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Spaces of Bounded Spherical Functions for Irreducible Nilpotent Gelfand Pairs: Part I

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Abstract. In prior work an orbit method, due to Pukanszky and Lipsman, was used to produce an injective mapping $\Psi\colon \Delta(K,N) \to \mathfrak{n}^*/K$ from the space of bounded K-spherical functions for a nilpotent Gelfand pair (K,N) into the space of K-orbits in the dual for the Lie algebra $\mathfrak{n}$ of N. We have conjectured that Ψ is a topological embedding. This has been proved for all pairs (K,N) with N a Heisenberg group. A nilpotent Gelfand pair (K,N) is said to be *irreducible* if K acts irreducibly on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$. In this paper and its sequel we will prove that Ψ is an embedding for all such irreducible pairs. Our proof involves careful study of the non-Heisenberg entries in Vinberg's classification of irreducible nilpotent Gelfand pairs. Part I concerns generalities and six related families of examples from Vinberg's list in which the center for $\mathfrak{n}$ can have arbitrarily large dimension.

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1. Introduction

Let N be a connected and simply connected nilpotent Lie group and K be a compact Lie group acting smoothly on N via automorphisms. One calls (K,N) a *nilpotent Gelfand pair* (n.G.p. for short) if the following equivalent conditions hold.

- The algebra $L^1_K(N)$ of integrable K-invariant functions on N commutes under convolution.
- The algebra $\mathbb{D}_K(N)$ of left-N and K-invariant differential operators on N is commutative.

In this case N is necessarily two-step or abelian [2]. The *spherical functions* for such a n.G.p. are the smooth K-invariant functions $N \to \mathbb{C}$ which

- are joint eigenfunctions for the operators $\mathbb{D}_K(N)$ and
- map the identity in N to 1.

We let $\Delta(K,N)$ denote the space of all *bounded* K-spherical functions on N with the topology of uniform convergence on compact sets. Integration against functions $\phi \in \Delta(K,N)$ gives the spectrum (or Gelfand space) for the commutative Banach $\star$-algebra $L^1_K(N)$. In [2] it is shown that each $\phi \in \Delta(K,N)$ is of positive type and expressible as the K-average of a matrix coefficient for some irreducible unitary representation $\pi \in \widehat{N}$.

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Letting $G = K \ltimes N$ one has that (K,N) is a n.G.p. if and only if (G,K) is a Gelfand pair in the usual sense. That is if and only if the algebra of integrable K-bi-invariant functions on G commutes under convolution. Equivalently the dimension $\dim(\rho^K)$ of the space of K-fixed vectors for each irreducible representation $\rho \in \widehat{G}$ is at most one [19]. We denote by $\widehat{G}_K$ the space

$$\widehat{G}_K := \{\rho \in \widehat{G} \ : \ \dim(\rho^K) = 1\}$$

of K*-spherical representations* of G equipped with the Fell topology. For $\rho \in \widehat{G}_K$ let $v_\rho \in \mathcal{H}_\rho$ be a unit K-fixed vector in the representation space for ρ. The function

$$\phi_\rho : N \to \mathbb{C}, \quad \phi_\rho(x) = \langle \rho(x)v_\rho, v_\rho \rangle_{\mathcal{H}_\rho}$$

is a bounded spherical function on N and the map

$$\widehat{G}_K \to \Delta(K,N), \quad \rho \mapsto \phi_\rho$$

is a homeomorphism.

Suppose that (K,N) is a n.G.p. and let $\mathfrak{n}$, $\mathfrak{k}$ and $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{n}$ denote the Lie algebras for K, N and $G = K \ltimes N$. An *orbit method* due to Pukanszky [25] and Lipsman [22, 23] associates to each $\rho \in \widehat{G}$ a coadjoint orbit $\mathcal{O}(\rho) \subset \mathfrak{g}^*$. The following result is proved in [4].

Theorem 1.1. *For each K-spherical representation $\rho \in \widehat{G}_K$ the set $\mathcal{O}(\rho) \cap \mathfrak{n}^*$ is a single* (*non-empty*) *K-orbit in $\mathfrak{n}^*$. The map $\rho \mapsto \mathcal{O}(\rho) \cap \mathfrak{n}^*$ from $\widehat{G}_K$ to $\mathfrak{n}^*/K$ is, moreover, injective.*

Here $\mathfrak{n}^*$ is identified with the annihilator of $\mathfrak{k}$ in $\mathfrak{g}^*$ and K acts on $\mathfrak{n}^*$ by the dual of its derived action by automorphisms on $\mathfrak{n}$. For $\phi \in \Delta(K,N)$ we let $\mathbf{O}_\phi \in \mathfrak{n}^*/K$ be defined as

$$\mathbf{O}_\phi := \mathcal{O}(\rho^\phi) \cap \mathfrak{n}^*$$

where $\rho^\phi \in \widehat{G}_K$ is the K-spherical representation for which $\phi_{\rho^\phi} = \phi$. We let

$$\mathcal{A}(K,N) := \{\mathbf{O}_\phi : \phi \in \Delta(K,N)\}$$

and call this the set of K*-spherical orbits* in $\mathfrak{n}^*$, with the quotient topology inherited from $\mathfrak{n}^*/K$. Thus we have a bijection

$$\Psi : \Delta(K,N) \to \mathcal{A}(K,N), \ \ \Psi(\phi) = \mathbf{O}_\phi. \tag{1}$$

An alternate description of the map Ψ, given below in Section 2.3, will be used for computation in examples. We conjectured in [4] that:

(O): The bijection $\Psi\colon \Delta(K,N) \to \mathcal{A}(K,N)$ is a homeomorphism.

Our aim is to provide a "geometric model" for the spectrum of $L^1_K(N)$. To date Conjecture (O) has been established for the following n.G.p.'s (K,N).

- all (K,N) with N abelian [4],
- all (K,N) with N a Heisenberg group [5, 6, 7],

- all (K,N) with N two-step subject to two conditions, discussed below in Section 1.4 [18],
- $(K,N) = (O(d), F(d))$ where $F(d) = \mathbb{R}^d \oplus \Lambda^2(\mathbb{R}^d)$ is the free two-step nilpotent Lie group on d generators [4].

In Section 8 we show that the closely related pair $(SO(d), F(d))$ also satisfies (O). This fact was announced without proof in [4].

A n.G.p. (K,N) is said to be *irreducible* if K acts irreducibly on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$. In the current paper and its sequel we will prove the following.

Theorem 1.2. *Every irreducible n.G.p. satisfies conjecture* (O)*.*

The proof of Theorem 1.2 boils down to verifying (O) for certain (families of) examples from Vinberg's classification of irreducible n.G.p.'s [29]. These are listed below in Table 1. Several results are required to reduce the proof of Theorem 1.2 to a detailed study of the entries in this table. To begin it suffices to consider only n.G.p.'s (K,N) with N two-step. For we know that (O) holds for all pairs with N abelian. So we assume for the remainder of this paper that

all nilpotent groups N under consideration are two-step.

Given an irreducible n.G.p. (K,N) we necessarily have $[\mathfrak{n},\mathfrak{n}] = \mathfrak{z}$, with $\mathfrak{z} = \mathfrak{z}(\mathfrak{n})$ the center of $\mathfrak{n}$. Fix a K-invariant inner product on $\mathfrak{n}$ and write $\mathfrak{n} = V \oplus \mathfrak{z}$, where $V = \mathfrak{z}^\perp$. The subspaces $V, \mathfrak{z} \subset \mathfrak{n}$ are K-invariant and the Lie bracket amounts to an anti-symmetric bilinear mapping $V \times V \to \mathfrak{z}$.

1.1. Central reduction. Suppose that (K,N) is an irreducible n.G.p. and that $\mathfrak{z}_\circ \subset \mathfrak{z}$ is a proper non-zero K-invariant subspace of the center of $\mathfrak{n}$. The pair $(K, N/\exp(\mathfrak{z}_\circ))$ is again an (irreducible) n.G.p. called a *central reduction* of (K,N) [28]. A n.G.p. is said to be *maximal* if it cannot be obtained from another n.G.p. via central reduction. The following result, proved in Section 3, shows that in proving Theorem 1.2 it suffices to consider maximal irreducible n.G.p.'s.

Theorem 1.3. *If a n.G.p. satisfies* (O) *then so does any of its central reductions.*

1.2. Normal extension. Suppose that we have compact groups $K \subset \widetilde{K}$ acting by automorphisms on N. If (K,N) is a n.G.p. then so is $(\widetilde{K},N)$. If K is a normal subgroup of $\widetilde{K}$ then the pair $(\widetilde{K},N)$ is called a *normal extension* of (K,N). We will prove the following theorem in Section 4.

Theorem 1.4. *If a n.G.p. satisfies* (O) *then so does any of its normal extensions.*

In view of this result it suffices, in proving Theorem 1.2, to consider irreducible n.G.p.'s (K,N) with K *minimal* in the sense that (K,N) cannot be obtained via normal extension from another pair.

1.3. Connectivity of K. Corollary 1.5 below is an immediate consequence of Theorem 1.4. This reduces the proof of Theorem 1.2 to consideration of irreducible n.G.p.'s (K,N) with K connected. Letting K° denote the identity component in K Proposition 2.5 in [3] asserts that (K,N) is a n.G.p. if and only if (K°,N) is a n.G.p.

Corollary 1.5. *If* (K,N) *is a n.G.p. and* (K°,N) *satisfies* (O) *then so does* (K,N).

Proof. As $K^\circ \lhd K$ the pair (K,N) is a normal extension of (K°,N). ■

1.4. List of examples. The above discussion has reduced the proof of Theorem 1.2 to the verification of (O) for maximal irreducible n.G.p.'s (K,N) with K minimal, in particular connected. Vinberg's classification [29] gives a list of all such pairs. In many of these N is a Heisenberg group. As (O) has been proved for all n.G.p.'s with N a Heisenberg group [5, 6, 7] we can remove these cases from Vinberg's list. The remaining pairs are given in Table 1.

	K	V	$\mathfrak{z}$	condition
1	$SO(d)$	$\mathbb{R}^d$	$\Lambda^2(\mathbb{R}^d)$	$d \geq 3$
2	$SU(d)$	$\mathbb{C}^d$	$\Lambda^2(\mathbb{C}^d) \oplus \mathbb{R}$	$d \geq 2$ even
3	$U(d)$	$\mathbb{C}^d$	$\Lambda^2(\mathbb{C}^d) \oplus \mathbb{R}$	$d \geq 3$ odd
4	$SU(d)$	$\mathbb{C}^d$	$\Lambda^2(\mathbb{C}^d)$	$d \geq 3$ odd
5	$U(d)$	$\mathbb{C}^d$	$H\Lambda^2(\mathbb{C}^d) = \mathfrak{u}(d)$	$d \geq 2$
6	$Sp(d)$	$\mathbb{H}^d$	$HS^2(\mathbb{H}^d) \oplus \mathbb{C}$	$d \geq 1$
7	$Sp(1) \times Sp(d)$	$\mathbb{H}^d$	$\mathbb{H}_0 = \mathfrak{sp}(1)$	$d \geq 2$
8	$Spin(7)$	$\mathbb{R}^8$	$\mathbb{R}^7$	
9	$SU(2) \times SU(d)$	$\mathbb{C}^2 \otimes \mathbb{C}^d$	$H\Lambda^2(\mathbb{C}^2) = \mathfrak{u}(2)$	$d \geq 3$
10	$U(2) \times SU(2)$	$\mathbb{C}^2 \otimes \mathbb{C}^2$	$H\Lambda^2(\mathbb{C}^2) = \mathfrak{u}(2)$	
11	$U(2) \times Sp(d)$	$\mathbb{C}^2 \otimes \mathbb{H}^d$	$H\Lambda^2(\mathbb{C}^2) = \mathfrak{u}(2)$	$d \geq 1$
12	$Sp(2) \times Sp(d)$	$\mathbb{H}^2 \otimes \mathbb{H}^d$	$H\Lambda^2(\mathbb{H}^2) = \mathfrak{sp}(2)$	$d \geq 1$
13	G_2	$\mathbb{R}^7$	$\mathbb{R}^7$	
14	$U(1) \times Spin(7)$	$\mathbb{C}^8$	$\mathbb{R}^7 \oplus \mathbb{R}$	

Table 1: The non-Heisenberg n.G.p.'s from Vinberg's list

The $\mathfrak{z}$ entries in the table follow Vinberg's notational conventions. We will discuss the actions of K on V and $\mathfrak{z}$ as well as the Lie bracket $V \times V \to \mathfrak{z}$ (which is in fact determined by the K-actions) as we verify conjecture (O) in each case. These can be found in Chapter 13 of [30] and in the papers [14, 15, 16, 17].

The first six entries in Table 1 contain families of examples in which $\dim(\mathfrak{z})$ increases without bound. The verification of conjecture (O) in each of these cases is carried out below. The issues involved in treating these examples are similar. The first table entry is $(K,N) = (SO(d), F(d))$ for which (O) was announced in [4]. The remaining table entries are exceptional cases involving a fixed center $\mathfrak{z}$. The result from [18] establishes (O) for the pairs in lines 7 and 8 of the table. In these examples K has spherical orbits on $\mathfrak{z}$ and the form $(v,w) \mapsto ([v,w],z)$ is non-degenerate on V for non-zero $z \in \mathfrak{z}$. The techniques in [18] are general and do not need a case-by-case analysis. The verification of conjecture (O) for the last 6 entries in Table 1 will be given in a sequel to this paper. Our approach to these exceptional examples will parallel that applied here but technicalities must be addressed in each case. In broad outline, the K-orbits in $\mathfrak{z}$ fall into "layers", each with a distinct stabilizer in K. This fact imposes a layering on $\Delta(K,N)$ and $\mathcal{A}(K,N)$. In studying convergence behavior of sequences in these spaces we need to consider sequences within various layers and the possibilities for the layers in which their limits lie.

1.5. Motivation and related work. Harmonic analysis with n.G.p.'s is a nascent and growing area [30]. We hope that the calculations given in this paper and its sequel are of interest in that they deepen our understanding of analysis with the specific pairs from Table 1. The papers [14, 15, 16, 17], concerning K-invariant Schwartz fnctions on n.G.p.'s, likewise involve detailed case-by-case analysis with these pairs. The restriction to *irreducible* pairs is, of course, traditional as a first step. But Vinberg's classification was substantially extended by Yakimova [31, 32] (see also [30]) to encompass many indecomposable but non-irreducible pairs. These provide a rich family of examples that merit additional study. A proof for Conjecture (O) that does not require case-by-case work from the classifications would, however, obviously be desirable. Although the proof given in [18] avoids case-by-case work and does not require irreducibility, restrictive conditions are imposed on the pairs involved.

Conjecture (O) is an instance of Kirillov's conjecture which asserts, roughly speaking, that the Orbit Method should topologically embed the unitary dual of a sufficiently nice Lie group in the space of coadjoint orbits on the dual of its Lie algebra. The Kirillov conjecture is known to hold for nilpotent [8] and for exponential solvable Lie groups [21]. For semi-direct products $G = K \ltimes N$ of compact with nilpotent Lie groups there is, however, to our knowledge, at present no clear understanding of how the topologies on $\widehat{G}$ and $\mathfrak{g}^*/Ad^*(G)$ interrelate. For motion groups $G = K \ltimes V$ (V a vector group) there has been recent progress on this problem [11, 1, 26] and reference [10] concerns the Heisenberg motion group $G = U(n) \ltimes H_n$. It seems, however, that a general proof for Conjecture (O) based on this line of research would require substantial additional progress.

1.6. Outline of the remainder. The rest of this paper is organized as follows. Section 2 summarizes background concerning spherical functions and the mapping $\Psi\colon \Delta(K,N) \to \mathfrak{n}^*/K$ for n.G.p.'s (K,N). The proofs for Theorems 1.3 and 1.4 are given in Sections 3 and 4. Section 5 gives a more detailed description of the dual and the map Ψ, and Section 6 describes how one creates K-invariant differential operators on N and computes their eigenvalues on spherical functions . The remaining sections of the paper verify Conjecture (O) for the first six entries in Table 1.

2. Background results

Throughout this section we assume that (K,N) is a n.G.p. with N two-step. Assume moreover that $[\mathfrak{n},\mathfrak{n}] = \mathfrak{z}$. As remarked in Section 1 this is the case for irreducible n.G.p.'s. Fix an $Ad(K)$-invariant inner product $(\cdot,\cdot)_{\mathfrak{k}}$ on $\mathfrak{k}$, a K-invariant inner product $(\cdot,\cdot)_{\mathfrak{n}}$ on $\mathfrak{n}$ and form the orthogonal decomposition $\mathfrak{n} = V \oplus \mathfrak{z}$ as in Section 1.

2.1. The space $\widehat{G}_K$. The group K acts on the unitary dual $\widehat{N}$ of N via

$$k \cdot \pi := \pi \circ k^{-1}$$

and there is a unitary representation

$$W_\pi\colon K_\pi \to U(\mathcal{H}_\pi)$$

of the stabilizer for $(\pi, \mathcal{H}_\pi) \in \widehat{N}$ intertwining $k \cdot \pi$ with π, i.e.

$$(k \cdot \pi)(x) = W_\pi(k)^{-1}\pi(x)W_\pi(k) \qquad \text{for all } k \in K_\pi,\ x \in N.$$

Given any irreducible unitary representation σ of K_π, Mackey theory ensures that

$$\rho_{\pi,\sigma} := Ind_{K_\pi \ltimes N}^{K\ltimes N}\Big((k,x) \mapsto \sigma(k) \otimes \pi(x) W_\pi(k)\Big)$$

is an irreducible unitary representation of $G = K \ltimes N$ and that all irreducible unitary representations of G have this form, up to unitary equivalence. Mackey theory dictates, moreover, that $\rho_{\pi,\sigma} = \rho_{\pi',\sigma'}$ if and only if (π,σ) and (π',σ') *differ by the action of* K. This means

$$\pi' = k_\circ \cdot \pi, \quad \sigma' = k_\circ \cdot \sigma$$

for some $k_\circ \in K$ where $K_{k_\circ \cdot \pi} = k_\circ K_\pi k_\circ^{-1}$ and $(k_\circ \cdot \sigma)(k) := \sigma(k_\circ^{-1} k k_\circ)$.

As (K,N) is a n.G.p. the intertwining representation W_π is multiplicity free for each $\pi \in \widehat{N}$ [9, 2]. An application of Frobenius reciprocity shows, moreover, that the space of K-spherical representations is

$$\widehat{G}_K = \{\rho_{\pi,\sigma^*} \ : \ \pi \in \widehat{N}, \ \sigma \text{ occurs in } W_\pi\}$$

where σ^* denotes the representation of K_π dual to σ.

2.2. The space $\Delta(K,N)$. As explained in Section 2.1 the space of bounded K-spherical functions on N is

$$\Delta(K,N) = \{\phi_\rho \ : \ \rho \in \widehat{G}_K\} = \{\phi_{\rho_{\pi,\sigma^*}} \ : \ \pi \in \widehat{N}, \ \sigma \text{ occurs in } W_\pi\}.$$

Let $\mathcal{O}^N(\pi) \subset \mathfrak{n}^*$ denote the $Ad^*(N)$-orbit associated with $\pi \in \widehat{N}$ via the Kirillov correspondence. Since N is a two-step group, one has type I representations which are non-trivial on the center Z, for which $\mathcal{O}^N(\pi)$ is an affine subspace. The type II representations have Z in their kernel, and act as characters on N/Z. The corresponding coadjoint orbit is a single point. We write $\widehat{N} = \widehat{N}^I \cup \widehat{N}^{II}$ to distinguish the two types of representations.

For a type I representation $\pi \in \widehat{N}^I$ let

$$\mathcal{H}_\pi = \bigoplus_{\alpha \in \Lambda_\pi} P_{\pi,\alpha} \tag{2}$$

denote the (canonical) decomposition of $\mathcal{H}_\pi$ into $W_\pi(K_\pi)$-irreducible subspaces. Here Λ_π is a countably infinite index set that depends on π. For $\alpha \in \Lambda_\pi$ we write $\phi_{\pi,\alpha}$ for the spherical function given by ρ_{π,σ^*} where $\sigma \in \widehat{K}_\pi$ is the representation of K_π on $P_{\pi,\alpha}$. This may be written as

$$\phi_{\pi,\alpha}(x) = \int_K \langle \pi(k \cdot x) v_{\pi,\alpha}, v_{\pi,\alpha}\rangle \, dk$$

where $v_{\pi,\alpha}$ is any unit vector in $P_{\pi,\alpha}$

For a type II point $w \in \mathfrak{z}^\perp \subset \mathfrak{n}^*$, one has spherical function ϕ_w given by

$$\phi_w(\exp X) = \int_K e^{iw(k\cdot X)} dk.$$

Using $(\cdot,\cdot)_\mathfrak{n}$ to identify $\mathfrak{n}^*$ with $\mathfrak{n}$ means we can take $w \in V$.

Thus we now have

$$\Delta(K,N) = \Delta^I(K,N) \cup \Delta^{II}(K,N) = \{\phi_{\pi,\alpha} : \pi \in \widehat{N}^I,\, \alpha \in \Lambda_\pi\} \cup \{\phi_w : w \in V\}.$$

Note that $\Delta^{II}(K,N) \cong V/K$.

2.3. The map $\Psi\colon \Delta(K,N) \to \mathcal{A}(K,N)$. Here we recall, from [4], an alternate definition for the orbit mapping (1). For type II spherical functions ϕ_w one has simply

$$\Psi(\phi_w) = K \cdot w.$$

For type I spherical functions $\phi_{\pi,\alpha}$ one can compute $\Psi(\phi_{\pi,\alpha})$ as follows.

Let $\pi \in \widehat{N}^I$ be a type I representation with corresponding coadjoint orbit $\mathcal{O}^N(\pi)$. Choose any $\ell \in \mathcal{O}^N(\pi)$ and set

$$\mathfrak{a}_\pi := \{X \in V \ : \ \ell([X,\mathfrak{n}]) = 0\}, \quad \mathfrak{w}_\pi := \mathfrak{a}_\pi^\perp \cap V.$$

As the notation suggests, these spaces do not depend on the choice of $\ell \in \mathcal{O}^N(\pi)$. The orbit $\mathcal{O}^N(\pi)$ contains, however, a unique point

$$\ell_\pi \in \mathcal{O}^N(\pi)$$

which is *aligned* in the sense that $\ell_\pi|_{\mathfrak{w}_\pi} = 0$. This aligned point has the property that the stabilizer K_π of π is the stabilizer of ℓ_π in K. The *moment map* $\tau_\pi\colon \mathcal{O}^N(\pi) \to \mathfrak{k}_\pi^*$ for $\pi \in \widehat{N}$ is defined as

$$\tau_\pi\big(Ad^*(\exp X)\ell_\pi\big)(A) := -\frac{1}{2}\ell_\pi[X, A\cdot X] \tag{3}$$

for $A \in \mathfrak{k}_\pi$, $X \in \mathfrak{n}$. At this point it is useful to take as index set Λ_π in (2) the set of highest weights occurring in W_π. That is, let

$$\Lambda_\pi = \{\alpha \in \mathfrak{k}_\pi^* \ : \ i\alpha \text{ is a highest weight for the representation of } K_\pi \text{ on } P_{\pi,\alpha}\}.$$

Here $\alpha \in \Lambda_\pi$ is defined on the Lie algebra $\mathfrak{t}_\pi$ for a maximal torus in K_π° and extended to $\mathfrak{k}_\pi$ as zero on the orthogonal complement to $\mathfrak{t}_\pi$ with respect to the $Ad(K)$-invariant inner product $(\cdot,\cdot)_\mathfrak{k}$.

The moment map $\tau_\pi\colon \mathcal{O}^N(\pi) \to \mathfrak{k}_\pi^*$ is one-to-one on K_π-orbits and each $\alpha \in \Lambda_\pi$ lies in the image of τ_π [4]. Choose any point $w_{\pi,\alpha} \in \mathcal{O}^N(\pi)$ such that $\tau_\pi(w_{\pi,\alpha}) = \alpha$. Such a point $w_{\pi,\alpha}$ is called a (π,α)-*spherical point.* One has

$$\Psi(\phi_{\pi,\alpha}) = \mathbf{O}_{\pi,\alpha} := K \cdot w_{\pi,\alpha}. \tag{4}$$

3. Central reduction

Our goal here is to prove Theorem 1.3. Let (K,N) be a n.G.p., $\mathfrak{z}_\circ$ a K-invariant subspace of $\mathfrak{z}$ and let $Z_\circ := \exp(\mathfrak{z}_\circ)$, $N_r := N/Z_\circ$. The pair (K,N_r) is a central reduction of (K,N). Let

$$G := K \ltimes N, \quad G_r := G/Z_\circ = K \ltimes N_r, \quad \mathfrak{n}_r := \mathfrak{n}/\mathfrak{z}_\circ, \quad \mathfrak{g}_r := \mathfrak{g}/\mathfrak{z}_\circ = \mathfrak{k} \ltimes \mathfrak{n}_r.$$

We can identify $\mathfrak{n}_r^*$ and $\mathfrak{g}_r^*$ with subspaces of $\mathfrak{n}^*$ and $\mathfrak{g}^*$, namely

$$\mathfrak{n}_r^* = \{\ell \in \mathfrak{n}^* \ : \ \ell|_{\mathfrak{z}_\circ} = 0\}, \quad \mathfrak{g}_r^* = \{\psi \in \mathfrak{g}^* \ : \ \psi|_{\mathfrak{z}_\circ} = 0\}.$$

Likewise we can identify $\widehat{N}_r$ and $\widehat{G}_r$ with subsets of $\widehat{N}$ and $\widehat{G}$, namely

$$\widehat{N}_r = \{\pi \in \widehat{N} \ : \ \pi|_{Z_\circ} = id\}, \quad \widehat{G}_r = \{\rho \in \widehat{G} \ : \ \rho|_{Z_\circ} = id\}.$$

As (K,N) is a n.G.p. we have $\dim(\rho^K) \leq 1$ for all $\rho \in \widehat{G}$. Thus also $\dim(\rho^K) \leq 1$ for all $\rho \in \widehat{G}_r$ since $\widehat{G}_r \subset \widehat{G}$ and the action of K on $\mathcal{H}_\rho$ via $\rho(G_r)$ coincides with its action via $\rho(G)$. This proves Vinberg's result that the central reduction (K,N_r) is itself a n.G.p. [28].

We note that $\Delta(N_r,K)$ is a subset of $\Delta(N,K)$, namely

$$\Delta(N_r,K) = \{\phi_{\pi,\alpha} : \pi \in \widehat{N}^I, \pi|_{Z_\circ} = id\} \cup \{\phi_w : w \in V\}.$$

Let π be a type I representation in $\widehat{N}_r$ and let ℓ be the aligned point in $O^N(\pi)$. We have the identification of $\mathfrak{n}_r^*$ as a subspace of $\mathfrak{n}^*$. Then, since $\mathrm{Ad}(Z)$ is trivial, $\mathcal{O}^{N_r}(\ell) = \mathcal{O}^N(\ell)$. Since $\mathfrak{z}_\circ$ is K-invariant and $\ell|_{\mathfrak{z}_\circ} = 0$, the stabilizer K_ℓ is the same for the action of K on both N and N_r. Thus the moment maps on $\mathcal{O}^{N_r}(\ell)$ and $\mathcal{O}^N(\ell)$ are the same, and hence the spherical points are the same. Thus $\mathcal{A}(N_r,K)$ is also a subset of $\mathcal{A}(N,K)$, namely

$$\mathcal{A}(N_r,K) = \{\mathbf{O}_{\pi,\alpha} : \pi \in \widehat{N}^I, \pi|Z_\circ = id\} \cup \{K \cdot w : w \in V\}.$$

Proof of Theorem 1.3. Denote the orbit maps for the pairs (K,N), (K,N_r) via $\Psi\colon \Delta(K,N) \to \mathcal{A}(K,N)$ and $\Psi_r\colon \Delta(K,N_r) \to \mathcal{A}(K,N_r)$. Then if Ψ is a homeomorphism onto its image then so is Ψ_r, for Ψ_r amounts to the restriction of the homeomorphism Ψ to the subspace $\Delta(K,N_r) \subset \Delta(K,N)$. ■

4. Normal extension

This section contains the proof of Theorem 1.4. Let (K,N) and $(\widetilde{K},N)$ be n.G.p.'s with K a normal subgroup of $\widetilde{K}$. We denote the orbit mappings for (K,N) and $(\widetilde{K},N)$ by $\Psi\colon \Delta(K,N) \to \mathcal{A}(K,N)$ and $\widetilde{\Psi}\colon \Delta(\widetilde{K},N) \to \mathcal{A}(\widetilde{K},N)$. We must show that if Ψ is a homeomorphism onto its image then so is $\widetilde{\Psi}$. We choose the Ad-invariant inner products on $\mathfrak{k}$ and $\widetilde{\mathfrak{k}}$ to ensure that $(\cdot,\cdot)_{\mathfrak{k}}$ is the restriction of $(\cdot,\cdot)_{\widetilde{\mathfrak{k}}}$ to $\mathfrak{k}$.

For a type I representation $\pi \in \widehat{N}$, let $\widetilde{\Lambda}_\pi \subset (\widetilde{\mathfrak{k}}_\pi)^*$ be the set of highest weights for irreducible representations occurring in the intertwining representation $W_\pi(\widetilde{K})$, and write the canonical multiplicity free decomposition of $\mathcal{H}_\pi$ under $W_\pi(\widetilde{K}_\pi)$ as

$$\mathcal{H}_\pi = \bigoplus_{\widetilde{\alpha}\in\widetilde{\Lambda}_\pi} P_{\pi,\widetilde{\alpha}}.$$

For each $\widetilde{\alpha} \in \widetilde{\Lambda}_\pi$ let $\Lambda_{\pi,\widetilde{\alpha}} \subset \mathfrak{k}_\pi^*$ be the set of highest weights of representations occurring in the restriction of W_π to $K_\pi = K \cap \widetilde{K}_\pi$. As $\dim(P_{\pi,\widetilde{\alpha}}) < \infty$, each $\Lambda_{\pi,\widetilde{\alpha}}$ is a finite set.

We write the canonical multiplicity free decomposition of $\mathcal{H}_\pi$ under $W_\pi(K_\pi)$ as

$$\mathcal{H}_\pi = \bigoplus_{\widetilde{\alpha}\in\widetilde{\Lambda}_\pi} \left(\bigoplus_{\alpha\in\Lambda_{\pi,\widetilde{\alpha}}} P_{\pi,\widetilde{\alpha},\alpha} \right) \quad \text{where} \quad P_{\pi,\widetilde{\alpha}} = \bigoplus_{\alpha\in\Lambda_{\pi,\widetilde{\alpha}}} P_{\pi,\widetilde{\alpha},\alpha}.$$

We note that $\widetilde{K}$ acts transitively on each $\Lambda_{\pi,\widetilde{\alpha}}$.

Now $\Delta(\widetilde{K},N) = \Delta^I(\widetilde{K},N) \cup \Delta^{II}(\widetilde{K},N)$ where

$$\Delta^I(\widetilde{K},N) = \left\{ \widetilde{\phi}_{\pi,\widetilde{\alpha}} \ : \ \pi \in \widehat{N}, \widetilde{\alpha} \in \widetilde{\Lambda}_\pi \right\}, \quad \Delta^{II}(\widetilde{K},N) \cong V/\widetilde{K}$$

and $\Delta(K,N) = \Delta^I(K,N) \cup \Delta^{II}(K,N)$ where

$$\Delta^I(K,N) = \left\{ \phi_{\pi,\widetilde{\alpha},\alpha} \ : \ \pi \in \widehat{N}, \widetilde{\alpha} \in \widetilde{\Lambda}_\pi, \alpha \in \Lambda_{\pi,\widetilde{\alpha}} \right\}, \quad \Delta^{II}(K,N) \cong V/K.$$

There are $\widetilde{K}$-actions on $\Delta(K,N)$ and $\mathcal{A}(K,N)$, namely $(\widetilde{k}\cdot\phi) = \phi(\widetilde{k}^{-1}\cdot n)$ for $\phi \in \Delta(K,N)$, and $\widetilde{k}\cdot(K\cdot w) = K\cdot(\widetilde{k}w)$ for $K\cdot w \in \mathcal{A}(K,N)$.

Lemma 4.1. $\Delta(\widetilde{K},N) \cong \Delta(K,N)/\widetilde{K}$.

Proof. As explained in Section 2.2 the spherical function $\widetilde{\phi}_{\pi,\widetilde{\alpha}}$ is the $\widetilde{K}$-average of a matrix coefficient $x \mapsto \langle \pi(x)v_{\pi,\widetilde{\alpha}}, v_{\pi,\widetilde{\alpha}}\rangle$ with $v_{\pi,\widetilde{\alpha}}$ a unit vector in $P_{\pi,\widetilde{\alpha}}$. and $\phi_{\pi,\widetilde{\alpha},\alpha}$ the K-average of $x \mapsto \langle \pi(x)v_{\pi,\widetilde{\alpha},\alpha}, v_{\pi,\widetilde{\alpha},\alpha}\rangle$ with $v_{\pi,\widetilde{\alpha},\alpha}$ a unit vector in $P_{\pi,\widetilde{\alpha},\alpha}$. Since $P_{\pi,\widetilde{\alpha},\alpha} \subset P_{\pi,\widetilde{\alpha}}$, we can use the $\widetilde{K}$-average of the unit vector $v_{\pi,\widetilde{\alpha},\alpha}$ to obtain the spherical function $\phi_{\pi,\widetilde{\alpha}}$. That is, for any $\alpha \in \Lambda_{\pi,\widetilde{\alpha}}$,

$$\widetilde{\phi}_{\pi,\widetilde{\alpha}}(n) = \int_{\widetilde{K}} \phi_{\pi,\widetilde{\alpha},\alpha}(\widetilde{k}\cdot n)\, d\widetilde{k}. \qquad \blacksquare$$

Lemma 4.2. *The map* $\Psi\colon \Delta(K,N) \to \mathcal{A}(K,N)$ *is* $\widetilde{K}$*-equivariant.*

Proof. Given $\phi_{\pi,\alpha} \in \Delta^I(K,N)$, we have $\Psi(\phi_{\pi,\alpha}) = K\cdot w_{\pi,\alpha}$, where $\tau_\pi(w_{\pi,\alpha}) = \alpha$.

Then $$\tau_{\widetilde{k}\cdot\pi}(\widetilde{k}(X\cdot\ell))(\widetilde{k}\cdot A) = \tau_\pi(X\cdot\ell)(A).$$

Thus $$\tau_{\widetilde{k}\cdot\pi}(\widetilde{k}\cdot w_{\pi,\alpha}) = \tau_\pi(w_{\pi,\alpha})\circ\widetilde{k}^{-1} = \alpha\circ\widetilde{k}^{-1} = \widetilde{k}\cdot\alpha,$$

and hence $\widetilde{k}\cdot w_{\pi,\alpha}$ is a $(\widetilde{k}\cdot\pi, \widetilde{k}\cdot\alpha)$-spherical point. That is,

$$\Psi(\widetilde{k}\cdot\phi_{\pi,\alpha}) = \Psi(\phi_{\widetilde{k}\pi,\widetilde{k}\alpha}) = K\cdot(\widetilde{k}\cdot w_{\pi,\alpha}) = \widetilde{k}\cdot\mathbf{O}_{\pi,\alpha}.$$

For $\phi_w \in \Delta^{II}(K,N)$, we have $\Psi(\widetilde{k}\cdot\phi_w) = \Psi(\phi_{\widetilde{k}w}) = K\cdot(\widetilde{k}w) = \widetilde{k}\cdot\Psi(\phi_w)$. $\blacksquare$

Proof of Theorem 1.4. The preceding lemmas show that $\widetilde{\Psi}\colon\Delta(\widetilde{K},N) \to \mathcal{A}(\widetilde{K},N)$ is topologically equivalent to the map

$$\Delta(K,N)/\widetilde{K} \to \mathcal{A}(K,N)/\widetilde{K}$$

obtained from the $\widetilde{K}$-equivariant map $\Psi\colon \Delta(K,N) \to \mathfrak{n}^*/K$ by passing to $\widetilde{K}$-orbits. Thus if Ψ is a homeomorphism onto its image then so is $\widetilde{\Psi}$. $\blacksquare$

5. A refined parametrization of $\Delta(K,N)$

Given $\pi \in \widehat{N}^I$, we regard the aligned point ℓ_π as an element in $\mathfrak{n}$ by using the K-invariant inner product $(\cdot,\cdot)_\mathfrak{n}$ to identify $\mathfrak{n}$ with $\mathfrak{n}^*$. Writing $\ell_\pi = a_\circ + z_\circ$ with $a_\circ \in V$ and $z_\circ \in \mathfrak{z}$ one has, in fact, $a_\circ \in \mathfrak{a}_\pi$ as ℓ_π is aligned. Noting that $\mathfrak{a}_\pi$ is determined by $z_\circ$ we write the decomposition $V = \mathfrak{a}_\pi \oplus \mathfrak{w}_\pi$ here as

$$V = \mathfrak{a}_{z_\circ} \oplus \mathfrak{w}_{z_\circ}$$

and let $$\Lambda_{z_\circ,a_\circ} := \Lambda_\pi, \quad \phi_{z_\circ,a_\circ,\alpha} := \phi_{\pi,\alpha} \ (\alpha \in \Lambda_{z_\circ,a_\circ}).$$

We may also write $\pi_{z_\circ,a_\circ}$ for the representation corresponding to the coadjoint orbit through $a_\circ + z_\circ$. We have established the following.

Proposition 5.1. *With notation as above*

$$\Delta^I(K,N) = \left\{\phi_{z_\circ,a_\circ,\alpha} \ : \ z_\circ \in \mathfrak{z}, z_\circ \neq 0, a_\circ \in \mathfrak{a}_{z_\circ}, \alpha \in \Lambda_{z_\circ,a_\circ}\right\}.$$

Moreover one has $\phi_{z_\circ,a_\circ,\alpha} = \phi_{z'_\circ,a'_\circ,\alpha'}$ whenever the data $(z'_\circ,a'_\circ,\alpha')$, $(z_\circ,a_\circ,\alpha)$ differ by the action of K.

Remark 5.1. In our subsequent treatment of pairs from Table 1 we will subsume type II spherical functions into the notational framework of Proposition 5.1. For $z_\circ = 0$ one has $\mathfrak{w}_0 = \{0\}$ and $\mathfrak{a}_0 = V$. In this case we adopt the convention that $\Lambda_{0,a_\circ} = \emptyset$ for each $a_\circ \in V$ and "$\phi_{0,a_\circ,-}$" (with empty α-parameter) is the type II spherical function $\phi_{a_\circ}$. ∎

For the following discussion, we fix the parameters $z_\circ \neq 0, a_\circ, \alpha$ with decomposition $V = \mathfrak{a}_{z_\circ} \oplus \mathfrak{w}_{z_\circ}$. Lemma 3.4 in [4] shows that the stabilizer of $a_\circ$ in $K_{z_\circ}$, namely

$$K_{z_\circ,a_\circ} := K_{z_\circ} \cap K_{a_\circ},$$

coincides with the stabilizer of the representation $\pi_{z_\circ,a_\circ}$. Let $J_{z_\circ} : V \to V$ denote the operator satisfying $(J_{z_\circ}(u),v)_\mathfrak{n} = ([u,v],z_\circ)_\mathfrak{n}$ for all $u,v \in V$.
The operator $J_{z_\circ}$ is skew-symmetric with respect to the inner product $(\cdot,\cdot)_\mathfrak{n}$ and has

$$Ker(J_{z_\circ}) = \mathfrak{a}_{z_\circ}, \qquad Image(J_{z_\circ}) = \mathfrak{w}_{z_\circ}$$

Since $z_\circ \neq 0$ and, by assumption, $[\mathfrak{n},\mathfrak{n}] = \mathfrak{z}$ one has $\mathfrak{w}_{z_\circ} \neq \{0\}$. The operator $J_{z_\circ}$ preserves $\mathfrak{w}_{z_\circ}$ and is non-degenerate and skew-symmetric on $\mathfrak{w}_{z_\circ}$. In particular $\mathfrak{w}_{z_\circ}$ is even dimensional and $J^2_{z_\circ} : \mathfrak{w}_\circ \to \mathfrak{w}_{z_\circ}$ is negative definite symmetric. Letting

$$\sigma^+(z_\circ) := \{\lambda > 0 \ : \ -\lambda^2 \text{ is an eigenvalue for } J^2_{z_\circ}\}. \tag{5}$$

we have an orthogonal direct sum decomposition for V into eigenspaces for $J^2_{z_\circ}$,

$$V = \mathfrak{a}_{z_\circ} \oplus \bigoplus_{\lambda \in \sigma^+(z_\circ)} \mathfrak{w}_{z_\circ,\lambda}, \tag{6}$$

where $J^2_{z_\circ} = -\lambda^2$ on $\mathfrak{w}_{z_\circ,\lambda}$. Note that if $\mathfrak{a}_{z_\circ} \neq 0$ then 0 is an eigenvalue for $J^2_{z_\circ}$ and $\mathfrak{a}_{z_\circ}$ is the 0-eigenspace for $J^2_{z_\circ}$. For $w \in \mathfrak{w}_{z_\circ}$, we write $w = \sum w_\lambda$ with $w_\lambda \in \mathfrak{w}_{z_\circ,\lambda}$.

The operator $J_{z_\circ}$ gives us a complex structure on $\mathfrak{w}_{z_\circ}$, namely

$$\widetilde{J}_{z_\circ}\colon \mathfrak{w}_{z_\circ} \to \mathfrak{w}_{z_\circ}, \quad \widetilde{J}_{z_\circ}\left(\sum w_\lambda\right) = \sum \frac{1}{\lambda} J_{z_\circ}(w_\lambda).$$

Let $\widetilde{\mathfrak{w}}_{z_\circ}$ denote the complex vector space $(\mathfrak{w}_{z_\circ}, \widetilde{J}_{z_\circ})$ and equip this with the hermitian inner product $\langle u, v\rangle_{z_\circ} := (u,v)_{\mathfrak{n}} + i(u, \widetilde{J}_{z_\circ}(v))_{\mathfrak{n}}$. The $\mathfrak{w}_{z_\circ,\lambda}$'s are complex subspaces of $\widetilde{\mathfrak{w}}_{z_\circ}$, orthogonal with respect to $\langle\cdot,\cdot\rangle_{z_\circ}$, and the action of $K_{z_\circ}$ on $(\widetilde{\mathfrak{w}}_{z_\circ}, \langle\cdot,\cdot\rangle_{z_\circ})$ is unitary. Moreover $(K_{z_\circ,a_\circ}, H_{z_\circ})$ is a n.G.p. where $H_{z_\circ}$ is the Heisenberg group whose Lie algebra is $\mathfrak{w}_{z_\circ} \oplus \mathbb{R}$ with bracket

$$\big[(u,s),(v,t)\big] = \big(0, -Im\,\langle u,v\rangle_{z_\circ}\big) = \big(0, -(u, \widetilde{J}_{z_\circ}(v))_{\mathfrak{n}}\big)$$

(see [2, 3, 28]). Equivalently $K_{z_\circ,a_\circ} : \widetilde{\mathfrak{w}}_{z_\circ}$ is a multiplicity free action of the compact group $K_{z_\circ,a_\circ}$ on the complex vector space $\widetilde{\mathfrak{w}}_{z_\circ}$ [9, 2].

The unnormalized moment map for the action $K_{z_\circ,a_\circ} : \widetilde{\mathfrak{w}}_{z_\circ}$ is

$$\eta\colon \widetilde{\mathfrak{w}}_{z_\circ} \to \mathfrak{k}^*_{z_\circ,a_\circ}, \quad \eta(w)(A) := Im\,\langle w, A\cdot w\rangle_{z_\circ} = (w, \widetilde{J}_{z_\circ}(A\cdot w))_{\mathfrak{n}}.$$

The following Lemma shows how the moment map τ_π from equation (3) is determined by η.

Lemma 5.2. *The moment maps τ_π on $\mathcal{O}^N(\pi) = a_\circ + \mathfrak{w}_{z_\circ} + z_\circ$ and η on $\widetilde{\mathfrak{w}}_{z_\circ}$ are related by*

$$\tau_\pi\left(a_\circ, \sum w_\lambda, z_\circ\right) = \eta\left(\sum \frac{1}{\sqrt{2\lambda}} w_\lambda\right).$$

Proof. Recall that by definition

$$\tau_\pi(Ad^*(\exp(w))\ell_\pi)(A) = -\frac{1}{2}\ell_\pi[w, A\cdot w] = -\frac{1}{2}([w, A\cdot w], z_\circ)_{\mathfrak{n}}$$

for $A \in \mathfrak{k}_{z_\circ,a_\circ}$ and $w \in \mathfrak{w}_{\mathfrak{z}_\circ}$. We have

$$\begin{aligned} Ad^*(\exp w)\ell_\pi(X) &= \ell_\pi(X) - \ell_\pi[w,X] = (a_\circ, X)_{\mathfrak{n}} + (z_\circ, X)_{\mathfrak{n}} - (z_\circ, [w,X])_{\mathfrak{n}} \\ &= (a_\circ, X)_{\mathfrak{n}} + (z_\circ, X)_{\mathfrak{n}} - (J_{z_\circ} w, X)_{\mathfrak{n}}. \end{aligned}$$

With our identification of $\mathfrak{n}$ with $\mathfrak{n}^*$, we have $Ad^*(\exp(w))\ell_\pi = (a_\circ, -J_{z_\circ}w, z_\circ)$. Therefore $\tau_\pi(a_\circ, w, z_\circ)(A) = -\frac{1}{2}\big([J_{z_\circ}^{-1}w, A\cdot J_{z_\circ}^{-1}w], z_\circ\big)_{\mathfrak{n}} = -\frac{1}{2}(w, A\cdot J_{z_\circ}^{-1}w)_{\mathfrak{n}}$.

For $w_\lambda \in \mathfrak{w}_{z_\circ,\lambda}$, we have $J_{z_\circ}w_\lambda = \lambda \widetilde{J}_{z_\circ}w_\lambda$, and hence $J_{z_\circ}^{-1}w_\lambda = -\frac{1}{\lambda}\widetilde{J}_{z_\circ}w_\lambda$. Since $J_{z_\circ}$ commutes with K_π, we now have

$$\begin{aligned} \tau_\pi(a_\circ, w_\lambda, z_\circ)(A) &= -\frac{1}{2}(w_\lambda, J_{z_\circ}^{-1}(A\cdot w_\lambda))_{\mathfrak{n}} = \frac{1}{2}\left(w_\lambda, \frac{1}{\lambda}\widetilde{J}_{z_\circ}(A\cdot w_\lambda)\right)_{\mathfrak{n}} \\ &= \eta\left(\frac{1}{\sqrt{2\lambda}}w_\lambda\right)(A). \end{aligned}$$

The result follows by noting that the $\mathfrak{w}_{z_\circ,\lambda}$'s are orthogonal subspaces, invariant under $K_{z_\circ,a_\circ}$ and $J_{z_\circ}$. Thus

$$\tau_\pi\left(a_\circ, \sum w_\lambda, z_\circ\right) = \sum_\lambda \tau_\pi(a_\circ, w_\lambda, z_\circ) = \sum_\lambda \eta\left(\frac{1}{\sqrt{2\lambda}}w_\lambda\right) = \eta\left(\sum \frac{1}{\sqrt{2\lambda}}w_\lambda\right). \quad \blacksquare$$

Lemma 5.2 together with equation (4) now yield the following.

Proposition 5.3. *Let $\pi \in \widehat{N}^I$ with aligned point $\ell_\pi = a_\circ + z_\circ, z_\circ \neq 0$. Decompose V with respect to $J_{z_\circ}$ as $V = \mathfrak{a}_{z_\circ} \oplus \sum \mathfrak{w}_{z_\circ,\lambda}$. Given $\alpha \in \Lambda_\pi = \Lambda_{z_\circ,a_\circ}$ let $w_\alpha \in \mathfrak{w}_{z_\circ}$ be any point for which $\eta(w_\alpha) = \alpha$. Write $w_\alpha = \sum w_\lambda$ with $w_\lambda \in \mathfrak{w}_{z_\circ,\lambda}$ and let $w'_\alpha := \sum (2\lambda)^{1/2} w_\lambda$. Then $\Psi(\phi_{z_\circ,a_\circ,\alpha}) = K \cdot (a_\circ, w'_\alpha, z_\circ)$.*

6. Eigenvalues for operators $D \in \mathbb{D}_K(N)$

Recall that $\mathbb{D}_K(N)$ denotes the set of differential operators on N that are invariant under both the action of K and left multiplication. The spherical functions are eigenfunctions for such operators. Given $D \in \mathbb{D}_K(N)$ and $\phi \in \Delta(K, N)$, we write $\widehat{D}(\phi)$ for the eigenvalue of D acting on ϕ, so that

$$D\phi = \widehat{D}(\phi)\phi.$$

For $D \in \mathbb{D}_K(N)$ and $\pi \in \widehat{N}$, the operator $\pi(D)$ commutes with the action of K_π on $\mathcal{H}_\pi$ and hence preserves the subspaces $P_{\pi,\alpha}$ in Decomposition (2). In this regard the following fact is fundamental.

Lemma 6.1. [4, Lemma 5.6] $\pi(D)|_{P_{\pi,\alpha}} = \widehat{D}(\phi_{\pi,\alpha})$.

Let $\{U_1, \ldots, U_{d_V}\}$ be an orthonormal basis for V and $\{Z_1, \ldots, Z_{d_\mathfrak{z}}\}$ an orthonormal basis for $\mathfrak{z}$. As the action of K on $\mathfrak{n}$ preserves $(\cdot,\cdot)_\mathfrak{n}$, V and $\mathfrak{z}$ the operators

$$\mathcal{L}_V := -(U_1^2 + \cdots + U_{d_V}^2), \qquad \mathcal{L}_\mathfrak{z} := -(Z_1^2 + \cdots + Z_{d_\mathfrak{z}}^2) \tag{7}$$

belong to $\mathbb{D}_K(N)$. These are, moreover, independent of the chosen orthonormal bases. The eigenvalues for $\mathcal{L}_V$ and $\mathcal{L}_\mathfrak{z}$ on type II spherical functions ϕ_w ($w \in V$) are clearly

$$\widehat{\mathcal{L}}_V(\phi_w) = |w|^2, \qquad \widehat{\mathcal{L}}_\mathfrak{z}(\phi_w) = 0. \tag{8}$$

The eigenvalues for $\mathcal{L}_V$ and $\mathcal{L}_\mathfrak{z}$ on spherical functions $\phi_{z_\circ,a_\circ,\alpha}$ with $z_\circ \neq 0$ are given below in Lemma 6.2.

Let $(z_\circ, a_\circ, \alpha)$ be spherical function parameters with $z_\circ \neq 0$. The representation $\pi = \pi_{z_\circ,a_\circ}$ can be realized in Fock space, a Hilbert space completion $\mathcal{H}_\pi$ of the polynomial ring $\mathbb{C}[\widetilde{\mathfrak{w}}_{z_\circ}]$. The subspaces $\mathfrak{w}_{z_\circ,\lambda}$ in the decomposition $\mathfrak{w}_{z_\circ} = \oplus_{\lambda\in\sigma^+(z_\circ)} \mathfrak{w}_{z_\circ,\lambda}$ are complex subspaces of $\widetilde{\mathfrak{w}}_{z_\circ}$ and we let, for $\lambda \in \sigma^+(z_\circ)$,

$$m(\lambda) = \dim(\mathfrak{w}_{z_\circ,\lambda})/2 = \dim_\mathbb{C}(\mathfrak{w}_{z_\circ,\lambda}).$$

The stabilizer $K_{z_\circ,a_\circ}$ acts on $\mathcal{H}_\pi$ as a subgroup of $O(\mathfrak{a}_{z_\circ}) \times \prod_{\lambda\in\sigma^+(z_\circ)} U(\mathfrak{w}_{z_\circ,\lambda})$ and hence preserves each subspace $\otimes_{\lambda\in\sigma^+(z_\circ)} \mathcal{P}_{\ell_\lambda}(\mathfrak{w}_{z_\circ,\lambda})$ for given non-negative integers $(\ell_\lambda : \lambda \in \sigma^+(z_\circ))$. Here $\mathcal{P}_{\ell_\lambda}(\mathfrak{w}_{z_\circ,\lambda}) \subset \mathbb{C}[\mathfrak{w}_{z_\circ,\lambda}]$ denotes the space of homogeneous polynomials of degree ℓ_λ on $\mathfrak{w}_{z_\circ,\lambda}$. As $K_{z_\circ,a_\circ} : \widetilde{\mathfrak{w}}_{z_\circ}$ is a multiplicity free action it follows that for each $\alpha \in \Lambda_{z_\circ,a_\circ}$ we have

$$P_{z_\circ,a_\circ,\alpha} \subset \bigotimes_{\lambda\in\sigma^+(z_\circ)} \mathcal{P}_{\alpha(\lambda)}(\mathfrak{w}_{z_\circ,\lambda})$$

for some non-negative integers $(\alpha(\lambda) : \lambda \in \sigma^+(z_\circ))$.

Lemma 6.2. *With notation as above one has*

$$\widehat{\mathcal{L}}_V(\phi_{z_\circ,a_\circ,\alpha}) = |a_\circ|^2 + \sum_{\lambda\in\sigma^+(z_\circ)} \lambda(2\alpha(\lambda)+m(\lambda)), \qquad \widehat{\mathcal{L}}_{\mathfrak{z}}(\phi_{z_\circ,a_\circ,\alpha}) = |z_\circ|^2.$$

Proof. The formula for $\widehat{\mathcal{L}}_{\mathfrak{z}}(\phi_{z_\circ,a_\circ,\alpha})$ is clear as $\pi_{z_\circ,a_\circ}$ has central character $z\mapsto e^{i(z,z_\circ)_{\mathfrak{n}}}$. To compute $\widehat{\mathcal{L}}_V(\phi_{z_\circ,a_\circ,\alpha})$ we reason as in the proof for [4, Lemma 9.1].
Let $V_0 := \mathfrak{a}_{z_\circ}$, $V_\lambda := \mathfrak{w}_{z_\circ,\lambda}$ for $\lambda\in\sigma^+(z_\circ)$ and choose an orthonormal basis $\{U_1,\ldots,U_{d_V}\}$ for V so as to ensure that each U_i belongs to some V_λ. This is possible since the eigenspaces for $J_{z_\circ}^2$ are mutually orthogonal. Thus now

$$\mathcal{L}_V = \sum_{\lambda\geq 0}\mathcal{L}_\lambda \quad\text{where}\quad \mathcal{L}_\lambda := -\Bigg(\sum_{\{i\,:\,U_i\in V_\lambda\}} U_i^2\Bigg).$$

For $\lambda>0$ the operator $\pi_{z_\circ,a_\circ}(\mathcal{L}_\lambda)$ acts on $\mathcal{P}_{\alpha(\lambda)}(V_\lambda)$ via the scalar $\lambda(2\alpha(\lambda)+m(\lambda))$ and annihilates $\mathcal{P}_{\alpha(\lambda')}(V_{\lambda'})$ for $\lambda'\neq\lambda$. Thus $\pi_{z_\circ,a_\circ}(\mathcal{L}_\lambda)$ acts on $P_{z_\circ,a_\circ,\alpha}\subset\otimes_{\lambda'>0}\mathcal{P}_{\alpha(\lambda')}(V_{\lambda'})$ as the scalar $\lambda(2\alpha(\lambda)+m(\lambda))$. For $a\in V_0$ the operator $\pi_{z_\circ,a_\circ}(a)$ acts on all of $\mathbb{C}[\widetilde{\mathfrak{w}}_{z_\circ}]$ via the scalar $e^{i(a,a_\circ)_{\mathfrak{n}}}$. Hence $\pi_{z_\circ,a_\circ}(\mathcal{L}_0)$ acts by $|a_\circ|^2$. We conclude that $\pi_{z_\circ,a_\circ}(\mathcal{L}_V)=\sum_{\lambda\geq0}\pi_{z_\circ,a_\circ}(\mathcal{L}_\lambda)$ acts on $P_{z_\circ,a_\circ,\alpha}$ by the scalar

$$|a_\circ|^2 + \sum_{\lambda>0}\lambda(2\alpha(\lambda)+m(\lambda)).$$

Thus also $\widehat{\mathcal{L}}_V(\phi_{z_\circ,a_\circ,\alpha}) = |a_\circ|^2+\sum_{\lambda>0}\lambda(2\alpha(\lambda)+m(\lambda))$, in view of Lemma 6.1. ∎

6.1. Symmetrization. Here we explain how operators $D\in\mathbb{D}_K(N)$ may be obtained via symmetrization from K-invariant polynomials $p\in\mathbb{R}[\mathfrak{n}]^K$.
Letting $S(\mathfrak{n})$ denote the (complex) symmetric algebra on $\mathfrak{n}$ the *symmetrization map*

$$Sym\colon S(\mathfrak{n})\to\mathbb{D}(N)$$

is the unique vector space isomorphism satisfying $Sym(X^n)=X^n$ for all $X\in\mathfrak{n}$ [20, Chapter II, Theorem 4.3]. The algebra $S(\mathfrak{n})$ is canonically isomorphic to $\mathbb{C}[\mathfrak{n}^*]$ and by using $(\cdot,\cdot)_{\mathfrak{n}}$ to identify $\mathfrak{n}^*\cong\mathfrak{n}$ we regard symmetrization as a map $\mathbb{C}[\mathfrak{n}]\to\mathbb{D}(N)$. Letting $\{U_1,\ldots,U_{d_V}\}$ and $\{Z_1,\ldots,Z_{d_{\mathfrak{z}}}\}$ be orthonormal bases for V and $\mathfrak{z}$ one has for $f\in C^\infty(N)$, $p\in\mathbb{C}[\mathfrak{n}]$, $(v,z)\in N$ the formula

$$Sym(p)f(v,z) = p\left(\nabla_t,\nabla_s\right)\big|_{t=0=s} f\big((v,z)\exp(t_1U_1+\cdots+t_{d_V}U_{d_V}+s_1Z_1+\cdots+s_{d_{\mathfrak{z}}}Z_{d_{\mathfrak{z}}})\big)$$

where $\nabla_t = \big(\partial_{t_1},\ldots,\partial_{t_{d_V}}\big)$, $\nabla_s=\big(\partial_{s_1},\ldots,\partial_{s_{d_{\mathfrak{z}}}}\big)$. Following [14, Section 2.2] the *modified* symmetrization map

$$Sym'\colon\mathbb{C}[\mathfrak{n}]\to\mathbb{D}(N)$$

is obtained by replacing (∇_t,∇_s) in this formula by $\big((1/i)\nabla_t,(1/i)\nabla_s\big)$. This variant of symmetrization has the virtue that for any polynomial $p\in\mathbb{R}[\mathfrak{n}]$ with *real* coefficients the operator $Sym'(p)\in\mathbb{D}(N)$ is formally self-adjoint.

Applying Sym' to a set of generators for the algebra $\mathbb{R}[\mathfrak{n}]^K$ of real K-invariant polynomials on $\mathfrak{n}$ yields a set of formally self-adjoint generators for the algebra $\mathbb{D}_K(N)$. For the irreducible n.G.p.'s in Table 1 explicit sets of generators for $\mathbb{R}[\mathfrak{n}]^K$ are given in [14, Theorem 7.5]. Each of these generating sets is *bi-homogeneous.* That is, each generator $p(v,z)$ has a fixed degree of homogeneity in the V- and $\mathfrak{z}$-variables.

Example 6.3. The polynomials

$$p_V(v,z) := |v|^2, \quad p_{\mathfrak{z}}(v,z) := |z|^2,$$

belong to $\mathbb{R}[\mathfrak{n}]^K$ and $Sym'(p_V) = \mathcal{L}_V$, $Sym'(p_{\mathfrak{z}}) = \mathcal{L}_{\mathfrak{z}}$, where $\mathcal{L}_V$, $\mathcal{L}_{\mathfrak{z}}$ are the sub- and central-Laplacian operators given in equation (7). ■

Symmetrization of polynomials $p \in \mathbb{R}[\mathfrak{z}]$ which depend only on central variables in $\mathfrak{n}$ produces operators in the center of $\mathbb{D}(N)$. The calculation given for the eigenvalues of $\mathcal{L}_{\mathfrak{z}}$ on spherical functions generalizes easily as follows.

Lemma 6.4. *For $p \in \mathbb{R}[\mathfrak{z}]^K$ the operator $(D_p := Sym'(p)) \in \mathbb{D}_K(N)$ has eigenvalues on spherical functions $\phi_{z_\circ,a_\circ,\alpha}$ $((z_\circ \neq 0) \in \mathfrak{z}, a_\circ \in \mathfrak{a}_{z_\circ}, \alpha \in \Lambda_{z_\circ,a_\circ})$ and ϕ_w $(w \in V)$ given by*

$$\widehat{D}_p(\phi_{z_\circ,a_\circ,\alpha}) = p(z_\circ), \qquad \widehat{D}_p(\phi_w) = 0.$$

Recalling the results $\Psi(\phi_{z_\circ,a_\circ,\alpha}) = K\cdot(a_\circ, w'_\alpha, z_\circ)$, and $\Psi(\phi_w) = K\cdot w$ together with Lemma 6.4 and using a K-invariant polynomial p on $\mathfrak{z}$, we obtain the following:

Corollary 6.5. *Letting $D_p = Sym'(p)$ for $p \in \mathbb{R}[\mathfrak{z}]^K$ one has $\widehat{D}_p(\phi) = p\big(\Psi(\phi)\big)$ for all $\phi \in \Delta(K,N)$.*

A closely related result is the following. We will make use of both Corollary 6.5 and Lemma 6.6 in our subsequent study of the entries in Table 1.

Lemma 6.6. *Let $D_p = Sym'(p)$ where $p \in \mathbb{R}[\mathfrak{n}]^K$ has $p(v,z)$ homogeneous of degree one in the V-variables. Then $\widehat{D}_p(\phi) = p\big(\Psi(\phi)\big)$ for all $\phi \in \Delta(K,N)$.*

Proof. Let $\phi = \phi_{z_\circ,a_\circ,\alpha}$ have spherical function parameters $(z_\circ, a_\circ, \alpha)$ with $z_\circ \neq 0$. Form the decomposition $V = \mathfrak{a}_{z_\circ} \oplus \mathfrak{w}_{z_\circ}$. For $X \in \mathfrak{a}_\circ \oplus \mathfrak{z}$, we see that $\pi_{z_\circ,a_\circ}(X)$ is the scalar $i(X, a_\circ + z_\circ)_{\mathfrak{n}}$, and hence for any polynomial q on $\mathfrak{a}_\circ \oplus \mathfrak{z}$,

$$\pi_{z_\circ,a_\circ}(D_q) = q(a_\circ, z_\circ).$$

Let $p_\circ \in \mathbb{R}[\mathfrak{w}_{z_\circ}]$ be the polynomial $p_\circ(w) := p(a_\circ, w, z_\circ)$. Then

$$\pi_{z_\circ,a_\circ}(D_p) = \pi_{z_\circ,a_\circ}(D_{p_\circ}).$$

Also, $p_\circ$ has degree one and is $K_{z_\circ,a_\circ}$-invariant. As $K_{z_\circ,a_\circ} : \widetilde{\mathfrak{w}}_{z_\circ}$ is a multiplicity free action, all invariants have even degree and thus $p_\circ$ is a constant polynomial. Thus $\pi_{z_\circ,a_\circ}(D_p)$ and $\widehat{D}_p(\phi)$ are equal to that constant value.

Now Proposition 5.3 shows that the spherical orbit $\Psi(\phi) \in \mathcal{A}(K,N)$ contains a point of the form $(a_\circ, w'_\alpha, z_\circ)$ with $w'_\alpha \in \mathfrak{w}_{z_\circ}$. Since $p(a_\circ, w, z_\circ)$ is independent of $w \in \mathfrak{w}_{z_\circ}$, we have

$$\widehat{D}_p(\phi) = p(a_\circ, w'_\alpha, z_\circ) = p(\Psi(\phi)).$$ ■

7. Proof strategy overview

To verify (O) for pairs (K,N) from Table 1 we will proceed as follows. The spaces $\Delta(K,N)$ and $\mathfrak{n}^*/K$ are metrizable and second countable. So convergence behavior of sequences determines the topologies on these spaces and sequence limits are unique.

Let $(\phi_n)_{n=1}^{\infty}$ be a sequence in $\Delta(K,N)$, $\phi \in \Delta(K,N)$ and set $\mathbf{O}_n := \Psi(\phi_n)$, $\mathbf{O} := \Psi(\phi)$. We need to show that $\phi_n \to \phi$ in $\Delta(K,N)$ if and only if $\mathbf{O}_n \to \mathbf{O}$ in $\mathfrak{n}^*/K$.

Let $\mathcal{G}$ be a finite set of generators for the algebra $\mathbb{R}[\mathfrak{n}]^K$ of real-valued K-invariant polynomials on $\mathfrak{n}$. Such generating sets are given for each example from Table 1 in an appendix to [14]. We will make use of these. One can identify $\mathfrak{n} \cong \mathfrak{n}^*$ by using the (fixed) K-invariant inner product on $\mathfrak{n}$ to regard each $g \in \mathcal{G}$ as a K-invariant polynomial on $\mathfrak{n}^*$. As the invariant polynomials for a compact action on a real vector space separate orbits [24, Chapter 3 §4.3] we have $\mathbf{O}_n \to \mathbf{O}$ in $\mathfrak{n}^*/K$ if and only if $g(\mathbf{O}_n) \to g(\mathbf{O})$ for each $g \in \mathcal{G}$. On the other hand, applying the modified symmetrization map yields a set $\{D_g = Sym'(g) \;:\; g \in \mathcal{G}\}$ of essentially self-adjoint generators for the algebra $\mathbb{D}_K(N)$. A fundamental result of Ferrari Ruffino shows that the map

$$\Delta(K,N) \to \mathbb{R}^{|\mathcal{G}|}, \quad \phi \mapsto \left(\widehat{D}_g(\phi)\right)_{g\in\mathcal{G}}$$

embeds $\Delta(K,N)$ into $\mathbb{R}^{|\mathcal{G}|}$ [12]. Thus $\phi_n \to \phi$ in $\Delta(K,N)$ if and only if we have $\widehat{D}_g(\phi_n) \to \widehat{D}_g(\phi)$ for each $g \in \mathcal{G}$. So to establish (O) it suffices to show that

$$\left(\widehat{D}_g(\phi_n) \to \widehat{D}_g(\phi) \text{ for all } g \in \mathcal{G}\right) \iff \left(g(\mathbf{O}_n) \to g(\mathbf{O}) \text{ for all } g \in \mathcal{G}\right).$$

Let ϕ and ϕ_n be given by data $(z_\circ, a_\circ, \alpha)$ and (z_n, a_n, α_n), as in Proposition 5.1. To establish (O) we will show that the conditions

- "$\widehat{D}_g(\phi_n) \to \widehat{D}_g(\phi)$ for all $g \in \mathcal{G}$" and
- "$g(\mathbf{O}_n) \to g(\mathbf{O})$ for all $g \in \mathcal{G}$"

each force a common set of conditions on the behavior of the sequence $((z_n, a_n, \alpha_n))_{n=1}^{\infty}$. In each example the generating set $\mathcal{G}$ includes a set of generators for $\mathbb{R}[\mathfrak{z}]^K$. For such central generators g one has $\widehat{D}_g(\phi_n) = g(\mathbf{O}_n)$ and $\widehat{D}_g(\phi) = g(\mathbf{O})$ (Corollary 6.5). The same holds for generators g that are homogeneous of degree one in the V-variables (Lemma 6.6). Let $\mathcal{C} \subset \mathcal{G}$ be the subset consisting of central generators together with any generators homogeneous of degree one in the V-variables. To prove (O) one need only show that

$$\left(\widehat{D}_g(\phi_n) \to \widehat{D}_g(\phi) \text{ for all } g \in \mathcal{G} - \mathcal{C}\right) \iff \left(g(\mathbf{O}_n) \to g(\mathbf{O}) \text{ for all } g \in \mathcal{G} - \mathcal{C}\right)$$

under the hypothesis that $g(\mathbf{O}_n) \to g(\mathbf{O})$ for all $g \in \mathcal{C}$.

In the examples treated in this paper the data (z_n, a_n, α_n) is used to produce formulas for $\widehat{D}_g(\phi_n)$ and $g(\mathbf{O}_n)$. The eigenvalues for $J_{z_n}^2$ intervene in these formulas. Using the hypothesis that $g(\mathbf{O}_n) \to g(\mathbf{O})$ for all $g \in \mathcal{C}$ we argue, after passing to a suitable subsequence, that the eigenvalues for $J_{z_n}^2$, together with their multiplicities, converge to those for $J_{z_\circ}^2$. This is a key technical step in showing, for each $g \in \mathcal{G} - \mathcal{C}$, that $\widehat{D}_g(\phi_n) \to \widehat{D}_g(\phi)$ if and only if $g(\mathbf{O}_n) \to g(\mathbf{O})$.

8. The pair $(SO(d), \mathbb{R}^d \oplus \Lambda^2(\mathbb{R}^d))$ with $d \geq 3$

This section concerns the first entry in Table 1. Here $\mathfrak{n} = \mathfrak{n}_d = V \oplus \mathfrak{z}$ where $V = \mathbb{R}^d$ (regarded as column vectors) and $\mathfrak{z} = \Lambda^2(\mathbb{R}^d)$ is to be identified with $so(d)$, the space of $d \times d$ skew-symmetric matrices.

The Lie bracket is given by $V \times V \to \mathfrak{z}$, $[u, v] := uv^t - vu^t$.

The group $K = SO(d)$ acts on $\mathfrak{n}$ via $k \cdot (v, A) = (kv, kAk^t)$.

The group $N_d := \exp(\mathfrak{n})$ is the free 2-step group on d generators. Spherical functions and aspects of analysis with the n.G.p.'s $(O(d), N_d)$ and $(SO(d), N_d)$ are discussed in [13] and [27]. We proved in [4] that $(O(d), N_p)$ satisfies (O). We also asserted that $(SO(d), N_d)$ satisfies (O) but gave no proof details. This will be done here. In fact Corollary 1.5 shows that if $(SO(d), N_d)$ satisfies (O) then so does $(O(d), N_d)$.

8.1. K-orbits in $\mathfrak{z}$. The action of $K = SO(d)$ on $\mathfrak{z} = so(d)$ is the adjoint action of K on its Lie algebra. As is well known each element of $\mathfrak{z}$ is conjugate via K to an element in the *standard form* given by (10) below. Let

$$\mathcal{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{J}_m = diag\big(\underbrace{\mathcal{J}, \ldots, \mathcal{J}}_{m \text{ times}}\big), \quad \mathcal{J}_m^- = diag\big(-\mathcal{J}, \underbrace{\mathcal{J}, \ldots, \mathcal{J}}_{m-1 \text{ times}}\big). \tag{9}$$

Each K-orbit contains a unique point of the form

$$B_\circ = diag(O_{m(0)}, \lambda_1 \mathcal{J}^{\pm}_{m(1)}, \ldots, \lambda_p \mathcal{J}_{m(p)}) \tag{10}$$

for some $p \geq 0$, integers $m(0) \geq 0$, $m(1), \ldots, m(p) \geq 1$ with

$$m(0) + 2m(1) + \cdots + 2m(p) = d$$

and some distinct positive real numbers $0 < \lambda_1 < \cdots < \lambda_p$. If $m(0) \geq 1$ then one can take $B_\circ = diag(O_{m(0)}, \lambda_1 \mathcal{J}_{m(1)}, \ldots, \lambda_p \mathcal{J}_{m(p)})$.

8.2. The space $\Delta(K, N)$. We equip $\mathfrak{n}$ with the (positive definite) K-invariant inner product

$$\Big((u, A), (v, B)\Big)_{\mathfrak{n}} := u^t v + \frac{1}{2} tr(A^t B) = u^t v - \frac{1}{2} tr(AB),$$

with respect to which V and $\mathfrak{z}$ are orthogonal. An easy calculation shows that for $u, v \in V$ and $B \in \mathfrak{z}$ one has

$$([u, v], B)_{\mathfrak{n}} = -(Bu, v)_{\mathfrak{n}}.$$

Thus the map $J_B : V \to V$ is simply

$$J_B(u) = -Bu. \tag{11}$$

So for given $B_\circ \in \mathfrak{z}$ eigenspace decomposition (6) with respect to $J^2_{B_\circ}$ here reads

$$V = \mathfrak{a}_{B_\circ} \oplus \mathfrak{w}_{B_\circ} = V_0 \oplus \bigoplus_{\lambda \in \sigma^+(B_\circ)} V_\lambda \tag{12}$$

where $\sigma^+(B_\circ) = \{\lambda > 0 : -\lambda^2 \text{ is an eigenvalue for } J^2_{B_\circ} = B^2_\circ\}$, $V_0 = \mathfrak{a}_{B_\circ} = Ker(B_\circ)$, $\mathfrak{w}_{B_\circ} = Image(B_\circ)$, and $J^2_{B_\circ} = B^2_\circ = -\lambda^2$ on V_λ. The stabilizer of $B_\circ$ in K is

$$K_{B_\circ} = SO(V_0) \times \prod_{\lambda \in \sigma^+(B_\circ)} U(V_\lambda)$$

and for $a_\circ \in V_0$ the stabilizer of $(a_\circ, B_\circ)$ is

$$K_{B_\circ,a_\circ} = SO(V_0 \cap a_\circ^\perp) \times \prod_{\lambda\in\sigma^+(B_\circ)} U(V_\lambda).$$

The space $\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ}]$ decomposes under the action of $K_{B_\circ,a_\circ}$ as

$$\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ}] = \bigoplus_\alpha \left(\bigotimes_{\lambda\in\sigma^+(B_\circ)} \mathcal{P}_{\alpha(\lambda)}(V_\lambda)\right)$$

where the parameter $\alpha = (\alpha(\lambda) \ : \ \lambda \in \sigma^+(B_\circ))$ is a list of non-negative integers, one for each $\lambda \in \sigma^+(B_\circ)$. According to Proposition 5.1 each bounded K-spherical function on N has the form $\phi_{B_\circ,a_\circ,\alpha}$ for some choice of data $(B_\circ, a_\circ, \alpha)$ as above.[1] Moreover two such spherical functions coincide if and only if the data differ by the action of K. So one may, in particular, assume that $B_\circ$ is of the form (10), in which case $\sigma^+(B_\circ) = \{\lambda_1, \dots, \lambda_p\}$, $K_{B_\circ} = SO(m(0)) \times U(m(1)) \times \cdots \times U(m(p))$ and $\widetilde{\mathfrak{w}}_{B_\circ} = \mathbb{C}^{m(1)} \oplus \cdots \oplus \mathbb{C}^{m(p)}$. An explicit parameterization for $\Delta(K,N)$ is given in [13], but the description given above will suffice for our purposes here.

8.3. The space $\mathcal{A}(K,N)$. The spherical orbit $\mathbf{O}(B_\circ, a_\circ, \alpha) := \Psi(\phi_{B_\circ,,a_\circ,\alpha})$ in $\mathfrak{n}^*/K$ corresponding to $\phi_{B_\circ,,a_\circ,\alpha} \in \Delta(K,N)$ is obtained using Proposition 5.3. Choosing a unit vector e_λ in each V_λ one has

$$\mathbf{O}(B_\circ, a_\circ, \alpha) = K \cdot \left(a_\circ + \sum_{\lambda\in\sigma^+(B_\circ)} (2\lambda\alpha(\lambda))^{1/2} e_\lambda, B_\circ \right). \tag{13}$$

8.4. Generators for $\mathbb{R}[\mathfrak{n}]^K$. Observe that for each integer $\ell \geq 0$ the following bi-homogeneous polynomials on $\mathfrak{n}$,

$$p_\ell(v, A) := v^t A^{2\ell} v, \quad q_\ell(v, A) := tr(A^{2\ell}),$$

are K-invariant. Likewise the Pfaffian polynomial $r(v, A)$ defined via

$$r(v,A) := \left\{ \begin{array}{cc} Pf(A) & \text{when } d \text{ is even} \\ Pf(A|v) := Pf\left[\begin{array}{c|c} A & v \\ \hline -v^t & 0 \end{array}\right] & \text{when } d \text{ is odd} \end{array} \right\}$$

belongs to $\mathbb{R}[\mathfrak{n}]^K$. Letting $d' := \lfloor d/2 \rfloor$ the set

$$\mathcal{G} := \{p_0, \dots, p_{d'-1},\ q_1, \dots, q_{d'},\ r\}$$

generates $\mathbb{R}[\mathfrak{n}]^K$ [14, Theorem 7.5]. (In fact $\{p_0, \dots, p_{d'-1},\ q_1, \dots, q_{d'-1},\ r\}$ suffices when $d = 2d'$ is even.) Note that $p_0(v, A) = |v|^2 = (v, v)_\mathfrak{n}$ and $q_1(v, A) = -2|A|^2 = -2(A, A)_\mathfrak{n}$.

8.5. Values $g(B_\circ, a_\circ, \alpha)$ and $\widehat{g}(B_\circ, a_\circ, \alpha)$ for $g \in \mathcal{G}$. The generators $g \in \mathcal{G}$ take the following values $g(B_\circ, a_\circ, \alpha) := g(\mathbf{O}(B_\circ, a_\circ, \alpha))$ on the spherical orbit $\mathbf{O}(B_\circ, a_\circ, \alpha)$ given in equation (13).

[1] For $B_\circ = 0$ we have $\sigma^+(B_\circ) = \emptyset$ and the α-parameter is empty. In this case $\phi_{0,a_\circ,-}$ is a spherical function of type II. See Remark 5.1.

Taking $B_\circ$ in standard form (9) one obtains

$$\left\{ \begin{array}{c} p_0(B_\circ, a_\circ, \alpha) = |a_\circ|^2 + \sum_{\lambda\in\sigma^+(B_\circ)} 2\lambda\alpha(\lambda), \quad \text{and} \\ p_\ell(B_\circ, a_\circ, \alpha) = (-1)^\ell \sum_{\lambda\in\sigma^+(B_\circ)} 2\lambda^{2\ell+1}\alpha(\lambda) \quad \text{for } \ell \geq 1 \end{array} \right\}, \tag{14}$$

$$\left\{ \begin{array}{c} q_\ell(B_\circ, a_\circ, \alpha) = tr(B_\circ^{2\ell}) \quad \text{and} \\ r(B_\circ, a_\circ, \alpha) = \left\{ \begin{array}{ll} Pf(B_\circ) & \text{when } d \text{ is even} \\ Pf(B_\circ|a_\circ) & \text{when } d \text{ is odd} \end{array} \right\} \end{array} \right\}. \tag{15}$$

Applying the modified symmetrization map to each generator $g \in \mathcal{G}$ yields a set of generators $\{D_g = Sym'(g) \ : \ g \in \mathcal{G}\}$ for the algebra $\mathbb{D}_K(N)$. Let $\widehat{g}(B_\circ, a_\circ, \alpha)$ denote the eigenvalue for D_g on a spherical function $\phi_{B_\circ,a_\circ,\alpha}$, i.e. $D_g(\phi_{B_\circ,a_\circ,\alpha}) = \widehat{g}(B_\circ, a_\circ, \alpha)\phi_{B_\circ,a_\circ,\alpha}$, and write

$$m(\lambda) := \dim(V_\lambda)/2 \text{ for } \lambda \in \sigma^+(B_\circ).$$

(For $B_\circ$ in standard form (10) one has $m(\lambda_j) = m(j)$.) We have

$$\left\{ \begin{array}{c} \widehat{p}_0(B_\circ, a_\circ, \alpha) = |a_\circ|^2 + \sum_{\lambda\in\sigma^+(B_\circ)} \lambda(2\alpha(\lambda) + m(\lambda)) \quad \text{and} \\ \widehat{p}_\ell(B_\circ, a_\circ, \alpha) = (-1)^\ell \sum_{\lambda\in\sigma^+(B_\circ)} \lambda^{2\ell+1}(2\alpha(\lambda) + m(\lambda)) \quad \text{for } \ell \geq 1 \end{array} \right\}, \tag{16}$$

$$\left\{ \begin{array}{c} \widehat{q}_\ell(B_\circ, a_\circ, \alpha) = q_\ell(B_\circ, a_\circ, \alpha) = tr(B_\circ^{2\ell}) \quad \text{and} \\ \widehat{r}(B_\circ, a_\circ, \alpha) = r(B_\circ, a_\circ, \alpha) \end{array} \right\}. \tag{17}$$

Indeed as the polynomials q_ℓ depend only on central variables we have $\widehat{q}_\ell(B_\circ, a_\circ, \alpha) = q_\ell(B_\circ, a_\circ, \alpha)$ by Corollary 6.5. Likewise polynomial r is central when d is even and homogeneous of degree one in the V-variables when d is odd. So $\widehat{r}_\ell(B_\circ, a_\circ, \alpha) = r_\ell(B_\circ, a_\circ, \alpha)$ by Corollary 6.5 and Lemma 6.6. The operator D_{p_0} is the sub-Laplacian $\mathcal{L}_V$ with eigenvalues given in Lemma 6.2. The value for $\widehat{p}_\ell(B_\circ, a_\circ, \alpha)$ with $\ell \geq 1$ is given (modulo sign conventions) in [4, Lemma 9.1]. The calculation parallels that given above in the proof for Lemma 6.2.

8.6. Condition (O) for (K, N)**.** We will establish condition (O) for the example at hand, following the proof strategy given in Section 7. This same proof will be adapted to encompass the subsequent examples treated in this paper. The proof draws on techniques from [4] while achieving some simplifications.

Proof. Let $(\phi_n = \phi_{B_n,a_n,\alpha_n})_{n=1}^\infty$ be a sequence in $\Delta(K, N)$ and $(\phi = \phi_{B_\circ,a_\circ,\alpha}) \in \Delta(K, N)$. We must show that $(\phi_n)_{n=1}^\infty$ converges to ϕ in $\Delta(K, N)$ if and only if the sequence $\big(\mathbf{O}_n := \mathbf{O}(B_n, a_n, \alpha_n)\big)_{n=1}^\infty$ converges to $\mathbf{O} := \mathbf{O}(B_\circ, a_\circ, \alpha)$ in $\mathcal{A}(K, N)$. As $\mathcal{G}$ generates $\mathbb{R}[\mathfrak{n}]^K$ we know, by [12], that $(\phi_n)_{n=1}^\infty$ converges to ϕ in $\Delta(K, N)$ if and only if $\widehat{g}(B_n, a_n, \alpha_n) \to \widehat{g}(B_\circ, a_\circ, \alpha)$ for each $g \in \mathcal{G}$. Likewise $(\mathbf{O}_n)_{n=1}^\infty$ converges to $\mathbf{O}$ in $\mathcal{A}(K, N)$ if and only if $g(B_n, a_n, \alpha_n) \to g(B_\circ, a_\circ, \alpha)$ for each $g \in \mathcal{G}$. This is the case as the invariants for a compact linear action on a finite dimensional real vector space separate orbits.

For the generators $g \in \{q_1, \ldots, q_{d'}, r\}$ we have $\widehat{g}(B_n, a_n, \alpha_n) = g(B_n, a_n, \alpha_n)$ and $\widehat{g}(B_\circ, a_\circ, \alpha) = g(B_\circ, a_\circ, \alpha)$. Suppose that

$$\lim_{n\to\infty} g(B_n, a_n, \alpha_n) = g(B_\circ, a_\circ, \alpha) \text{ for each } g \in \{q_1, \ldots, q_{d'}, r\}. \tag{18}$$

Under these hypotheses it now suffices to verify that, for $\ell = 0, \ldots, d'-1$,

$$\lim_{n\to\infty} \widehat{p}_\ell(B_n, a_n, \alpha_n) = \widehat{p}_\ell(B_\circ, a_\circ, \alpha) \iff \lim_{n\to\infty} p_\ell(B_n, a_n, \alpha_n) = p_\ell(B_\circ, a_\circ, \alpha). \quad (19)$$

The polynomials $\{q_1, \ldots, q_{d'}, r\}$ generate $\mathbb{R}[\mathfrak{z}]^K$ when d is even and $\{q_1, \ldots, q_{d'}\}$ generates $\mathbb{R}[\mathfrak{z}]^K$ when d is odd.

So for each n, the values $q_1(B_n, a_n, \alpha_n), \ldots, q_{d'}(B_n, a_n, \alpha_n)$, $r(B_n, a_n, \alpha_n)$ determine the K-orbit through B_n. As we may assume that B_n is of the standard form (9) it follows that the values $q_1(B_n, a_n, \alpha_n), \ldots, q_{d'}(B_n, a_n, \alpha_n)$, $r(B_n, a_n, \alpha_n)$ determine the eigenvalues for $J_{B_n}^2 = B_n^2$ together with their multiplicities. Thus hypothesis (18) implies that the eigenvalues for $J_{B_n}^2 = B_n^2$, together with their multiplicities, converge to those for $J_{B_\circ}^2 = B_\circ^2$.

Let $\sigma^+(B_n) = \{\mu_1(n), \ldots, \mu_{I(n)}(n)\}$ where $0 < \mu_1(n) < \mu_2(n) < \cdots < \mu_{I(n)}(n)$ and write the decomposition for V into eigenspaces for $J_{B_n}^2$ as

$$V = \mathcal{V}_0(n) \oplus \bigoplus_{\mu\in\sigma^+(B_n)} \mathcal{V}_\mu(n) = \bigoplus_{j=0}^{I(n)} \mathcal{V}_j(n),$$

where $\mathcal{V}_0(n) = Ker(J_{B_n}) = Ker(J_{B_n}^2) = Ker(B_n^2)$ and $J_{B_n}^2 = B_n^2 = -\mu_j(n)^2$ on $\mathcal{V}_j(n)$ for $j = 1, \ldots, I(n)$. We can partition a tail of the sequence $(\phi_n)_{n=1}^\infty$ into finitely many subsequences in which the values $I(n)$ and $\dim(\mathcal{V}_j(n))$ are constant in n. It suffices to verify (19) for each of these subsequences. Thus we suppose henceforth that

$$I = I(n), \quad m_j = m(\mu_j(n)) := \frac{1}{2}\dim(\mathcal{V}_j(n)) \ \ (j = 1, \ldots, I), \quad (20)$$

independent of n.

Let the eigenspace decomposition for V with respect to $J_{B_\circ}^2 = B_\circ^2$ be as in (12) and recall that $m(\lambda) := \dim(V_\lambda)/2$ for $\lambda \in \sigma^+(B_\circ)$ We have now the following facts.

- $\lim_{n\to\infty} \mu_j(n) \in \sigma^+(B_\circ) \cup \{0\}$ for $j = 1, \ldots, I$.
- If $\lambda \in \sigma^+(B_\circ)$ then $\lambda = \lim_{n\to\infty} \mu_j(n)$ for some $j \in \{1, \ldots, I\}$. We write $S_\lambda = \{j \in \{1, \ldots, I\} \ : \ \mu_j(n) \to \lambda\}$.
- For each $\lambda \in \sigma^+(B_\circ)$ one has $m(\lambda) = \sum_{j\in S_\lambda} m_j$.

Together these imply that

$$\lim_{n\to\infty} \sum_{\mu\in\sigma^+(B_n)} \mu^{2\ell+1} m(\mu) = \sum_{\lambda\in\sigma^+(B_\circ)} \lambda^{2\ell+1} m(\lambda) \quad \text{for all } \ell \geq 0. \quad (21)$$

Indeed

$$\lim_{n\to\infty} \sum_{\mu\in\sigma^+(B_n)} \mu^{2\ell+1} m(\mu) = \lim_{n\to\infty} \sum_{j=1}^{I} \mu_j(n)^{2\ell+1} m_j = \lim_{n\to\infty} \sum_{\lambda\in\sigma^+(B_\circ)} \sum_{j\in S_\lambda} \mu_j(n)^{2\ell+1} m_j$$
$$= \sum_{\lambda\in\sigma^+(B_\circ)} \lambda^{2\ell+1} \sum_{j\in S_\lambda} m_j = \sum_{\lambda\in\sigma^+(B_\circ)} \lambda^{2\ell+1} m(\lambda).$$

Recall that the parameters α and α_n for the spherical functions $\phi = \phi_{B_\circ, a_\circ, \alpha}$ and $\phi_n = \phi_{B_n, a_n, \alpha_n}$ are lists of non-negative integers $\alpha = (\alpha(\lambda) \ : \ \lambda \in \sigma^+(B_\circ))$ and $\alpha_n = (\alpha_n(\mu) \ : \ \mu \in \sigma^+(B_n))$. Using the first equation from (14) we have that

- $\lim_{n\to\infty} p_0(B_n, a_n, \alpha_n) = p_0(B_\circ, a_\circ, \alpha)$ if and only if

$$\lim_{n\to\infty}\left[|a_n|^2 + \sum_{\mu\in\sigma^+(B_n)} 2\mu\alpha_n(\mu)\right] = |a_\circ|^2 + \sum_{\lambda\in\sigma^+(B_\circ)} 2\lambda\alpha(\lambda). \tag{22}$$

On the other hand the first of equations (16) gives

- $\lim_{n\to\infty} \widehat{p}_0(B_n, a_n, \alpha_n) = \widehat{p}_0(B_\circ, a_\circ, \alpha)$ if and only if

$$\lim_{n\to\infty}\left[|a_n|^2 + \sum_{\mu\in\sigma^+(B_n)} \mu(2\alpha_n(\mu) + m(\mu))\right] = |a_\circ|^2 + \sum_{\lambda\in\sigma^+(B_\circ)} \lambda(2\alpha(\lambda) + m(\lambda)). \tag{23}$$

Applying (21) with $\ell = 0$ it is transparent that (22) holds if and only if (23) holds. Likewise for $\ell \geq 1$ the second of equations (14) shows

- $\lim_{n\to\infty} p_\ell(B_n, a_n, \alpha_n) = p_\ell(B_\circ, a_\circ, \alpha)$ if and only if

$$\lim_{n\to\infty}\left[\sum_{\mu\in\sigma^+(B_n)} 2\mu^{2\ell+1}\alpha_n(\mu)\right] = \sum_{\lambda\in\sigma^+(B_\circ)} 2\lambda^{2\ell+1}\alpha(\lambda), \tag{24}$$

whereas the second of the equations (16) yields

- $\lim_{n\to\infty} \widehat{p}_0(B_n, a_n, \alpha_n) = \widehat{p}_0(B_\circ, a_\circ, \alpha)$ if and only if

$$\lim_{n\to\infty}\left[\sum_{\mu\in\sigma^+(B_n)} \mu^{2\ell+1}(2\alpha_n(\mu) + m(\mu))\right] = \sum_{\lambda\in\sigma^+(B_\circ)} \lambda^{2\ell+1}(2\alpha(\lambda) + m(\lambda)). \tag{25}$$

Again (24) and (25) are equivalent in view of equation (21). This completes the proof that (K, N) satisfies (O). ■

9. The pair $(SU(d), \mathbb{C}^d \oplus \Lambda^2(\mathbb{C}^d))$ with $d \geq 2$

Here $\mathfrak{n} = \mathfrak{n}_d = V \oplus \mathfrak{z}$ where $V = \mathbb{C}^d$ and $\mathfrak{z} = \Lambda^2(\mathbb{C}^d)$ is to be identified with the space of $d \times d$ skew-symmetric matrices with complex entries. For d odd this is entry 4 in Table 1. For d even the example is a central reduction from table entry 2. Both V and $\mathfrak{z}$ are to be viewed as *real* vector spaces of (even) dimensions $2d$ and $d(d-1)$ respectively. But the usual complex structure on $V = \mathbb{C}^d$ will also also play a role. As in the previous example the Lie bracket is given by

$$V \times V \to \mathfrak{z}, \qquad [u, v] := uv^t - vu^t$$

and $K = SU(d)$ acts on $\mathfrak{n}$ via

$$k \cdot (v, A) = (kv, kAk^t).$$

Our treatment of this example closely parallels that given in Section 8.

9.1. K-orbits in $\mathfrak{z}$. With notation as in (9) each K-orbit in $\mathfrak{z}$ contains a point of the form

$$B_\circ = \gamma\, diag(O_{\ell(0)}, \lambda_1\mathcal{J}_{\ell(1)} \ldots, \lambda_p\mathcal{J}_{\ell(p)}) \tag{26}$$

for some $p \geq 0$, scalar $\gamma \in \mathbb{T}$, integers $\ell(0) \geq 0$, $\ell(1), \ldots, \ell(p) \geq 1$ with

$$\ell(0) + 2\ell(1) + \cdots + 2\ell(p) = d$$

and some distinct positive real numbers $0 < \lambda_1 < \cdots < \lambda_p$. The values p, $\ell(j)$ and λ_j are uniquely determined and γ is unique up to multiplication by a d'th root of unity. If $m(0) \geq 1$ then one can take $\gamma = 1$.

9.2. The space $\Delta(K, N)$. We equip $\mathfrak{n}$ with the K-invariant inner product

$$\Big((u,A),(v,B)\Big)_{\mathfrak{n}} := Re\,\langle u,v\rangle + \frac{1}{2}Re(tr(AB^*)) = Re\,\langle u,v\rangle - \frac{1}{2}Re(tr(A\,\overline{B})).$$

Here $\langle u, v\rangle := v^*u$ is the usual hermitian inner product on $V = \mathbb{C}^d$. For $u, v \in V$ and $B \in \mathfrak{z}$ one has

$$([u,v],B)_{\mathfrak{n}} = -(B\overline{u}, v)_{\mathfrak{n}}$$

and hence $J_B : V \to V$ is given by

$$J_B(u) = -B\overline{u}. \tag{27}$$

Note that J_B is $\mathbb{R}$-linear but $\mathbb{C}$-*conjugate* linear. From (27) one obtains

$$J_B^2(u) = B\overline{B}u = -BB^*u. \tag{28}$$

So for given $B_\circ \in \mathfrak{z}$ one has $J_{B_\circ}^2$-eigenspace decomposition

$$V = \mathfrak{a}_{B_\circ} \oplus \mathfrak{w}_{B_\circ} = V_0 \oplus \bigoplus_{\lambda\in\sigma^+(B_\circ)} V_\lambda \tag{29}$$

where $\sigma^+(B_\circ) := \{\lambda > 0 \mid -\lambda^2 \text{ is an eigenvalue for } B_\circ\overline{B}_\circ\}$, $V_0 = \mathfrak{a}_{B_\circ} = Ker(B_\circ^*)$, $\mathfrak{w}_{B_\circ} = Image(B_\circ)$ and $B_\circ\overline{B}_\circ = -\lambda^2$ on V_λ. The stabilizer of $B_\circ$ in K is

$$K_{B_\circ} = SU(V_0) \times \prod_{\lambda\in\sigma^+(B_\circ)} Sp(V_\lambda)$$

and for $a_\circ \in V_0$ the stabilizer of $(a_\circ, B_\circ)$ is

$$K_{B_\circ,a_\circ} = SU(V_0 \cap a_\circ^\perp) \times \prod_{\lambda\in\sigma^+(B_\circ)} Sp(V_\lambda).$$

Just as in the previous example $\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ}]$ decomposes under the action of $K_{B_\circ,a_\circ}$ as

$$\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ}] = \bigoplus_\alpha \Big(\bigotimes_{\lambda\in\sigma^+(B_\circ)} \mathcal{P}_{\alpha(\lambda)}(V_\lambda)\Big)$$

where the parameter $\alpha = (\alpha(\lambda)\ :\ \lambda \in \sigma^+(B_\circ))$ is a list of non-negative integers. Each bounded K-spherical function on N has the form $\phi_{B_\circ,a_\circ,\alpha}$ for some choice of data $(B_\circ, a_\circ, \alpha)$ as above. (See Proposition 5.1 and Remark 5.1). Two such spherical functions coincide if and only if the data differ by the action of K. So one can assume that $B_\circ$ is of the form (26).

In this case $\sigma^+(B_\circ) = \{\lambda_1, \ldots, \lambda_p\}$, $K_{B_\circ} = SU(\ell(0)) \times Sp(\ell(1)) \times \cdots \times Sp(\ell(p))$, where $Sp(\ell) = \{k \in U(2\ell)\ :\ k\mathcal{J}_\ell k^t = \mathcal{J}_\ell\}$ and $\widetilde{\mathfrak{w}}_{B_\circ} = \mathbb{C}^{2\ell(1)} \oplus \cdots \oplus \mathbb{C}^{2\ell(p)}$.

9.3. The space $\mathcal{A}(K,N)$. Proposition 5.3 yields the spherical orbit $\mathbf{O}(B_\circ, a_\circ, \alpha) := \Psi(\phi_{B_\circ,,a_\circ,\alpha})$ corresponding to $\phi_{B_\circ,,a_\circ,\alpha} \in \Delta(K,N)$. As in the previous example

$$\mathbf{O}(B_\circ, a_\circ, \alpha) = K \cdot \left(a_\circ + \sum_{\lambda>0} (2\lambda\alpha(\lambda))^{1/2} e_\lambda, B_\circ \right) \tag{30}$$

where $e_\lambda \in V_\lambda$ is any chosen unit vector.

9.4. Generators for $\mathbb{R}[\mathfrak{n}]^K$. For each integer $\ell \geq 0$ the following bi-homogeneous polynomials on $\mathfrak{n}$,

$$p_\ell(v, A) := v^t (A\overline{A})^\ell v, \quad q_\ell(v, A) := tr((A\overline{A})^\ell),$$

are real valued and K-invariant. Likewise the polynomials $r(v, A)$, $s(v, A)$ defined as the real and imaginary parts of $Pf(A)$ when d is even and as the real and imaginary parts of $Pf(A|v)$ when d is odd belong to $\mathbb{R}[\mathfrak{n}]^K$. Letting $d' := \lfloor d/2 \rfloor$ the set

$$\mathcal{G} := \{p_0, \ldots, p_{d'-1},\, q_1, \ldots, q_{d'},\, r,\, s\}$$

generates $\mathbb{R}[\mathfrak{n}]^K$ [14, Theorem 7.5]. (In fact $\{p_0, \ldots, p_{d'-1},\, q_1, \ldots, q_{d'-1},\, r,\, s\}$ suffices when $d = 2d'$ is even.)

9.5. Values $g(B_\circ, a_\circ, \alpha)$ and $\widehat{g}(B_\circ, a_\circ, \alpha)$ for $g \in \mathcal{G}$. With notation as in Section 8.5 the values $g(B_\circ, a_\circ, \alpha) := g(\mathbf{O}(B_\circ, a_\circ, \alpha))$ and eigenvalues $\widehat{g}(B_\circ, a_\circ, \alpha)$ for each generator $g \in \mathcal{G}$ are as follows.

- The values $p_\ell(B_\circ, a_\circ, \alpha)$ and $\widehat{p_\ell}(B_\circ, a_\circ, \alpha)$ are given in equations (14) and (16). (The values $m(\lambda)$ in equation (16) are $m(\lambda_j) = 2\ell(j)$ for $B_\circ$ of the form (26).)
- $\widehat{q_\ell}(B_\circ, a_\circ, \alpha)) = q_\ell(B_\circ, a_\circ, \alpha)) = tr((B_\circ\overline{B}_\circ)^\ell)$,
- $\widehat{r}(B_\circ, a_\circ, \alpha) = r(B_\circ, a_\circ, \alpha) = \left\{ \begin{array}{ll} Re(Pf(B_\circ)) & \text{for } d \text{ even} \\ Re(Pf(B_\circ|a_\circ)) & \text{for } d \text{ odd} \end{array} \right\}$,
- $\widehat{s}(B_\circ, a_\circ, \alpha) = s(B_\circ, a_\circ, \alpha) = \left\{ \begin{array}{ll} Im(Pf(B_\circ)) & \text{for } d \text{ even} \\ Im(Pf(B_\circ|a_\circ)) & \text{for } d \text{ odd} \end{array} \right\}$.

9.6. Condition (O) for (K,N). The values of the central invariants $q_1, \ldots, q_{d'} \in \mathbb{R}[\mathfrak{z}]^K$, together with those for $r, s \in \mathbb{R}[\mathfrak{z}]^K$ when d is even, determine the $SU(d)$-orbit through any given $B \in \mathfrak{z}$ and hence, in view of the standard form (26), determine the eigenvalues for $J_B^2 = B\overline{B}$ together with their multiplicites. So the proof that $(SO(d), \mathbb{R}^d \oplus \Lambda^2(\mathbb{R}^d))$ satisfies (O), given in Section 8.6, goes through essentially verbatim in the current context. ∎

10. The pair $((S)U(d), \mathbb{C}^d \oplus (\Lambda^2(\mathbb{C}^d) \oplus \mathbb{R}))$ with $d \geq 2$

Next consider entries 2 and 3 in Table 1. Here $\mathfrak{n} = \mathfrak{n}_d = V \oplus \mathfrak{z}$ where $V = \mathbb{C}^d$ and $\mathfrak{z} = \Lambda^2(\mathbb{C}^d) \oplus \mathbb{R}$. The Lie bracket is given by

$$V \times V \to \mathfrak{z}, \qquad [u, v] := (uv^t - vu^t, -Im\,\langle u, v\rangle).$$

The group $U(d)$ acts as before on $\mathbb{C}^d \oplus \Lambda^2(\mathbb{C}^d)$ and acts trivially on $\mathbb{R}$. The notation "$K = (S)U(d)$" indicates $SU(d)$ or $U(d)$. When d is even we can take $K = SU(d)$ but for d odd require the full unitary group, $K = U(d)$, in order to have a Gelfand pair.

10.1. The space $\Delta(K, N)$. The K-invariant inner product from Section 9 is extended in the obvious way.

$$\Big((u, A, t), (v, B, s)\Big)_{\mathfrak{n}} := Re\,\langle u, v\rangle + \frac{1}{2}Re(tr(AB^*)) + ts = Re\,\langle u, v\rangle - \frac{1}{2}Re(tr(A\overline{B})) + ts.$$

For $u, v \in V$ and $(B, t) \in \mathfrak{z}$ one has $([u, v], (B, t))_{\mathfrak{n}} = (-B\overline{u} + itu, v)_{\mathfrak{n}}$ and hence $J_{B,t}\colon V \to V$ satisfies

$$J_{B,t}(u) = -B\overline{u} + itu, \qquad J^2_{B,t}(u) = B\overline{B}u - t^2u. \tag{31}$$

Just as in the previous example we have, for given $B_\circ \in \Lambda^2(\mathbb{C}^d)$, decomposition

$$V = \mathfrak{a}_{B_\circ} \oplus \mathfrak{w}_{B_\circ} = V_0 \oplus \bigoplus_{\lambda \in \sigma^+(B_\circ)} V_\lambda. \tag{32}$$

into eigenspaces for $J^2_{B_\circ,0} = J^2_{B_\circ} = B_\circ\overline{B}_\circ$. Moreover for any $a_\circ \in V_0$ the stabilizer of $(a_\circ, B_\circ, 0)$ in K is

$$K_{(B_\circ,0),a_\circ} = (S)U(V_0 \cap a_\circ^\perp) \times \textstyle\prod_{\lambda\in\sigma^+(B_\circ)} Sp(V_\lambda)$$

and the space $\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ}]$ decomposes under $K_{(B_\circ,0),a_\circ}$ as

$$\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ}] = \textstyle\bigoplus_\alpha \Big(\bigotimes_{\lambda\in\sigma^+(B_\circ)} \mathcal{P}_{\alpha(\lambda)}(V_\lambda)\Big)$$

where $\alpha = (\alpha(\lambda) \;:\; \lambda \in \sigma^+(B_\circ))$ is a list of non-negative integers, one for each $\lambda \in \sigma^+(B_\circ)$. On the other hand (31) shows that for $t_\circ \in \mathbb{R}^\times$ the eigenspace decomposition for V under $J^2_{B_\circ,t_\circ}$ coincides with (32) but now

- the operator $J^2_{B_\circ,t_\circ}$ acts on $\mathfrak{a}_{B_\circ} = V_0$ by $-t_\circ^2$ and acts on V_λ by $-(\lambda^2 + t_\circ^2)$ for each $\lambda \in \sigma^+(B_\circ)$.

That is the eigenvalues are shifted and $J^2_{B_\circ,t_\circ}$ has trivial kernel. Let

$$\sigma(B_\circ) := \{\lambda \geq 0 \;:\; -\lambda^2 \text{ is an eigenvalue for } J^2_{B_\circ} = B_\circ\overline{B}_\circ\}$$

so that $\sigma^+(B_\circ) = \{\lambda \in \sigma(B_\circ) \;:\; \lambda > 0\}$. One has $\mathfrak{w}_{B_\circ,t_\circ} = V$ and $\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ,t_\circ}]$ decomposes under $K_{(B_\circ,t_\circ),0} = (S)U(V_0) \times \prod_{\lambda\in\sigma^+(B_\circ)} Sp(V_\lambda)$ as

$$\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ,t_\circ}] = \bigoplus_\alpha \Bigg(\bigotimes_{\lambda\in\sigma(B_\circ)} \mathcal{P}_{\alpha(\lambda)}(V_\lambda)\Bigg)$$

where $\mathcal{P}_{\alpha(\lambda)}(V_\lambda)$ is the space of polynomials of degree $\alpha(\lambda)$ on V_λ, and therefore $\alpha = (\alpha(\lambda) \;:\; \lambda \in \sigma(B_\circ))$ is a list of non-negative integers, one for each $\lambda \in \sigma(B_\circ)$.

We see that spherical functions $\phi \in \Delta(K, N)$ are determined by data "$(B_\circ, t_\circ, a_\circ, \alpha)$" where at most one of $t_\circ$, $a_\circ$ can be non-zero. More precisely let

$$\Gamma^0 := \big\{(B_\circ, a_\circ, \alpha) \;:\; B_\circ \in \Lambda^2(\mathbb{C}^d),\ a_\circ \in Ker(J^2_{B_\circ}),\ \alpha \in (\mathbb{Z}^+)^{|\sigma^+(B_\circ)|}\big\},$$
$$\Gamma^\times := \big\{(B_\circ, t_\circ, \alpha) \;:\; B_\circ \in \Lambda^2(\mathbb{C}^d),\ t_\circ \in \mathbb{R}^\times,\ \alpha \in (\mathbb{Z}^+)^{|\sigma(B_\circ)|}\big\}.$$

Then $\Delta(K,N)=\Delta^0\cup\Delta^\times$ where

$$\Delta^0 := \{\phi_{B_\circ,a_\circ,\alpha} \ : \ (B_\circ,a_\circ,\alpha)\in\Gamma^0\}, \quad \Delta^\times := \{\phi_{B_\circ,t_\circ,\alpha} \ : \ (B_\circ,t_\circ,\alpha)\in\Gamma^\times\}.$$

10.2. The space $\mathcal{A}(K,N)$. The spherical orbits $\mathbf{O}(B_\circ,a_\circ,\alpha):=\Psi(\phi_{B_\circ,,a_\circ,\alpha})$ and $\mathbf{O}(B_\circ,t_\circ,\alpha):=\Psi(\phi_{B_\circ,,t_\circ,\alpha})$ for data $(B_\circ,a_\circ,\alpha)\in\Gamma^0$, $(B_\circ,t_\circ,\alpha)\in\Gamma^\times$ are as follows. Choose for each $\lambda\in\sigma(B_\circ)$ a unit vector $e_\lambda\in V_\lambda$. One has

$$\mathbf{O}(B_\circ,a_\circ,\alpha)=K\cdot\left(a_\circ+\sum_{\lambda\in\sigma^+(B_\circ)}(2\lambda\alpha(\lambda))^{1/2}e_\lambda,B_\circ,0\right). \tag{33}$$

$$\mathbf{O}(B_\circ,t_\circ,\alpha)=K\cdot\left(\sum_{\lambda\in\sigma(B_\circ)}(2(\lambda^2+t_\circ^2)^{1/2}\alpha(\lambda))^{1/2}e_\lambda,B_\circ,t_\circ\right). \tag{34}$$

10.3. Generators for $\mathbb{R}[\mathfrak{n}]^K$. As in Section 9 we consider the polynomials

$$p_\ell(v,A,t):=v^t(A\overline{A})^\ell v, \quad q_\ell(v,A,t):=tr((A\overline{A})^\ell),$$

together with $r(v,A,t):=Re(Pf(A))$ and $s(v,A,t):=Im(Pf(A))$ in the case that d is even. Letting $\tau(v,A,t):=t$ we have that

$$\mathcal{G}:=\left\{\begin{array}{cl}\{p_0,\dots,p_{d'-1},\ q_1,\dots,q_{d'-1},\ r,\ s,\ \tau\} & \text{for } d \text{ even}\\ \{p_0,\dots,p_{d'-1},\ q_1,\dots,q_{d'},\tau\} & \text{for } d \text{ odd}\end{array}\right\} \quad (d':=\lfloor d/2\rfloor)$$

generates $\mathbb{R}[\mathfrak{n}]^K$. That is, $\mathcal{G}$ generates $\mathbb{R}[\mathfrak{n}_d]^{SU(d)}$ when d is even and $\mathbb{R}[\mathfrak{n}_d]^{U(d)}$ when d is odd.

10.4. Values $g(B_\circ,a_\circ,\alpha)$, $\widehat{g}(B_\circ,a_\circ,\alpha)$, $g(B_\circ,t_\circ,\alpha)$, $\widehat{g}(B_\circ,t_\circ,\alpha)$ for $g\in\mathcal{G}$. Each generator $g\in\mathcal{G}$ takes values $g(B_\circ,a_\circ,\alpha):=g(\mathbf{O}(B_\circ,a_\circ,\alpha))$ and $g(B_\circ,t_\circ,\alpha):=g(\mathbf{O}(B_\circ,t_\circ,\alpha))$ on the spherical orbits given in equations (33) and (34). Also the associated operator $D_g\in\mathbb{D}_K(N)$ has eigenvalues $\widehat{g}(B_\circ,a_\circ,\alpha)$ and $\widehat{g}(B_\circ,t_\circ,\alpha)$ on the spherical functions $\phi_{B_\circ,a_\circ,\alpha},\phi_{B_\circ,t_\circ,\alpha}\in\Delta(K,N)$. For the central generators q_ℓ, τ and r, s (when d is even) we have

$$\left\{\begin{array}{c}\widehat{q}_\ell(B_\circ,a_\circ,\alpha)=tr((B_\circ\overline{B}_\circ)^\ell)=q_\ell(B_\circ,a_\circ,\alpha)\\ \widehat{q}_\ell(B_\circ,t_\circ,\alpha)=tr((B_\circ\overline{B}_\circ)^\ell)=q_\ell(B_\circ,t_\circ,\alpha)\\ \widehat{r}(B_\circ,a_\circ,\alpha)=Re(Pf(B_\circ))=r(B_\circ,a_\circ,\alpha)\\ \widehat{r}(B_\circ,t_\circ,\alpha)=Re(Pf(B_\circ))=r(B_\circ,t_\circ,\alpha)\\ \widehat{s}(B_\circ,a_\circ,\alpha)=Im(Pf(B_\circ))=s(B_\circ,a_\circ,\alpha)\\ \widehat{s}(B_\circ,t_\circ,\alpha)=Im(Pf(B_\circ))=s(B_\circ,t_\circ,\alpha)\\ \widehat{\tau}(B_\circ,a_\circ,\alpha)=0=\tau(B_\circ,a_\circ,\alpha)\\ \widehat{\tau}(B_\circ,t_\circ,\alpha)=t_\circ=\tau(B_\circ,t_\circ,\alpha)\end{array}\right\}.$$

For the non-central generators p_ℓ one has

$$\left\{\begin{array}{c}p_0(B_\circ,a_\circ,\alpha)=|a_\circ|^2+\sum_{\lambda\in\sigma^+(B_\circ)}2\lambda\alpha(\lambda)\\ \widehat{p}_0(B_\circ,a_\circ,\alpha)=|a_\circ|^2+\sum_{\lambda\in\sigma^+(B_\circ)}\lambda(2\alpha(\lambda)+m(\lambda))\\ p_\ell(B_\circ,a_\circ,\alpha)=(-1)^\ell\sum_{\lambda\in\sigma^+(B_\circ)}2\lambda^{2\ell+1}\alpha(\lambda)\quad\text{for } \ell\ge 1\\ \widehat{p}_\ell(B_\circ,a_\circ,\alpha)=(-1)^\ell\sum_{\lambda\in\sigma^+(B_\circ)}\lambda^{2\ell+1}(2\alpha(\lambda)+m(\lambda)\quad\text{for } \ell\ge 1 \text{ and}\\ p_\ell(B_\circ,t_\circ,\alpha)=(-1)^\ell\sum_{\lambda\in\sigma(B_\circ)}2((\lambda^2+t_\circ^2)^{1/2})^{2\ell+1}\alpha(\lambda)\quad\text{for } \ell\ge 0\\ \widehat{p}_\ell(B_\circ,t_\circ,\alpha)=(-1)^\ell\sum_{\lambda\in\sigma(B_\circ)}((\lambda^2+t_\circ^2)^{1/2})^{2\ell+1}(2\alpha(\lambda)+m(\lambda))\quad\text{for } \ell\ge 0\end{array}\right\},$$

where $m(\lambda) := \dim(V_\lambda)/2$ for all $\lambda \in \sigma(B_\circ)$. Note that as $J^2_{B_\circ} = B_\circ\overline{B}_\circ$ is $\mathbb{C}$-linear on $V = \mathbb{C}^d$ all eigenspaces V_λ have even (real) dimension. In particular V_0 is also even dimensional when $0 \in \sigma(B_\circ)$. We also note that the invariant τ reveals that $\Delta^\circ \subset \Delta(K,N)$ is a closed subset.

10.5. Condition (O) for (K,N).

We adapt the proof given in Section 8.5 to establish condition (O) for the pairs at hand. Let $(\phi_n)_{n=1}^\infty$ be a sequence in $\Delta(K,N)$ and $\phi \in \Delta(K,N)$. We will show that $(g(\Psi(\phi_n)))_{n=1}^\infty$ converges to $g(\Psi(\phi))$ for each $g \in \mathcal{G}$ if and only if $(\widehat{D}_g(\phi_n))_{n=1}^\infty$ converges to $\widehat{D}_g(\phi)$ for each $g \in \mathcal{G}$. There are a number of cases to consider. We have either $\phi \in \Delta^0$ or $\phi \in \Delta^\times$ and by passing to subsequences we can assume that either $(\phi_n)_{n=1}^\infty \subset \Delta^0$ or $(\phi_n)_{n=1}^\infty \subset \Delta^\times$.

Case 1: $\phi \in \Delta^0$ and $(\phi_n)_{n=1}^\infty \subset \Delta^0$:

In this case the argument in Section 8.6 goes through verbatim, just as in Section 9.6.

Case 2: $\phi \in \Delta^0$ and $(\phi_n)_{n=1}^\infty \subset \Delta^\times$:

Suppose now that $(\phi = \phi_{B_\circ,a_\circ,\alpha}) \in \Delta^0$ and $(\phi_n = \phi_{B_n,t_n,\alpha_n})_{n=1}^\infty \subset \Delta^\times$. Suppose moreover that for each central generator $g = q_\ell$, τ and r, s (when d is even) the sequence $\big(\widehat{g}(B_n,t_n,\alpha_n) = g(B_n,t_n,\alpha_n)\big)_{n=1}^\infty$ converges to $\big(\widehat{g}(B_\circ,a_\circ,\alpha) = g(B_\circ,a_\circ,\alpha)\big)$.

Under these assumptions it suffices to verify that $\lim_{n\to\infty} \widehat{p}_\ell(B_n,t_n,\alpha_n) = \widehat{p}_\ell(B_\circ,a_\circ,\alpha)$ if and only if $\lim_{n\to\infty} p_\ell(B_n,t_n,\alpha_n) = p_\ell(B_\circ,a_\circ,\alpha)$ for $\ell = 0,\ldots,d'-1$.

Our hypotheses concerning the central generators $g \in \mathcal{G}$ yield the following. The assumption that $\tau(B_n,t_n,\alpha_n) \to \tau(B_\circ,a_\circ,\alpha)$ gives $\lim_{n\to\infty} t_n = 0$.

Moreover using the invariants q_ℓ together with r, s (when d is even) the argument from Section 8.6 concerning behavior of eigenspace decompositions with respect to the operators $J^2_{B_n}$ applies. In particular we can assume, as in equation (21), that

$$\lim_{n\to\infty} \sum_{\mu\in\sigma^+(B_n)} \mu^{2\ell+1} m(\mu) = \sum_{\lambda\in\sigma^+(B_\circ)} \lambda^{2\ell+1} m(\lambda)$$

for $\ell \geq 0$. As $t_n \to 0$ this also implies that

$$\lim_{n\to\infty} \sum_{\mu\in\sigma(B_n)} ((\mu^2+t_n^2)^{1/2})^{2\ell+1} m(\mu) = \sum_{\lambda\in\sigma^+(B_\circ)} \lambda^{2\ell+1} m(\lambda) \quad \text{for } \ell \geq 0. \tag{35}$$

The condition that $p_0(B_n,t_n,\alpha_n) \to p_0(B_\circ,a_\circ,\alpha)$ reads

$$\lim_{n\to\infty} \sum_{\mu\in\sigma(B_n)} 2(\mu^2+t_n^2)^{1/2}\alpha_n(\mu) = |a_\circ|^2 + \sum_{\lambda\in\sigma^+(B_\circ)} 2\lambda\alpha(\lambda).$$

whereas $\widehat{p}_0(B_n,t_n,\alpha_n) \to \widehat{p}_0(B_\circ,a_\circ,\alpha)$ means

$$\lim_{n\to\infty} \sum_{\mu\in\sigma(B_n)} (\mu^2+t_n^2)^{1/2}(2\alpha_n(\mu)+m(\mu)) = |a_\circ|^2 + \sum_{\lambda\in\sigma^+(B_\circ)} \lambda(2\alpha(\lambda)+m(\lambda)).$$

Using (35) with $\ell = 0$ shows that $\lim_{n\to\infty} p_0(B_n,t_n,\alpha_n) = p_0(B_\circ,a_\circ,\alpha)$ if and only if $\lim_{n\to\infty} \widehat{p}_0(B_n,t_n,\alpha_n) = \widehat{p}_0(B_\circ,a_\circ,\alpha)$. Likewise for $\ell \geq 1$ the condition $p_\ell(B_n,t_n,\alpha_n) \to p_\ell(B_\circ,a_\circ,\alpha)$ reads

$$\lim_{n\to\infty} \sum_{\mu\in\sigma(B_n)} 2((\mu^2+t_n^2)^{1/2})^{2\ell+1}\alpha_n(\mu) = \sum_{\lambda\in\sigma^+(B_\circ)} 2\lambda^{2\ell+1}\alpha(\lambda)$$

whereas $\widehat{p}_0(B_n, t_n, \alpha_n) \to \widehat{p}_0(B_\circ, a_\circ, \alpha)$ means

$$\lim_{n\to\infty} \sum_{\mu\in\sigma(B_n)} ((\mu^2 + t_n^2)^{1/2})^{2\ell+1}(2\alpha_n(\mu) + m(\mu)) = \sum_{\lambda\in\sigma^+(B_\circ)} \lambda^{2\ell+1}(2\alpha(\lambda) + m(\lambda)).$$

Again these are equivalent conditions in view of (35).

Case 3: $\phi \in \Delta^\times$ and $(\phi_n)_{n=1}^\infty \subset \Delta^0$:
Suppose that $(\phi = \phi_{B_\circ,t_\circ,\alpha}) \in \Delta^\times$ and $(\phi_n = \phi_{B_n,a_n,\alpha_n})_{n=1}^\infty \subset \Delta^0$. As the sequence $\left(\tau(B_n, a_n, \alpha_n) = 0 = \widehat{\tau}(B_n, a_n, \alpha_n)\right)_{n=1}^\infty$ fails to converge to $\tau(B_\circ, t_\circ, \alpha) = t_\circ = \widehat{\tau}(B_\circ, t_\circ, \alpha)$ this case poses no issue.

Case 4: $\phi \in \Delta^\times$ and $(\phi_n)_{n=1}^\infty \subset \Delta^\times$:
Suppose that $(\phi = \phi_{B_\circ,t_\circ,\alpha}) \in \Delta^\times$ and $(\phi_n = \phi_{B_n,t_n,\alpha_n})_{n=1}^\infty \subset \Delta^\times$. The proof is as in Case 2 with the condition

$$\lim_{n\to\infty} \sum_{\mu\in\sigma(B_n)} ((\mu^2 + t_n^2)^{1/2})^{2\ell+1} m(\mu) = \sum_{\lambda\in\sigma(B_\circ)} ((\lambda^2 + t_\circ^2)^{1/2})^{2\ell+1} m(\lambda) \quad \text{for } \ell \geq 0.$$

in place of equation (35). ■

11. The pair $(U(d), \mathbb{C}^d \oplus H\Lambda^2(\mathbb{C}^d))$ with $d \geq 2$

This is entry 5 in Table 1. One has $\mathfrak{n} = \mathfrak{n}_d = V \oplus \mathfrak{z}$ with $V = \mathbb{C}^d$ and $\mathfrak{z} = H\Lambda^2(\mathbb{C}^d) = \mathfrak{u}(d) = \{A \in M_d(\mathbb{C}) \ : \ A^* = -A\}$, the $d \times d$ skew-hermitian matrices. The Lie bracket is

$$V \times V \to \mathfrak{z}, \qquad [u, v] := uv^* - vu^*$$

and $K = U(n)$ acts via $k \cdot (v, A) = (kv, kAk^*)$. The K-action on $\mathfrak{z}$ is the adjoint action of K on its Lie algebra.

11.1. K-orbits in $\mathfrak{z}$. Each K-orbit in $\mathfrak{z}$ contains a unique diagonal matrix of the form

$$B_\circ = diag(O_{m(0)}, i\lambda_1 I_{m(1)} \ldots, i\lambda_p I_{m(p)}) \tag{36}$$

for some $p \geq 0$, integers $m(0) \geq 0$, $m(1), \ldots, m(p) \geq 1$ with

$$m(0) + m(1) + \cdots + m(p) = d$$

and some distinct positive real numbers $0 < \lambda_1 < \cdots < \lambda_p$.

11.2. The space $\Delta(K, N)$. We put the following K-invariant inner product on $\mathfrak{n}$,

$$\Big((u, A), (v, B)\Big)_\mathfrak{n} := Re\,\langle u, v\rangle + \frac{1}{2}Re(tr(AB^*)) = Re(v^*u) - \frac{1}{2}Re(tr(AB)).$$

This gives $([u, v], B)_\mathfrak{n} = -(Bu, v)_\mathfrak{n}$ and hence $J_B \colon V \to V$ satisfies

$$J_B(u) = -Bu, \qquad J_B^2(u) = B^2u.$$

Given $B_\circ \in \Lambda^2(\mathbb{C}^d)$, the space V decomposes into eigenspaces for $J_{B_\circ}^2 = B_\circ^2$ as $V = \mathfrak{a}_{B_\circ} \oplus \mathfrak{w}_{B_\circ} = V_0 \oplus \bigoplus_{\lambda\in\sigma^+(B_\circ)} V_\lambda$, just as in previous examples. Given any $a_\circ \in V_0$ the stabilizer of $(a_\circ, B_\circ)$ in K is $K_{B_\circ,a_\circ} = U(V_0 \cap a_\circ^\perp) \times \prod_{\lambda\in\sigma^+(B_\circ)} U(V_\lambda)$ and the space $\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ}]$ decomposes under $K_{B_\circ,a_\circ}$ as $\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ}] = \bigoplus_\alpha \left(\bigotimes_{\lambda\in\sigma^+(B_\circ)} \mathcal{P}_{\alpha(\lambda)}(V_\lambda)\right)$ where $\alpha = (\alpha(\lambda) \;:\; \lambda \in \sigma^+(B_\circ))$ is a list of non-negative integers. Each bounded K-spherical function on N has the form $\phi_{B_\circ,a_\circ,\alpha}$ for some choice of data $(B_\circ, a_\circ, \alpha)$ as above (see Proposition 5.1 and Remark 5.1). Two such spherical functions coincide if and only if the data differ by the action of K. For $B_\circ$ in the diagonal form (36) one has $\sigma^+(B_\circ) = \{\lambda_1, \ldots, \lambda_p\}$, $K_{B_\circ} = U(m(0)) \times \cdots \times U(m(p))$ and $\widetilde{\mathfrak{w}}_{B_\circ} = \mathbb{C}^{m(1)} \oplus \cdots \oplus \mathbb{C}^{m(p)}$.

11.3. The space $\mathcal{A}(K,N)$. Proposition 5.3 yields the spherical orbit $\mathbf{O}(B_\circ, a_\circ, \alpha) := \Psi(\phi_{B_\circ,,a_\circ,\alpha})$ corresponding to $\phi_{B_\circ,,a_\circ,\alpha} \in \Delta(K,N)$. As in previous examples

$$\mathbf{O}(B_\circ, a_\circ, \alpha) = K \cdot \left(a_\circ + \sum_{\lambda>0} (2\lambda\alpha(\lambda))^{1/2} e_\lambda, B_\circ\right) \tag{37}$$

where $e_\lambda \in V_\lambda$ is any chosen unit vector.

11.4. Generators for $\mathbb{R}[\mathfrak{n}]^K$. Letting

$$p_\ell(v,A) := i^\ell v^* A^\ell v, \quad q_\ell(v,A) := i^\ell tr(A^\ell),$$

the set $\mathcal{G} = \{p_0, \ldots, p_{d-1}, q_1, \ldots, q_d\}$ generates $\mathbb{R}[\mathfrak{n}]^K$ [14, Theorem 7.5].

11.5. Values $g(B_\circ, a_\circ, \alpha)$ and $\widehat{g}(B_\circ, a_\circ, \alpha)$ for $g \in \mathcal{G}$.
With notation as in Section 8.5 the values $g(B_\circ, a_\circ, \alpha) := g(\mathbf{O}(B_\circ, a_\circ, \alpha))$ and eigenvalues $\widehat{g}(B_\circ, a_\circ, \alpha)$ for each generator $g \in \mathcal{G}$ are as follows.

$$\left\{\begin{array}{c} \widehat{q}_\ell(B_\circ, a_\circ, \alpha) = i^\ell tr(B_\circ^\ell) = q_\ell(B_\circ, a_\circ, \alpha) \\ p_0(B_\circ, a_\circ, \alpha) = |a_\circ|^2 + \sum_{\lambda\in\sigma^+(B_\circ)} 2\lambda\alpha(\lambda) \\ \widehat{p}_0(B_\circ, a_\circ, \alpha) = |a_\circ|^2 + \sum_{\lambda\in\sigma^+(B_\circ)} \lambda(2\alpha(\lambda) + m(\lambda)) \\ p_\ell(B_\circ, a_\circ, \alpha) = (-1)^\ell \sum_{\lambda\in\sigma^+(B_\circ)} 2\lambda^{\ell+1}\alpha(\lambda) \quad \text{for } \ell \geq 1 \\ \widehat{p}_\ell(B_\circ, a_\circ, \alpha) = (-1)^\ell \sum_{\lambda\in\sigma^+(B_\circ)} \lambda^{\ell+1}(2\alpha(\lambda) + m(\lambda)) \quad \text{for } \ell \geq 1 \end{array}\right\}.$$

where $m(\lambda) := \dim(V_\lambda)/2$ for all $\lambda \in \sigma^+(B_\circ)$. (For $B_\circ$ in standard form (36) one has $m(\lambda_j) = m(j)$.)

11.6. Condition (O) for (K,N). The proof that (K,N) satisfies condition (O) using these values for $g(B_\circ, a_\circ, \alpha)$, $\widehat{g}(B_\circ, a_\circ, \alpha)$ $(g \in \mathcal{G})$ proceeds exactly as that given in Section 8.6 for the pair $(SO(d), \mathbb{R}^d \oplus \Lambda^2(\mathbb{R}^d))$. ■

12. The pair $(Sp(d), \mathbb{H}^d \oplus (HS^2(\mathbb{H}^d) \oplus \mathbb{C}))$ with $d \geq 1$

Entry 6 in Table 1 has $\mathfrak{n} = \mathfrak{n}_d = V \oplus \mathfrak{z}$ with $V = \mathbb{H}^d$ and $\mathfrak{z} = HS^2(\mathbb{H}^d) \oplus \mathbb{C}$. Here $HS^2(\mathbb{H}^d)$ denotes the space of hermitian symmetric matrices of size $d \times d$ with quaternion entries, $HS^2(\mathbb{H}^d) = \{A \in M_d(\mathbb{H}) \;:\; A^* = A\}$. As always V and $\mathfrak{z}$ are regarded as real vector spaces but we will also view V as a vector space over the division algebra $\mathbb{H}$. In this regard we follow the conventions in [30, Chapter 2], letting quaternion scalars act on V from the right and $d \times d$ quaternion matrices act from the left. Real scalars can act from either side.

For imaginary quaternions $q = a\,i + b\,j + c\,k$ let $q_{\mathbb{C}} \in \mathbb{C}$ be defined as

$$q_{\mathbb{C}} := b + c\,i$$

so that $q = a\,i + q_{\mathbb{C}} j$. With this convention the Lie bracket is given by

$$V \times V \to \mathfrak{z}, \qquad [u,v] := \big(uiv^* - viu^*, (u^*v - v^*u)_{\mathbb{C}}\big).$$

In this example $Sp(d) = \{k \in M_d(\mathbb{H}) \ : \ kk^* = I\}$ is the group of $\mathbb{H}$-linear transformations $V \to V$ preserving the hermitian form $\langle u, v\rangle = v^*u$. The group $K = Sp(d)$ acts on $\mathfrak{n}$ via

$$k \cdot (v, A, c) = (kv, kAk^*, c).$$

Remark 12.1. Note that the action of K on $\mathfrak{z}$ is trivial on the $\mathbb{C}$-factor and also trivial on the scalar matrices $\{tI \ : \ t \in \mathbb{R}\} \subset HS^2(\mathbb{H}^d)$. A different model for the pair $(K, \mathfrak{n})$ is found in [29] and elsewhere, including [30] and [14]. In these references the center is written as $\mathfrak{z}' = HS^2_0(\mathbb{H}^d) \oplus Im(\mathbb{H})$ with

$$HS^2_0(\mathbb{H}^d) = \{A \in HS^2(\mathbb{H}^d) : tr(A) = 0\}$$

and the Lie bracket $V \times V \to \mathfrak{z}'$ becomes

$$[u,v]' := \big((uiv^* - viu^*)_\circ, u^*v - v^*u\big)$$

where $A_\circ := A - \frac{tr(A)}{d} I$ for $A \in HS^2(\mathbb{H}^d)$. The group $K = Sp(d)$ acts on $\mathfrak{n}' = V \oplus \mathfrak{z}'$ as $k \cdot (v, A, q) = (kv, kAk^*, q)$. To reconcile the two models one can check that the map

$$\varphi \colon V \oplus \mathfrak{z}' \to V \oplus \mathfrak{z}, \qquad \big(v, A, (q = a\,i + q_{\mathbb{C}} j)\big) \mapsto \left(v, A + \frac{a}{d} I, q_{\mathbb{C}}\right)$$

is a K-equivariant Lie algebra isomorphism. ■

12.1. K-orbits in $\mathfrak{z}$. Each K-orbit in $HS^2(\mathbb{H}^d)$ contains a unique diagonal matrix of the form

$$B_\circ = diag(O_{\ell(0)}, \lambda_1 I_{\ell(1)} \ldots, \lambda_p I_{\ell(p)}) \tag{38}$$

for some $p \geq 0$, integers $\ell(0) \geq 0$, $\ell(1), \ldots, \ell(p) \geq 1$ with

$$\ell(0) + \ell(1) + \cdots + \ell(p) = d$$

and some distinct positive real numbers $0 < \lambda_1 < \cdots < \lambda_p$.

12.2. The space $\Delta(K, N)$. We equip $\mathfrak{n}$ with the following K-invariant inner product.

$$\begin{aligned}\Big((u, A, c), (v, B, c')\Big)_{\mathfrak{n}} &:= Re\,\langle u, v\rangle + \frac{1}{2} Re(tr(AB^*)) + \frac{1}{2} Re(c\,\overline{c'}) \\ &= Re(v^*u) + \frac{1}{2} Re(tr(AB)) + \frac{1}{2} Re(c\,\overline{c'}).\end{aligned}$$

For $u, v \in V$ and $(B, c) \in \mathfrak{z}$ one can check that $([u,v], B)_{\mathfrak{n}} = (Bui, v)_n$ and $([u,v], c)_{\mathfrak{n}} = (u(cj), v)_{\mathfrak{n}}$. Thus $J_{B,c} \colon V \to V$ is given by

$$J_{B,c}(u) = Bui + u(cj).$$

From this one obtains

$$J_{B,c}^2(u) = -B^2u - |c|^2u \qquad \text{(where } |c|^2 = c\,\overline{c}, \text{ not } (c,c)_{\mathfrak{n}} = c\,\overline{c}/2). \tag{39}$$

Thus for fixed $B_\circ \in HS^2(\mathbb{H}^d)$ the decomposition for V into eigenspaces for $J^2_{B_\circ,c_\circ}$ does not depend on $c_\circ \in \mathbb{C}$. But if $v \in V$ is a $(-\lambda^2)$-eigenvector for $J^2_{B_\circ} = J^2_{B_\circ,0}$ then v becomes a $-(\lambda^2 + |c_\circ|^2)$-eigenvector for $J^2_{B_\circ,c_\circ}$. This situation parallels that for the pairs $((S)U(d), \mathbb{C}^d \oplus (\Lambda^2(\mathbb{C}^d) \oplus \mathbb{R}))$, discussed in Section 10.

For fixed $B_\circ \in HS^2(\mathbb{H}^d)$ let

$$\sigma(B_\circ) := \{\lambda \geq 0 \ : \ -\lambda^2 \text{ is an eigenvalue for } J^2_{B_\circ} = -B_\circ^2\},$$

$\sigma^+(B_\circ) := \{\lambda \in \sigma(B_\circ) \ : \ \lambda > 0\}$ and write $V = \mathfrak{a}_{B_\circ} \oplus \mathfrak{w}_{B_\circ} = V_0 \oplus \bigoplus_{\lambda \in \sigma^+(B_\circ)} V_\lambda$, a sum of eigenspaces for $J^2_{B_\circ}$ as usual. As $J^2_{B_\circ}$ is $\mathbb{H}$-linear all eigenspaces have real dimension divisible by four. For any $a_\circ \in V_0$ the stabilizer of $(a_\circ, B_\circ, 0)$ in K is $K_{(B_\circ,0),a_\circ} = Sp(V_0 \cap a_\circ^\perp) \times \prod_{\lambda \in \sigma^+(B_\circ)} Sp(V_\lambda)$ and the space $\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ}]$ decomposes under $K_{(B_\circ,0),a_\circ}$ as $\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ}] = \bigoplus_\alpha \left(\bigotimes_{\lambda \in \sigma^+(B_\circ)} \mathcal{P}_{\alpha(\lambda)}(V_\lambda) \right)$ where $\alpha = (\alpha(\lambda) : \lambda \in \sigma^+(B_\circ))$ is a list of non-negative integers, one for each $\lambda \in \sigma^+(B_\circ)$. For $B_\circ$ in standard form form (38) one has $\sigma^+(B_\circ) = \{\lambda_1, \ldots, \lambda_p\}$, $K_{B_\circ} = Sp(\ell(0)) \times \cdots \times Sp(\ell(p))$ and $\widetilde{\mathfrak{w}}_{B_\circ} = \mathbb{H}^{\ell(1)} \oplus \cdots \oplus \mathbb{H}^{\ell(p)}$.

For $c_\circ \in \mathbb{C}^\times$ the operator $J^2_{B_\circ,c_\circ}$ has trivial kernel, $\mathfrak{w}_{B_\circ,c_\circ} = V$ and $\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ,c_\circ}]$ decomposes under $K_{(B_\circ,c_\circ),0} = \prod_{\lambda \in \sigma(B_\circ)} Sp(V_\lambda)$ as $\mathbb{C}[\widetilde{\mathfrak{w}}_{B_\circ,c_\circ}] = \bigoplus_\alpha \left(\bigotimes_{\lambda \in \sigma(B_\circ)} \mathcal{P}_{\alpha(\lambda)}(V_\lambda) \right)$ where $\alpha = (\alpha(\lambda) \ : \ \lambda \in \sigma(B_\circ))$ is a list of non-negative integers, one for each $\lambda \in \sigma(B_\circ)$.

As in Section 10, the spherical functions $\phi \in \Delta(K,N)$ are determined by the data "$(B_\circ, c_\circ, a_\circ, \alpha)$" where at most one of $c_\circ$, $a_\circ$ can be non-zero. Letting

$$\Gamma^0 := \left\{ (B_\circ, a_\circ, \alpha) : B_\circ \in HS^2(\mathbb{H}^d),\ a_\circ \in Ker(J^2_{B_\circ}),\ \alpha \in (\mathbb{Z}^+)^{|\sigma^+(B_\circ)|} \right\},$$
$$\Gamma^\times := \left\{ (B_\circ, c_\circ, \alpha) : B_\circ \in HS^2(\mathbb{H}^d),\ c_\circ \in \mathbb{C}^\times,\ \alpha \in (\mathbb{Z}^+)^{|\sigma(B_\circ)|} \right\},$$

one has $\Delta(K,N) = \Delta^0 \cup \Delta^\times$ where

$$\Delta^0 := \left\{ \phi_{B_\circ,a_\circ,\alpha} : (B_\circ, a_\circ, \alpha) \in \Gamma^0 \right\}, \quad \Delta^\times := \left\{ \phi_{B_\circ,c_\circ,\alpha} : (B_\circ, c_\circ, \alpha) \in \Gamma^\times \right\}.$$

12.3. The space $\mathcal{A}(K,N)$. The spherical orbits $\mathbf{O}(B_\circ, a_\circ, \alpha) := \Psi(\phi_{B_\circ,,a_\circ,\alpha})$ and $\mathbf{O}(B_\circ, c_\circ, \alpha) := \Psi(\phi_{B_\circ,,c_\circ,\alpha})$ for data $(B_\circ, a_\circ, \alpha) \in \Gamma^0$, $(B_\circ, c_\circ, \alpha) \in \Gamma^\times$ are as follows. Choose for each $\lambda \in \sigma(B_\circ)$ a unit vector $e_\lambda \in V_\lambda$. One has

$$\mathbf{O}(B_\circ, a_\circ, \alpha) = K \cdot \left(a_\circ + \sum_{\lambda \in \sigma^+(B_\circ)} (2\lambda\alpha(\lambda))^{1/2} e_\lambda, B_\circ, 0 \right). \tag{40}$$

$$\mathbf{O}(B_\circ, c_\circ, \alpha) = K \cdot \left(\sum_{\lambda \in \sigma(B_\circ)} (2(\lambda^2 + |c_\circ|^2)^{1/2}\alpha(\lambda))^{1/2} e_\lambda, B_\circ, c_\circ \right). \tag{41}$$

12.4. Generators for $\mathbb{R}[\mathfrak{n}]^K$. In terms of our model for $\mathfrak{n}$, a generating set $\mathcal{G}$ for $\mathbb{R}[\mathfrak{n}]^K$, given in [14, Theorem 7.5], is $\mathcal{G} = \{p_0, \ldots, p_{d-1}, q_1, \ldots, q_d, r, s\}$, where

$$p_\ell(v, A, c) := v^* A^\ell v,\ q_\ell(v, A, c) := tr(A^\ell),\ r(v, A, c) = Re(c),\ s(v, A, c) := Im(c).$$

12.5. Values $g(B_\circ, a_\circ, \alpha)$, $\widehat{g}(B_\circ, a_\circ, \alpha)$, $g(B_\circ, c_\circ, \alpha)$, $\widehat{g}(B_\circ, c_\circ, \alpha)$ for $g \in \mathcal{G}$.

Each generator $g \in \mathcal{G}$ takes values $g(B_\circ, a_\circ, \alpha) := g(\mathbf{O}(B_\circ, a_\circ, \alpha))$ and $g(B_\circ, c_\circ, \alpha) := g(\mathbf{O}(B_\circ, c_\circ, \alpha))$ on the spherical orbits given in equations (40) and (41). Also the associated operator $D_g \in \mathbb{D}_K(N)$ has eigenvalues $\widehat{g}(B_\circ, a_\circ, \alpha)$ and $\widehat{g}(B_\circ, c_\circ, \alpha)$ on the spherical functions $\phi_{B_\circ,a_\circ,\alpha}, \phi_{B_\circ,c_\circ,\alpha} \in \Delta(K,N)$. For the central generators q_ℓ, r, s we have

$$\left\{\begin{array}{c} \widehat{q_\ell}(B_\circ, a_\circ, \alpha) = tr(B_\circ^\ell) = q_\ell(B_\circ, a_\circ, \alpha) \\ \widehat{q_\ell}(B_\circ, c_\circ, \alpha) = tr(B_\circ^\ell) = q_\ell(B_\circ, c_\circ, \alpha) \\ \widehat{r}(B_\circ, a_\circ, \alpha) = 0 = r(B_\circ, a_\circ, \alpha) \\ \widehat{r}(B_\circ, c_\circ, \alpha) = Re(c_\circ) = r(B_\circ, c_\circ, \alpha) \\ \widehat{s}(B_\circ, a_\circ, \alpha) = 0 = s(B_\circ, a_\circ, \alpha) \\ \widehat{s}(B_\circ, c_\circ, \alpha) = Im(c_\circ) = s(B_\circ, c_\circ, \alpha) \end{array}\right\}.$$

For the non-central generators p_ℓ one has

$$\left\{\begin{array}{c} p_0(B_\circ, a_\circ, \alpha) = |a_\circ|^2 + \sum_{\lambda\in\sigma^+(B_\circ)} 2\lambda\alpha(\lambda) \\ \widehat{p_0}(B_\circ, a_\circ, \alpha) = |a_\circ|^2 + \sum_{\lambda\in\sigma^+(B_\circ)} \lambda(2\alpha(\lambda) + m(\lambda)) \\ p_\ell(B_\circ, a_\circ, \alpha) = (-1)^\ell \sum_{\lambda\in\sigma^+(B_\circ)} 2\lambda^{\ell+1}\alpha(\lambda) \quad \text{for } \ell \geq 1 \\ \widehat{p_\ell}(B_\circ, a_\circ, \alpha) = (-1)^\ell \sum_{\lambda\in\sigma^+(B_\circ)} \lambda^{\ell+1}(2\alpha(\lambda) + m(\lambda)) \quad \text{for } \ell \geq 1 \text{ and} \\ p_\ell(B_\circ, c_\circ, \alpha) = (-1)^\ell \sum_{\lambda\in\sigma(B_\circ)} 2((\lambda^2 + |c_\circ|^2)^{1/2})^{\ell+1}\alpha(\lambda) \quad \text{for } \ell \geq 0 \\ \widehat{p_\ell}(B_\circ, c_\circ, \alpha) = (-1)^\ell \sum_{\lambda\in\sigma(B_\circ)}((\lambda^2 + |c_\circ|^2)^{1/2})^{\ell+1}(2\alpha(\lambda) + m(\lambda)) \quad \text{for } \ell \geq 0 \end{array}\right\},$$

where $m(\lambda) := \dim(V_\lambda)/2$ for all $\lambda \in \sigma(B_\circ)$. (For $B_\circ$ in standard form (38) one has $m(\lambda_j) = 2\ell(j)$.)

12.6. Condition (O) for (K,N). The proof that (K,N) satisfies condition (O) using these values for $g(B_\circ, a_\circ, \alpha)$, $\widehat{g}(B_\circ, a_\circ, \alpha)$, $g(B_\circ, c_\circ, \alpha)$, $\widehat{g}(B_\circ, c_\circ, \alpha)$ $(g \in \mathcal{G})$ closely parallels that given in Section 10.5 for the pairs $((S)U(d), \mathbb{C}^d \oplus (\Lambda^2(\mathbb{C}^d) \oplus \mathbb{R}))$. In the *Case 2* portion of the proof one has a sequence $(\phi_n = \phi_{B_n,c_n,\alpha_n})_{n=1}^\infty \subset \Delta^\times$ and function $(\phi = \phi_{B_\circ,a_\circ,\alpha}) \in \Delta^0$. One assumes that $\big(g(B_n, c_n, \alpha_n) = \widehat{g}(B_n, c_n, \alpha_n)\big)_{n=1}^\infty$ converges to $g(B_\circ, a_\circ, \alpha) = \widehat{g}(B_\circ, a_\circ, \alpha)$ for the central generators $g \in \{q_1, \ldots, q_d, r, s\}$. These assumptions imply, using invariants $\{r, s\}$, that $c_n \to 0$ and enable one to obtain

$$\lim_{n\to\infty} \sum_{\mu\in\sigma(B_n)} ((\mu^2 + |c_n|^2)^{1/2})^{\ell+1} m(\mu) = \sum_{\lambda\in\sigma^+(B_\circ)} \lambda^{\ell+1} m(\lambda) \quad \text{for } \ell \geq 0,$$

the analog for equation (35) in the current context. Using this it is easy to show that $p_\ell(B_n, c_n, \alpha_n) \to p_\ell(B_\circ, a_\circ, \alpha)$ if and only if $\widehat{p_\ell}(B_n, c_n, \alpha_n) \to \widehat{p_\ell}(B_\circ, a_\circ, \alpha)$ for $\ell \geq 0$. ■

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