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The Doubly Warped Product of Holomorphic Lie Algebroids

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Abstract. We define the doubly warped product of holomorphic Finsler Lie algebroids. We consider a complex Finsler function and the Chern-Finsler connection of the product bundle and we investigate its relation with the Chern-Finsler connections of each bundle. In the geometrical setting of the prolongations of two Finsler algebroids, we obtain similar and also different properties from the ones of the doubly warped product of Finsler manifolds.

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Key Words: Lie algebroid, doubly warped product, prolongation bundle, nonlinear connection, Finsler bundle.

1. Introduction

The concept of Lie algebroid is a generalization of tangent bundles and Lie algebras. Lie algebroids have been intensely studied in the past decades as a theory with special applications in mechanics. Weinstein [30] developed a generalized theory of Lagrangians on Lie algebroids and obtained the Euler-Lagrange equations using the structure of the dual of a Lie algebroid. He opened the problem of developing a similar formalism to Klein's in ordinary Lagrangian mechanics, but without using the structure of the dual. Two approaches were taken on this problem. First, Martinez [21, 22] used prolongations of Lie algebroids, a setting introduced under another name by Higgins and Mackenzie in [10], which proved to be adequate for studying many aspects of the classical theory [20, 23]. Another approach on Weinstein's problem uses the Tulczyjew triple structure [9, 8]. Lie algebroids were recently applied in topics such as optimal control, interpolation problems, trajectory planning and more [7, 20, 27].

In [13, 14, 15, 16], Lie algebroids in complex Finsler geometry were studied. In this paper, we investigate the notion of doubly warped product $E_1 \times E_2$ of two algebroids E_1 and E_2. Warped products were considered in Riemannian geometry, in [2, 6, 28]. Applications of warped products in cosmology were given in [11, 12]. Asanov [3, 4] generalized the Schwarzschild metric to Finsler geometry and developed relativistic models using warped products of Finsler metrics.

In the literature, there are warped product bundles, named after warped manifolds, studied for instance in [18], where the Finsler fundamental function was used to define

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the warped product. Then, there are doubly warped products, as the ones studied also for products of manifolds in [25, 19], where two warping functions are involved. This is the case we analyze for holomorphic Finsler algebroids. Some applications of Riemannian and Lorentzian doubly warped products were described in [29]. If the function f depends also on the vector variables, then the product is called twisted. Such product of manifolds were studied, for instance, in [26].

The paper is structured as follows. In the second section, we briefly recall holomorphic Lie algebroids with Finsler structures and the construction of the prolongation bundles. We also define the gradient and the Hessian of a function on an algebroid. In the third section, we introduce and study the doubly warped product of two holomorphic Finsler algebroids. We prove that the doubly warped product is a Finsler bundle by defining a Finsler function on it, using the fundamental Finsler functions of the two algebroids, as well as the two warping functions. In two subsections of the third section, considering vertical subbundles, we study the relation between the Chern-Finsler connections of the factor algebroids and that of their doubly warped product. We also study the properties of the vertical curvature on the warped product, following the line of the study from the case of Finsler manifolds [18, 19].

2. Preliminaries on holomorphic Finsler algebroids

First, let us give the definition of a Finsler Lie algebroid as in [14]. Consider M to be a complex manifold and E – a holomorphic anchored vector bundle, with the holomorphic anchor map $\rho_E : E \to T'M$. Denote by $\widetilde{E}$ the open submanifold of E consisting in the nonzero sections. The local coordinates in a chart in $z \in M$ are $\{z^k\}_{k=\overline{1,n}}$, while a section of E is $u = u^\alpha e_\alpha$. Here, $\{e_\alpha\}_{\alpha=\overline{1,m}}$ is a basis of local holomorphic sections of E.

Definition 2.1. [1] A *complex Finsler structure* F on E is a real-valued function $F : E \to \mathbb{R}$ satisfying the following conditions:

(1) F is C^∞-class on $\widetilde{E}$;

(2) $F(z,u) \geq 0$ and $F(z,u) = 0$ iff $u = 0$;

(3) $F(z,\lambda u) = |\lambda|^2 F(z,u)$ for all $\lambda \in \mathbb{C}$.

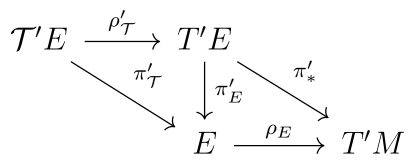
A Finsler structure F is said to be *convex* if the Hermitian matrix defined by

$$h_{\alpha\bar{\beta}} = \frac{\partial^2 F}{\partial u^\alpha \partial \bar{u}^\beta} \tag{1}$$

is positive-definite. In the following, we assume that the function F is convex and we call the pair (E,F) a complex Finsler Lie algebroid [14].

In the following, we are mostly interested in the properties of a linear connection on the algebroid. We consider the most well-known connection in Finsler geometry, the Chern-Finsler connection, introduced in [14] for the Lie algebroid E. We have obtained the Chern-Finsler linear connection defined on the prolongation of the algebroid by inducing it from a vertical connection on E.

We have defined and studied in [13, 14] the holomorphic prolongation $\mathcal{T}'E$ of a holomorphic algebroid E. By complexification, in [15] we have obtained the complexified prolongation $\mathcal{T}_{\mathbb{C}}E$ of the algebroid. We now briefly recall the construction, as we will need it as a setting for introducing the doubly warped product of two holomorphic Lie algebroids. The holomorphic prolongation of E is defined using the tangent mapping $\pi'_* : T'E \to T'M$ between the holomorphic tangent bundles of E and M, respectively, and the holomorphic anchor map $\rho_E : E \to T'M$. We first define the subset $\mathcal{T}'E$ of $E \times T'E$ by $\mathcal{T}'E = \{(e,v) \in E \times T'E \mid \rho_E(e) = \pi'_*(v)\}$ and the mapping $\pi'_{\mathcal{T}} : \mathcal{T}'E \to E$, given by $\pi'_{\mathcal{T}}(e,v) = \pi_E(v)$, where $\pi'_E : T'E \to E$ is the holomorphic tangent projection. Then, $(\mathcal{T}'E, \pi'_{\mathcal{T}}, E)$ is a holomorphic vector bundle over E, of rank $2m$, called the *holomorphic prolongation* of E:



The holomorphic vector bundle structure of the holomorphic Lie algebroid E allows us to further consider the complexified bundle $E_{\mathbb{C}}$, as well as $T_{\mathbb{C}}E = T'E \oplus T''E$, its complexified tangent bundle. The definition of the complexified prolongation $\mathcal{T}_{\mathbb{C}}E$ of E then follows from a similar idea to that of Martinez [21, 22]. More precisely, by extending $\mathbb{C}$-linearly the tangent mapping $\pi'_* : T'E \to T'M$ and the anchor $\rho_E : E \to T'M$, we obtain $\pi_{*,\mathbb{C}} : T_{\mathbb{C}}E \to T_{\mathbb{C}}M$ and $\rho_{E,\mathbb{C}} : E_{\mathbb{C}} \to T_{\mathbb{C}}M$, respectively. Denote by $\pi_{E,\mathbb{C}} : T_{\mathbb{C}}E \to E_{\mathbb{C}}$ the tangent projection extended to the complexified spaces. The *complexified prolongation* of E is then the complex vector bundle $(\mathcal{T}_{\mathbb{C}}E, \pi_{\mathcal{T},\mathbb{C}}, E_{\mathbb{C}})$ over $E_{\mathbb{C}}$, where the subset $\mathcal{T}_{\mathbb{C}}E$ of $E_{\mathbb{C}} \times T_{\mathbb{C}}E$ is

$$\mathcal{T}_{\mathbb{C}}E = \{(e,v) \in E_{\mathbb{C}} \times T_{\mathbb{C}}E \mid \rho_{E,\mathbb{C}}(e) = \pi_{*,\mathbb{C}}(v)\},$$

and the mapping $\pi_{\mathcal{T},\mathbb{C}} : \mathcal{T}_{\mathbb{C}}E \to E_{\mathbb{C}}$ is $\pi_{\mathcal{T},\mathbb{C}}(e,v) = \pi_{E,\mathbb{C}}(v)$. Also, the anchor of the complexified prolongation is the projection onto the second factor,

$$\rho_{\mathcal{T},\mathbb{C}} : \mathcal{T}_{\mathbb{C}}E \to T_{\mathbb{C}}E, \quad \rho_{\mathcal{T},\mathbb{C}}(e,v) = v.$$

Note that the complexified prolongation coincides with the complexification of the prolongation $\mathcal{T}'E$ (as a complex manifold), that is $\mathcal{T}_{\mathbb{C}}E = \mathcal{T}'E \oplus \mathcal{T}''E$, where $\mathcal{T}''E = \overline{\mathcal{T}'E} = E'' \times T''E$, with the required restrictions $\rho'_E(e) = \pi'_*(v)$ and its conjugate. In the following, we will denote the prolongation bundle simply by $\mathcal{T}E$. We further use the well-known abbreviations $\partial_k = \frac{\partial}{\partial z^k}$, $\dot{\partial}_\alpha = \frac{\partial}{\partial u^\alpha}$ and their conjugates, and we denote by $\rho_\alpha^k = \rho_\alpha^k(z)$, the local holomorphic function coefficients of the anchor map ρ_E. Let $F : E \to \mathbb{R}_+$ be a Finsler function on E, as defined above, with the complex Finsler metric tensor $h_{\alpha\bar{\beta}} = \dot{\partial}_\alpha \dot{\partial}_{\bar{\beta}} F$. We consider the Chern-Finsler nonlinear connection on E, $N_k^\beta = h^{\bar{\sigma}\beta} \partial_k \dot{\partial}_{\bar{\sigma}} F$, therefore the coefficients of the Chern-Finsler nonlinear connection of the prolongation $\mathcal{T}'E$ are

$$N_\alpha^\beta = \rho_\alpha^k N_k^\beta. \tag{2}$$

The *local basis of sections* of the prolongation algebroid is defined by the formula $\{\mathcal{Z}_\alpha := (e_\alpha, \rho_\alpha^k \partial_k), \mathcal{V}_\alpha := (0, \dot{\partial}_\alpha)\}$ and their conjugates, $\{\mathcal{Z}_{\bar{\alpha}}, \mathcal{V}_{\bar{\alpha}}\}$. The dual basis is denoted by $\{d\mathcal{Z}^\alpha, d\mathcal{V}^\alpha, d\mathcal{Z}^{\bar{\alpha}}, d\mathcal{V}^{\bar{\alpha}}\}$. We consider

$$\mathcal{X}_\alpha = \mathcal{Z}_\alpha - N_\alpha^\beta \mathcal{V}_\alpha \tag{3}$$

and its conjugate to be the adapted frames on the holomorphic prolongation $\mathcal{T}'E$, with respect to the Chern-Finsler nonlinear connection (2). Also, $\{d\mathcal{Z}^\alpha, \delta\mathcal{V}^\alpha := d\mathcal{V}^\alpha + N^\alpha_\beta d\mathcal{Z}^\beta\}$ are the dual adapted frames of $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$. By the anchor of the prolongation algebroid, the adapted fields $\mathcal{X}_\alpha$ are mapped to adapted fields on the complex algebroid E [16], that is, we have $\rho_{\mathcal{T}}(\mathcal{X}_\alpha) = \delta_\alpha =: \rho^k_\alpha \delta_k$, where $\delta_k = \partial_k - N^\alpha_k \dot{\partial}_\alpha$.

As in the well-known cases of a Finsler manifold or bundle, the nonlinear connection (2) leads to a decomposition of the holomorphic prolongation $\mathcal{T}'E$ into vertical and horizontal bundles, i.e., $\mathcal{T}'E = H\mathcal{T}'E \oplus V\mathcal{T}'E$. Also, in [14] we have defined a Chern-Finsler complex linear connection of type $(1,0)$ on $\mathcal{T}E$ by the connection form

$$\omega^\gamma_\alpha = L^\gamma_{\alpha\beta}\mathcal{Z}^\beta + C^\gamma_{\alpha\beta}\delta\mathcal{V}^\beta,$$

where the coefficients are given by:

$$L^\gamma_{\alpha\beta} = h^{\bar{\sigma}\gamma}\delta_\beta(h_{\alpha\bar{\sigma}}), \quad C^\gamma_{\alpha\beta} = h^{\bar{\sigma}\gamma}\dot{\partial}_\beta(h_{\alpha\bar{\sigma}}). \tag{4}$$

The vertical connection coefficients are symmetrical in the lower indices, that is, $C^\gamma_{\alpha\beta} = C^\gamma_{\beta\alpha}$, while the horizontal ones satisfy the identity $L^\gamma_{\alpha\beta} = \dot{\partial}_\alpha N^\gamma_\beta$. Using these properties, we find that the components of the torsion of the Chern-Finsler connection on $\mathcal{T}E$ are:

$$\begin{gathered}
\mathcal{T}(\mathcal{X}_\alpha, \mathcal{X}_\beta) = \big(L^\gamma_{\beta\alpha} - L^\gamma_{\alpha\beta} - \mathcal{C}^\gamma_{\alpha\beta}\big)\mathcal{X}_\gamma - \mathcal{R}^\gamma_{\alpha\beta}\mathcal{V}_\gamma,\\
\mathcal{T}(\mathcal{X}_\alpha, \mathcal{X}_{\bar{\beta}}) = -(\delta_{\bar{\beta}}N^\gamma_\alpha)\mathcal{V}_\gamma + (\delta_\alpha N^{\bar{\gamma}}_{\bar{\beta}})\mathcal{V}_{\bar{\gamma}},\\
\mathcal{T}(\mathcal{X}_\alpha, \mathcal{V}_\beta) = -C^\gamma_{\alpha\beta}\mathcal{X}_\gamma, \quad \mathcal{T}(\mathcal{X}_\alpha, \mathcal{V}_{\bar{\beta}}) = -(\dot{\partial}_{\bar{\beta}}N^\gamma_\alpha)\mathcal{V}_\gamma,\\
\mathcal{T}(\mathcal{V}_\alpha, \mathcal{V}_\beta) = 0, \quad \mathcal{T}(\mathcal{V}_\alpha, \mathcal{V}_{\bar{\beta}}) = 0.
\end{gathered}$$

The tensor (1) can be used to define a Hermitian metric structure on the prolongation $\mathcal{T}E$ by

$$\mathcal{G} = h_{\alpha\bar{\beta}}d\mathcal{Z}^\alpha \otimes d\bar{\mathcal{Z}}^\beta + h_{\alpha\bar{\beta}}\delta\mathcal{V}^\alpha \otimes \delta\bar{\mathcal{V}}^\beta. \tag{5}$$

The Chern-Finsler connection (4) satisfies the identity

$$U\mathcal{G}(V,W) = \mathcal{G}(\mathcal{D}_U V, W) + \mathcal{G}(V, \mathcal{D}_U W), \quad \forall U, V, W \in \Gamma(\mathcal{T}E),$$

which shows that it is metric with respect to the hermitian structure $\mathcal{G}$. Also, a straightforward computation in which we use the vanishing of the torsion of two vertical fields leads to the Koszul formula on the vertical subbundle of the prolongation, with respect to the Chern-Finsler connection.

Lemma 2.2. *Let (E, F) be a holomorphic Finsler Lie algebroid with the Chern-Finsler connection $\mathcal{D}$. For $U, V, W \in V\mathcal{T}E$, the following identity holds:*

$$\begin{aligned}
2\mathcal{G}(\mathcal{D}_U V, W) &= U\mathcal{G}(V,W) + V\mathcal{G}(W,U) - W\mathcal{G}(U,V)\\
&\quad - \mathcal{G}(U,[V,W]) + \mathcal{G}(V,[W,U]) + \mathcal{G}(W,[U,V]).
\end{aligned} \tag{6}$$

We further investigate two notions which will prove to be very useful in our study of the doubly warped product, i.e., the *gradient* and the *Hessian* of a function.

Definition 2.3. The *gradient* of a function on the prolongation bundle $\mathcal{T}E$ is the differential operator ∇ given by

$$\mathcal{G}(Z, \nabla f) = Zf, \ \forall Z \in \mathcal{T}E.$$

In coordinates, the gradient is defined by

$$\nabla f = h^{\bar{\beta}\alpha}(\delta_{\bar{\beta}} f)\mathcal{X}_\alpha + h^{\bar{\beta}\alpha}(\delta_\alpha f)\mathcal{X}_{\bar{\beta}} + h^{\bar{\beta}\alpha}(\dot{\partial}_{\bar{\beta}} f)\mathcal{V}_\alpha + h^{\bar{\beta}\alpha}(\dot{\partial}_\alpha f)\mathcal{V}_{\bar{\beta}}. \tag{7}$$

As we stated above, we study the vertical prolongation bundle, hence we are interested in the vertical part of the gradient,

$$\nabla^{v\bar{v}} f = h^{\bar{\beta}\alpha}(\dot{\partial}_{\bar{\beta}} f)\mathcal{V}_\alpha + h^{\bar{\beta}\alpha}(\dot{\partial}_\alpha f)\mathcal{V}_{\bar{\beta}}. \tag{8}$$

Definition 2.4. The *Hessian* of a function f with respect to the Chern-Finsler connection $\mathcal{D}$ on $\mathcal{T}E$ is the second covariant differential, $\mathcal{H}f = \mathcal{D}(\mathcal{D}f)$.

In [18], a study of the Hessian of a function on a Finsler manifold was made. In our case, we can obtain after some computations in local coordinates the following properties of the Hessian of a function on a Finsler algebroid.

Lemma 2.5. *The Hessian $\mathcal{H}f$ satisfies the identities:*

$$\begin{aligned}\mathcal{H}f(V, W) &= VWf - (\mathcal{D}_V W)f + (\mathcal{D}_{V^v} W^v + \mathcal{D}_{V^{\bar{v}}} W^{\bar{v}})f \\ &= \mathcal{G}(\mathcal{D}_V(\nabla^{v\bar{v}} f), W) + (\mathcal{D}_{V^v} W^v + \mathcal{D}_{V^{\bar{v}}} W^{\bar{v}})f,\end{aligned} \tag{9}$$

where $$V = V^\alpha \mathcal{V}_\alpha + V^{\bar{\alpha}} \mathcal{V}_{\bar{\alpha}}, \quad W = W^\beta \mathcal{V}_\beta + W^{\bar{\beta}} \mathcal{V}_{\bar{\beta}},$$
$$(\mathcal{D}_{V^v} W^v)f = (\mathcal{D}_{V^\alpha \mathcal{V}_\alpha} W^\beta \mathcal{V}_\beta)f = V^\alpha(\dot{\partial}_\alpha W^\beta)(\dot{\partial}_\beta f)$$

and $(\mathcal{D}_{V^{\bar{v}}} W^{\bar{v}})f$ *is its conjugate.*

3. The doubly warped product of algebroids

Let (E_1, F_1) and (E_2, F_2) be two holomorphic Finsler algebroids, where E_1 and E_2 are holomorphic vector bundles over the complex manifolds M_1 and M_2, respectively. We take $\mathcal{T}E_1$ and respectively $\mathcal{T}E_2$ to be their prolongation bundles, $\mathcal{D}^1$ and $\mathcal{D}^2$ – the Chern-Finsler connections, and the bundle projections $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$. Also, we consider two holomorphic functions defined on each corresponding manifold, $f_1 : M_1 \to \mathbb{R}_+$ and $f_2 : M_2 \to \mathbb{R}_+$.

Denote the local coordinates on E_1 ($\dim M_1 = n_1$, $\operatorname{rank} E_1 = m_1$) by (z^k, u^α), $k \in \{1, ..., n_1\}$, $\alpha \in \{1, ..., m_1\}$, and on E_2 ($\dim M_2 = n_2$, $\operatorname{rank} E_2 = m_2$), by (z^h, u^a), $h \in \{1, ..., n_2\}$, $a \in \{1, ..., m_2\}$.

Following the ideas from [18] and [25], we introduce the function $F : E_1 \times E_2 \to \mathbb{R}$ on the product bundle $E_1 \times E_2$ by

$$F(\mathrm{u}_1, \mathrm{u}_2) = f_2^2(\pi_2(\mathrm{u}_2))F_1(\mathrm{u}_1) + f_1^2(\pi_1(\mathrm{u}_1))F_2(\mathrm{u}_2), \tag{10}$$

where, locally, $\mathrm{u}_1 = (z_1, u_1) \in E_1$, $\mathrm{u}_2 = (z_2, u_2) \in E_2$. We note that this definition is inspired by Aikou's definition of a Finsler metric on a vector bundle [1].

Since F_1 and F_2 are smooth, it follows that the function F is smooth on $\widetilde{E}_1 \times \widetilde{E}_2$. Since F is not necessarily smooth on vector fields of the form $(\mathrm{u}_1, 0)$ and $(0, \mathrm{u}_2) \in E_1 \times E_2$, which is a similar impediment to the case of real Finsler manifolds [18], we find ourselves forced to restrict our study on $\widetilde{E}_1 \times \widetilde{E}_2$. Also, F is homogeneous with respect to the vector variables, as it can be easily seen from the homogeneity properties of F_1 and F_2. The Hessian of F with respect to the vector variables,

$$\begin{pmatrix} f_2^2 \dot{\partial}_\alpha \dot{\partial}_{\bar{\beta}} F_1 & 0 \\ 0 & f_1^2 \dot{\partial}_a \dot{\partial}_{\bar{b}} F_2 \end{pmatrix},$$

is positive, as the Hessians of F_1 and F_2 are positive. All the above mentioned properties prove that F is a Finsler function on the bundle $\widetilde{E}_1 \times \widetilde{E}_2$.

We will call the product bundle $E_1 \times E_2$ the *doubly warped product* of the algebroids E_1 and E_2, with the warping functions f_1 and f_2 and the Finsler metric $F = F(\mathrm{u}_1, \mathrm{u}_2)$ defined in (10). Hence, $(E_1 \times E_2, F)$ is a complex Finsler bundle. We further construct the prolongation of the doubly warped product $E_1 \times E_2$ to find that it is the direct sum of the prolongations of the algebroids E_1 and E_2. We shall also study the adapted frames and the Chern-Finsler connection.

3.1. The prolongation bundle of the doubly warped product bundle

Let $E_1 \times E_2$ be the doubly warped product of two holomorphic Lie algebroids E_1 and E_2, as it was previously defined. Consider $\mathcal{T}E_1$ and $\mathcal{T}E_2$ to be, respectively, the prolongations of E_1 and E_2, whose constructions were presented in the preliminary section of the paper. As in the case of real algebroids [21, 22] and as we have found in some previous papers [13, 14, 15] on holomorphic algebroids, the prolongation bundle seems to be an adequate setting for a study of similar properties of vector bundles as those of the tangent bundle of a manifold. We therefore make the following considerations on the prolongation bundles $\mathcal{T}E_1$ and $\mathcal{T}E_2$ instead of the tangent bundles TE_1 and TE_2 of the two algebroids.

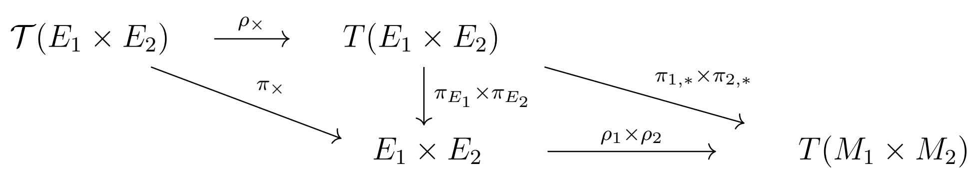
As discussed in [14], the vertical subbundles of the two prolongation bundles are defined using the projections $\tau_i : \mathcal{T}E_i \to E_i,\ \tau_i(e_i, v_i) = e_i \in E_i$ as follows:

$$V\mathcal{T}E_i = \ker \tau_i = \{(e_i, v_i) \in \mathcal{T}E_i \mid \tau_i(e_i, v_i) = 0\},\ i = 1, 2.$$

We now define the map $\rho_\times : \mathcal{T}(E_1 \times E_2) \to T(E_1 \times E_2),\ \rho_\times := \rho_{\mathcal{T}_1} \times \rho_{\mathcal{T}_2}$, whose action for $(e_1, v_1, e_2, v_2) \in \mathcal{T}(E_1 \times E_2)$ is

$$\begin{aligned} \rho_{\mathcal{T}_\times}(e_1, v_1, e_2, v_2) &= (\rho_{\mathcal{T}_1}(e_1, v_1), \rho_{\mathcal{T}_2}(e_2, v_2)) \\ &= (v_1, v_2) \in T(E_1 \times E_2) \equiv TE_1 \oplus TE_2. \end{aligned}$$

This is the anchor map of the prolongation algebroid $\mathcal{T}(E_1 \times E_2)$ of the doubly warped product $E_1 \times E_2$ and we find that $\mathcal{T}(E_1 \times E_2) = \mathcal{T}E_1 \oplus \mathcal{T}E_2$:



The vertical bundle of the prolongation $\mathcal{T}E_1 \oplus \mathcal{T}E_2$ is $V\mathcal{T}E_1 \oplus V\mathcal{T}E_2$, which is proved by the relation $\ker(\tau_1 \times \tau_2) = \ker \tau_1 \oplus \ker \tau_2$. In the following sections, we will work on this vertical bundle, which we denote by $V\mathcal{T}$.

The following step is to define the lifts of vector fields on $\mathcal{T}E_1$ or on $\mathcal{T}E_2$ to $\mathcal{T}(E_1 \times E_2)$. We need the projections $p_1 : \mathcal{T}(E_1 \times E_2) \to \mathcal{T}E_1$ and $p_2 : \mathcal{T}(E_1 \times E_2) \to \mathcal{T}E_2$. Hence, the lift of a vector field $U_1 \in \Gamma(\mathcal{T}E_1)$ to $\mathcal{T}(E_1 \times E_2)$ is the vector field $\widehat{U}_1 \in \Gamma(\mathcal{T}(E_1 \times E_2))$ defined by the identities $p_1(\widehat{U}_1) = U_1$, $p_2(\widehat{U}_1) = 0$. Analogously, the lift of a vector field $U_2 \in \Gamma(\mathcal{T}E_2)$ to $\mathcal{T}(E_1 \times E_2)$ is the vector field $\widehat{U}_2 \in \Gamma(\mathcal{T}(E_1 \times E_2))$ given by the identities $p_1(\widehat{U}_2) = 0$, $p_2(\widehat{U}_2) = U_2$.

Further, we locally denote by $N_\alpha^\beta = N_\alpha^\beta(\mathrm{u}_1)$ and $N_a^b = N_a^b(\mathrm{u}_2)$ the coefficients of the Chern-Finsler nonlinear connections on the two prolongation bundles $\mathcal{T}E_1$ and $\mathcal{T}E_2$, respectively, i.e.,

$$N_\alpha^\beta = h^{\bar{\gamma}\beta}\partial_\alpha\partial_{\bar{\gamma}}F_1, \quad N_a^b = h^{\bar{c}b}\partial_a\partial_{\bar{c}}F_2. \tag{11}$$

These nonlinear connections lead to the the decomposition

$$\mathcal{T}(E_1 \times E_2) = \mathcal{T}E_1 \oplus \mathcal{T}E_2 = H\mathcal{T}E_1 \oplus V\mathcal{T}E_1 \oplus H\mathcal{T}E_2 \oplus V\mathcal{T}E_2. \tag{12}$$

Each distribution is spanned by a corresponding adapted frames of fields, i.e.,

$$\mathcal{T}E_1 = \mathrm{span}\{\mathcal{X}_\alpha, \mathcal{V}_\alpha, \mathcal{X}_{\bar{\alpha}}, \mathcal{V}_{\bar{\alpha}}\} \text{ and } \mathcal{T}E_2 = \mathrm{span}\{\mathcal{X}_a, \mathcal{V}_a, \mathcal{X}_{\bar{a}}, \mathcal{V}_{\bar{a}}\}, \text{ respectively.}$$

The following basic properties of the Lie brackets $[\cdot,\cdot]_{\mathcal{T}_\times}$ on $\mathcal{T}(E_1 \times E_2)$ give the relations with the brackets of the factors and can be proved straightforward:

$$[U_1, V_2]_{\mathcal{T}_\times} = [U_2, V_1]_{\mathcal{T}_\times} = 0, \quad [U_1, V_1]_{\mathcal{T}_\times} = [U_1, V_1]_{\mathcal{T}E_1}, \quad [U_2, V_2]_{\mathcal{T}_\times} = [U_2, V_2]_{\mathcal{T}E_2}, \tag{13}$$

for all $U_1, V_1 \in \Gamma(\mathcal{T}E_1)$ and $U_2, V_2 \in \Gamma(\mathcal{T}E_2)$. The *Chern-Finsler linear connections* will be defined by the coefficients $(L_{\alpha\beta}^\gamma, C_{\alpha\beta}^\gamma)$ on $\mathcal{T}E_1$ and (L_{ab}^c, C_{ab}^c) on $\mathcal{T}E_2$, where

$$L_{\alpha\beta}^\gamma = h^{\bar{\sigma}\gamma}\delta_\beta(h_{\alpha\bar{\sigma}}), \quad C_{\alpha\beta}^\gamma = h^{\bar{\sigma}\gamma}\dot{\partial}_\beta(h_{\alpha\bar{\sigma}}), \quad L_{ab}^c = h^{\bar{d}c}\delta_b(h_{a\bar{d}}), \quad C_{ab}^c = h^{\bar{d}c}\dot{\partial}_b(h_{a\bar{d}}).$$

3.2. The Chern-Finsler connection of the doubly warped product bundle

We are now interested in the relation between the connections on each prolongation bundle and the connection on their product. All the considerations will be made with respect to the Chern-Finsler connections. For this study, carried in [18] in the case of Finsler manifolds, the Koszul formula (6) holds only for vertical fields on a prolongation bundle, hence we must restrict our considerations to the vertical part $V\mathcal{T} = V\mathcal{T}E_1 \oplus V\mathcal{T}E_2$ of the decomposition (12).

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be Hermitian metrics defined by (5) on the prolongation bundles $\mathcal{T}E_1$ and $\mathcal{T}E_2$, respectively. Their restrictions $\mathcal{G}_1^v$ and $\mathcal{G}_2^v$ on the vertical bundles can be used to define a (vertical) Hermitian metric $\mathcal{G}^v$ on $V\mathcal{T}$ by

$$\mathcal{G}^v(\cdot,\cdot) = f_2^2(\pi_2(\mathrm{u}_2))\mathcal{G}_1^v(\cdot,\cdot) + f_1^2(\pi_1(\mathrm{u}_1))\mathcal{G}_2^v(\cdot,\cdot).$$

Locally, we have $\quad \mathrm{h}_{i\bar{j}}(\mathrm{u}_1, \mathrm{u}_2) = f_2^2(\pi_2(\mathrm{u}_2))h_{\alpha\bar{\beta}}(\mathrm{u}_1) + f_1^2(\pi_1(\mathrm{u}_1))h_{a\bar{b}}(\mathrm{u}_2),$

hence

$$\mathrm{h}^{\bar{\gamma}\beta} = \frac{1}{f_2^2}h^{\bar{\gamma}\beta}, \qquad \mathrm{h}^{\bar{c}b} = \frac{1}{f_1^2}h^{\bar{c}b}. \tag{14}$$

If we define locally the Chern-Finsler nonlinear connection of the doubly warped product by the coefficients

$$\mathrm{N}_i^j = \mathrm{h}^{\bar{m}j} \frac{\partial^2 F}{\partial \mathrm{z}^i \partial \bar{\mathrm{u}}^m}, \tag{15}$$

where $\mathrm{z} = (z_1, z_2)$, $\mathrm{u} = (u_1, u_2)$, $i, j = \overline{1, m_1 + m_2}$, then we can obtain the relations with the Chern-Finsler connections of $\mathcal{T}E_1$ and $\mathcal{T}E_2$ as follows. First, taking $j = \beta$ in (15) yields

$$\mathrm{N}_\alpha^\beta = \mathrm{h}^{\bar{\gamma}\beta} \frac{\partial^2 F}{\partial z^k \partial \bar{u}^\gamma}.$$

Then, we have
$$\frac{\partial^2 F}{\partial z^k \partial \bar{u}^\gamma} = f_2^2 \frac{\partial^2 F_1}{\partial z^k \partial \bar{u}^\gamma},$$

which together with (14) gives $\mathrm{N}_\alpha^\beta = N_\alpha^\beta$. Similarly, we can obtain $\mathrm{N}_a^b = N_a^b$.

We further investigate the Chern-Finsler linear connections to find that they satisfy some less straightforward identities. Moreover, the proofs can be made independent of local coordinates.

Theorem 3.1. *Let $\mathcal{D}^1$ and $\mathcal{D}^2$ be the Chern-Finsler linear connections of the algebroids E_1 and E_2, respectively, and let $\mathcal{D}$ be the Chern-Finsler connection on the doubly warped product bundle. For the vertical vector fields $U_1, V_1 \in V\mathcal{T}E_1$ and $U_2, V_2 \in V\mathcal{T}E_2$, the following identities hold:*

(1) $\mathcal{D}_{U_1} V_1 = \mathcal{D}^1_{U_1} V_1 - \frac{1}{f_2}\mathcal{G}(U_1, V_1)\nabla^v f_2$, *i.e., $\mathcal{D}$ restricted to $V\mathcal{T}$ is the lift of $\mathcal{D}^1_{U_1} V_1$ on $V\mathcal{T}E_1$ and the lift of $-\frac{1}{f_2}\mathcal{G}(U_1, V_1)\nabla^v f_2$ on $V\mathcal{T}E_2$ to $V\mathcal{T}E_1 \oplus V\mathcal{T}E_2$;*

(2) $\mathcal{D}_{U_2} V_2 = \mathcal{D}^2_{U_2} V_2 - \frac{1}{f_1}\mathcal{G}(U_2, V_2)\nabla^v f_1$;

(3) $\mathcal{D}_{U_1} U_2 = \mathcal{D}_{U_2} U_1 = \frac{U_2 f_2}{f_2} U_1 + \frac{U_1 f_1}{f_1} U_2$;

(4) $\mathcal{T}(U_1, U_2) = \mathcal{T}(U_2, U_1) = 0$.

Proof. Following the line of the proof from [18], we use (6), (13) and the fact that $\mathcal{G}(\cdot_1, \cdot_2) = \mathcal{G}(\cdot_2, \cdot_1) = 0$ to obtain

$$\begin{aligned} 2\mathcal{G}(\mathcal{D}_{U_1} V_1, U_2) &= -U_2 \mathcal{G}(U_1, V_1) = -U_2(f_2^2)\mathcal{G}_1(U_1, V_1) = -2f_2 U_2(f_2))\mathcal{G}_1(U_1, V_1) \\ &= -2f_2 \mathcal{G}\left(U_2, \nabla^v f_2 \frac{\mathcal{G}(U_1, V_1)}{f_2^2}\right) = -2\mathcal{G}\left(\frac{1}{f_2}\mathcal{G}(U_1, V_1)\nabla^v f_2, U_2\right), \end{aligned}$$

which concludes the proof of the first statement. The second one is analogous. For the third assertion, we have again from (6)

$$\begin{aligned} 2\mathcal{G}(\mathcal{D}_{U_1} U_2, U_2) &= U_1 \mathcal{G}(U_2, U_2) = U_1\left(f_1^2 \mathcal{G}_2(U_2, U_2)\right) = 2f_1 U_1(f_1)\mathcal{G}_2(U_2, U_2) \\ &= 2f_1 U_1(f_1)\frac{\mathcal{G}(U_2, U_2)}{f_1^2} = 2\mathcal{G}\left(\frac{U_1(f_1)}{f_1} U_2, U_2\right). \end{aligned}$$

Next, using also the first assertion,

$$\begin{aligned} \mathcal{G}(\mathcal{D}_{U_1} U_2, V_1) &= -\mathcal{G}(U_2, \mathcal{D}_{U_1} V_1) = -\mathcal{G}\left(U_2, -\frac{1}{f_2}\mathcal{G}(U_1, V_1)\nabla^v f_2\right) \\ &= \mathcal{G}(U_1, V_1)\mathcal{G}\left(U_2, \frac{\nabla^v f_2}{f_2}\right) = \mathcal{G}\left(\frac{U_2 f_2}{f_2} U_1, V_1\right), \end{aligned}$$

which yields the third identity. The last one is obvious and the proof is complete. ∎

The above theorem helps us obtain the vertical coefficients of the connection $\mathcal{D}$ on the doubly warped product as follows. We use the fact that $\mathcal{G}$ is a Hermitian metric and also that the Chern-Finsler connection is of $(1,0)$-type. We have:

$$\mathcal{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta = \mathcal{D}^1_{\mathcal{V}_\alpha}\mathcal{V}_\beta - \frac{1}{f_2}\mathcal{G}(\mathcal{V}_\alpha,\mathcal{V}_\beta) = \mathcal{D}^1_{\mathcal{V}_\alpha}\mathcal{V}_\beta,$$
$$\mathcal{D}_{\mathcal{V}_\alpha}\mathcal{V}_{\bar\beta} = \mathcal{D}^1_{\mathcal{V}_\alpha}\mathcal{V}_{\bar\beta} - \frac{1}{f_2}\mathcal{G}(\mathcal{V}_\alpha,\mathcal{V}_{\bar\beta}) = -f_2 h_{\alpha\bar\beta}\nabla^v f_2,$$
$$\mathcal{D}_{\mathcal{V}_\alpha}\mathcal{V}_b = \mathcal{D}_{\mathcal{V}_b}\mathcal{V}_\alpha = \frac{1}{f_2}(\dot\partial_b f_2)\mathcal{V}_\alpha + \frac{1}{f_1}(\dot\partial_\alpha f_1)\mathcal{V}_b,$$
$$\mathcal{D}_{\mathcal{V}_\alpha}\mathcal{V}_{\bar b} = \mathcal{D}_{\mathcal{V}_{\bar b}}\mathcal{V}_\alpha = \frac{1}{f_2}(\dot\partial_{\bar b} f_2)\mathcal{V}_\alpha + \frac{1}{f_1}(\dot\partial_\alpha f_1)\mathcal{V}_{\bar b},$$
$$\mathcal{D}_{\mathcal{V}_a}\mathcal{V}_b = \mathcal{D}^2_{\mathcal{V}_\alpha}\mathcal{V}_b - \frac{1}{f_1}\mathcal{G}(\mathcal{V}_a,\mathcal{V}_b)\nabla^v f_1 = \mathcal{D}^2_{\mathcal{V}_a}\mathcal{V}_b,$$
$$\mathcal{D}_{\mathcal{V}_a}\mathcal{V}_{\bar b} = \mathcal{D}^2_{\mathcal{V}_a}\mathcal{V}_{\bar b} - \frac{1}{f_1}\mathcal{G}(\mathcal{V}_a,\mathcal{V}_{\bar b})\nabla^v f_1 = -f_1 h_{a\bar b}\nabla^v f_1,$$

together with their conjugates.

Let us denote the local coefficients of the connection $\mathcal{D}$ on the vertical prolongation according to the decomposition $V\mathcal{T} = V\mathcal{T}'E_1 \oplus V\mathcal{T}''E_1 \oplus V\mathcal{T}'E_2 \oplus V\mathcal{T}''E_2$ as

$$\mathcal{D}_{\mathcal{V}_\alpha}\mathcal{V}_\beta = \mathrm{C}^{\gamma}_{\beta\alpha}\mathcal{V}_\gamma + \mathrm{C}^{\bar\sigma}_{\beta\alpha}\mathcal{V}_{\bar\sigma} + \mathrm{C}^{c}_{\beta\alpha}\mathcal{V}_c + \mathrm{C}^{\bar d}_{\beta\alpha}\mathcal{V}_{\bar d},$$
$$\mathcal{D}_{\mathcal{V}_\alpha}\mathcal{V}_{\bar\beta} = \mathrm{C}^{\gamma}_{\bar\beta\alpha}\mathcal{V}_\gamma + \mathrm{C}^{\bar\sigma}_{\bar\beta\alpha}\mathcal{V}_{\bar\sigma} + \mathrm{C}^{c}_{\bar\beta\alpha}\mathcal{V}_c + \mathrm{C}^{\bar d}_{\bar\beta\alpha}\mathcal{V}_{\bar d},$$
$$\mathcal{D}_{\mathcal{V}_\alpha}\mathcal{V}_b = \mathrm{C}^{\gamma}_{b\alpha}\mathcal{V}_\gamma + \mathrm{C}^{\bar\sigma}_{b\alpha}\mathcal{V}_{\bar\sigma} + \mathrm{C}^{c}_{b\alpha}\mathcal{V}_c + \mathrm{C}^{\bar d}_{b\alpha}\mathcal{V}_{\bar d},$$
$$\mathcal{D}_{\mathcal{V}_\alpha}\mathcal{V}_{\bar b} = \mathrm{C}^{\gamma}_{\bar b\alpha}\mathcal{V}_\gamma + \mathrm{C}^{\bar\sigma}_{\bar b\alpha}\mathcal{V}_{\bar\sigma} + \mathrm{C}^{c}_{\bar b\alpha}\mathcal{V}_c + \mathrm{C}^{\bar d}_{\bar b\alpha}\mathcal{V}_{\bar d},$$
$$\mathcal{D}_{\mathcal{V}_a}\mathcal{V}_b = \mathrm{C}^{\gamma}_{ba}\mathcal{V}_\gamma + \mathrm{C}^{\bar\sigma}_{ba}\mathcal{V}_{\bar\sigma} + \mathrm{C}^{c}_{ba}\mathcal{V}_c + \mathrm{C}^{\bar d}_{ba}\mathcal{V}_{\bar d},$$
$$\mathcal{D}_{\mathcal{V}_a}\mathcal{V}_{\bar b} = \mathrm{C}^{\gamma}_{\bar ba}\mathcal{V}_\gamma + \mathrm{C}^{\bar\sigma}_{\bar ba}\mathcal{V}_{\bar\sigma} + \mathrm{C}^{c}_{\bar ba}\mathcal{V}_c + \mathrm{C}^{\bar d}_{\bar ba}\mathcal{V}_{\bar d}.$$

The above identities yield:

$$\mathrm{C}^{\gamma}_{\beta\alpha} = C^{\gamma}_{\beta\alpha} = h^{\bar\sigma\gamma}(\dot\partial_\alpha h_{\beta\bar\sigma}), \quad \mathrm{C}^{\gamma}_{\bar\beta\alpha} = -f_2(\dot\partial_{\bar d} f_2)h^{\bar d c}h_{\alpha\bar\beta}, \quad \mathrm{C}^{\bar d}_{\bar\beta\alpha} = -f_2(\dot\partial_c f_2)h^{\bar d c}h_{\alpha\bar\beta},$$
$$\mathrm{C}^{\gamma}_{b\alpha} = \frac{1}{f_2}\delta^{\gamma}_{\alpha}(\dot\partial_b f_2), \quad \mathrm{C}^{c}_{b\alpha} = \frac{1}{f_1}\delta^{c}_{b}(\dot\partial_\alpha f_1), \quad \mathrm{C}^{\gamma}_{\bar b\alpha} = \frac{1}{f_2}\delta^{\gamma}_{\alpha}(\dot\partial_{\bar b} f_2),$$
$$\mathrm{C}^{\bar d}_{\bar b\alpha} = \frac{1}{f_1}\delta^{\bar d}_{\bar b}(\dot\partial_\alpha f_1), \quad \mathrm{C}^{c}_{ba} = C^{c}_{ba} = h^{\bar d c}(\dot\partial_a h_{b\bar d}),$$
$$\mathrm{C}^{\gamma}_{\bar ba} = -f_1(\dot\partial_{\bar\sigma} f_1)h^{\bar\sigma\gamma}h_{a\bar b}, \quad \mathrm{C}^{\bar\sigma}_{\bar ba} = -f_1(\dot\partial_\gamma f_1)h^{\bar\sigma\gamma}h_{a\bar b},$$

where $\delta^{\cdot}_{\cdot}$ denotes the Kronecker symbol. All the other coefficients are zero.

We now investigate the curvature of the warped product and its relations with the curvatures of each prolongation bundle in the product. The curvature of the product bundle $\mathcal{T}(E_1 \times E_2)$ is $\mathcal{R}(U,V)W = \mathcal{D}_U\mathcal{D}_V W - \mathcal{D}_V\mathcal{D}_U W - \mathcal{D}_{[U,V]}W$. We obtain

Theorem 3.2. *Let $\mathcal{T}E_1 \times \mathcal{T}E_2$ be the product of the prolongations of two holomorphic vector bundles, with the curvatures denoted by $\mathcal{R}$ on the product bundle and respectively by $\mathcal{R}^1$ and $\mathcal{R}^2$ on $\mathcal{T}E_1$ and $\mathcal{T}E_2$. For the vertical fields $U_1, V_1, W_1 \in V\mathcal{T}E_1$ and $U_2, V_2, W_2 \in V\mathcal{T}E_2$, the following identities hold:*

(1) $\mathcal{R}(U_1, V_1)W_1 = \mathcal{R}^1(U_1, V_1)W_1 + \frac{\nabla f_2}{f_1 f_2}\mathcal{G}\left(U_1(V_1 f_1) - V_1(U_1 f_1), W_1\right);$

(2) $\mathcal{R}(U_1, V_1)W_2 = \frac{W_2 f_2}{f_1 f_2}\left[U_1(V_1 f_1) - V_1(U_1 f_1)\right];$

(3) $\mathcal{R}(U_1, V_2)W_1 = \frac{1}{f_2}\mathcal{G}(U_1, W_1)\left(\mathcal{D}^2_{V_2}(\nabla f_2) - 2\frac{\nabla f_2}{f_2}V_2 f_2\right) + {}$
$+ \frac{V_2}{f_1}\left(\mathcal{H}^{f_1}(U_1, V_1) - (\mathcal{D}^1_{U_1^v}V_1^v + \mathcal{D}^1_{U_1^{\bar v}}V_1^{\bar v})f\right);$

(4) $\mathcal{R}(U_1, V_2)W_2 = \frac{U_1 f_1}{f_1}\mathcal{G}(V_2, W_2)\left(\frac{\nabla f_1}{f_1} + \frac{\nabla f_2}{f_2} - \frac{\mathcal{D}^1_{U_1}(\nabla f_1)}{U_1 f_1}\right) - {}$
$- \frac{(U_1 f_1)(W_2 f_2)}{f_1 f_2}V_2 - \frac{U_1}{f_2}\left(\mathcal{H}^{f_2}(V_2, W_2) - (\mathcal{D}^2_{V_2^v}W_2^v + \mathcal{D}^2_{V_2^{\bar v}}W_2^{\bar v})f\right);$

(5) $\mathcal{R}(U_2, V_2)W_1 = \frac{W_1 f_1}{f_1}\left(\frac{V_2 f_2}{f_2}U_2 - \frac{U_2 f_2}{f_2}V_2\right);$

(6) $\mathcal{R}(U_2, V_2)W_2 = \mathcal{R}^2(U_2, V_2)W_2 + \frac{\nabla f_1}{f_1 f_2}\mathcal{G}\left(U_2(V_2 f_2) - V_2(U_2 f_2), W_2\right).$

Proof. All of the identities in the theorem can be proved by computations using Theorem 3.1. We only give here the proof of the first one, as the others follow in a similar manner. We have:

$$\begin{aligned}
\mathcal{R}(U_1, V_1)W_1 &= \mathcal{D}_{U_1}\left(\mathcal{D}^1_{V_1}W_1 - \frac{1}{f_2}\mathcal{G}(V_1, W_1)\nabla f_2\right) - \mathcal{D}_{V_1}\left(\mathcal{D}^1_{U_1}W_1 - \frac{1}{f_2}\mathcal{G}(U_1, W_1)\nabla f_2\right)\\
&= \mathcal{D}^1_{U_1}\mathcal{D}^1_{V_1}W_1 - \frac{1}{f_2}\mathcal{G}(U_1, \mathcal{D}^1_{V_1}W_1)\nabla f_2 - U_1\left(\frac{1}{f_2}\mathcal{G}(V_1, W_1)\right)\nabla f_2\\
&\quad - \frac{1}{f_2}\mathcal{G}(V_1, W_1)\mathcal{D}_{U_1}(\nabla f_2) - \mathcal{D}^1_{V_1}\mathcal{D}^1_{U_1}W_1 + \frac{1}{f_2}\mathcal{G}(V_1, \mathcal{D}^1_{U_1}W_1)\nabla f_2\\
&\quad + V_1\left(\frac{1}{f_2}\mathcal{G}(U_1, W_1)\right)\nabla f_2 + \frac{1}{f_2}\mathcal{G}(U_1, W_1)\mathcal{D}_{V_1}(\nabla f_2)\\
&= \mathcal{R}^1(U_1, V_1)W_1 - \frac{\nabla f_2}{f_2}\left[\mathcal{G}(U_1, \mathcal{D}^1_{V_1}W_1) - V_1\mathcal{G}(U_1, W_1) - \mathcal{G}(V_1, \mathcal{D}^1_{U_1}W_1) + U_1\mathcal{G}(V_1, W_1)\right]\\
&\quad - \frac{1}{f_2}\left[\mathcal{G}(V_1, W_1)\left(\frac{U_1 f_1}{f_1}\nabla f_2 + \frac{\nabla f_2(f_2)}{f_2}U_1\right) - \mathcal{G}(U_1, W_1)\left(\frac{V_1 f_1}{f_1}\nabla f_2 + \frac{\nabla f_2(f_2)}{f_2}V_1\right)\right]\\
&= \mathcal{R}^1(U_1, V_1)W_1 - \frac{\nabla f_2}{f_2}\left[\mathcal{G}(\mathcal{D}^1_{U_1}V_1, W_1) - \mathcal{G}(\mathcal{D}^1_{V_1}U_1, W_1)\right]\\
&\quad - \frac{1}{f_2}\left[\frac{\nabla f_2(f_2)}{f_2}\left(\mathcal{G}(V_1, W_1)U_1 - \mathcal{G}(U_1, W_1)V_1\right) + \frac{\nabla f_2}{f_1}\mathcal{G}(V_1 U_1(f_1) - U_1 V_1(f_1), W_1)\right]\\
&= \mathcal{R}^1(U_1, V_1)W_1 + \frac{\nabla f_2}{f_1 f_2}\mathcal{G}\left(U_1(V_1 f_1) - V_1(U_1 f_1), W_1\right).
\end{aligned}$$

■

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