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Kobayashi's Conjecture on Associated Varieties for Klein Four Symmetric Pairs $(\mathbf{E}_{6(-14)}, \mathbf{Spin}(8,1))$

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Abstract. We confirm a conjecture on associated varieties by Toshiyuki Kobayashi for the Klein four symmetric pair $(\mathrm{E}_{6(-14)}, \mathrm{Spin}(8,1))$, which provides an alternative way to confirm the conjecture for the symmetric pair $(\mathrm{Spin}(8,2), \mathrm{Spin}(8,1))$. Also, for Klein four symmetric pairs (G, G^Γ) with the exceptional simple Lie groups G of Hermitian type, there exists a discrete series representation of G which is G^Γ-admissible if and only if (G, G^Γ) is of holomorphic type.

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Key Words: Associated variety, discrete branching law, discrete series representation, Klein four symmetric pair.

1. Introduction and main results

In his celebrated paper [12], Toshiyuki Kobayashi introduced associated varieties to the study of the restrictions of simple $(\mathfrak{g}, K)$-modules, which we recall now. Let G be a reductive Lie group with Lie algebra $\mathfrak{g}$, and G' a closed reductive subgroup with subalgebra $\mathfrak{g}'$. We fix a maximal compact subgroup K of G such that $K' := K \cap G'$ is a maximal compact subgroup of G'. Denote by $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{g}'_\mathbb{C}$ the complexified Lie algebras of $\mathfrak{g}$ and $\mathfrak{g}'$ respectively. The inclusion $\mathfrak{g}'_\mathbb{C} \hookrightarrow \mathfrak{g}_\mathbb{C}$ gives a projection of the dual spaces $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'} : \mathfrak{g}_\mathbb{C}^* \twoheadrightarrow \mathfrak{g}'^*_\mathbb{C}$.

For a unitarizable simple $(\mathfrak{g}, K)$-module X, we denote by $\mathcal{V}_{\mathfrak{g}_\mathbb{C}}(X)$ the associated variety of X in the dual space $\mathfrak{g}_\mathbb{C}^*$. Kobayashi established in [12, Theorem 3.1] that, if Y is a simple $(\mathfrak{g}', K')$-module such that $\mathrm{Hom}_{(\mathfrak{g}',K')}(Y, X) \neq \{0\}$, then $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}\mathcal{V}_{\mathfrak{g}_\mathbb{C}}(X) \subseteq \mathcal{V}_{\mathfrak{g}'_\mathbb{C}}(Y)$. Moreover, Kobayashi proposed a conjecture in [15] that the inclusion is actually an equality. Namely,

Conjecture 1.1 (Kobayashi [15, Conjecture 5.11])**.** Let X be a simple $(\mathfrak{g}, K)$-module. If Y is a simple $(\mathfrak{g}', K')$-module such that $\mathrm{Hom}_{(\mathfrak{g}',K')}(Y, X) \neq \{0\}$, then $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}\mathcal{V}_{\mathfrak{g}_\mathbb{C}}(X) = \mathcal{V}_{\mathfrak{g}'_\mathbb{C}}(Y)$.

For convenience, if $\mathfrak{g}$, $\mathfrak{g}'$, and X satisfy Conjecture 1.1, we will say that Conjecture 1.1 is true for the triple $(\mathfrak{g}, \mathfrak{g}', X)$. Conjecture 1.1 is known to be true for $(\mathfrak{g}, \mathfrak{g}', X)$ in the following cases:

(1) X is a generalized Verma module and $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair ([16, Theorem 4.12]);

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(2) X is the underlying $(\mathfrak{g}, K)$-module of the minimal representation of $\mathrm{O}(p, q)$ with $p+q$ even and $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair ([18]);

(3) $X = A_{\mathfrak{q}}(\lambda)$ and $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair ([22, Theorem 8.5]);

(4) X is a highest / lowest weight simple $(\mathfrak{g}, K)$-module and the natural embedding $G'/K' \hookrightarrow G/K$ is holomorphic ([16, Theorem 4.12], cf. [21, Theorem 7.4]);

(5) X is the minimal holomorphic representation and $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair ([21, Theorem 7.6]).

These verifications for Conjecture 1.1 were made mostly for symmetric pairs. However, Conjecture 1.1 still holds for some non-symmetric pairs as is seen from explicit discrete branching laws in the following cases:

(6) $X = A_{\mathfrak{q}}(\lambda)$ and $(G, G') = (\mathrm{U}(p, q; \mathbb{F}), \mathrm{U}(p, r; \mathbb{F}))$ with $q > r$ and $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ ([9, Theorem 3.3 & Theorem 3.4 & Theorem 3.5]);

(7) $X = A_{\mathfrak{q}}(\lambda)$ and $(G, G') = (\mathrm{SO}_0(4p, 4q), \mathrm{Sp}(p, q))$ ([10, Theorem 6.1]);

(8) X is a discrete series representation and $(G, G') = (\mathrm{SO}_0(8, 8), \mathrm{Spin}(8, 1))$ ([17, Theorem 5.5]).

It can be computed from [7] that $(\mathrm{SO}_0(4p, 4q), \mathrm{Sp}(p, q))$ is a Klein four symmetric pair, but $(\mathrm{SO}_0(8, 8), \mathrm{Spin}(8, 1))$ is not. As for $(\mathrm{U}(p, q; \mathbb{F}), \mathrm{U}(p, r; \mathbb{F}))$, whether it is a Klein four symmetric pair depends on p, q, r, and $\mathbb{F}$. In this article, we will confirm Conjecture 1.1 for the Klein four symmetric pair $(\mathrm{E}_{6(-14)}, \mathrm{Spin}(8, 1))$.

Definition 1.2. Let G be a reductive Lie group, and Γ a Klein four subgroup of its automorphism group $\mathrm{Aut}G$. Write $G^\Gamma := \{g \in G \mid \sigma(g) = g \text{ for all } \sigma \in \Gamma\}$. Then (G, G^Γ) is called a *Klein four symmetric pair*. In particular, if G is a simple Lie group of Hermitian type and every nonidentity element $\sigma \in \Gamma$ defines a symmetric pair of holomorphic type (cf. [14, Definition 1.4]), then (G, G^Γ) is called a Klein four symmetric pair of holomorphic type. Respectively, let $\mathfrak{g}$ be a reductive Lie algebra, and Γ a Klein four subgroup of its automorphism group $\mathrm{Aut}\mathfrak{g}$. Write $\mathfrak{g}^\Gamma := \{X \in \mathfrak{g} \mid \sigma(X) = X \text{ for all } \sigma \in \Gamma\}$. Then $(\mathfrak{g}, \mathfrak{g}^\Gamma)$ is called a Klein four symmetric pair (on the Lie algebra level). In particular, if $\mathfrak{g}$ is a simple Lie algebra of Hermitian type and every nonidentity element $\sigma \in \Gamma$ defines a symmetric pair of holomorphic type, then $(\mathfrak{g}, \mathfrak{g}^\Gamma)$ is called a Klein four symmetric pair of holomorphic type (on the Lie algebra level). ■

Discrete branching laws were studied in the past three decades by some mathematicians. Kobayashi initiated a study of the restriction of $A_{\mathfrak{q}}(\lambda)$ to reductive subgroups in [9]. Since then, he developed a general theory of discretely decomposable restrictions of irreducible representations of reductive groups to their reductive subgroups, and showed many significant results in [10], [11], and [12]. Recently, discrete branching laws for Klein four symmetric pairs were studied in [3], [4], [5], and [6]. When G is an exceptional Lie group of Hermitian type, the Klein four symmetric pairs (G, G^Γ) of holomorphic type were classified, making full use of the classification of elementary abelian 2-subgroups of compact Lie groups [24] and the classification of symmetric pairs of holomorphic / anti-holomorphic type [14, Table 3.4.2 & Table 3.4.1]. In this case, because of the fourth bullet item listed above, Conjecture 1.1

is automatically true for $(\mathfrak{g}, \mathfrak{g}^\Gamma, X)$ whenever X is a highest / lowest weight simple $(\mathfrak{g}, K)$-module. As for Klein four symmetric pairs of non-holomorphic type, $(G, G^\Gamma) = (\mathrm{E}_{6(-14)}, \mathrm{Spin}(8,1))$ is a very special Klein four symmetric pair for exceptional Lie group G of Hermitian type in the sense of [5, Theorem 16]. Precisely, $(G, G^\Gamma) = (\mathrm{E}_{6(-14)}, \mathrm{Spin}(8,1))$ is the only Klein four symmetric pair such that there exists a nontrivial unitarizable simple $(\mathfrak{g}, K)$-module π_K which is both discretely decomposable as a $(\mathfrak{g}^\Gamma, K^\Gamma)$-module and is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module for some $\sigma \in \Gamma$ of anti-holomorphic type. Now we may state the first main result of this article.

Theorem 1.3. *Let X be the minimal holomorphic representation of $\mathfrak{e}_{6(-14)}$. Then* Conjecture 1.1 *is true for the triple $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(8,1), X)$.*

The definition for minimal holomorphic representations will be made precise at the end of Section 2. Moreover, for $(\mathfrak{g}, \mathfrak{g}^\Gamma) = (\mathfrak{e}_{6(-14)}, \mathfrak{so}(8,1))$, it is known from [5, Lemma 12] that there exists an involution $\sigma \in \Gamma$ such that

$$(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{6(-14)}, \mathfrak{so}(8,2) \oplus \mathfrak{so}(2)).$$

Then the first direct summand $\mathfrak{so}(8,2)$ together with $\mathfrak{so}(8,1)$ forms another symmetric pair $(\mathfrak{so}(8,2), \mathfrak{so}(8,1))$. We will confirm Conjecture 1.1 for the symmetric pair $(\mathfrak{so}(8,2), \mathfrak{so}(8,1))$ with a series of unitarizable simple $(\mathfrak{so}(8,2), \mathrm{Spin}(8) \times \mathrm{Spin}(2))$-modules. Although this result is a special case of the general theory established by Kobayashi [16, Theorem 4.12], this provides a new way to study discrete branching laws and Conjecture 1.1 for symmetric pairs by means of Klein four symmetric pairs.

In previous articles involving discrete branching laws for Klein four symmetric pairs, we mainly discussed the discrete decomposability of the restrictions of $(\mathfrak{g}, K)$-modules. In the final part of this article, we will discuss, for Klein four symmetric pairs (G, G^Γ) with exceptional simple Lie groups G of Hermitian type, G^Γ-admissibility of the restrictions of discrete series representations of G. Thus, here is the second result of this article.

Theorem 1.4. *Let G be an exceptional simple Lie group of Hermitian type. If (G, G^Γ) is a Klein four symmetric pair of non-holomorphic type, then there is no discrete series representation of G which is G^Γ-admissible.*

The following corollary follows from Theorem 1.4 immediately.

Corollary 1.5. *Let G be an exceptional simple Lie group of Hermitian type, and (G, G^Γ) a Klein four symmetric pair. Then there exists a discrete series representation π of G which is G^Γ-admissible, if and only if (G, G^Γ) is of holomorphic type.*

Proof. The conclusion follows from Theorem 1.4, and the fact that any holomorphic / anti-holomorphic discrete series of G is G^Γ-admissible. ■

2. Preliminary on associated varieties

We recall a definition of the associated varieties and the general theory of its behavior under the restriction to reductive subalgebra from [12]. Let $\mathfrak{g}$ be a reductive Lie algebra with its complexification $\mathfrak{g}_\mathbb{C}$, and let $\{U_j(\mathfrak{g}_\mathbb{C})\}_{j \in \mathbb{Z}_{\geq 0}}$ be the standard increasing filtration of the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$.

Suppose that X is a finitely generated $\mathfrak{g}_{\mathbb{C}}$-module. A filtration $X = \bigcup_{i\in\mathbb{Z}_{\geq 0}} X_i$ is called a *good filtration* if it satisfies the following conditions:

(C1) X_i is finite dimensional for any $i \in \mathbb{Z}_{\geq 0}$;

(C2) $U_j(\mathfrak{g}_{\mathbb{C}})X_i \subseteq X_{i+j}$ for any $i, j \in \mathbb{Z}_{\geq 0}$;

(C3) there exists $n \in \mathbb{Z}_{\geq 0}$ such that $U_j(\mathfrak{g}_{\mathbb{C}})X_i = X_{i+j}$ for any $i \geq n$ and $j \in \mathbb{Z}_{\geq 0}$.

The graded algebra $\mathrm{gr}U(\mathfrak{g}_{\mathbb{C}}) := \bigoplus_{j\in\mathbb{Z}_{\geq 0}} U_j(\mathfrak{g}_{\mathbb{C}})/U_{j-1}(\mathfrak{g}_{\mathbb{C}})$ is isomorphic to the symmetric algebra $S(\mathfrak{g}_{\mathbb{C}})$ by the Poincaré-Birkhoff-Witt theorem and one may regard the graded module $\mathrm{gr}X := \bigoplus_{i\in\mathbb{Z}_{\geq 0}} X_i/X_{i-1}$ as an $S(\mathfrak{g}_{\mathbb{C}})$-module.
Let $\mathrm{Ann}_{S(\mathfrak{g}_{\mathbb{C}})}(\mathrm{gr}X) := \{f \in S(\mathfrak{g}_{\mathbb{C}}) \mid fv = 0 \text{ for any } v \in \mathrm{gr}X\}$ and define

$$\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X) := \{x \in \mathfrak{g}_{\mathbb{C}}^* \mid f(x) = 0 \text{ for any } f \in \mathrm{Ann}_{S(\mathfrak{g}_{\mathbb{C}})}(\mathrm{gr}X)\}$$

which does not depend on the choice of good filtration. Then $\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X)$ is called the associated variety of X.
Let $\mathfrak{g}'$ be a reductive subalgebra of $\mathfrak{g}$, and then the inclusion $\mathfrak{g}' \hookrightarrow \mathfrak{g}$ gives a projection of the complexified dual spaces $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'} : \mathfrak{g}_{\mathbb{C}}^* \twoheadrightarrow \mathfrak{g}'^*_{\mathbb{C}}$.

Fact 2.1 (Kobayashi [12, Theorem 3.1 & Theorem 3.7]).
Let X be a simple $(\mathfrak{g}, K)$-module.

(1) *If Y is a simple $(\mathfrak{g}', K')$-module such that $\mathrm{Hom}_{(\mathfrak{g}',K')}(Y, X) \neq \{0\}$, then $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}\mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(X) \subseteq \mathcal{V}_{\mathfrak{g}'_{\mathbb{C}}}(Y)$.*

(2) *If Y_i are simple $(\mathfrak{g}', K')$-modules such that $\mathrm{Hom}_{(\mathfrak{g}',K')}(Y_i, X) \neq \{0\}$ for $i = 1, 2$, then $\mathcal{V}_{\mathfrak{g}'_{\mathbb{C}}}(Y_1) = \mathcal{V}_{\mathfrak{g}'_{\mathbb{C}}}(Y_2)$.*

As the preparation for the next section, we recalls the constructions of the lowest weight modules $L(\lambda)$ and the precise definition for minimal holomorphic representations.

Suppose that G is a simple Lie group of Hermitian type, and then the Lie algebra $\mathfrak{k}$ of K has the one-dimensional center $Z(\mathfrak{k})$. A maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ becomes a Cartan subalgebra of $\mathfrak{g}$. Moreover, there exists a characteristic element $Z \in Z(\mathfrak{k})$ such that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+ + \mathfrak{p}_-$ is a decomposition with respect to the eigenspaces of $\mathrm{ad}Z$ on $\mathfrak{g}_{\mathbb{C}}$ corresponding to the eigenvalues 0, $\sqrt{-1}$, and $-\sqrt{-1}$ respectively.

Suppose that X is a simple $(\mathfrak{g}, K)$-module. Then set

$$X^{\mathfrak{p}_-} = \{v \in X \mid Yv = 0 \text{ for any } Y \in \mathfrak{p}^-\}.$$

Since K normalizes $\mathfrak{p}_-$, $X^{\mathfrak{p}_-}$ is a K-submodule. Further, $X^{\mathfrak{p}_-}$ is either zero or an irreducible finite-dimensional representation of K. A $(\mathfrak{g}, K)$-module X is called a *lowest weight module* if $X^{\mathfrak{p}_-} \neq \{0\}$. Any lowest weight simple $(\mathfrak{g}, K)$-module is constructed as follows. Denote by $F(\lambda)$ the irreducible representation of K with the highest weight λ. Let $\mathfrak{p}_-$ act as zero on $F(\lambda)$ and the generalized Verma module $M(\lambda) = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}}+\mathfrak{p}_-)} F(\lambda)$ is a $(\mathfrak{g}, K)$-module. Then the unique simple quotient $L(\lambda)$ of $M(\lambda)$ is a lowest weight simple $(\mathfrak{g}, K)$-module.

Let $Z' := -\sqrt{-1}Z \in \sqrt{-1}Z(\mathfrak{k})$ such that $\mathrm{ad}Z'$ acts as 0 and ± 1 on $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\pm}$ respectively. We take the weight $\zeta \in \sqrt{-1}\mathfrak{t}^*$ such that $\zeta(Z') > 0$ and $\zeta([\mathfrak{k}, \mathfrak{k}] \cap \mathfrak{t}) = 0$. According to the well-known classification results of unitarizable highest / lowest weight modules proved by [2] and [8] independently, for any $(\mathfrak{k}, \mathfrak{t})$-dominant integral weight $\lambda_0 \in \sqrt{-1}\mathfrak{t}^*$, there exist $a \in \mathbb{R}$, $c \in \mathbb{R}_{>0}$, and $r \in \mathbb{Z}_{>0}$ such that $L(\lambda_0 + x\zeta)$ for $x \in \mathbb{R}$ is unitarizable if and only if $x \in \{a - ic \mid 0 \le i \le r-1, i \in \mathbb{Z}\} \cup (a, +\infty)$. If $\lambda_0 = 0$, then $a = (r-1)c$ and r is equal to the real rank of $\mathfrak{g}$. Thus, $L(x\zeta)$ is unitarizable if and only if $x \in \{ic \mid 0 \le i \le r-1, i \in \mathbb{Z}\} \cup ((r-1)c, +\infty)$.

Definition 2.2. Let $\mathfrak{g}$ be a simple Lie algebra of Hermitian type such that the real rank of $\mathfrak{g}$ is greater than 1. Then $L(c\zeta)$ is called the *minimal holomorphic representation* of $\mathfrak{g}$.

3. Proof for Theorem 1.3

In this section, let $G = \mathrm{E}_{6(-14)}$ with the Lie algebra $\mathfrak{g} = \mathfrak{e}_{6(-14)}$, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{e}_6$ the complex simple Lie algebra of type E_6. It is known from [5, Proposition 10] that there is a Klein four subgroup Γ of $\mathrm{Aut}G$ such that $\mathfrak{g}^{\Gamma} = \mathfrak{so}(8,1)$. By [5, Lemma 12 & Lemma 14], Γ is generated by two involutive automorphisms σ and τ with $\mathfrak{g}^{\sigma} \cong \mathfrak{f}_{4(-20)}$ and $\mathfrak{g}^{\tau} \cong \mathfrak{so}(8,2) \oplus \mathfrak{so}(2)$.

Let K be a Γ-stable maximal compact subgroup of G, and $\mathfrak{k}$ the corresponding compact subalgebra with its complexification $\mathfrak{k}_{\mathbb{C}}$. We fix a Γ-stable Cartan subalgebra of the complexified Lie algebra $\mathfrak{k}_{\mathbb{C}}$, which is automatically a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ because $\mathfrak{g}$ is of Hermitian type. We choose a simple root system $\{\alpha_i \mid 1 \le i \le 6\}$ such that α_6 is the unique noncompact simple root corresponding to the real form $\mathfrak{g}$. The corresponding Dynkin diagram is given in Figure 1.

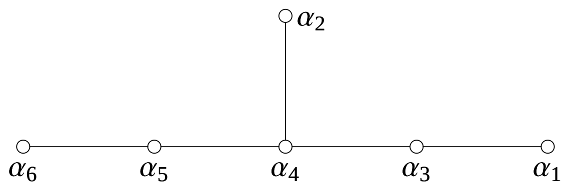

Figure 1: Dynkin diagram of E_6.

For each simple root α_i, we denote by ω_i the fundamental weight corresponding to α_i. As described in [21, 3.11], put $\beta_i := \alpha_{7-i}$ for $1 \le i \le 5$, and then $\{\beta_i\}_{i=1}^5$ form a set of simple roots for $\mathfrak{so}(10,\mathbb{C})$, the simple summand of $\mathfrak{g}_{\mathbb{C}}^{\tau}$. We write μ_i for the fundamental weights of β_i for $1 \le i \le 5$.

It is known from [2, Theorem 12.4] that the lowest weight simple $(\mathfrak{g}, K)$-module $L(3\omega_6)$ is unitarizable, and affords the minimal holomorphic representation of $\mathfrak{g}$. The general theory of Kobayashi ensures that the restriction of $L(3\omega_6)$ to $\mathfrak{g}^{\tau}$ is discretely decomposable ([10], [11], and [12]) and is multiplicity-free ([14, Theorem A]) when restricted to $(\mathfrak{g}^{\tau}, K^{\tau})$. The explicit branching law in this special case was given in [21, Definition 1.3 & Setting 2.6], see also [14, Chapter 8] for the general algorithm, as follows:

$$L(3\omega_6) \cong \bigoplus_{k=0}^{+\infty} L'(3\mu_1 + k\mu_5) \boxtimes \mathbb{C}_{k+2}$$

as $(\mathfrak{g}^{\tau}, K^{\tau})$-modules, where $L'(3\mu_1 + k\mu_5)$ denotes the lowest weight simple $(\mathfrak{so}(8,2)$, $\mathrm{Spin}(8) \times \mathrm{Spin}(2))$-module with lowest weight $3\mu_1 + k\mu_5$ and $\mathbb{C}_{k+2}$ is an one-dimensional module of $\mathfrak{so}(2)$.

If one forgets the action of $\mathfrak{so}(2)$, then one has the decomposition

$$L(3\omega_6) \cong \bigoplus_{k=0}^{+\infty} L'(3\mu_1 + k\mu_5)$$

as $(\mathfrak{so}(8,2), \mathrm{Spin}(8) \times \mathrm{Spin}(2))$-modules.

Proof for Theorem 1.3. Since $\mathfrak{g}^\tau \cong \mathfrak{so}(8,2) \oplus \mathfrak{so}(2)$ is not compact, the center of $\mathfrak{k}$ does not centralize the whole $\mathfrak{g}^\tau$. It follows that the center of $\mathfrak{k}$ is contained in $\mathfrak{so}(8,2)$. For convenience, write $\mathfrak{h}$ for $\mathfrak{so}(8,2)$ and $\mathfrak{h}_{\mathbb{C}}$ for its complexification. By [21, Theorem 7.4], $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}} \mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(L(3\omega_6)) = \mathcal{V}_{\mathfrak{h}_{\mathbb{C}}}(L'(3\mu_1 + k\mu_5))$ for any $k \in \mathbb{Z}_{\geq 0}$. On the other hand, it is known from [21, Setting 2.2] that $L'(3\mu_1)$ is the minimal holomorphic representation of $\mathfrak{h} = \mathfrak{so}(8,2)$. Moreover, by [23, Theorem 19] in this special case or by the general theory of [15, Section 3], $L'(3\mu_1)$ is simple as a $(\mathfrak{g}^\Gamma, K^\Gamma)$-module. By [21, Theorem 7.6], $\mathrm{pr}_{\mathfrak{h}\to\mathfrak{g}^\Gamma} \mathcal{V}_{\mathfrak{h}_{\mathbb{C}}}(L'(3\mu_1)) = \mathcal{V}_{\mathfrak{g}^\Gamma_{\mathbb{C}}}(L'(3\mu_1))$ because $(\mathfrak{h}, \mathfrak{g}^\Gamma) = (\mathfrak{so}(8,2), \mathfrak{so}(8,1))$ is a symmetric pair of anti-holomorphic type. One has

$$\begin{aligned}\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}^\Gamma} \mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(L(3\omega_6)) &= \mathrm{pr}_{\mathfrak{h}\to\mathfrak{g}^\Gamma} \circ \mathrm{pr}_{\mathfrak{g}\to\mathfrak{h}} \mathcal{V}_{\mathfrak{g}_{\mathbb{C}}}(L(3\omega_6)) = \mathrm{pr}_{\mathfrak{h}\to\mathfrak{g}^\Gamma} \mathcal{V}_{\mathfrak{h}_{\mathbb{C}}}(L'(3\mu_1)) \\ &= \mathcal{V}_{\mathfrak{g}^\Gamma_{\mathbb{C}}}(L'(3\mu_1)).\end{aligned}$$

The conclusion follows from the fact that $L(3\omega_6)$ is the minimal holomorphic representation and Fact 2.1(2). ■

4. An alternative way to study symmetric pairs

Let G be a noncompact reductive Lie group, G' a reductive subgroup of G, and G'' a reductive subgroup of G'. Needless to say, let $\mathfrak{g}$, $\mathfrak{g}'$, and $\mathfrak{g}''$ be the corresponding Lie algebras with their complexifications $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}'_{\mathbb{C}}$, and $\mathfrak{g}''_{\mathbb{C}}$. Take a maximal compact subgroup K of G such that $K' := K \cap G'$ and $K'' := K \cap G''$ are maximal compact subgroups of G' and G'' respectively.

Definition 4.1 (Kobayashi [12, Definition 1.1])**.** A $(\mathfrak{g}, K)$-module X is said to be discretely decomposable as a $(\mathfrak{g}', K')$-module if there exists an increasing filtration $\{X_i\}_{i\in\mathbb{Z}_{>0}}$ of $(\mathfrak{g}', K')$-modules such that $\bigcup_{i\in\mathbb{Z}_{>0}} X_i = X$ and X_i is of finite length as a $(\mathfrak{g}', K')$-module for any $i \in \mathbb{Z}_{>0}$.

Fact 4.2 (Kobayashi [12, Lemma 1.3 & Lemma 1.5])**.** *Let X be a simple $(\mathfrak{g}, K)$-module. Then X is discretely decomposable as a $(\mathfrak{g}', K')$-module if and only if there exists a simple $(\mathfrak{g}', K')$-module Y such that $\mathrm{Hom}_{(\mathfrak{g}',K')}(Y, X) \neq \{0\}$. Moreover, suppose that X is a unitarizable simple $(\mathfrak{g}, K)$-module. Then X is discretely decomposable as a $(\mathfrak{g}', K')$-module if and only if it is isomorphic to a direct sum of simple $(\mathfrak{g}', K')$-modules.*

Lemma 4.3. *Let $\mathfrak{g}$ be a reductive Lie algebra with reductive subalgebras $\mathfrak{g}'$ and $\mathfrak{g}''$ satisfying $\mathfrak{g}'' \subseteq \mathfrak{g}'$. Let X be a unitarizable simple $(\mathfrak{g}, K)$-module, which is discretely decomposable as a $(\mathfrak{g}', K')$-module and is also discretely decomposable as a $(\mathfrak{g}'', K'')$-module. Suppose that Y is a simple $(\mathfrak{g}', K')$-module such that $\mathrm{Hom}_{(\mathfrak{g}',K')}(Y, X) \neq \{0\}$.*

(1) *Then Y is discretely decomposable as a $(\mathfrak{g}'', K'')$-module.*
(2) *If* Conjecture 1.1 *is true for $(\mathfrak{g}, \mathfrak{g}'', X)$, then* Conjecture 1.1 *is also true for $(\mathfrak{g}', \mathfrak{g}'', Y)$.*

Proof. Since the unitarizable simple $(\mathfrak{g}, K)$-module X is discretely decomposable as $(\mathfrak{g}', K')$-module and Y is a simple $(\mathfrak{g}', K')$-module such that $\mathrm{Hom}_{(\mathfrak{g}',K')}(Y, X) \neq \{0\}$, $X \cong Y \oplus Y'$ as $(\mathfrak{g}', K')$-modules for some unitarizable $(\mathfrak{g}', K')$-module Y' by Fact 4.2. Then the natural projection shows that

$$\{0\} \neq \mathrm{Hom}_{(\mathfrak{g}',K')}(X, Y) \subseteq \mathrm{Hom}_{(\mathfrak{g}'',K'')}(X, Y).$$

On the other hand, since X is also discretely decomposable as a $(\mathfrak{g}'', K'')$-module, $X \cong \bigoplus_i m(i) Z_i$ is a direct sum of simple $(\mathfrak{g}'', K'')$-modules with multiplicities $m(i) \in \mathbb{Z}_{>0} \cup \{+\infty\}$ by Fact 4.2. Now

$$\{0\} \neq \mathrm{Hom}_{(\mathfrak{g}'',K'')}(X, Y) \cong \mathrm{Hom}_{(\mathfrak{g}'',K'')}(\bigoplus_i m(i) Z_i, Y) \subseteq \prod_i m(i) \mathrm{Hom}_{(\mathfrak{g}'',K'')}(Z_i, Y).$$

Hence, there must be some Z_i such that $\mathrm{Hom}_{(\mathfrak{g}'',K'')}(Z_i, Y) \neq \{0\}$. Therefore, Y is discretely decomposable as a $(\mathfrak{g}'', K'')$-module by Fact 4.2. This proves (1).

If Conjecture 1.1 is true for $(\mathfrak{g}, \mathfrak{g}'', X)$, then $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}''} \mathcal{V}_{\mathfrak{g}_\mathbb{C}}(X) = \mathcal{V}_{\mathfrak{g}''_\mathbb{C}}(Z)$ for any simple $(\mathfrak{g}'', K'')$-module Z with $\mathrm{Hom}_{(\mathfrak{g}'',K'')}(Z, X) \neq \{0\}$. If Y is simple $(\mathfrak{g}', K')$-module with $\mathrm{Hom}_{(\mathfrak{g}',K')}(Y, X) \neq \{0\}$, then Y is discretely decomposable as a $(\mathfrak{g}'', K'')$-module by (1), and $\mathrm{pr}_{\mathfrak{g}'\to\mathfrak{g}''} \mathcal{V}_{\mathfrak{g}'_\mathbb{C}}(Y) \subseteq \mathcal{V}_{\mathfrak{g}''_\mathbb{C}}(Z)$ by Fact 2.1. Thus we have $\mathcal{V}_{\mathfrak{g}''_\mathbb{C}}(Z) = \mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}''} \mathcal{V}_{\mathfrak{g}_\mathbb{C}}(X) = \mathrm{pr}_{\mathfrak{g}'\to\mathfrak{g}''} \circ \mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'} \mathcal{V}_{\mathfrak{g}_\mathbb{C}}(X) \subseteq \mathrm{pr}_{\mathfrak{g}'\to\mathfrak{g}''} \mathcal{V}_{\mathfrak{g}'_\mathbb{C}}(Y) \subseteq \mathcal{V}_{\mathfrak{g}''_\mathbb{C}}(Z)$, and thus $\mathrm{pr}_{\mathfrak{g}'\to\mathfrak{g}''} \mathcal{V}_{\mathfrak{g}'_\mathbb{C}}(Y) = \mathcal{V}_{\mathfrak{g}''_\mathbb{C}}(Z)$. Since any simple $(\mathfrak{g}'', K'')$-submodule of Y is also a simple $(\mathfrak{g}'', K'')$-submodule of X, (2) follows from Fact 2.1(2). ■

Retain the notations as in the last section, we obtain the following result.

Corollary 4.4. *For $k \in \mathbb{Z}_{\geq 0}$, the lowest weight simple $(\mathfrak{so}(8,2), \mathrm{Spin}(8) \times \mathrm{Spin}(2))$ module $L'(3\mu_1 + k\mu_5)$ is discretely decomposable as a $(\mathfrak{so}(8,1), \mathrm{Spin}(8))$-module, and Conjecture 1.1 is true for $(\mathfrak{so}(8,2), \mathfrak{so}(8,1), L'(3\mu_1 + k\mu_5))$.*

Proof. Since $L(3\omega_6)$ is discretely decomposable as a $(\mathfrak{so}(8,2), \mathrm{Spin}(8) \times \mathrm{Spin}(2))$-module with direct summands $L'(3\mu_1 + k\mu_5)$ for $k \in \mathbb{Z}_{\geq 0}$ and $L(3\omega_6)$ is also discretely decomposable as a $(\mathfrak{so}(8,1), \mathrm{Spin}(8))$-module, the first statement follows from Lemma 4.3(1). The second statement follows from Theorem 1.3 and Lemma 4.3(2). ■

Remark 4.5. Notice that $(\mathfrak{so}(8,2), \mathfrak{so}(8,1))$ is a symmetric pair and each lowest weight simple module $L'(3\mu_1 + k\mu_5)$ is actually a simple generalized Verma module. The result of Corollary 4.4 is a special case of the general theory due to Kobayashi on branching laws of generalized Verma modules [16, Theorem 4.1 & Theorem 4.12]. However, Lemma 4.3 offers an alternative way to study discrete branching laws and Conjecture 1.1 for symmetric pairs through Klein four symmetric pairs, and Corollary 4.4 can be regarded as an example.

5. Proof for Theorem 1.4

In the final part of this article, we will discuss the restrictions of discrete series representations for Klein four symmetric pairs (G, G^Γ) with exceptional simple Lie groups G of Hermitian type.

Let G be a noncompact reductive Lie group, and G' a reductive subgroup of G. The notion of admissible restriction to reductive subgroup was introduced by Kobayashi in [10], which we recall now.

Definition 5.1 (Kobayashi [10])**.** Let π be an irreducible unitary representation of G on a Hilbert space. Then π is said to be G'-admissible if π decomposes as a Hilbert direct sum of irreducible unitary representations of G' with finite multiplicities, i.e.,

$$\pi \cong \widehat{\bigoplus_{\tau \in \widehat{G'}}} m(\tau)\tau$$

with the multiplicities $m(\tau) \in \mathbb{Z}_{>0}$, where $\widehat{G'}$ denotes the unitary dual of G'.

Fact 5.2. *Let π be a discrete series representation of G. Then the following conditions are equivalent:*
(1) *π is K'-admissible;*
(2) *π is G'-admissible;*
(3) *the underlying $(\mathfrak{g}, K)$-module π_K is discretely decomposable as a $(\mathfrak{g}', K')$-module with finite multiplicities.*

Proof. The directions (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) follow from [12, Proposition 1.6] and [13, Theorem 2.7] respectively, which hold for general unitary representations π. If π is a discrete series representation of G, then the direction (2) $\Rightarrow$ (1) follows from [1, Corollary 2.5]. ■

Remark 5.3. The direction (3) $\Rightarrow$ (1) was also proved in [25, Theorem 1.3].

Lemma 5.4. *Let (G, G^Γ) be a Klein four symmetric pair. Suppose that π is a discrete series representation of G. If π is G^Γ-admissible, then its underlying $(\mathfrak{g}, K)$-module π_K is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module for any $\sigma \in \Gamma$.*

Proof. This follows from [10, Theorem 1.2] and Fact 5.2 immediately. ■

Proof for Theorem 1.4. Since (G, G^Γ) is supposed to be of non-holomorphic type, there exists a nonidentity element $\sigma \in \Gamma$ such that (G, G^σ) is a symmetric pair of anti-holomorphic type.

Now assume that there exists a discrete series representation π of G which is G^Γ-admissible, then by Lemma 5.4, its underlying $(\mathfrak{g}, K)$-module π_K is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module. According to the classification result [20, Theorem 5.2 & Table 1], the only possible symmetric pair of anti-holomorphic type is the symmetric pair $(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$. It is well known that the underlying $(\mathfrak{g}, K)$-module of a discrete series representation is equal to $A_\mathfrak{b}(\lambda)$ for some θ-stable Borel subalgebra of $\mathfrak{g}$.

However, according to the classification result of Kobayashi and Oshima in [19, Table C.3], $(\mathfrak{g}, \mathfrak{g}^\sigma) = (\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$ is not of discrete series type in the sense of [19]; in other words, there does not exist a Zuckman's derived functor module $A_\mathfrak{b}(\lambda)$ for a θ-stable Borel subalgebra of $\mathfrak{g}$ which is discretely decomposable as a $(\mathfrak{g}^\sigma, K^\sigma)$-module. Thus, we obtain a contradiction. ■

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