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Conformal Vector Fields on Lorentzian Lie Groups of Dimension 5

Hui Zhang, Zhiqi Chen, Shaoxiang Zhang*

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Abstract. We classify Lorentzian Lie groups of dimension 5 which admit left-invariant non-Killing conformal vector fields.

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1. Introduction

Let (M, g) be a pseudo-Riemannnian manifold. A vector field $X \in \Gamma(TM)$ is *conformal* if

$$L_X g = 2\rho g, \tag{1}$$

where L_X is the Lie derivation and ρ is a smooth function on M. If $\rho = 0$, we call X a *Killing* vector field which provides a close link between the geometry of a manifold M and the algebra of $I(M)$, the set of all isometries in M (see [13]). If g is a Riemannian metric, the existence of the function ρ might give some information about the topological structure of the Riemannian manifold (see [5, 12]).

To our knowledge, there are only a few results about the classification of left-invariant conformal vector fields on Lorentzian Lie groups which are Lie groups with left-invariant Lorentzian metrics. Firstly, the authors in [4] showed the existence of left-invariant conformal vector fields on three dimensional Lorentzian Lie groups. Then Araujo, Chen and Leandro proved a necessary condition for a Lorentzian Lie group to admit left-invariant non-Killing conformal vector fields and classified such Lorentzian Lie groups of dimension 2 and 3 in [2]. Also in [2], it was proved that there are some left-invariant non-Killing conformal vector fields on Lorentzian Lie groups in higher dimensions. In [7], Tan, Chen and Xu classified all Lorentzian Lie groups of dimension 4 admitting left-invariant non-Killing conformal vector fields. Recently left-invariant conformal Killing-Yano forms (a higher-degree generalization of conformal vector fields) have been treated by Andrada and Dotti in [1].

In this paper, we denote by $\mathfrak{g}$ the Lie algebra of the connected Lie group G. It is clear that G is solvable if and only if $\mathfrak{g}$ is solvable, so we can abuse them without

* Corresponding author.

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ambiguity. We focus on Lorentzian Lie groups of dimension 5 admitting left-invariant non-Killing conformal vector fields, and first prove the following theorem.

Theorem 1.1. *Let G be a Lorentzian Lie group of dimension 5. If G admits a left-invariant non-Killing conformal vector field, then G is solvable.*

Finally, based on the discussion on solvable Lorentzian Lie groups of dimension 5, we get the classification theorem.

Theorem 1.2. *Let G be a Lorentzian Lie group of dimension 5 and $\mathfrak{g}$ be the corresponding Lie algebra. If G admits a left-invariant non-Killing conformal vector field, then there is a basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{e_1, e_2, e_3, e_4\}$, the Lorentzian metric associated with the basis is defined by*

$$\langle\cdot,\cdot\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

and the non-zero brackets are one of the following cases:
(1) $[e_1, e_2] = ke_4$, $[e_2, e_3] = k\frac{\gamma}{\beta}e_4$, $[e_1, e_5] = \alpha e_1 + \beta e_2 - \delta(1 - \frac{k\beta}{\alpha^2+\gamma^2+\beta^2})e_4$, $[e_2, e_5] = -\beta e_1 + \alpha e_2 + \gamma e_3 + \frac{k\alpha\delta}{\alpha^2+\gamma^2+\beta^2}e_4$, $[e_3, e_5] = -\gamma e_2 + \alpha e_3 - \frac{k\gamma\delta}{\alpha^2+\gamma^2+\beta^2}e_4$, $[e_4, e_5] = 2\alpha e_4$, *where γ, δ are arbitrary and $k\alpha\beta \neq 0$.*
For this case, the left-invariant non-Killing conformal vector field is a non-zero constant multiple of $\delta(\alpha^2 + \gamma^2)e_1 - \alpha\beta\delta e_2 + \gamma\beta\delta e_3 + \frac{\delta^2(\alpha^2+\gamma^2)}{2\alpha}e_4 + \alpha(\alpha^2 + \gamma^2 + \beta^2)e_5$.
(2) $[e_2, e_3] = ke_4$, $[e_1, e_5] = \alpha e_1 - \delta e_4$, $[e_2, e_5] = \alpha e_2 + \gamma e_3$, $[e_3, e_5] = -\gamma e_2 + \alpha e_3$, $[e_4, e_5] = 2\alpha e_4$, *where $k\alpha\gamma\delta \neq 0$. For this case, the left-invariant non-Killing conformal vector field is a non-zero constant multiple of* $\delta e_1 + \frac{\delta^2}{2\alpha}e_4 + \alpha e_5$.
(3) $[e_1, e_2] = le_4$, $[e_1, e_3] = me_4$, $[e_2, e_3] = ne_4$, $[e_1, e_5] = \alpha e_1 - \delta e_4$, $[e_2, e_5] = \alpha e_2 + l\frac{\delta}{\alpha}e_4$, $[e_3, e_5] = \alpha e_3 + m\frac{\delta}{\alpha}e_4$, $[e_4, e_5] = 2\alpha e_4$, *where δ is arbitrary and $\alpha(l^2 + m^2 + n^2) \neq 0$. For this case, the left-invariant non-Killing conformal vector field is a non-zero constant multiple of* $\delta e_1 + \frac{\delta^2}{2\alpha}e_4 + \alpha e_5$.
(4) $[e_1, e_5] = \alpha e_1 + \beta e_2 - \delta e_4$, $[e_2, e_5] = -\beta e_1 + \alpha e_2 + \gamma e_3$, $[e_3, e_5] = -\gamma e_2 + \alpha e_3$, $[e_4, e_5] = 2\alpha e_4$, *where γ, δ, β are arbitrary and $\alpha \neq 0$. For this case, the left-invariant non-Killing conformal vector field is a non-zero constant multiple of* $\delta(\alpha^2 + \gamma^2)e_1 - \alpha\beta\delta e_2 + \gamma\beta\delta e_3 + \frac{\delta^2(\alpha^2+\gamma^2)}{2\alpha}e_4 + \alpha(\alpha^2 + \gamma^2 + \beta^2)e_5$.

The paper is organized as follows. We recall some facts about left-invariant non-Killing conformal vector fields on pseudo-Riemannian Lie groups in Section 2. In Section 3, we prove that the Lorentzian Lie group of dimension 5 is solvable if it admits a left-invariant non-Killing conformal vector field. We devote Section 4 to classify five dimensional Lorentzian Lie groups G admitting left-invariant non-Killing conformal vector fields, i.e. Theorem 1.2.

2. Preliminaries

Let G be a Lie group with the Lie algebra $\mathfrak{g}$ consisting of left-invariant vector fields and $\langle\cdot,\cdot\rangle$ be a pseudo-Riemannian metric on G. Let ∇ denote the Levi-Civita connection associated with $\langle\cdot,\cdot\rangle$.

Then
$$[X,Y]=\nabla_X Y-\nabla_Y X. \tag{2}$$
If $\langle\cdot,\cdot\rangle$ is left-invariant on G, we have
$$\langle\nabla_Z X,Y\rangle+\langle X,\nabla_Z Y\rangle=0, \tag{3}$$
for any $X,Y,Z\in\mathfrak{g}$. By (2) and (3),
$$\langle\nabla_X Y,Z\rangle=\frac{1}{2}(\langle[X,Y],Z\rangle-\langle[Y,Z],X\rangle+\langle[Z,X],Y\rangle), \tag{4}$$
where X,Y,Z are left-invariant vector fields. Assume that $X\in\mathfrak{g}$ is a conformal vector field. By (1), we have $0=L_X\langle X,X\rangle=2\rho|X|^2$. It follows that any left-invariant non-Killing conformal vector field X is null, i.e., $\langle X,X\rangle=0$. Moreover, if $\langle\cdot,\cdot\rangle$ is a Riemannian metric, then $\rho=0$ or $X=0$. That is, X is Killing or trivial. For the left-invariant pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$, we have

Lemma 2.1 ([2]). *Let G be a unimodular pseudo-Riemannian Lie group. Then any left-invariant conformal vector field on G is a Killing vector field.*

If G is a non-unimodular pseudo-Riemannian Lie group, we have the following results.

Lemma 2.2 ([2]). *Let G be a Lorentzian Lie group admitting left-invariant non-Killing conformal vector fields. Then the restriction of $\langle\cdot,\cdot\rangle$ on $[\mathfrak{g},\mathfrak{g}]$ is degenerate.*

Lemma 2.3 ([7]). *Let G be a Lorentzian Lie group admitting left-invariant non-Killing conformal vector fields. Then* $\dim[\mathfrak{g},\mathfrak{g}]=\dim\mathfrak{g}-1$.

From now on, we always suppose that G is a Lorentzian Lie group of dimension 5 with the Lie algebra $\mathfrak{g}$ and G admits a left-invariant non-Killing conformal vector field. By Lemma 2.2 and Lemma 2.3, we know that there exists a basis $\{e_1,e_2,e_3,e_4,e_5\}$ of $\mathfrak{g}$ such that $[\mathfrak{g},\mathfrak{g}]=\mathrm{span}\{e_1,e_2,e_3,e_4\}$. The matrix of the Lorentzian metric associated with this basis is defined by
$$\langle\cdot,\cdot\rangle=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&-1\\0&0&0&-1&0\end{pmatrix}. \tag{5}$$
Assume that X is a left-invariant non-Killing conformal vector field on G. Set $adX(e_j)=\sum_{i=1}^5 a_{ij}e_i, a_{ij}\in\mathbb{R}, i,j\in\{1,2,3,4,5\}$. Since the metric (5) is left-invariant, by (1), we have
$$\langle[X,e_i],e_j\rangle+\langle e_i,[X,e_j]\rangle=-2\rho\langle e_i,e_j\rangle.$$
where $0\neq\rho\in\mathbb{R}$ is a constant. After a routine calculation, we know that the matrix of adX with respect to the basis $\{e_1,e_2,e_3,e_4,e_5\}$ is
$$\begin{pmatrix}-\rho&r_1&r_2&0&r_3\\-r_1&-\rho&r_4&0&r_5\\-r_2&-r_4&-\rho&0&r_6\\r_3&r_5&r_6&-2\rho&0\\0&0&0&0&0\end{pmatrix}, \tag{6}$$

where $r_i \in \mathbb{R}$, $i \in \{1,2,3,4,5,6\}$. In the following we find another basis of $\mathfrak{g}$ such that

(1) the matrix of the metric under the basis is the same as (5),

(2) the matrix of adX with respect to the basis is simpler.

Firstly, assume that r_3, r_5, r_6 are not all zero. Without loss of generality, we can assume $r_3 \neq 0$. Set $\tilde{e}_1 = \frac{1}{k}(e_1 + \frac{r_6}{r_3}e_3)$, $\tilde{e}_2 = e_2$, $\tilde{e}_3 = \frac{1}{k}(e_3 - \frac{r_6}{r_3}e_1)$, $\tilde{e}_4 = e_4$, $\tilde{e}_5 = e_5$, where $k = \sqrt{1 + (\frac{r_6}{r_3})^2}$. With respect to the basis $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5\}$, the matrix of the metric is (5) and the matrix of adX is

$$\begin{pmatrix} -\rho & \frac{1}{k}(r_1 - \frac{r_6 r_4}{r_3}) & r_2 & 0 & \frac{1}{k}(r_3 + \frac{{r_6}^2}{r_3}) \\ -\frac{1}{k}(r_1 - \frac{r_6 r_4}{r_3}) & -\rho & \frac{1}{k}(r_4 + \frac{r_6 r_1}{r_3}) & 0 & r_5 \\ -r_2 & -\frac{1}{k}(r_4 + \frac{r_6 r_1}{r_3}) & -\rho & 0 & 0 \\ \frac{1}{k}(r_3 + \frac{{r_6}^2}{r_3}) & r_5 & 0 & -2\rho & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Denote $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5\}$ by $\{e_1, e_2, e_3, e_4, e_5\}$, the matrix of adX with respect to the basis $\{e_1, e_2, e_3, e_4, e_5\}$ is of the form

$$\begin{pmatrix} -\rho & r_1 & r_2 & 0 & r_3 \\ -r_1 & -\rho & r_4 & 0 & r_5 \\ -r_2 & -r_4 & -\rho & 0 & 0 \\ r_3 & r_5 & 0 & -2\rho & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

Secondly, assume that r_3, r_5 are not all zero. Without loss of generality, we can assume $r_3 \neq 0$. Set $\tilde{e}_1 = \frac{1}{k}(e_1 + \frac{r_5}{r_3}e_2)$, $\tilde{e}_2 = \frac{1}{k}(e_2 - \frac{r_5}{r_3}e_1)$, $\tilde{e}_3 = e_3$, $\tilde{e}_4 = e_4$, $\tilde{e}_5 = e_5$, where $k = \sqrt{1 + (\frac{r_5}{r_3})^2}$. With respect to the basis $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5\}$, the matrix of the metric is (5) and the matrix of adX is

$$\begin{pmatrix} -\rho & r_1 & \frac{1}{k}(r_2 + \frac{r_5 r_4}{r_3}) & 0 & \frac{1}{k}(r_3 + \frac{{r_5}^2}{r_3}) \\ -r_1 & -\rho & \frac{1}{k}(r_4 - \frac{r_5 r_2}{r_3}) & 0 & 0 \\ -\frac{1}{k}(r_2 + \frac{r_5 r_4}{r_3}) & -\frac{1}{k}(r_4 - \frac{r_5 r_2}{r_3}) & -\rho & 0 & 0 \\ \frac{1}{k}(r_3 + \frac{{r_5}^2}{r_3}) & 0 & 0 & -2\rho & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Denote $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5\}$ by $\{e_1, e_2, e_3, e_4, e_5\}$, the matrix of adX with respect to the basis $\{e_1, e_2, e_3, e_4, e_5\}$ is of the form

$$\begin{pmatrix} -\rho & r_1 & r_2 & 0 & r_3 \\ -r_1 & -\rho & r_4 & 0 & 0 \\ -r_2 & -r_4 & -\rho & 0 & 0 \\ r_3 & 0 & 0 & -2\rho & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Lastly, assume that r_1, r_2 are not all zero. Without loss of generality, we can assume $r_1 \neq 0$. Set $\tilde{e}_1 = e_1$, $\tilde{e}_2 = \frac{1}{k}(e_2 + \frac{r_2}{r_1}e_3)$, $\tilde{e}_3 = \frac{1}{k}(e_3 - \frac{r_2}{r_1}e_2)$, $\tilde{e}_4 = e_4$, $\tilde{e}_5 = e_5$, where

$k=\sqrt{1+(\frac{r_2}{r_1})^2}$. With respect to the basis $\{\tilde{e}_1,\tilde{e}_2,\tilde{e}_3,\tilde{e}_4,\tilde{e}_5\}$, the matrix of the metric is (5) and the matrix of adX is

$$\begin{pmatrix} -\rho & \frac{1}{k}(r_1+\frac{{r_2}^2}{r_1}) & 0 & 0 & r_3 \\ -\frac{1}{k}(r_1+\frac{{r_2}^2}{r_1}) & -\rho & r_4 & 0 & 0 \\ 0 & -r_4 & -\rho & 0 & 0 \\ r_3 & 0 & 0 & -2\rho & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In summary, there is a basis $\{e_1,e_2,e_3,e_4,e_5\}$ of $\mathfrak{g}$ with $[\mathfrak{g},\mathfrak{g}]=\mathrm{span}\{e_1,e_2,e_3,e_4\}$, the matrix of the Lorentzian metric is unchanged, while the matrix of adX is

$$\begin{pmatrix} -\rho & \lambda & 0 & 0 & \theta \\ -\lambda & -\rho & \eta & 0 & 0 \\ 0 & -\eta & -\rho & 0 & 0 \\ \theta & 0 & 0 & -2\rho & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

where $\lambda,\theta,\eta\in\mathbb{R}$. Then we can write the left-invariant non-Killing conformal field X as

$$X=x_1e_1+x_2e_2+x_3e_3+x_4e_4+x_5e_5, \tag{10}$$

where $x_i\in\mathbb{R}$. Using Jacobi identity and (9), we get the following important equations

$$adX([e_1,e_2])=-2\rho[e_1,e_2]-\eta[e_1,e_3]-\theta[e_2,e_4], \tag{11}$$
$$adX([e_1,e_3])=-2\rho[e_1,e_3]+\eta[e_1,e_2]-\lambda[e_2,e_3]-\theta[e_3,e_4], \tag{12}$$
$$adX([e_2,e_3])=-2\rho[e_2,e_3]+\lambda[e_1,e_3], \tag{13}$$
$$adX([e_1,e_4])=-3\rho[e_1,e_4]-\lambda[e_2,e_4], \tag{14}$$
$$adX([e_2,e_4])=-3\rho[e_2,e_4]+\lambda[e_1,e_4]-\eta[e_3,e_4], \tag{15}$$
$$adX([e_3,e_4])=-3\rho[e_3,e_4]+\eta[e_2,e_4]. \tag{16}$$

By (9), the characteristic polynomial of adX on $[\mathfrak{g},\mathfrak{g}]$ is

$$f(x)=(x+\rho)(x+2\rho)((x+\rho)^2+\lambda^2+\eta^2).$$

So the real eigenvalues of adX on $[\mathfrak{g},\mathfrak{g}]$ are $-\rho$ and -2ρ. If $\lambda^2+\eta^2\neq 0$, it is easy to see that the eigenspaces of adX on $[\mathfrak{g},\mathfrak{g}]$ are

$$V_{-\rho}=\mathrm{span}\{\eta e_1+\lambda e_3+\frac{\theta\eta}{\rho}e_4\},\quad V_{-2\rho}=\mathrm{span}\{e_4\};$$

if $\lambda^2+\eta^2=0$, the eigenspaces of adX on $[\mathfrak{g},\mathfrak{g}]$ are

$$V_{-\rho}=\mathrm{span}\{e_1+\frac{\theta}{\rho}e_4,e_2,e_3\},\quad V_{-2\rho}=\mathrm{span}\{e_4\}.$$

3. The proof of Theorem 1.1

This section aims at proving Theorem 1.1. Consider the Levi decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}, \tag{17}$$

where $\mathfrak{s}$ is a semisimple Lie subalgebra and $\mathfrak{r}$ is the radical. Since $\mathfrak{g}$ is of dimension 5, we only need to consider two kinds, i.e., $\mathfrak{g} = \mathfrak{r}$ is solvable or $\dim \mathfrak{s} = 3$, $\mathfrak{s}$ is simple. If $\dim \mathfrak{s} = 3$, then $\dim \mathfrak{r} = 2$. Since $\mathfrak{r}$ is solvable, it is well-known that $\mathfrak{r}$ is either defined by $[x, y] = 0$ or defined by $[x, y] = x$, where $\{x, y\}$ is a basis of $\mathfrak{r}$. If $[x, y] = 0$, it is easy to see that $\mathfrak{g}$ is unimodular, by Lemma 2.1, we know that $\mathfrak{g}$ admits no left-invariant non-Killing conformal vector fields.

Lemma 3.1. *If $\mathfrak{r}$ is defined by $[x, y] = x$, then all derivations of $\mathfrak{r}$ are inner and $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ is a direct sum of ideals.*

Proof. Let $\sigma \in Der(\mathfrak{r})$ be any derivation of $\mathfrak{r}$, and suppose

$$\sigma(x, y) = (x, y) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then by $\sigma([x, y]) = [\sigma(x), y] + [x, \sigma(y)]$, we conclude that $a_{21} = a_{22} = 0$. It is easy to verify that $\sigma = a_{12} ad_{\mathfrak{r}} x - a_{11} ad_{\mathfrak{r}} y$, hence σ is inner, so the first statement is proved. For the second statement, we consider the following representation of $\mathfrak{s}$

$$\varphi\colon \mathfrak{s} \to \mathfrak{gl}(\mathfrak{r}), \quad z \mapsto (ad_{\mathfrak{g}} z)|_{\mathfrak{r}}$$

Since $\mathfrak{s}$ is simple, we have that $\varphi(\mathfrak{s}) = 0$ or $\varphi(\mathfrak{s})$ is simple. Notice that for any $z \in \mathfrak{s}$, $\varphi(z)$ is a derivation of $\mathfrak{r}$, then by the proof of the first statement, we know

$$\varphi(\mathfrak{s}) \subset \left\{ \begin{pmatrix} c_1 & c_2 \\ 0 & 0 \end{pmatrix} : c_1, c_2 \in \mathbb{R} \right\},$$

with respect to the basis $\{x, y\}$ of $\mathfrak{r}$. That is $\dim \varphi(\mathfrak{s}) \leq 2$, so we must have $\varphi(\mathfrak{s}) = 0$, i.e. $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ is a direct sum of ideals. ∎

By Lemma 3.1, we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{s} \oplus \mathrm{span}\{x\}$ with the center $\mathrm{span}\{x\}$. Then, it is clear that $adX(x) = cx$ for some $c \in \mathbb{R}$. Namely the center of $[\mathfrak{g}, \mathfrak{g}]$ are eigenvectors of adX. In the following, we split (9) into two cases: $\lambda^2 + \eta^2 = 0$ and $\lambda^2 + \eta^2 \neq 0$.

Case 1. $\lambda^2+\eta^2=0$. In this case, the eigenvalues of adX on $[\mathfrak{g}, \mathfrak{g}]$ are $-\rho$ and -2ρ, then by (14), (15) and (16), we know that $[e_1, e_4] = [e_2, e_4] = [e_3, e_4] = 0$; by (11), (12) and (13), we know that $[e_1, e_2], [e_1, e_3], [e_2, e_3] \in V_{-2\rho}$. Since we have $\mathfrak{g}^{(2)} = \mathrm{span}\{[e_i, e_j]\}_{1 \leq i,j \leq 4}$, it follows that $\dim \mathfrak{g}^{(2)} \leq 1$, which contradicts $\dim \mathfrak{g}^{(2)} = 3$.

Case 2. $\lambda^2+\eta^2 \neq 0$. The center of $[\mathfrak{g}, \mathfrak{g}]$ is either $V_{-\rho}= \mathrm{span}\{\eta e_1 + \lambda e_3 + \frac{\eta\theta}{\rho} e_4\}$ or $V_{-2\rho}= \mathrm{span}\{e_4\}$ in this case. If $V_{-2\rho}= \mathrm{span}\{e_4\}$ is the center of $[\mathfrak{g}, \mathfrak{g}]$, then $[e_1, e_4] = [e_2, e_4] = [e_3, e_4] = 0$. Since $\mathfrak{g}^{(2)} = \mathrm{span}\{[e_i, e_j]\}_{1 \leq i,j \leq 4}$ and $\dim \mathfrak{g}^{(2)} = 3$, it follows that $[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{[e_1, e_2], [e_1, e_3], [e_2, e_3]\} \oplus \mathrm{span}\{e_4\}$. By (11), (12) and (13), it is easy to see that $\mathrm{Trace}(adX|_{[\mathfrak{g},\mathfrak{g}]}) = -8\rho$. But from (9) we know that $\mathrm{Trace}(adX|_{[\mathfrak{g},\mathfrak{g}]}) = -5\rho$, hence $\rho = 0$, which is a contradiction.

Therefore, $V_{-\rho}= \mathrm{span}\{\eta e_1 + \lambda e_3 + \frac{\eta\theta}{\rho} e_4\}$ is the center of $[\mathfrak{g}, \mathfrak{g}]$. If $\eta = 0$, then $V_{-\rho}= \mathrm{span}\{e_3\}$, so $[e_1, e_3] = [e_2, e_3] = [e_3, e_4] = 0$. Similarly, it is easy to see that $\mathrm{Trace}(adX|_{[\mathfrak{g},\mathfrak{g}]}) = -9\rho$ by (11), (14) and (15), which is a contradiction. If $\eta \neq 0$, by (24), (25), (26), we directly get $\mathrm{Trace}(adX|_{[\mathfrak{g},\mathfrak{g}]}) = -9\rho$, which is also a contradiction. This completes the proof of Theorem 1.1.

4. The proof of Theorem 1.2

By Theorem 1.1, it is enough to study the case that $\mathfrak{g}$ is a solvable Lie algebra of dimension 5. By Lemma 2.3, $[\mathfrak{g},\mathfrak{g}]$ is a nilpotent Lie algebra of dimension 4. It is known that a four-dimensional nilpotent Lie algebra is one of the three types of Table 1 (see [8, 15]).

Table 1: Four-dimensional nilpotent Lie algebras

Type	Basis	Non-zero Brackets	Center
1	$\{y_1, y_2, y_3, y_4\}$	$[y_1, y_2] = y_3$	$\mathrm{span}\{y_3, y_4\}$
2	$\{y_1, y_2, y_3, y_4\}$	none	$\mathrm{span}\{y_1, y_2, y_3, y_4\}$
3	$\{y_1, y_2, y_3, y_4\}$	$[y_1, y_2] = y_3, [y_1, y_3] = y_4$	$\mathrm{span}\{y_4\}$

We will complete Theorem 1.2 by three subsections based on the type of $[\mathfrak{g},\mathfrak{g}]$.

4.1. $[\mathfrak{g},\mathfrak{g}]$ is of Type 1 in Table 1

We know that $[\mathfrak{g},\mathfrak{g}] = \mathrm{span}\{e_1, e_2, e_3, e_4\} = \mathrm{span}\{y_1, y_2, y_3, y_4\}$, and the 2-step derived algebra $\mathfrak{g}^{(2)} = \mathrm{span}\{[e_i, e_j]\}_{1\le i,j\le 4} = \mathrm{span}\{y_3\}$ is of dimension 1 and it is contained in the center of $[\mathfrak{g},\mathfrak{g}]$. Clearly $adX(y_3) = cy_3$ for some $c \in \mathbb{R}$, so elements of $\mathfrak{g}^{(2)}$ are eigenvectors of adX. In the following, we will discuss two cases: $\lambda^2+\eta^2 \neq 0$ and $\lambda^2 + \eta^2 = 0$.

Case 1: $\lambda^2 + \eta^2 \neq 0$.

Lemma 4.1. *Let the notations be as above. Then e_4 lies in the center of $[\mathfrak{g},\mathfrak{g}]$.*

Proof. Suppose $[e_1, e_4] \neq 0$, then $\mathfrak{g}^{(2)} = \mathrm{span}\{[e_1, e_4]\}$. Since $[e_1, e_4]$ is an eigenvector of adX, we have

$$adX([e_1, e_4]) = -3\rho[e_1, e_4] - \lambda[e_2, e_4] = -2\rho[e_1, e_4], \tag{18}$$

or
$$adX([e_1, e_4]) = -3\rho[e_1, e_4] - \lambda[e_2, e_4] = -\rho[e_1, e_4]. \tag{19}$$

If (18) is true, then $\lambda[e_2, e_4] = -\rho[e_1, e_4]$. Using this relation and (15), (18), we can easily get the following equation

$$\begin{aligned}(adX)^2([e_1, e_4]) &= -3\rho(-3\rho[e_1, e_4] - \lambda[e_2, e_4]) - \lambda(-3\rho[e_2, e_4] + \lambda[e_1, e_4] - \eta[e_3, e_4])\\ &= 9\rho^2[e_1, e_4] - 3\rho^2[e_1, e_4] - 3\rho^2[e_1, e_4] - \lambda^2[e_1, e_4] + \lambda\eta[e_3, e_4]\\ &= (3\rho^2 - \lambda^2)[e_1, e_4] + \lambda\eta[e_3, e_4] = 4\rho^2[e_1, e_4],\end{aligned}$$

then $\lambda\eta[e_3, e_4] = (\rho^2 + \lambda^2)[e_1, e_4]$. Using this relation and (16), (18), we have

$$\begin{aligned}(adX)^3([e_1, e_4]) &= (3\rho^2 - \lambda^2)(-3\rho[e_1, e_4] - \lambda[e_2, e_4]) + \lambda\eta(-3\rho[e_3, e_4] + \eta[e_2, e_4])\\ &= (-9\rho^3 - \rho\lambda^2 - \rho\eta^2)[e_1, e_4] = (-8\rho^3)[e_1, e_4].\end{aligned}$$

It follows that $-\rho^2 = \lambda^2 + \eta^2 > 0$, which is absurd. Following the same reasoning, one can easily show that (19) is also impossible. This means $[e_1, e_4] = 0$. Similarly, one can show $[e_3, e_4] = 0$.

Combining $[e_1, e_4] = [e_3, e_4] = 0$ with (15), we have $adX([e_2, e_4]) = -3\rho[e_2, e_4]$. Since the real eigenvalues of adX on $[\mathfrak{g},\mathfrak{g}]$ are $-\rho$ and -2ρ, we conclude $[e_2, e_4] = 0$. So e_4 lies in the center of $[\mathfrak{g},\mathfrak{g}]$. ∎

Lemma 4.2. *Let the notations be as above. Then* $[e_1, e_3] = 0$ *and* $\eta[e_1, e_2] = \lambda[e_2, e_3]$.

Proof. We first claim $\lambda[e_1, e_3] = 0$. Suppose $\lambda[e_1, e_3] \neq 0$, we have $adX([e_2, e_3]) = -2\rho[e_2, e_3]+\lambda[e_1, e_3]$ by (13). Clearly, $[e_2, e_3] \neq 0$ (otherwise, it contradicts the initial assumption), since elements of $\mathfrak{g}^{(2)} = \mathrm{span}\{[e_i, e_j]\}_{1\leq i,j\leq 4}$ are eigenvectors of adX, we know $[e_2, e_3]$ is an eigenvector of adX. Then either $adX([e_2, e_3]) = -\rho[e_2, e_3]$ or $adX([e_2, e_3]) = -2\rho[e_2, e_3]$. If the former is true, one can easily follow the proof of Lemma 4.1 to get $-\rho^2 = \lambda^2 + \eta^2 > 0$, which is absurd; if the latter is true, we have $\lambda[e_1, e_3] = 0$, which contradicts the initial assumption. So $\lambda[e_1, e_3] = 0$, the claim is proved.
Similarly one can prove $\eta[e_1, e_3] = 0$ by Lemma 4.1 and (11). Since $\lambda^2 + \eta^2 \neq 0$, we must have $[e_1, e_3] = 0$. Then by Lemma 4.1 and (12), we have $\eta[e_1, e_2] = \lambda[e_2, e_3]$. ∎

After these preparations we can discuss further:

(★) $\lambda \neq 0$. For this case, by Lemma 4.1, Lemma 4.2 and (11), there is a non-zero constant $k \in \mathbb{R}$ such that $[e_1, e_2] = ke_4, [e_2,\ e_3] = k\frac{\eta}{\lambda}e_4$. Using (9) and (10), we have

$$\begin{aligned}(adX)(e_1) &= [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5, e_1] = -kx_2e_4 + x_5[e_5, e_1]\\ &= -\rho e_1 - \lambda e_2 + \theta e_4.\end{aligned}$$

Since $\rho \neq 0$, we know $x_5 \neq 0$. So

$$[e_1, e_5] = \frac{1}{x_5}\{\rho e_1 + \lambda e_2 - (\theta + kx_2)e_4\}.$$

Similarly,

$$\begin{aligned}[e_2, e_5] &= \frac{1}{x_5}\{-\lambda e_1 + \rho e_2 + \eta e_3 + k(x_1 - x_3\frac{\eta}{\lambda})e_4\},\\ [e_3, e_5] &= \frac{1}{x_5}\{-\eta e_2 + \rho e_3 + kx_2\frac{\eta}{\lambda}e_4\},\\ [e_4, e_5] &= \frac{1}{x_5}\{2\rho e_4\}.\end{aligned}$$

We know $adX(e_5) = \sum_{i=1}^{4} x_i[e_i, e_5] = \theta e_1$ by (9), and using those brackets above, we can easily get the following identities:

$$x_1 = \frac{\theta(\rho^2+\eta^2)}{\rho(\rho^2+\eta^2+\lambda^2)}x_5,\quad x_2 = \frac{-\lambda\theta}{(\rho^2+\eta^2+\lambda^2)}x_5,\quad x_3 = \frac{\eta\lambda\theta}{\rho(\rho^2+\eta^2+\lambda^2)}x_5,\quad x_4 = \frac{\theta^2(\rho^2+\eta^2)}{2\rho^2(\rho^2+\lambda^2+\eta^2)}x_5.$$

Define $\alpha = \frac{\rho}{x_5},\ \beta = \frac{\lambda}{x_5},\ \gamma = \frac{\eta}{x_5},\ \delta = \frac{\theta}{x_5}$. The non-zero brackets of $\mathfrak{g}$ are

$$\begin{aligned}&[e_1, e_2] = ke_4,\ [e_2, e_3] = k\frac{\gamma}{\beta}e_4,\\ &[e_1, e_5] = \alpha e_1 + \beta e_2 - \delta(1 - \frac{k\beta}{\alpha^2 + \gamma^2 + \beta^2})e_4,\\ &[e_2, e_5] = -\beta e_1 + \alpha e_2 + \gamma e_3 + \frac{k\alpha\delta}{\alpha^2 + \gamma^2 + \beta^2}e_4,\\ &[e_3, e_5] = -\gamma e_2 + \alpha e_3 - \frac{k\gamma\delta}{\alpha^2 + \gamma^2 + \beta^2}e_4,\\ &[e_4, e_5] = 2\alpha e_4,\end{aligned}$$

where γ, δ are arbitrary and $k\alpha\beta \neq 0$. Then the left-invariant non-Killing conformal vector field is a non-zero constant multiple of

$$\delta(\alpha^2+\gamma^2)e_1 - \alpha\beta\delta e_2 + \gamma\beta\delta e_3 + \tfrac{\delta^2(\alpha^2+\gamma^2)}{2\alpha}e_4 + \alpha(\alpha^2+\gamma^2+\beta^2)e_5.$$

This is case (1) of Theorem 1.2.

($\blacklozenge$) $\lambda = 0$. Since $\lambda^2+\eta^2 \neq 0$, for this case, we have $\eta \neq 0$, moreover, we can require $\delta \neq 0$ because $\lambda = \delta = 0$, $\eta \neq 0$ is equivalent to $\eta = \delta = 0$, $\lambda \neq 0$ in ($\bigstar$) after reordering the basis. By Lemma 4.1, Lemma 4.2 and (13), there is a constant $k \neq 0$ such that $[e_2, e_3] = ke_4$, $[e_1, e_2] = 0$. Following the same procedure of ($\bigstar$), we can get the non-zero brackets of $\mathfrak{g}$ as

$$\begin{aligned} &[e_2,e_3] = ke_4, \quad [e_1,e_5] = \alpha e_1 - \delta e_4, \quad [e_2,e_5] = \alpha e_2 + \gamma e_3, \\ &[e_3,e_5] = -\gamma e_2 + \alpha e_3, \quad [e_4,e_5] = 2\alpha e_4, \end{aligned}$$

where $k\alpha\gamma\delta \neq 0$. Then the left-invariant non-Killing conformal vector field is a non-zero constant multiple of $\delta e_1 + \frac{\delta^2}{2\alpha}e_4 + \alpha e_5$. This is case (2) of Theorem 1.2.

Case 2: $\lambda^2 + \eta^2 = 0$.

Recall that the only real eigenvalues of adX on $[\mathfrak{g}, \mathfrak{g}]$ are $-\rho$ and -2ρ. By (14), (15) and (16), $[e_1, e_4] = [e_2, e_4] = [e_3, e_4] = 0$. Then by (11), (12) and (13), $[e_1, e_2], [e_1, e_3], [e_2, e_3] \in V_{-2\rho}$, which means there exist $l, m, n \in \mathbb{R}$, $l^2 + m^2 + n^2 \neq 0$ such that $[e_1, e_2] = le_4, [e_1, e_3] = me_4, [e_2, e_3] = ne_4$. In the same way with ($\bigstar$), we can get the non-zero brackets of $\mathfrak{g}$ as

$$\begin{aligned} &[e_1,e_2] = le_4, \quad [e_1,e_3] = me_4, \quad [e_2,e_3] = ne_4, \quad [e_1,e_5] = \alpha e_1 - \delta e_4 \\ &[e_2,e_5] = \alpha e_2 + l\frac{\delta}{\alpha}e_4, \quad [e_3,e_5] = \alpha e_3 + m\frac{\delta}{\alpha}e_4, \quad [e_4,e_5] = 2\alpha e_4, \end{aligned}$$

where δ is arbitrary and $\alpha(l^2+m^2+n^2) \neq 0$. The left-invariant non-Killing conformal vector field is a non-zero constant multiple of $\delta e_1 + \frac{\delta^2}{2\alpha}e_4 + \alpha e_5$. This is the case (3) of Theorem 1.2.

4.2. $[\mathfrak{g}, \mathfrak{g}]$ is of Type 2 in Table 1

Following the procedure of ($\bigstar$) again we calculate the non-zero brackets of $\mathfrak{g}$ as

$$\begin{aligned} &[e_1,e_5] = \alpha e_1 + \beta e_2 - \delta e_4, \quad [e_2,e_5] = -\beta e_1 + \alpha e_2 + \gamma e_3, \\ &[e_3,e_5] = -\gamma e_2 + \alpha e_3, \quad [e_4,e_5] = 2\alpha e_4, \end{aligned}$$

where γ, δ, β are arbitrary and $\alpha \neq 0$. Then the left-invariant non-Killing conformal vector field is a non-zero constant multiple of

$$\delta(\alpha^2+\gamma^2)e_1 - \alpha\beta\delta e_2 + \gamma\beta\delta e_3 + \tfrac{\delta^2(\alpha^2+\gamma^2)}{2\alpha}e_4 + \alpha(\alpha^2+\gamma^2+\beta^2)e_5.$$

This is case (4) of Theorem 1.2.

4.3. $[\mathfrak{g}, \mathfrak{g}]$ is of Type 3 in Table 1

For this case, we will show $\mathfrak{g}$ admits no left-invariant non-Killing conformal vector fields. We first prove a simple lemma as follows.

Lemma 4.3. *Any inner derivation of $\mathfrak{g}$ leaves the center of $[\mathfrak{g}, \mathfrak{g}]$ invariant.*

Proof. For any $X, U, V \in \mathfrak{g}$ and Y in the center of $[\mathfrak{g},\mathfrak{g}]$ we get by Jacobi's identity:

$$[[U,V],[X,Y]] = [[[U,V],X],Y] + [X,[[U,V],Y]] = 0.$$

Hence $[X,Y]$ lies in the center of $[\mathfrak{g},\mathfrak{g}]$, which means adX leaves the center of $[\mathfrak{g},\mathfrak{g}]$ invariant and the Lemma is proved. ∎

By Table 1, we know that $[\mathfrak{g},\mathfrak{g}] = \mathrm{span}\{e_1,e_2,e_3,e_4\} = \mathrm{span}\{y_1,y_2,y_3,y_4\}$ and the 2-step derived algebra $\mathfrak{g}^{(2)} = \mathrm{span}\{[e_i,e_j]\}_{1\le i,j\le 4} = \mathrm{span}\{y_3,y_4\}$ is of dimension 2. The center of $[\mathfrak{g},\mathfrak{g}]$ is $\mathrm{span}\{y_4\}$ of dimension 1. So by Lemma 4.3, we know that elements in center of $[\mathfrak{g},\mathfrak{g}]$ are eigenvectors of adX, which is a very important fact in the following discussion. Similarly, we discuss the following two cases: $\lambda^2+\eta^2 = 0$ and $\lambda^2+\eta^2 \neq 0$.

If $\lambda^2+\eta^2 = 0$, by (14), (15) and (16), we know that $[e_1,e_4] = [e_2,e_4] = [e_3,e_4] = 0$. Combining this with (11), (12) and (13), we conclude

$$[e_1,e_2],[e_1,e_3],[e_2,e_3] \in V_{-2\rho} = \mathrm{span}\{e_4\},$$

which contradicts the fact that $\dim\mathfrak{g}^{(2)} = 2$. So we must have $\lambda^2+\eta^2 \neq 0$. Notice that the center of $[\mathfrak{g},\mathfrak{g}]$ is either $V_{-2\rho}=\mathrm{span}\{e_4\}$ or $V_{-\rho}=\mathrm{span}\{\eta e_1+\lambda e_3+\frac{\eta\theta}{\rho}e_4\}$.

Proposition 4.4. *Let the notations be as above. If e_4 lies in the center of $[\mathfrak{g},\mathfrak{g}]$, then $\mathfrak{g}$ admits no left-invariant non-Killing conformal vector fields.*

Proof. Since e_4 is in the center of $[\mathfrak{g},\mathfrak{g}]$, we have $[e_1,e_4] = [e_2,e_4] = [e_3,e_4] = 0$.

Case 1. $\eta = 0$. Then $\lambda \neq 0$ since $\lambda^2+\eta^2 \neq 0$.

First, we claim that $[e_1,e_3]$ and $[e_2,e_3]$ are non-zero. Otherwise, for example, $[e_1,e_3] = 0$, then by (12) and $\lambda \neq 0$, $\eta = 0$, we have $[e_2,e_3] = 0$, which means e_3 also lies in the center of $[\mathfrak{g},\mathfrak{g}]$. This is a contradiction to the fact that center of $[\mathfrak{g},\mathfrak{g}]$ is of dimension 1.

Secondly, we claim that $[e_1,e_3]$, $[e_2,e_3]$ are linearly independent, so they are not the eigenvectors of adX. Otherwise, there is a non-zero $t \in \mathbb{R}$ such that $[e_1,e_3] = t[e_2,e_3]$. Then by (12) and (13), we have

$$adX([e_1,e_3]) = (-2\rho - \tfrac{\lambda}{t})[e_1,e_3] \quad \text{and} \quad adX([e_2,e_3]) = (-2\rho+\lambda t)[e_2,e_3],$$

so $-2\rho - \frac{\lambda}{t} = -2\rho + \lambda t$, which means $t^2 = -1$. It is impossible. So they are linearly independent. Suppose $[e_1,e_3]$ (or $[e_2,e_3]$) is an eigenvector of adX , from (12) (or (13)), it is easy to see that they are linearly dependent.

Finally, since $\dim\mathfrak{g}^{(2)} = 2$, we have $\mathfrak{g}^{(2)} = \mathrm{span}\{e_4,[e_1,e_3]\} = \mathrm{span}\{e_4,[e_2,e_3]\}$. Let $[e_2,e_3] = ke_4 + l[e_1,e_3], (k,l \in \mathbb{R}, kl \neq 0)$. By the action of adX, we have $\lambda(1+l^2) = -\lambda lk = 0$, which violates the hypothesis.

Case 2. $\eta \neq 0$. Using (11), (12), (13) and the fact that $\dim\mathfrak{g}^{(2)} = 2$, one can similarly follow the calculation of Case 1 to prove Case 2. Here, we just give a sketch of the proof.

Firstly, we prove that $[e_1,e_2]$, $[e_1,e_3]$ are both non-zero because of $\dim\mathfrak{g}^{(2)} = 2$.

Secondly, we prove that $[e_1,e_2]$, $[e_1,e_3]$ are linearly independent, hence neither $[e_1,e_2]$ nor $[e_1,e_3]$ is the eigenvector of adX.

Finally, we have $\mathfrak{g}^{(2)} = \mathrm{span}\{e_4, [e_1, e_2]\} = \mathrm{span}\{e_4, [e_1, e_3]\}$. Let $[e_1, e_3] = ke_4 + l[e_1, e_2], (k, l \in \mathbb{R},\ kl \neq 0)$. By the action of adX, we conclude that $[e_1, e_3]$ is an eigenvector of adX, which is a contradiction.
So for this case, $\mathfrak{g}$ admits no left-invariant non-Killing conformal vector fields. ∎

Similar to Proposition 4.4, we have the following proposition.

Proposition 4.5. *Let the notations be as above. If $v = \eta e_1 + \lambda e_3 + \frac{\theta\eta}{\rho} e_4$ lies in the center of $[\mathfrak{g}, \mathfrak{g}]$, then $\mathfrak{g}$ admits no left-invariant non-Killing conformal vector fields.*

Proof. Since $v = \eta e_1 + \lambda e_3 + \frac{\theta\eta}{\rho} e_4$ lies in the center of $[\mathfrak{g}, \mathfrak{g}]$, we have

$$[v, e_1] = -\lambda[e_1, e_3] - \frac{\theta\eta}{\rho}[e_1, e_4] = 0, \tag{20}$$

$$[v, e_2] = \eta[e_1, e_2] - \lambda[e_2, e_3] - \frac{\theta\eta}{\rho}[e_2, e_4] = 0, \tag{21}$$

$$[v, e_3] = \eta[e_1, e_3] - \frac{\theta\eta}{\rho}[e_3, e_4] = 0, \tag{22}$$

$$[v, e_4] = \eta[e_1, e_4] + \lambda[e_3, e_4] = 0. \tag{23}$$

Since $\lambda^2 + \eta^2 \neq 0$, we have the following cases:

Case 1. $\eta = 0$. Then $\lambda \neq 0$. For this case, we have $v = \lambda e_3$, then $[e_1, e_3] = [e_2, e_3] = [e_3, e_4] = 0$. Then there is a contradiction similar to the discussion in Proposition 4.4 for $[e_1, e_4], [e_2, e_4]$.

Case 2. $\eta \neq 0$. In this case, by (20), (21), (22) and (23) we obtain $[e_1, e_3] = -\frac{\theta}{\rho}[e_3, e_4]$, $[e_1, e_4] = -\frac{\lambda}{\eta}[e_3, e_4]$, $[e_3, e_4] = \frac{\lambda}{\eta}[e_2, e_3] + \frac{\theta}{\rho}[e_2, e_4]$. By these relations and (13), (15), (16), we have

$$adX([e_2, e_3]) = -2\rho[e_2, e_3] + \frac{\theta\lambda}{\rho}[e_3, e_4], \tag{24}$$

$$adX([e_2, e_4]) = -3\rho[e_2, e_4] - \frac{1}{\eta}(\lambda^2 + \eta^2)[e_3, e_4], \tag{25}$$

$$adX([e_3, e_4]) = -3\rho[e_3, e_4] + \eta[e_2, e_4]. \tag{26}$$

Furthermore, we get a contradiction similar to the discussion in Proposition 4.4 for $[e_2, e_4]$, $[e_3, e_4]$. Summing up, we have proved Theorem 1.2. ∎

Remark 4.6. (Yamabe soliton) A vector field X on a pseudo-Riemannian manifold (M, g) is called a *Yamabe soliton* vector field if $L_X g = 2(R - \lambda)g$, where L_X is the Lie derivative, R is the scalar curvature of the metric g and λ is a constant. It plays an important role in exploring the Yamabe flow (see [2, 3, 4, 6, 14]). Note that the scalar curvature of any Lorentzian Lie group is a constant. So by the definitions of a conformal vector field and a Yamabe soliton vector field, any left-invariant non-Killing conformal vector field is a Yamabe soliton vector field.

Remark 4.7. (Some notes on the Lie algebras in Theorem 1.2) Denote the Lie algebras in Theorem 1.2 defined by the brackets in

(1) the first case by $\mathfrak{A}(k,\alpha,\beta,\gamma,\delta)$, where γ, δ are arbitrary and $k\alpha\beta\neq 0$;
(2) the second case by $\mathfrak{B}(k,\alpha,\gamma,\delta)$, where $k\alpha\gamma\delta\neq 0$;
(3) the third case by $\mathfrak{C}(l,m,n,\alpha,\delta)$, where δ is arbitrary and $\alpha(l^2+m^2+n^2)\neq 0$;
(4) the fourth case by $\mathfrak{D}(\alpha,\beta,\gamma,\delta)$, where γ, δ, β are arbitrary and $\alpha\neq 0$.

Clearly $\mathfrak{A}$ and $\mathfrak{D}$ have non-isomorphic derived algebras, thus they are non-isomorphic to each other as Lie algebras.

Lemma 4.8. *For given k,α,γ and arbitrary $\delta\neq 0$, $\mathfrak{B}(k,\alpha,\gamma,\delta)$ is isomorphic to $\mathfrak{A}(k,\alpha,\gamma,0,0)$.*

Proof. For any δ, the Lie algebra $\mathfrak{B}(k,\alpha,\gamma,\delta)$ is isomorphic to $\mathfrak{B}(k,\alpha,\gamma,0)$ (here, we allow $\delta=0$). In fact, set

$$e_1{}'=e_1+\tfrac{\delta}{\alpha}e_4,\quad e_2{}'=e_2,\quad e_3{}'=e_3,\quad e_4{}'=e_4,\quad e_5{}'=e_5.$$

Then the non-zero brackets are

$$[e_2{}',e_3{}']=ke_4{}',\ \ [e_1{}',e_5{}']=\alpha e_1{}',\ \ [e_2{}',e_5{}']=\alpha e_2{}'+\gamma e_3{}',$$
$$[e_3{}',e_5{}']=-\gamma e_2{}'+\alpha e_3{}',\ \ [e_4{}',e_5']=2\alpha e_4{}',$$

where $k\alpha\gamma\neq 0$. This is $\mathfrak{B}(k,\alpha,\gamma,0)$.

In the Lie algebra $\mathfrak{A}(k,\alpha,\gamma,0,0)$, set $\tilde e_1=e_3,\ \tilde e_2=e_1,\ \tilde e_3=e_2,\ \tilde e_4=e_4,\ \tilde e_5=e_5$. Then the non-zero brackets are

$$[\tilde e_2,\tilde e_3]=k\tilde e_4,\ \ [\tilde e_1,\tilde e_5]=\alpha\tilde e_1,\ \ [\tilde e_2,\tilde e_5]=\alpha\tilde e_2+\gamma\tilde e_3,$$
$$[\tilde e_3,\tilde e_5]=-\gamma\tilde e_2+\alpha\tilde e_3,\ \ [\tilde e_4,\tilde e_5]=2\alpha\tilde e_4,$$

where $k\alpha\gamma\neq 0$. Then the linear map φ defined by $\varphi(e_i{}')=\tilde e_i$ is an isomorphism between $\mathfrak{B}(k,\alpha,\gamma,0)$ and $\mathfrak{A}(k,\alpha,\gamma,0,0)$. ■

By the isomorphisms of Lemma 4.8, we have a family of Lorentzian metrics on $\mathfrak{B}(k,\alpha,\gamma,0)\cong\mathfrak{A}(k,\alpha,\gamma,0,0)$. The matrices of the family of Lorentzian metrics are defined by

$$\langle\cdot,\cdot\rangle_\delta=\begin{pmatrix}1&0&0&0&-\frac{\delta}{\alpha}\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&-1\\-\frac{\delta}{\alpha}&0&0&-1&0\end{pmatrix},\tag{27}$$

and the corresponding left-invariant non-Killing conformal vector field is a non-zero constant multiple of $\delta e_1{}'-\frac{\delta^2}{2\alpha}e_4{}'+\alpha e_5{}'$ with respect to each δ. We have the following conjecture.

Conjecture 4.9. *For given k,α,γ, the metrics $\langle\cdot,\cdot\rangle_\delta$ on the Lie algebra $\mathfrak{B}(k,\alpha,\gamma,0)$ are non-isometric to each other.*

For the study on isometry, we refer to [9], [10].

In summary, we can say that the Lie algebras listed in Theorem 1.2 can be isomorphic to each other. One can distinguish them by [11]. But the isomorphism will induce different metrics on the Lie algebras. Here, we use the notation in this paper to keep the metric form (5) simplest.

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Hui Zhang, Zhiqi Chen, Shaoxiang Zhang
School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, P. R. China
2120160023@mail.nankai.edu.cn, chenzhiqi@nankai.edu.cn, zhangshaoxiang@mail.nankai.edu.cn

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