcal{F}^l$ be the irreducible representation of $Sl\,(2,\mathbb{C})$ of dimension $l+1$. Then, the restriction to $Sl\,(2,\mathbb{C})\times Gl\,(k,\mathbb{C})$ decomposes as

$$\mathcal{P}\left(\mathbb{C}^2\otimes\mathbb{C}^k\right)=\oplus_{l,d}\mathcal{F}^l\otimes\mathcal{F}_k^{\mu},\quad l,d\geq 0$$

with $\mu_1+\mu_2=d$ and $\mu_1-\mu_2=l$. The corresponding set of spherical functions is given by (3) and it is parametrized by $\{\lambda,d,l\}$ with $\lambda\neq 0$ and $l,d\geq 0$.

• **Case 9**. In this case $\mathfrak{g}=\mathfrak{u}(n)$, $V=\mathbb{C}^n$ and the action is the standard one. Thus $G=SU(n)$, $U=T^1$ and $K=U(n)$. For $x_\lambda\in\mathfrak{g}'$ the corresponding spherical functions are as in case **V**.

For the elements in the center of $\mathfrak{g}$, the stabilizer is K and $\omega\downarrow_K=\oplus_{r\geq 0}v_r$, where v_r denotes the irreducible representation of $U(n)$ on the space of homogeneous polynomials of degree r. The set of associated spherical functions is

$$\phi_{\lambda,r}(t,v)=e^{i|\lambda|t}L_r^{n-1}\left(\frac{\lambda}{2}|v|^2\right)e^{-\frac{|\lambda|}{4}|v|^2},$$

where L_r^{n-1} is a Laguerre polynomial of degree r.

• **Case 10.** In this case $\mathfrak{g}=\mathfrak{su}\,(m_1)\oplus...\oplus\mathfrak{su}\,(m_r)\oplus\mathfrak{c}$ where $\mathfrak{c}$ is its center and there are α copies of $\mathfrak{su}(2)$. The abelian component satisfies $1\leq dim(\mathfrak{c})\leq r-1$; $V=V_1\oplus...\oplus V_r$, and the representation π of $\mathfrak{g}$ on V is defined as follows:

For each $1\leq j\leq r,\mathfrak{su}\,(m_j)$ acts non trivially only on V_j, $\mathfrak{c}$ has a unique subspace $\mathfrak{c}_j$ acting non trivially on V_j and $\dim(\mathfrak{c}_j)=1$. If $m_j\geq 3$, $V_j=\mathbb{C}^{m_j}$ and $\mathfrak{su}\,(m_j)\oplus\mathfrak{c}_j$ (which is isomorphic to $\mathfrak{u}\,(m_j)$) acts in the standard way on V_j; we denote by S^1 the group of intertwining operators of this action. If $m_j=2$, $V_j=(\mathbb{C}^2)^k\oplus(\mathbb{C}^2)^n$ and $\mathfrak{su}\,(2)\oplus\mathfrak{c}_j$ acts on V_j as in Case 8, therefore the group of intertwining operators is $U\,(k)\times Sp\,(n)$.

We first consider the case $\mathfrak{g}=\mathfrak{su}\,(m)\oplus\mathfrak{su}\,(2)\oplus\mathfrak{c}$, $m\geq 3$, $n>0$. Thus $\dim\mathfrak{c}=\mathfrak{1}$, $V=\mathbb{C}^m\oplus(\mathbb{C}^2)^k\oplus(\mathbb{C}^2)^n$, $G=SU\,(m)\times SU\,(2)$, $K=G\times S^1\times U\,(k)\times Sp\,(n)$. Let T_{m-1} be a maximal torus of $SU\,(m)$, thus $T_{m-1}\times S^1$ is (isomorphic to) an n-dimensional torus acting on $\mathbb{C}^m$ in the standard way, $K_\lambda=T_{m-1}\times S^1\times T_1\times U\,(k)\times Sp\,(n)$ and $T_1\times U\,(k)\times Sp\,(n)$ acts on $(\mathbb{C}^2)^k\oplus(\mathbb{C}^2)^n$ as in case 8. Thus

$$\begin{aligned}\varpi\downarrow K_\lambda=&\left(\oplus_{k_1,\ldots,k_m\in\mathbb{Z}\geq 0}\chi_{k_1,\ldots,k_m}\left(\theta_1,...,\theta_m\right)\right)\\ &\otimes\oplus_{r,s}\oplus_{i=1}^{\min(r,s)}\chi_{r-s}\left(\theta\right)\oplus_{i=1}^{\min(r,s)}v_{r+s-2i,i}\otimes\left(\oplus_j\chi_j\left(\theta\right)\eta_j\right).\end{aligned}$$

For $v=(u,v_1,v_2)\,,u\in\mathbb{C}^m,v_1\in\mathbb{C}^{2k},v_2\in\mathbb{C}^{2n}$, the spherical functions associated to the pair (K_λ,N_λ) are

$$\psi_{\lambda,\mathbf{k},,r,s,i,j}\left(t,u,v_1,v_2\right)=e^{i|\lambda|t}e^{-\frac{|\lambda|}{4}|v|^2}\prod_{j=1}^{m}L_{k_j}^0\left(\frac{|\lambda|}{2}\left|u_j\right|^2\right)q_{i,r,s}\left(\frac{|\lambda|}{2}v_1\right)L_j^{2n-1}\left(\frac{|\lambda|}{2}\left|v_2\right|^2\right),$$

where $\mathbf{k}=(k_1,...,k_m)\,, i = 1,...,\min(r,s)$ and $r,s,\ j \geq 0$. The action of G is componentwise, so writing

$$q_{\mathbf{k},,r,s,i}\left(u,v_1\right)=\prod_{j=1}^{m}L_{k_j}^{0}\left(\frac{|\lambda|}{2}\left|u_j\right|^2\right)q_{i,r,s}\left(\frac{|\lambda|}{2}v_1\right)$$

we obtain $\phi_{|\lambda|,\mathbf{k},,r,s,i,j}\left(z,v\right) =$

$$=\left(\int_{G/T}e^{i|\lambda|t\langle Ad(g)y_\lambda,z\rangle}q_{\mathbf{k},,r,s,i,}\left(\pi\left(g\right)\left(u,v_1\right)\right)dg\right)L_j^{2n-1}\left(\frac{|\lambda|}{2}\left|v_2\right|^2\right)e^{-\frac{|\lambda|}{4}|v|^2}.$$

When $n=0$, the description of the set of spherical functions corresponding to the elements of the center $\mathfrak{c}$ of $\mathfrak{g}$ follows from Case 9 and Case 8 with $n=0$, since $U(m)$ acts on $\mathcal{P}(\mathbb{C}^m)$ and $SU(2)\times U(k)$ acts on $\mathcal{P}\left(\mathbb{C}^{2k}\right)$. This completes the simplest case. The general case follows similar lines.

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Andrea L. Gallo and Linda V. Saal
FaMAF, Universidad Nacional de Córdoba, CIEM (CONICET), Córdoba 5000, Argentina
andregallo88@gmail.com, saal@mate.uncor.edu

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