

Three-Dimensional Metric Lie Algebras and Ricci Flow

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Abstract. Three-dimensional metric Lie algebras (\mathfrak{g}, Q) , where \mathfrak{g} is a three-dimensional Lie algebra and Q is an inner product on \mathfrak{g} , are studied. We first complete a unified study of both the unimodular and non-unimodular cases, classifying all three-dimensional metric Lie algebras up to two notions of equivalence: isomorphism and isomorphism and scaling. For both notions of equivalence we parametrize the equivalence classes of three-dimensional metric Lie algebras using the Lie algebra structure constants determined by a particular choice of orthonormal frame, providing topologies on these sets and showing their structure as stratified sets (of dimensions 3 and 2 respectively). We then study the Ricci flow on the parameter spaces of equivalence classes of three-dimensional metric Lie algebras by expressing the equations governing the Ricci flow in terms of the Lie algebra structure constants. In the case of equivalence up to isomorphism and scaling, we analyze the trajectories of the Ricci flow and classify the fixed points of the flow. Metric Lie algebras corresponding to these fixed points give algebraic Ricci solitons for the Ricci flow on the associated simply-connected three-dimensional Lie groups, and we show that every left invariant Ricci soliton metric on such a simply-connected Lie group arises from one of these fixed points of the Ricci flow.

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1. Introduction

Three-dimensional metric Lie algebras have been well understood since the remarkable paper by Milnor [21]. A metric Lie algebra is a pair (\mathfrak{g}, Q) where \mathfrak{g} is a real Lie algebra and Q is an inner product on \mathfrak{g} . We define two metric Lie algebras (\mathfrak{g}_1, Q_1) and (\mathfrak{g}_2, Q_2) to be isomorphic if there is a Lie algebra isomorphism $L : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ which preserves the inner product: $L^*Q_2 = Q_1$. We say (\mathfrak{g}_1, Q_1) and (\mathfrak{g}_2, Q_2) are equivalent up to isomorphism and scaling if there is a Lie algebra isomorphism $L : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such that $Q_1 = cL^*Q_2$ for some constant $c > 0$. We write \mathfrak{M} for the set of isomorphism classes of three-dimensional metric Lie algebras and \mathfrak{N} for the set of equivalence classes of non-abelian three-dimensional metric Lie algebras up to isomorphism and scaling. In this paper we parameterize \mathfrak{M} and \mathfrak{N} (using Lie algebra structure constants relative to special orthonormal frames) providing topologies for these sets and showing their structure as stratified sets (of dimensions three and two respectively).

Using the work of Milnor in [21], we will analyze \mathfrak{M} (and \mathfrak{N}) in two pieces. Let \mathfrak{M}_u consist of isomorphism classes of all unimodular metric Lie algebras, where a

Lie algebra is unimodular if $\text{trace}(\text{ad}(x)) = 0$ for every $x \in \mathfrak{g}$. Let \mathfrak{M}_2 consist of isomorphism classes of all metric Lie algebras for which the derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ has dimension at most two; these are exactly the solvable Lie algebras. The Jacobi identity yields $\text{ad}(z) = \text{ad}([x, y]) = \text{ad}(x)\text{ad}(y) - \text{ad}(y)\text{ad}(x)$ for any $z = [x, y] \in \mathfrak{g}'$. But then $\text{trace}(\text{ad}(z)) = 0$ for any $z = [x, y] \in \mathfrak{g}'$, and we must have \mathfrak{g} unimodular whenever $\mathfrak{g}' = \mathfrak{g}$ has dimension three. So $\mathfrak{M}_u \cup \mathfrak{M}_2 = \mathfrak{M}$. The points in the intersection $\mathfrak{M}_u \cap \mathfrak{M}_2$ correspond to four types of (unimodular, solvable) metric Lie algebras as described in Section 2 below. We also write \mathfrak{N}_u and \mathfrak{N}_2 for the corresponding equivalence classes up to isomorphism and scaling.

In Section 2 we outline the parameterization of \mathfrak{M}_u and \mathfrak{N}_u . Note that Glickenstein and Payne in [7] essentially covered this unimodular case (although they used a different definition of metric Lie algebra isomorphism as discussed in Section 7 below). In Section 3 we cover the structure of \mathfrak{M}_2 and \mathfrak{N}_2 and their intersections with the unimodular cases, thus giving the total pictures of \mathfrak{M} and \mathfrak{N} as stratified sets.

We next describe the Ricci flow on the spaces \mathfrak{M} and \mathfrak{N} . Up to isomorphism, our metric Lie algebras correspond one-to-one to left invariant Riemannian metrics on three-dimensional, simply-connected Lie groups. The usual Ricci flow starting with a left invariant metric on such a simply-connected Lie group flows through left invariant metrics on the group and therefore corresponds to a flow through the corresponding metric Lie algebras. The usual Ricci flow equation $\frac{\partial g(t)}{\partial t} = -2R(t)$ for metrics on the group then becomes an ordinary differential equation $\frac{dg(t)}{dt} = -2R(t)$ for metrics on the Lie algebra.

This Ricci flow on Lie groups and Lie algebras has been widely investigated, for example by Lauret [16], Arroyo and Lafuente [1], and Di Cerbo [5] and, in the case of three-dimensional Lie groups, by Isenberg and Jackson [9] and Glickenstein [6]. However, the standard approach in those investigations was to examine the Ricci flow through metrics on a fixed Lie group or algebra. In this paper we instead take the approach of Glickenstein and Payne in [7] by treating the Ricci flow as occurring on the parameter spaces \mathfrak{M} and \mathfrak{N} . Note that Lauret in [18] treated the Ricci flow as occurring on a space of Lie brackets rather than on the metrics for a fixed Lie algebra. This approach was applied by Lauret in [19] and, more recently, by Böhmer and Lafuente [2], [3], [4] to a large class of solvable Lie algebras.

Isomorphic metric Lie algebras give rise to isomorphic flows, so we have a Ricci flow on the space \mathfrak{M} . While rescaling a metric does not simply rescale the flow, the projections into \mathfrak{N} of the orbits starting from two metrics in \mathfrak{M} which differ by a scaling lie on the same trajectory in \mathfrak{N} . (The rescaling just alters the velocity along the trajectory.) So we can also study the Ricci flow on \mathfrak{N} .

We determine the differential equations for these Ricci flows in terms of the parameters on \mathfrak{M} and \mathfrak{N} . We find the fixed points for the flow on \mathfrak{N} and study the trajectories of the flow between the fixed points. These fixed points correspond to Ricci solitons, that is, to left invariant metrics on the associated simply-connected Lie groups which evolve under the Ricci flow by diffeomorphisms and rescaling. In fact, they give “special” solitons which evolve by Lie group isomorphisms and rescaling. These special solitons are *algebraic solitons* as introduced by Lauret [16].

Note that Glickenstein and Payne in the paper [7], which motivated this current work, carried out this project for the unimodular case, that is, for the flow on \mathfrak{N}_u .

In Sections 5 and 6 we cover the flow for \mathfrak{M}_u , \mathfrak{N}_u and \mathfrak{M}_2 , \mathfrak{N}_2 respectively and check that the flow agrees on the intersections $\mathfrak{M}_u \cap \mathfrak{M}_2$ and $\mathfrak{N}_u \cap \mathfrak{N}_2$. Finally, in Section 7, we comment on the difference between our definition of isomorphism of metric Lie algebras and that used by Glickenstein and Payne in [7]. We show how using their alternative definition would alter the parameter spaces and modify our results. We also verify that any left invariant Ricci soliton metric on a simply-connected three-dimensional Lie group is actually a special Ricci soliton (an algebraic soliton) corresponding to a fixed point of the Ricci flow on \mathfrak{N} . (For certain solvable Lie algebras one could give an alternative proof of this fact using results in Böhm and Lafuente [3].)

2. Structure of \mathfrak{M}_u

We begin by outlining the results for the structure of \mathfrak{M}_u and \mathfrak{N}_u . This work largely duplicates the work of Glickenstein and Payne in [7]. (The differences caused by their use of an alternative definition of metric Lie algebra isomorphism are discussed in Section 7 below.)

Define \mathfrak{S}_u as a quotient of a certain subset of \mathbb{R}^3 : Take the subset

$$S_u = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 \geq a_2 \geq 0 \text{ and } a_2 \geq a_3\}$$

and then define \mathfrak{S}_u to be S_u modulo the identification $(a_1, 0, a_3) \sim (-a_3, 0, -a_1)$. Also let T_u be the subset of the projective plane corresponding to S_u , that is

$$T_u = \{[a_1, a_2, a_3] \in P^2 : a_1 \geq a_2 \geq 0 \text{ and } a_2 \geq a_3\}$$

and put $\mathfrak{T}_u = T_u$ modulo the identification $[a_1, 0, a_3] \sim [-a_3, 0, -a_1]$.

Proposition 2.1. *There are bijections between \mathfrak{M}_u and \mathfrak{S}_u and between \mathfrak{N}_u and \mathfrak{T}_u . For any unimodular metric Lie algebra \mathfrak{g} , there exists an orthonormal basis $\{e_1, e_2, e_3\}$, which we call a Milnor basis, and a point $(a_1, a_2, a_3) \in S_u$, unique up to equivalence in \mathfrak{S}_u , which we call the triple defined by the Milnor basis, such that $[e_i, e_j] = a_k e_k$ for i, j, k any cyclic permutation of 1, 2, 3.*

\mathfrak{S}_u and \mathfrak{T}_u have natural topologies and stratifications as (quotients of) subsets of \mathbb{R}^3 and P^2 . We will use the proposition to identify \mathfrak{M}_u with \mathfrak{S}_u and \mathfrak{N}_u with \mathfrak{T}_u , thus giving a topology and stratification to \mathfrak{M}_u and to \mathfrak{N}_u . We will often identify a triple $(a_1, a_2, a_3) \in S_u$ with its equivalence class in \mathfrak{S}_u and refer to it as “a point in \mathfrak{M}_u ”. Similarly, we may refer to a point $[a_1, a_2, a_3] \in T_u$ as “a point in \mathfrak{N}_u ”. Notice that the “triple” defined by a Milnor basis consists of the (nontrivial) Lie algebra structure constants relative to the given orthonormal basis.

We now sketch the proof of Proposition 2.1.

Proof. As shown in Milnor [21], for any three-dimensional unimodular metric Lie algebra (\mathfrak{g}, Q) there exists an orthonormal basis $\{e_1, e_2, e_3\}$ and a triple of real numbers (a_1, a_2, a_3) such that

$$[e_2, e_3] = a_1 e_1, [e_3, e_1] = a_2 e_2, [e_1, e_2] = a_3 e_3. \quad (1)$$

We call such a basis a *Milnor basis* and say the Milnor basis *defines* the triple (a_1, a_2, a_3) . The triple (a_1, a_2, a_3) is determined up to ordering and to change of all

signs by the metric Lie algebra (\mathfrak{g}, Q) . That is, each (\mathfrak{g}, Q) corresponds to a unique equivalence class of triples, which we think of as points $(a_1, a_2, a_3) \in \mathbb{R}^3$. The equivalence is given by $(-a_1, -a_2, -a_3) \sim (a_1, a_2, a_3)$ and $(a_1, a_2, a_3) \sim (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$ for any permutation σ of $\{1, 2, 3\}$. Different Milnor bases for a given metric Lie algebra define equivalent triples. It is easy to check that every equivalence class of triples corresponds to exactly one point in \mathfrak{S}_u , so every unimodular metric Lie algebra can be mapped to a well-defined point in \mathfrak{S}_u . We claim this map gives rise to a bijection between \mathfrak{M}_u and \mathfrak{S}_u .

First, every triple in S_u (and therefore every equivalence class in \mathfrak{S}_u) arises from some metric Lie algebra: Just take any basis $\{e_1, e_2, e_3\}$ for \mathbb{R}^3 , define a metric by declaring the basis to be orthonormal, and then define a bracket operation by equations (1). It is easy to verify the Jacobi identity, so we have a metric Lie algebra with a Milnor basis defining the triple (a_1, a_2, a_3) . So the map from unimodular metric Lie algebras to \mathfrak{S}_u is surjective.

Next, if $F : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is an isomorphism of metric Lie algebras and $\{e_1, e_2, e_3\}$ is a Milnor basis for \mathfrak{g}_1 defining the triple (a_1, a_2, a_3) , then clearly $\{F(e_1), F(e_2), F(e_3)\}$ is a Milnor basis for \mathfrak{g}_2 defining the same triple (a_1, a_2, a_3) . So isomorphic metric Lie algebras map to the same point of \mathfrak{S}_u giving a well-defined (surjective) map $\mathfrak{M}_u \rightarrow \mathfrak{S}_u$.

Finally, we claim that the map $\mathfrak{M}_u \rightarrow \mathfrak{S}_u$ is injective. Suppose $\mathfrak{g}_1, \mathfrak{g}_2$ are metric Lie algebras with Milnor bases $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ which both define the same triple $(a_1, a_2, a_3) \in \mathfrak{S}_u$. Define $F : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ to be the linear map such that $F(e_i) = f_i$. Then F is an isometry (preserves orthonormal bases) and we claim F is also a Lie algebra isomorphism (preserves brackets). For example, $[F(e_2), F(e_3)] = [f_2, f_3] = a_1 f_1 = F(a_1 e_1) = F([e_2, e_3])$. So $\mathfrak{g}_1, \mathfrak{g}_2$ are in the same isomorphism class of \mathfrak{M}_u . We conclude that there is in fact a one-to-one correspondence between the sets \mathfrak{M}_u and \mathfrak{S}_u .

Turning to \mathfrak{N}_u , notice that if we scale the metric on a metric Lie algebra by replacing the inner product Q by Q/c^2 for some c , then a (Milnor) orthonormal basis $\{e_1, e_2, e_3\}$ for the original metric is replaced by $\{ce_1, ce_2, ce_3\}$ as a (Milnor) orthonormal basis for the scaled metric. This has the effect of replacing the defined triple (a_1, a_2, a_3) by (ca_1, ca_2, ca_3) . Thus, on a line through the origin in \mathbb{R}^3 , all points except the origin correspond to the same metric Lie algebra up to scaling, i.e. to a single point of \mathfrak{N}_u .

Since for every point in \mathfrak{M}_u there is at least one corresponding point (a_1, a_2, a_3) in S_u , it follows that for every point in \mathfrak{N}_u there is at least one corresponding line $[a_1, a_2, a_3] \in T_u$. Points $(a_1, 0, a_3)$ and $(-a_3, 0, -a_1)$ correspond to the same point of \mathfrak{M}_u , so lines $[a_1, 0, a_3]$ and $[-a_3, 0, -a_1]$ correspond to the same point of \mathfrak{N}_u . That is, \mathfrak{N}_u can be identified with \mathfrak{T}_u . ■

We can identify T_u with a subset of the real plane together with one “point at infinity”. Every line in T_u except $[0, 0, c]$ intersects the plane $a_1 = 1$, so we can identify T_u with a subset of this plane together with one point at infinity, $\infty = [0, 0, c]$. The line $[a_1, a_2, a_3]$ intersects the plane $a_1 = 1$ in the point $(1, m_2, m_3)$ where $m_2 = a_2/a_1, m_3 = a_3/a_1$. Then T_u is identified with the point ∞ together with the following region in the $m_2 m_3$ -plane: $\{(m_2, m_3) \in \mathbb{R}^2 : 0 \leq m_2 \leq 1, m_3 \leq m_2\}$.

To obtain the parameter space \mathfrak{T}_u , we must identify lines $[a_1, 0, a_3]$ and $[-a_3, 0, -a_1]$, so the line $[1, 0, m_3]$ is identified with the line $[-m_3, 0, -1] = [1, 0, 1/m_3]$. Then if $A = (0, -1), B = (0, 0)$ in the m_2m_3 -plane, we must identify points on the m_3 -axis: $(0, m_3) \in A\infty$ is equivalent to $(0, 1/m_3) \in AB$. (See Figure 1 below.) Similarly, the line $[0, 0, c]$ must agree with the line $[-c, 0, 0] = [1, 0, 0]$, so the point ∞ is equivalent to the point $B = (0, 0)$.

The diagram in Figure 1 illustrates $\mathfrak{N}_u = \mathfrak{T}_u$.

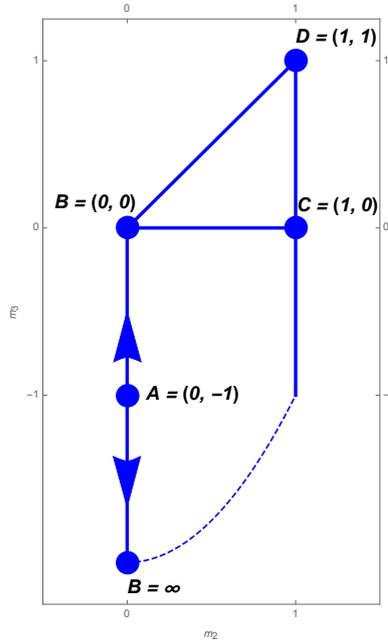


Figure 1: \mathfrak{N}_u

As stated previously, we will identify \mathfrak{T}_u and \mathfrak{N}_u and also treat a point $(m_2, m_3) \in T_u$ as “a point in \mathfrak{N}_u ”.

Up to Lie algebra isomorphism, there are six unimodular three-dimensional Lie algebras. In [21] it is shown that the signs of the parameters a_1, a_2, a_3 determine which of the six algebras corresponds to a given point of \mathfrak{S}_u . We describe here the subregions of \mathfrak{M}_u or \mathfrak{N}_u corresponding to each of the six Lie algebras:

- (1) Sign pattern $(+, +, +)$. Regions $a_1 \geq a_2 \geq a_3 > 0$ in \mathfrak{M}_u or $1 \geq m_2 \geq m_3 > 0$ in \mathfrak{N}_u . The Lie algebra for these regions, $\mathfrak{su}(2)$, is simple and has three-dimensional derived algebra $\mathfrak{g}' = \mathfrak{g}$.
- (2) Sign pattern $(+, +, -)$. Regions $a_1 \geq a_2 > 0, a_3 < 0$ in \mathfrak{M}_u or $1 \geq m_2 > 0, m_3 < 0$ in \mathfrak{N}_u . The Lie algebra for these regions, $\mathfrak{sl}(2, R)$, is also simple with three-dimensional derived algebra $\mathfrak{g}' = \mathfrak{g}$.
- (3) Sign pattern $(+, +, 0)$. Regions $a_1 \geq a_2 > 0, a_3 = 0$ in \mathfrak{M}_u or $1 \geq m_2 > 0, m_3 = 0$ in \mathfrak{N}_u . For these regions, the Lie algebra $\mathfrak{e}(2)$ is solvable with a two-dimensional derived algebra.
- (4) Sign pattern $(+, -, 0)$. Regions $a_1 > 0, a_2 = 0, a_3 < 0$ in \mathfrak{M}_u or $m_2 = 0, m_3 < 0$ in \mathfrak{N}_u . For these regions, the Lie algebra, $\mathfrak{e}(1, 1)$, is again solvable with a two-dimensional derived algebra.

- (5) Sign pattern $(+, 0, 0)$. Regions $a_1 > 0$, $a_2 = a_3 = 0$ (equivalent to $a_1 = a_2 = 0$, $a_3 < 0$) in \mathfrak{M}_u or $m_2 = m_3 = 0$ (equivalent to ∞) in \mathfrak{N}_u . For these regions, the Lie algebra is the Heisenberg algebra, a nilpotent Lie algebra whose one-dimensional derived algebra coincides with its center.
- (6) Sign pattern $(0, 0, 0)$. For the point $a_1 = a_2 = a_3 = 0$ in \mathfrak{M}_u , the Lie algebra is abelian.

3. Structure of \mathfrak{M}_2

In this section we deal with three-dimensional metric Lie algebras (\mathfrak{g}, Q) for which the derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ has dimension at most two. As remarked in the introduction, these are exactly the solvable Lie algebras and they include all non-unimodular Lie algebras. In fact the only three-dimensional Lie algebras with dimension of \mathfrak{g}' equal to three are the unimodular algebras $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, R)$. We also note that up to Lie algebra isomorphism there are only three algebras for which the dimension of \mathfrak{g}' is less than two: A non-unimodular algebra for which \mathfrak{g}' and the center of \mathfrak{g} are both one dimensional and unequal; the Heisenberg algebra for which \mathfrak{g}' equals the center and is one dimensional; and the abelian algebra.

For a metric Lie algebra (\mathfrak{g}, Q) , let $\mathbf{e} = \{e_1, e_2, e_3\}$ be an orthonormal basis and let $P(\mathbf{e})$ be the plane spanned by e_2, e_3 . For any except the abelian case, call \mathbf{e} an *adapted basis* if $P(\mathbf{e})$ contains both \mathfrak{g}' and the center of \mathfrak{g} . (For the abelian case, call any orthonormal basis adapted.) Notice that $P(\mathbf{e}) = \mathfrak{g}'$ in all cases except when dimension \mathfrak{g}' is less than two and that the plane $P(\mathbf{e})$ is determined independently of the basis \mathbf{e} except for the Heisenberg and abelian cases.

For an adapted basis \mathbf{e} , define a linear map $L = L(\mathbf{e}): P(\mathbf{e}) \rightarrow P(\mathbf{e})$ by $L(x) = [e_1, x]$. (L is just $\text{ad}(e_1)$ restricted to $P(\mathbf{e})$.) Notice that, up to sign, the map L is determined independently of the basis \mathbf{e} , again except for the Heisenberg and abelian cases. Then let $[L]_{\mathbf{e}}$ be the matrix of L relative to the basis e_2, e_3 . That is, $[L]_{\mathbf{e}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ where $L(e_2) = [e_1, e_2] = ae_2 + be_3$, $L(e_3) = [e_1, e_3] = ce_2 + de_3$.

We will say that the adapted basis \mathbf{e} *defines* the matrix $[L]_{\mathbf{e}}$. We want to obtain a normal form for this matrix $[L]_{\mathbf{e}}$. Our first result is the following lemma.

Lemma 3.1. *For any metric Lie algebra (\mathfrak{g}, Q) with dimension \mathfrak{g}' at most two, there exists a unique matrix of the form $\begin{bmatrix} a & -b \\ b & d \end{bmatrix}$ with $a \geq |d|$ and $b \geq 0$ which is defined by some adapted basis for \mathfrak{g} .*

Proof. We first show existence of such a matrix. Take any adapted basis \mathbf{b} for (\mathfrak{g}, Q) and let the defined matrix be $A = [L]_{\mathbf{b}} = \begin{bmatrix} a_0 & c_0 \\ b_0 & d_0 \end{bmatrix}$. Write A as a sum of symmetric and anti-symmetric parts, $A = A^s + A^a$. We can diagonalize the symmetric matrix A^s by a rotation $R_\theta: R_\theta^T \circ A^s \circ R_\theta = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ where a, d are the eigenvalues of A^s : $a, d = \frac{1}{2}((a_0 + d_0) \pm \sqrt{(a_0 - d_0)^2 + (b_0 + c_0)^2})$. Notice that A^a is unchanged by any rotation of coordinates, so $R_\theta^T \circ A^a \circ R_\theta = A^a = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$ where $b = (b_0 - c_0)/2$. Thus if $\mathbf{f} = (f_1, f_2, f_3)$ is a new adapted basis with $f_1 = b_1$ and

f_2, f_3 obtained from b_2, b_3 by an appropriate rotation R_θ , then the basis \mathbf{f} defines the matrix $R_\theta^T \circ (A^s + A^a) \circ R_\theta = \begin{bmatrix} a & -b \\ b & d \end{bmatrix}$.

By an extra rotation through 90 degrees, we can interchange the eigenvalues a, d while leaving b unchanged. Thus we can assume $|a| \geq |d|$. Replacing f_1 by $-f_1$ changes the sign of all matrix entries, so we can assume a is positive and $a \geq |d|$. Finally, a reflection replacing f_2 by $-f_2$, changes the sign of the off-diagonal entries while leaving the diagonal entries unchanged. Thus we can assume $b \geq 0$. This shows the existence of the desired matrix.

To show uniqueness, we first assume that \mathfrak{g} is not the Heisenberg or abelian algebra. Then the plane P and, up to sign, the map $L : P \rightarrow P$ are determined independently of the adapted basis \mathbf{b} . Then up to sign and order, a and d are determined as the eigenvalues of the symmetric part A^s of the matrix of L relative to any orthonormal basis for P . But b is also determined up to sign as $b = \frac{b_0 - c_0}{2}$ where b_0, c_0 are from the matrix A of L relative to any orthonormal basis. Then the restrictions $a \geq |d|$ and $b \geq 0$ determine a, b, d uniquely.

If \mathfrak{g} is abelian, then any orthonormal basis is adapted and always defines the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so uniqueness is trivial.

Finally, assume \mathfrak{g} is the Heisenberg algebra. Then $\mathfrak{g}' = \text{center}(\mathfrak{g})$ is one-dimensional. Choose any plane P containing \mathfrak{g}' and any adapted frame \mathbf{b} for which the plane spanned by b_2, b_3 is P . Let $L : P \rightarrow P$ be the associated map. We can modify the basis so that $b_3 \in \mathfrak{g}'$. Then since b_3 is also in the center, the basis defines a matrix of the form $A = \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}$ for some real x . Arguing as above with the symmetric

and anti-symmetric parts of A , we find a unique matrix of the form $\begin{bmatrix} a & -b \\ b & d \end{bmatrix}$ with $a \geq |d|$ and $b \geq 0$ defined by an adapted basis with b_2, b_3 spanning P . In fact, we must have $a = |x|/2$, $d = -|x|/2$ and $b = |x|/2$. So to prove uniqueness it suffices to show that any plane P containing \mathfrak{g}' must produce the same value of x up to sign. But if $(\bar{b}_1, \bar{b}_2, \bar{b}_3)$ is any other adapted basis with $\bar{b}_3 \in \mathfrak{g}'$ then b_1, b_2 and \bar{b}_1, \bar{b}_2 are both orthonormal bases for the plane orthogonal to \mathfrak{g}' and are thus related by an orthogonal transformation (rotation or rotation and reflection). A computation then shows that $[b_1, b_2] = \pm[\bar{b}_1, \bar{b}_2]$. But $[b_1, b_2] = xb_3$, $[\bar{b}_1, \bar{b}_2] = \bar{x}\bar{b}_3$ and $b_3 = \pm\bar{b}_3$. It follows that $\bar{x} = \pm x$ as desired. ■

We could use the matrix entries a, b, d given by Lemma 3.1 as parameters for \mathfrak{M}_2 , but we prefer to transform to a different normal form for the matrix A .

Lemma 3.2. *For any metric Lie algebra (\mathfrak{g}, Q) with dimension \mathfrak{g}' at most two, there exists a unique matrix of the form $\begin{bmatrix} a & c \\ b & a \end{bmatrix}$ with $b \geq |c|$ and $a \geq 0$ which is defined by some adapted basis for \mathfrak{g} .*

Proof. To show existence of such a matrix, first apply Lemma 3.1 to obtain an adapted basis $\mathbf{f} = (f_1, f_2, f_3)$ which defines a matrix $\begin{bmatrix} a_0 & -b_0 \\ b_0 & d_0 \end{bmatrix}$ with $a_0 \geq |d_0|$ and $b_0 \geq 0$. So $[f_1, f_2] = a_0 f_2 + b_0 f_3$ and $[f_1, f_3] = -b_0 f_2 + d_0 f_3$.

Change to a new orthonormal basis \mathbf{e} where

$$e_1 = f_1, \quad e_2 = \frac{f_2 - f_3}{\sqrt{2}}, \quad e_3 = \frac{f_2 + f_3}{\sqrt{2}} \quad (2)$$

$$f_1 = e_1, \quad f_2 = \frac{e_2 + e_3}{\sqrt{2}}, \quad f_3 = \frac{e_3 - e_2}{\sqrt{2}}. \quad (3)$$

(\mathbf{f} is related to \mathbf{e} by a 45 degree rotation in the plane P followed by a reflection.)

Computing brackets yields $[e_1, e_2] = ae_2 + be_3$ and $[e_1, e_3] = ce_2 + ae_3$ where

$$a = \frac{a_0 + d_0}{2}, \quad b = b_0 + \frac{a_0 - d_0}{2}, \quad c = -b_0 + \frac{a_0 - d_0}{2}. \quad (4)$$

It is not hard to check that $a_0 \geq |d_0|, b_0 \geq 0 \Rightarrow a \geq 0, b \geq |c|$, so $\begin{bmatrix} a & c \\ b & a \end{bmatrix}$ is the desired matrix defined by the adapted basis \mathbf{e} .

To show uniqueness, suppose that \mathbf{e} is an orthonormal basis defining a matrix $\begin{bmatrix} a & c \\ b & a \end{bmatrix}$ with $a \geq 0, b \geq |c|$. Then computing brackets we find that the equations

(2) and (3) give an orthonormal basis \mathbf{f} defining a matrix of the form $\begin{bmatrix} a_0 & -b_0 \\ b_0 & d_0 \end{bmatrix}$ with $a_0 \geq |d_0|$ and $b_0 \geq 0$. In fact we have

$$a_0 = a + \frac{b + c}{2}, \quad b_0 = \frac{b - c}{2}, \quad d_0 = a - \frac{b + c}{2}. \quad (5)$$

The maps $(b_0, a_0, d_0) \rightarrow (a, b, c)$ and $(a, b, c) \rightarrow (b_0, a_0, d_0)$ given by equations (4) and (5) are inverses of each other and define a one-to-one correspondence from the region $x_1 \geq 0, x_2 \geq |x_3|$ in \mathbb{R}^3 to itself. So distinct matrices of the form $\begin{bmatrix} a & b \\ c & a \end{bmatrix}$ defined by adapted bases for (\mathfrak{g}, Q) would result in distinct matrices of the form $\begin{bmatrix} a_0 & -b_0 \\ b_0 & d_0 \end{bmatrix}$ defined by adapted bases, contradicting the uniqueness assertion in Lemma 3.1. ■

Definition 3.3. An adapted basis \mathbf{e} is a *standard basis* for (\mathfrak{g}, Q) if the matrix $[L]_{\mathbf{e}}$ has equal diagonal entries, that is, $[L]_{\mathbf{e}} = \begin{bmatrix} a & c \\ b & a \end{bmatrix}$. We say the basis \mathbf{e} *defines the triple* (a, b, c) .

Lemma 3.2 can then be rephrased to give

Corollary 3.4. *For every metric Lie algebra (\mathfrak{g}, Q) with dimension \mathfrak{g}' at most two there is a unique triple (a, b, c) with $a \geq 0, b \geq |c|$ such that some standard basis defines the triple.*

Define a subset of \mathbb{R}^3 by $\mathfrak{S}_2 = \{(a, b, c) \in \mathbb{R}^3 : a \geq 0, b \geq |c|\}$ and let \mathfrak{T}_2 be the corresponding region of the projective plane, $\mathfrak{T}_2 = \{[a, b, c] \in P^2 : a \geq 0, b \geq |c|\}$.

Proposition 3.5. *There are bijections $\mathfrak{M}_2 \iff \mathfrak{S}_2$ and $\mathfrak{N}_2 \iff \mathfrak{T}_2$.*

Similarly to the unimodular case, \mathfrak{S}_2 and \mathfrak{T}_2 have natural topologies and stratifications as subsets of \mathbb{R}^3 and P^2 . We will again use the proposition to identify \mathfrak{M}_2 with \mathfrak{S}_2 and \mathfrak{N}_2 with \mathfrak{T}_2 , thus giving a topology and stratification to \mathfrak{M}_2 and \mathfrak{N}_2 .

We will again often refer to a triple $(a, b, c) \in \mathfrak{S}_2$ as “a point in \mathfrak{M}_2 ” and a point $[a, b, c] \in \mathfrak{T}_2$ as “a point in \mathfrak{N}_2 ”. The “triple” defined by a standard basis again consists of the (nontrivial) Lie algebra structure constants relative to the given orthonormal basis. We now prove the proposition.

Proof. Writing $\mathfrak{S}_2 = \{(a, b, c) \in \mathbb{R}^3 : a \geq 0, b \geq |c|\}$, Corollary 3.4 says that each metric Lie algebra defines a triple corresponding to a unique point in \mathfrak{S}_2 . Also, for any point in \mathfrak{S}_2 there exists a metric Lie algebra defining the triple: Take any basis $\{e_1, e_2, e_3\}$ for \mathbb{R}^3 , define a metric by declaring the basis to be orthonormal, and define a bracket by $[e_1, e_2] = ae_2 + be_3$, $[e_1, e_3] = ce_2 + ae_3$, $[e_2, e_3] = 0$. The Jacobi identity is easily checked so we have a metric Lie algebra which defines (a, b, c) . Thus the map from our Lie algebras to \mathfrak{S}_2 is surjective.

Next, let $F : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ be an isomorphism of metric Lie algebras and $\{e_1, e_2, e_3\}$ be a standard basis for \mathfrak{g}_1 defining the triple $(a, b, c) \in \mathfrak{S}_2$. Then, since F is an isometry, $\{F(e_1), F(e_2), F(e_3)\}$ is an orthonormal basis for \mathfrak{g}_2 . But it must also be a standard basis for \mathfrak{g}_2 defining the same triple (a, b, c) (since F is a Lie algebra isomorphism). So isomorphic metric Lie algebras map to the same point of \mathfrak{S}_2 and there is a well-defined (surjective) map $\mathfrak{M}_2 \rightarrow \mathfrak{S}_2$.

Finally, we claim that the map $\mathfrak{M}_2 \rightarrow \mathfrak{S}_2$ is injective. Suppose $\mathfrak{g}_1, \mathfrak{g}_2$ are metric Lie algebras with standard bases $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ which both define the same triple $(a, b, c) \in \mathfrak{S}_2$. Define $F : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ to be the linear map such that $F(e_i) = f_i$. Then F is an isometry (preserves orthonormal bases) and we claim F is also a Lie algebra isomorphism (preserves brackets). For example, $[F(e_1), F(e_2)] = [f_1, f_2] = af_2 + bf_3 = F(ae_2 + be_3) = F([e_1, e_2])$. So $\mathfrak{g}_1, \mathfrak{g}_2$ are in the same isomorphism class of \mathfrak{M}_2 .

We conclude that there is a one to one correspondence between the sets \mathfrak{M}_2 and \mathfrak{S}_2 . Use this identification to transfer the natural topology and stratification of \mathfrak{S}_2 to \mathfrak{M}_2 . This proves the first part of the proposition.

As in the unimodular case, scaling the metric on a metric Lie algebra (by $\frac{1}{k^2}$) replaces a standard orthonormal basis $\{e_1, e_2, e_3\}$ by $\{ke_1, ke_2, ke_3\}$ for some positive k , which has the effect of replacing the defined triple (a, b, c) by (ka, kb, kc) . So all nonzero triples on a ray through the origin correspond to equivalent metric Lie algebras up to scaling. If a triple (a, b, c) is in \mathfrak{S}_2 , then so is (ka, kb, kc) for any positive k . So \mathfrak{N}_2 can be identified with the set \mathfrak{T}_2 in the projective plane. This completes the proof of the proposition. ■

We can give a better description of the set \mathfrak{T}_2 as a subset of the ordinary plane with some “points at infinity”. A ray through (a, b, c) which does not lie in the plane $a = 0$ will intersect the plane $a = 1$ in the point $(1, r, s) = (1, b/a, c/a)$. So these rays in \mathfrak{T}_2 correspond to the region $\{(r, s) : r \geq |s|\}$ in the plane $a = 1$. The remaining rays in \mathfrak{T}_2 lie in the plane $a = 0$ and can be thought of as points on “the line at infinity”. These rays have the form $\{(0, b, mb) : b > 0\}$ where $1 \geq |m|$, so they can be identified with the interval $-1 \leq m \leq +1$. Notice that for $-1 \leq m \leq +1$, points $(r, mr) \in \mathfrak{T}_2$ converge (in the projective plane) to the ray $\{(0, b, mb) : b > 0\}$ as $r \rightarrow \infty$. \mathfrak{T}_2 is illustrated in the right hand picture in Figure 2 below.

We next examine the intersection $\mathfrak{M}_2 \cap \mathfrak{M}_u$ (and $\mathfrak{N}_2 \cap \mathfrak{N}_u$). First note that for a standard orthonormal basis $\{e_1, e_2, e_3\}$ for a point $(\mathfrak{g}, Q) \in \mathfrak{M}_2$,

$$e_2, e_3 \in \mathfrak{g}' \Rightarrow \text{trace}(\text{ad}(e_2)) = \text{trace}(\text{ad}(e_3)) = 0.$$

So \mathfrak{g} is unimodular if and only if $\text{trace}(\text{ad}(e_1)) = 0$. But if the basis defines a triple (a, b, c) , then $\text{trace}(\text{ad}(e_1)) = 2a$. So if the equivalence class of (\mathfrak{g}, Q) is in \mathfrak{M}_2 , then \mathfrak{g} is unimodular if and only if (\mathfrak{g}, Q) defines a triple $(0, b, c)$ with $a = 0$ (and $b \geq |c|$). Then computing brackets shows that, as a *unimodular* metric Lie algebra, (\mathfrak{g}, Q) defines the triple $(a_1, a_2, a_3) = (b, -c, 0)$ when $c < 0$ and the triple $(a_1, a_2, a_3) = (b, 0, -c)$ when $c \geq 0$. That is, the point $(0, b, c) \in \mathfrak{S}_2$ is identified with the point $(a_1, a_2, a_3) = (b, -c, 0) \in \mathfrak{S}_u$ when $c < 0$ and the point $(a_1, a_2, a_3) = (b, 0, -c) \in \mathfrak{S}_u$ when $c \geq 0$. Similarly, the point m (that is, the point $[0, 1, m]$) in \mathfrak{T}_2 is identified with the point $(m_2, m_3) = (-m, 0)$ (that is, the point $[a_1, a_2, a_3] = [1, -m, 0]$) in \mathfrak{T}_u when $m < 0$ and the point $(m_2, m_3) = (0, -m)$ (that is, $[a_1, a_2, a_3] = [1, 0, -m]$) in \mathfrak{T}_u when $m \geq 0$.

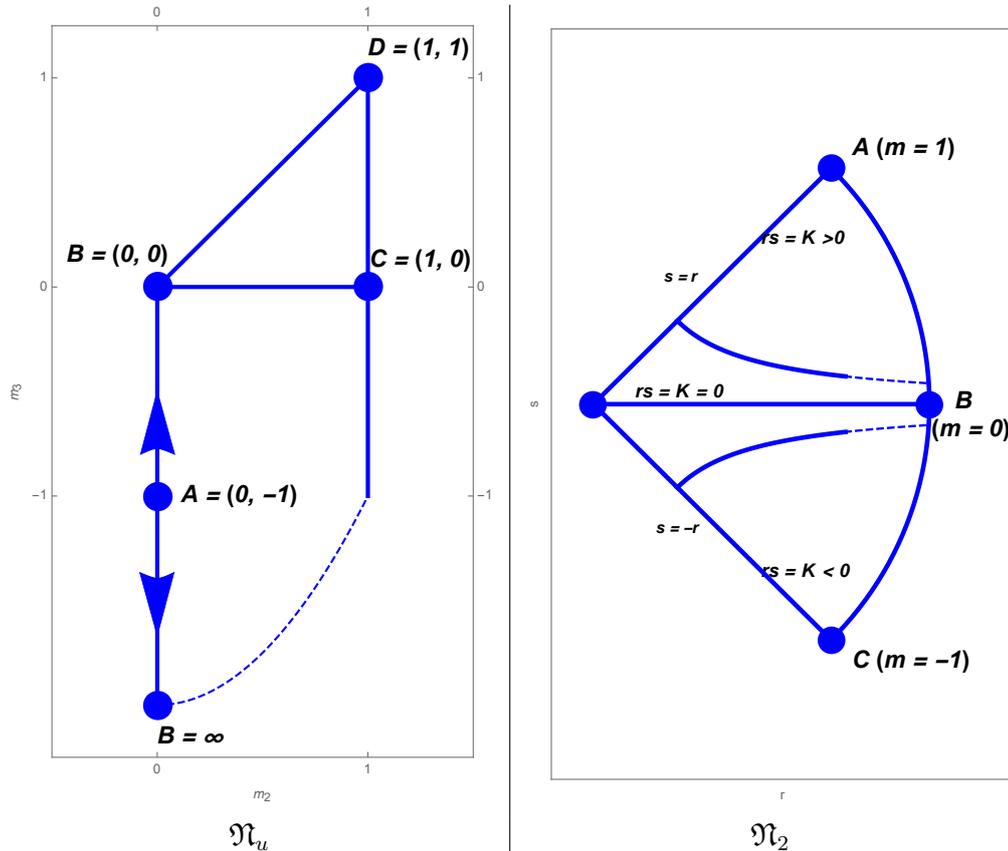


Figure 2: Diagrams for \mathfrak{N}

We use these identifications to obtain \mathfrak{M} as the union of \mathfrak{M}_u and \mathfrak{M}_2 with appropriate strata identified. Similarly, we can write \mathfrak{N} as the union of \mathfrak{N}_u and \mathfrak{N}_2 with appropriate strata identified.

In the two-dimensional case we can illustrate \mathfrak{N} with the diagrams in Figure 2. Recall that in the diagram for \mathfrak{N}_u the point ∞ is to be identified with the point $B = (0, 0)$ and each point $(0, -c)$ for $c > 1$ is identified with the point $(0, -1/c)$ in the interval AB . Then the points $A = (0, -1), B = (0, 0), C = (1, 0)$ in \mathfrak{N}_u are identified with the corresponding points $A(m = 1), B(m = 0), C(m = -1)$ in \mathfrak{N}_2 . A point $(0, -c)$ on AB in \mathfrak{N}_u is identified with the point $m = c$ on AB in \mathfrak{N}_2 . Finally, a point $(b, 0)$ BC in \mathfrak{N}_u is identified with the point $m = -b$ on BC in \mathfrak{N}_2 .

Now consider the Lie algebras corresponding to various strata of \mathfrak{M}_2 and \mathfrak{N}_2 . First, for triples $(0, b, mb)$ with $b > 0$ and $-1 \leq m \leq +1$ in \mathfrak{M}_2 or for “points

at infinity” with $-1 \leq m \leq +1$ in \mathfrak{N}_2 , the Lie algebra is one of the unimodular algebras described in Section 2 above. For $-1 \leq m < 0$ the Lie algebra is $\mathfrak{e}(2)$. For $0 < m \leq +1$, the Lie algebra is $\mathfrak{e}(1,1)$. For $m = 0$, the Lie algebra is the Heisenberg algebra. Of course if $a = b = c = 0$ the algebra is abelian.

Next, consider a non-unimodular metric Lie algebra \mathfrak{g} defining a triple (a, b, c) in \mathfrak{M}_2 with $a > 0, b \geq |c|$. For the case $b = c = 0$, \mathfrak{g} is the special case of a (nonunimodular) Lie algebra with a basis $\{b_1, b_2, b_3\}$ for which $[b_1, b_2] = b_2, [b_1, b_3] = b_3, [b_2, b_3] = 0$. We will denote this Lie algebra by \mathfrak{g}_I .

Turning to the cases where $b > 0$, define $K = K(\mathfrak{g}) = \frac{bc}{a^2}$.

Lemma 3.6. *For $i = 1, 2$, let (\mathfrak{g}_i, Q_i) be non-unimodular metric Lie algebras defining triples (a_i, b_i, c_i) with $a_i > 0, b_i > 0$. Then \mathfrak{g}_1 and \mathfrak{g}_2 are isomorphic as Lie algebras if and only if $K(\mathfrak{g}_1) = K(\mathfrak{g}_2)$.*

Thus for each real K there is a Lie algebra $\mathfrak{g}(K)$ such that the Lie algebras isomorphic to $\mathfrak{g}(K)$ are exactly those with $K(\mathfrak{g}) = K$. (This was essentially noticed by Milnor in [21].)

Proof. For $i = 1, 2$, assume the metric Lie algebra \mathfrak{g}_i has a standard basis $\{e_{i,1}, e_{i,2}, e_{i,3}\}$ defining a triple (a_i, b_i, c_i) so that $K_i = K(\mathfrak{g}_i) = \frac{b_i c_i}{a_i^2}$. Let P_i be the standard abelian subalgebra of \mathfrak{g}_i spanned by $e_{i,2}, e_{i,3}$. Put $f_{i,1} = e_{i,1}/a_i$ and define a linear map $L_i : P_i \rightarrow P_i$ by $L_i(x) = [f_{i,1}, x]$. Define a new basis for P_i by $f_{i,2} = e_{i,2}, f_{i,3} = [f_{i,1}, e_{i,2}] = e_{i,2} + \frac{b_i}{a_i} e_{i,3}$. Then a little computation gives $[f_{i,1}, f_{i,2}] = f_{i,3}, [f_{i,1}, f_{i,3}] = (K_i - 1) f_{i,2} + 2f_{i,3}$. It follows at once that if $K_1 = K_2$, then the map $F : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ defined by $F(f_{1,j}) = f_{2,j}$ is a Lie algebra isomorphism.

To prove the converse, suppose $F : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a Lie algebra isomorphism and define $\bar{f}_i = F(f_{1,i}) \in \mathfrak{g}_2$. F must map P_1 onto P_2 . Define $\bar{L} : P_2 \rightarrow P_2$ by $\bar{L}(x) = [\bar{f}_1, x]$. Then since F is a Lie algebra isomorphism, we have $\bar{L}(F(x)) = [F(f_{1,1}), F(x)] = F([f_{1,1}, x]) = F(L_1(x))$ for $x \in M_1$. Now the matrix for L_i relative to the basis $f_{i,2}, f_{i,3}$ given above is $A_i = \begin{bmatrix} 0 & 1 \\ K_i - 1 & 2 \end{bmatrix}$, which has trace 2 and determinant $1 - K_i$. The matrix for \bar{L} relative to the basis \bar{f}_2, \bar{f}_3 is A_1 , so it also has trace 2 and determinant $1 - K_1$. We claim that in fact $\bar{L} = L_2$ and then $\det(\bar{L}) = \det(L_2) \Rightarrow 1 - K_1 = 1 - K_2 \Rightarrow K_1 = K_2$ as desired. To see that $\bar{L} = L_2$, notice that we must have $\bar{f}_1 = \alpha f_{2,1} + v$ for some nonzero $\alpha \in \mathbb{R}$ and $v \in P_2$. Then for $x \in P_2$ we have $\bar{L}(x) = [\alpha f_{2,1} + v, x] = \alpha [f_{2,1}, x] = \alpha L_2(x)$. So $\bar{L} = \alpha L_2$ is a nonzero multiple of L_2 . But since both have trace 2, we must have $\alpha = 1$, so $\bar{L} = L_2$ as claimed. ■

By the lemma, for points (a, b, c) in \mathfrak{M}_2 with $a > 0$ and $\frac{bc}{a^2} = K$, or for points (r, s) in \mathfrak{N}_2 on the hyperbola $rs = K$, the Lie algebra is the (non-unimodular) algebra $\mathfrak{g}(K)$ when $K \neq 0$.

When $K = 0$, the Lie algebra is still $\mathfrak{g}(0)$ as long as $b > 0$ (or $r > 0$). Points $(a, 0, 0), a > 0$ in \mathfrak{M}_2 or $(0, 0)$ in \mathfrak{N}_2 correspond to the special Lie algebra \mathfrak{g}_I .

All of these Lie algebras are non-unimodular. The derived algebras are two-dimensional except for the case $K = 1$. The Lie algebra $\mathfrak{g}(1)$ has a one-dimensional derived algebra and a disjoint one-dimensional center.

4. Ricci flow and Ricci solitons

The Ricci flow equation on a manifold M is $\frac{\partial g(t)}{\partial t} = -2R(t)$ where $g(t)$ is a (Riemannian) metric on M and $R(t)$ is the Ricci curvature tensor corresponding to $g(t)$. The Ricci flow on M with starting metric g_0 is a solution $g(t)$, $t \in I$, of the Ricci flow equation where I is some open interval containing 0 and $g(0) = g_0$. If $g(t)$ is a left invariant metric on a Lie group, then both $g(t)$ and the corresponding Ricci tensor $R(t)$ are determined by their values on the tangent space at the identity, that is, on the Lie algebra \mathfrak{g} . So they are determined by bilinear functions $g(t): \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$ and $R(t): \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$. (Caution: Note that we use g and R for both the tensors on the Lie group and the corresponding functions on the Lie algebra. The context should determine which is meant.) The bilinear function $g(t)$ on \mathfrak{g} defines the inner product $Q(t)$ for the corresponding metric Lie algebra and we will often refer to $g(t)$ (rather than $Q(t)$) as the metric on \mathfrak{g} . For a Ricci flow with left invariant starting metric g_0 on the Lie group, each $g(t)$ will be left invariant and hence will be determined by the corresponding function $g(t)$ on the Lie algebra. The Lie group flow then corresponds to the flow on the Lie algebra, governed by the ordinary differential equation $\frac{dg(t)}{dt} = -2R(t)$, $g(0) = g_0$, defined on the algebra \mathfrak{g} .

For our three-dimensional Lie algebras, such a Ricci flow should give a smoothly varying family of standard orthonormal bases $\{e_i(t), i = 1, 2, 3\}$ and then a smoothly varying family of standard parameters, either $\{a_i(t), i = 1, 2, 3\}$ or $\{a(t), b(t), c(t)\}$. We can then translate the Ricci flow DE into a system of DE's involving these standard parameters giving the Ricci flow on $\mathfrak{M}_u, \mathfrak{N}_u$ or on $\mathfrak{M}_2, \mathfrak{N}_2$.

A Ricci soliton on a manifold M is a metric g which remains “constant up to diffeomorphism and scaling” under the Ricci flow on M . That is, it is a metric g for which there exists a one-parameter family of diffeomorphisms φ_t of M and a smooth scalar function c such that $g(t) = c(t)\varphi_t^*g$ is a solution of the Ricci equation with initial value $g(0) = g$. If M is a Lie group G and the Ricci soliton g is a left invariant metric on G , then the corresponding Ricci flow $g(t)$ on the Lie algebra \mathfrak{g} with initial value $g(0) = g$ will satisfy $g(t) = c(t)d\varphi_t^*g$ where $d\varphi_t: \mathfrak{g} \rightarrow \mathfrak{g}$ is the differential of φ_t . So we define a metric g (corresponding to an inner product Q) on a Lie algebra \mathfrak{g} to be a *Ricci soliton metric* if there exists a one-parameter family of diffeomorphisms φ_t of the corresponding simply-connected Lie group G and a scalar function c such that $g(t) = c(t)d\varphi_t^*g$ is a solution of the Ricci equation $\frac{dg(t)}{dt} = -2R(t)$ on \mathfrak{g} with initial condition $g(0) = g$.

So g is a Ricci soliton metric on \mathfrak{g} if and only if the corresponding left invariant metric g is a Ricci soliton on the simply-connected G . Many authors have dealt with these left invariant Ricci solitons on Lie groups, including Jablonski [10], Lauret [19], [20], and Payne [22], [23].

Now suppose (\mathfrak{g}, Q) is a metric Lie algebra corresponding to a fixed point of the Ricci flow on \mathfrak{N} . Then as the metric g (corresponding to the inner product Q) evolves, it stays constant up to Lie algebra isomorphism and scaling. So there exists a smoothly varying family of Lie algebra isomorphisms $f_t: \mathfrak{g} \rightarrow \mathfrak{g}$ and a scalar function c such that $g(t) = c(t)f_t^*g$ is a solution of the Ricci equation with $g(0) = g$. But a Lie algebra isomorphism f_t lifts uniquely to a Lie group isomorphism $F_t: G \rightarrow G$ where G is the simply-connected Lie group with Lie algebra \mathfrak{g} and $f_t = dF_t$. Also each metric $g(t)$ lifts to a left invariant metric $g(t)$ on G such that $g(t) = c(t)F_t^*g$ and

the family $g(t)$ satisfies the Ricci equation on G with initial condition g . Since a Lie group isomorphism is in particular a diffeomorphism, g is a Ricci soliton on G and g is a Ricci soliton metric on \mathfrak{g} . We will call a Ricci soliton on a Lie group G a *special* Ricci soliton if the diffeomorphisms $\varphi_t : G \rightarrow G$ are actually Lie group isomorphisms.

For Lie algebras, we define a metric g on \mathfrak{g} to be a *special* Ricci soliton metric if there exists a one-parameter family of Lie algebra isomorphisms f_t of \mathfrak{g} and a scalar function c such that $g(t) = c(t)f_t^*g$ is a solution of the Ricci equation $\frac{dg(t)}{dt} = -2R(t)$ on \mathfrak{g} with initial condition $g(0) = g$.

Then a metric g on \mathfrak{g} is a special Ricci soliton if and only if the corresponding left invariant metric on the simply-connected Lie group G is a special Ricci soliton. We have just seen that special Ricci solitons occur for metric Lie algebras corresponding to fixed points of the Ricci flow on \mathfrak{N} . We will argue in Section 7 that every Ricci soliton metric on a three-dimensional Lie algebra is in fact a special soliton (and so every left invariant Ricci soliton on a simply-connected three dimensional Lie group is a special soliton).

Remark 4.1. For certain solvable Lie algebras this last fact can also be proved using results in Böhm and Lafuente [3]. Specifically, all solvable, non-abelian, three-dimensional Lie algebras except for $\mathfrak{c}(2)$ are of so-called *real type*. Then for each of these algebras of real type except $\mathfrak{g}(0)$ (the one corresponding to $K = 0$) we will find in Section 6 a non-flat, special Ricci soliton (a fixed point of the Ricci flow on \mathfrak{N}_2 , called a *solvsoliton* in this context). A result in [3] then implies that any Ricci soliton on such a Lie algebra must be special (and in fact is unique up to scaling). ■

A metric g on the Lie algebra \mathfrak{g} is called *semi-algebraic* if it satisfies an equation $\hat{R} = cI + D + D^*$ where \hat{R} is the Ricci operator corresponding to g , c is a constant, I is the identity, the linear operator D is a derivation of the Lie algebra \mathfrak{g} (that is, $D : \mathfrak{g} \rightarrow \mathfrak{g}$ is a linear map with $D([x, y]) = [Dx, y] + [x, Dy]$ for all $x, y \in \mathfrak{g}$) and D^* is the adjoint of D (that is, $g(D^*x, y) = g(x, Dy)$ for all x, y). If the derivation D is self-adjoint ($D^* = D$), then the metric g is defined to be *algebraic* (Lauret [16]).

It can be shown that semi-algebraic metrics correspond exactly to special Ricci solitons as defined above (and are therefore usually called semi-algebraic solitons). A theorem of Jablonski [13] shows that any semi-algebraic metric is actually algebraic, so special Ricci solitons in fact correspond to algebraic metrics (therefore called algebraic solitons). (One can also verify by direct computation that all the special Ricci solitons corresponding to fixed points of the Ricci flow on \mathfrak{N} are actually algebraic solitons.)

We observe that a weaker concept of algebraic or semi-algebraic soliton is often defined (see for example Jablonski [11],[12]). Jablonski [11] shows that all Ricci solitons on simply-connected Lie groups are algebraic in this weaker sense, not in our stronger sense of algebraic.

Among the many authors dealing with solvsolitons and algebraic Ricci solitons are Jablonski [10], [11], [12], [13], Lauret [17], [18], and Lafuente and Lauret [14], [15].

5. Ricci flow on \mathfrak{M}_u

Consider a unimodular metric Lie algebra with a standard basis $\{e_i, i = 1, 2, 3\}$ defining a triple (a_1, a_2, a_3) in \mathfrak{S}_u . (This corresponds to the case worked out by Glickenstein and Payne in [7].) As noted by Milnor [21], the Ricci matrix is diagonal relative to the standard basis: $R = \text{diag}(R_{11}, R_{22}, R_{33})$, where

$$R_{11} = \frac{a_1^2 - (a_2 - a_3)^2}{2}, \quad R_{22} = \frac{a_2^2 - (a_3 - a_1)^2}{2}, \quad R_{33} = \frac{a_3^2 - (a_1 - a_2)^2}{2}. \quad (6)$$

Choose a fixed initial time u (often time 0), and let t be a variable time. The diagonal form of the Ricci matrix suggests trying a diagonal transition matrix $M(u, t) = \text{diag}(\alpha_1, \alpha_2, \alpha_3)$ relating the standard bases at the two times u, t . That is, $e_i(u) = \alpha_i(t)e_i(t)$. Then the metric matrix $g^u(t)$ at each time t expressed in terms of the basis at time u is diagonal: $g_{ii}^u(t) = \alpha_i^2(t)$ for $i = 1, 2, 3$. Let $R(t)$ be the (diagonal) Ricci matrix a time t expressed in terms of the standard basis at time t and $R^u(t)$ be this matrix expressed in terms of the basis at time u . Then $R^u(t) = M^T R(t)M$ is also diagonal with entries $R_{ii}^u(t) = \alpha_i^2 R_{ii}(t)$ (where the entries $R_{ii}(t)$ are given in equations (6)). The Ricci flow equation $\frac{dg^u(t)}{dt} = -2R^u(t)$ then gives three equations: For $i = 1, 2, 3$, $2\alpha_i \alpha_i' = -2\alpha_i^2 R_{ii}$ or

$$2 \log(\alpha_i)' = -2R_{ii}. \quad (7)$$

To relate the transition functions $\alpha_i(t)$ to the parameters $a_i(t)$ we calculate the bracket $[e_2(u), e_3(u)]$ in two ways. On the one hand,

$$[e_2(u), e_3(u)] = a_1(u)e_1(u) = \alpha_1(t)a_1(u)e_1(t).$$

But also $[e_2(u), e_3(u)] = \alpha_2(t)\alpha_3(t)[e_2(t), e_3(t)] = \alpha_2(t)\alpha_3(t)a_1(t)e_1(t)$.

So $\alpha_1(t)a_1(u) = \alpha_2(t)\alpha_3(t)a_1(t)$ or $a_1(t) = \left(\frac{\alpha_1}{\alpha_2\alpha_3}\right)a_1(u)$. Similarly we find

$$a_i(t) = \left(\frac{\alpha_i}{\alpha_j\alpha_k}\right)a_i(u), \quad (8)$$

where i, j, k is any permutation of 1, 2, 3. Differentiating (8) gives

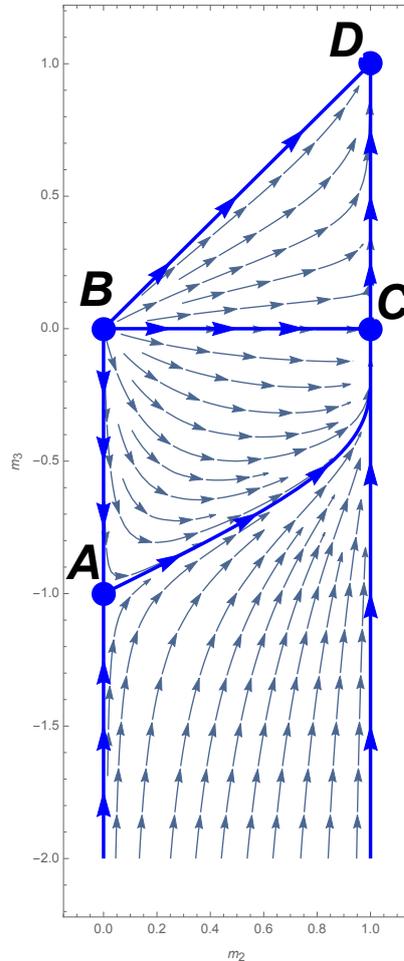
$$a_i'(t) = \left(\frac{\alpha_i}{\alpha_j\alpha_k}\right) \left(\log\left(\frac{\alpha_i}{\alpha_j\alpha_k}\right)\right)' a_i(u) = a_i(t) (\log(\alpha_i)' - \log(\alpha_j)' - \log(\alpha_k)').$$

Then (7) gives $a_i' = a_i(-R_{ii} + R_{jj} + R_{kk})$ (9)

(i, j, k any permutation of 1, 2, 3). Then, using equations (6) to express the R_{ii} in terms of the a_i , the equations (9) give the desired system of three DE's governing the Ricci flow on \mathfrak{S}_u .

To investigate the flow on $\mathfrak{N}_u = \mathfrak{T}_u$ we assume $a_1(t) \neq 0$ and put $m_i = a_i/a_1$ for $i = 2, 3$. Using (8) we find $m_i = \frac{\alpha_i^2 a_i(u)}{\alpha_1^2 a_1(u)}$. Then

$$m_i' = \left(\frac{\alpha_i^2}{\alpha_1^2}\right)' \left(\frac{a_i(u)}{a_1(u)}\right) = m_i \left(\log\left(\frac{\alpha_i^2}{\alpha_1^2}\right)\right)' = m_i (2 \log(\alpha_i)' - 2 \log(\alpha_1)').$$

Figure 3: Ricci flow on \mathfrak{N}_u

By (7), this gives $m'_i = 2m_i(R_{11} - R_{ii})$ for $i = 2, 3$. Using (6) and simplifying then yields $m'_i = 2m_i(a_1 - a_i)(a_1 + a_i - a_j) = 2a_1^2 m_i(1 - m_i)(1 + m_i - m_j)$. Explicitly, this gives the system found in [7] (up to a factor of 2, which does not affect the results) :

$$\begin{aligned} \frac{dm_2}{dt} &= 2a_1^2 m_2(1 - m_2)(1 + m_2 - m_3) \\ \frac{dm_3}{dt} &= 2a_1^2 m_3(1 - m_3)(1 + m_3 - m_2) \end{aligned} \tag{10}$$

This pair of DE's describes the Ricci flow on $\mathfrak{T}_u = \mathfrak{N}_u$. Recall that this is the region of the $m_2 m_3$ plane given by $\{(m_2, m_3) \in \mathbb{R}^2 : 0 \leq m_2 \leq 1, m_3 \leq m_2\}$.

Glickenstein and Payne in [7] examine the dynamics of this system in detail. As they point out, the positive factor a_1^2 simply alters the velocity with which the metric moves along a trajectory in the $m_2 m_3$ plane. There are fixed points at $A = (0, -1)$, $B = (0, 0)$, $C = (1, 0)$, and $D = (1, 1)$. Lines preserved by the flow are $m_2 = 0$, $m_3 = 0$, $m_2 = 1$, and $m_2 = m_3$. The flow on $\mathfrak{N}_u = \mathfrak{T}_u$ is summarized in Figure 3 above (similar to the flow diagrams found in [7]). Note that the point $\infty \in \mathfrak{T}_u$ for which $a_1 = 0$ is equivalent to the fixed point $(m_2, m_3) = (0, 0) = B$. Also, as discussed above, the fixed points of the flow on $\mathfrak{N}_u = \mathfrak{T}_u$ correspond to Ricci solitons on the associated simply-connected Lie groups.

6. Ricci flow on \mathfrak{M}_2

We now consider the Ricci flow for the metric Lie algebras in \mathfrak{M}_2 . For a (solvable) metric Lie algebra (\mathfrak{g}, Q) with the dimension of the derived algebra \mathfrak{g}' at most two, recall the definition from Section 3 of a standard basis for (\mathfrak{g}, Q) defining a triple (a, b, c) . The first result is the following lemma.

Lemma 6.1. *Let (\mathfrak{g}, Q) be a solvable metric Lie algebra. If a standard basis for (\mathfrak{g}, Q) defines the triple (a, b, c) , then the Ricci matrix relative to this standard basis is*

$$R = \begin{bmatrix} -2a^2 - \frac{(b+c)^2}{2} & 0 & 0 \\ 0 & -2a^2 - \frac{b^2-c^2}{2} & -a(b+c) \\ 0 & -a(b+c) & -2a^2 - \frac{c^2-b^2}{2} \end{bmatrix}.$$

The proof is a straight-forward but tedious exercise which we omit.

We again consider a fixed initial time u and a variable time t . The form of the Ricci matrix suggests the transition matrix $M(u, t)$ between two standard bases at times

t and u should have the form $M(u, t) = \begin{bmatrix} \tau & 0 & 0 \\ 0 & \alpha & \gamma \\ 0 & \beta & \delta \end{bmatrix}$.

That is $e_1(u) = \tau e_1(t)$, $e_2(u) = \alpha e_2(t) + \beta e_3(t)$, $e_3(u) = \gamma e_2(t) + \delta e_3(t)$.

We must relate the triples (a, b, c) at times u and t to these transition parameters. As in Section 3, for each time t we write P_t for the special plane spanned by $(e_2(t), e_3(t))$ and L_t for the linear map $L_t : P_t \rightarrow P_t$ defined by $L_t(x) = [e_1(t), x]$. (L_t is $\text{ad}(e_1(t))$ restricted to P_t .) Evidently L_t has trace $2a(t)$, while, for the fixed time u , L_u has trace $2a(u)$. Then $e_1(u) = \tau e_1(t) \Rightarrow L_u = \tau L_t \Rightarrow 2a(u) = \tau 2a(t)$, so

$$a(u) = \tau a(t). \tag{11}$$

Next we compute $[e_1(u), e_2(u)]$ in two ways:

$$\begin{aligned} [e_1(u), e_2(u)] &= [\tau e_1(t), \alpha e_2(t) + \beta e_3(t)] \\ &= (\tau \alpha a(t) + \tau \beta c(t)) e_2(t) + (\tau \alpha b(t) + \tau \beta a(t)) e_3(t). \end{aligned}$$

But also

$$\begin{aligned} [e_1(u), e_2(u)] &= a(u) e_2(u) + b(u) e_3(u) \\ &= (a(u) \alpha + b(u) \gamma) e_2(t) + (a(u) \beta + b(u) \delta) e_3(t). \end{aligned}$$

Then $\tau \alpha a(t) + \tau \beta c(t) = a(u) \alpha + b(u) \gamma$ and $\tau \alpha b(t) + \tau \beta a(t) = a(u) \beta + b(u) \delta$, or, using $a(u) = \tau a(t)$, just

$$c(t) = \frac{\gamma}{\tau \beta} b(u), \quad b(t) = \frac{\delta}{\tau \alpha} b(u). \tag{12}$$

A similar computation of $[e_1(u), e_3(u)]$ yields

$$c(t) = \frac{\alpha}{\tau \delta} c(u), \quad b(t) = \frac{\beta}{\tau \gamma} c(u). \tag{13}$$

Equating the expressions for either $c(t)$ or $b(t)$ gives an identity

$$\alpha \beta c(u) = \gamma \delta b(u) \tag{14}$$

relating the transition parameters and the constants $b(u), c(u)$. Also note that in light of the identity

$$\frac{b(t)c(t)}{a(t)^2} = K = \frac{b(u)c(u)}{a(u)^2},$$

the equations (12) are equivalent to (13). For example:

$$\begin{aligned} \tau\beta c(t) = b(u)\gamma &\Leftrightarrow \beta \frac{c(t)}{a(t)} = \frac{b(u)}{a(u)}\gamma \Leftrightarrow \beta K \frac{a(t)}{b(t)} = \gamma K \frac{a(u)}{c(u)} \\ &\Leftrightarrow \beta c(u) = \gamma b(t) \frac{a(u)}{a(t)} = \tau\gamma b(t). \end{aligned}$$

We can now write the Ricci flow equation in terms of the transition parameters with the basis $\{e_i(u), i = 1, 2, 3\}$ at the fixed time u . The metric tensor is

$$g^u(t) = M^T M = \begin{bmatrix} \tau^2 & 0 & 0 \\ 0 & \alpha^2 + \beta^2 & \alpha\gamma + \beta\delta \\ 0 & \alpha\gamma + \beta\delta & \gamma^2 + \delta^2 \end{bmatrix}, \tag{15}$$

Evaluate the derivative $\frac{dg^u}{dt}$ at time $t = u$ using $\tau(u) = \alpha(u) = \delta(u) = 1, \beta(u) = \gamma(u) = 0$. The result is

$$\frac{dg^u(u)}{dt} = \begin{bmatrix} 2\tau'(u) & 0 & 0 \\ 0 & 2\alpha'(u) & \gamma'(u) + \beta'(u) \\ 0 & \gamma'(u) + \beta'(u) & 2\delta'(u) \end{bmatrix}.$$

By the Ricci flow equation, this must equal $-2R^u(u)$, where the Ricci tensor at time u for this basis is given by Lemma 6.1 (evaluated at time u). Equating components gives the system

$$2\tau' = -2R_{11}, \quad 2\alpha' = -2R_{22}, \quad 2\delta' = -2R_{33}, \quad \gamma' + \beta' = -2R_{23} = -2R_{32}. \tag{16}$$

To express the transition functions in terms of the standard triples, we differentiate the equations (11), (12) and (13). Equation (11) gives

$$a'(t) = \frac{-\tau'(t)}{\tau^2(t)} a(u) = \frac{-\tau'(t)}{\tau(t)} a(t),$$

or, evaluating at $t = u$ and using (16) and $\tau(u) = 1$,

$$a'(u) = -\tau'(u)a(u) = R_{11}(u)a(u).$$

Equation (12) gives

$$\begin{aligned} b'(t) &= \left(\frac{\delta}{\tau\alpha}\right)'(t) b(u) = \left(\frac{\delta}{\tau\alpha}\right)(t) \left(\log\left(\frac{\delta}{\tau\alpha}\right)\right)'(t) b(u) \\ &= \left(\frac{\delta'(t)}{\delta(t)} - \frac{\tau'(t)}{\tau(t)} - \frac{\alpha'(t)}{\alpha(t)}\right) b(t). \end{aligned}$$

Evaluating at $t = u$ and using (16) and $\alpha(u) = \tau(u) = \delta(u) = 1$,

$$b'(u) = (-R_{33}(u) + R_{11}(u) + R_{22}(u)) b(u).$$

Similarly, equation (13) gives

$$c'(u) = (-R_{22}(u) + R_{11}(u) + R_{33}(u)) c(u).$$

Using the values for $R_{ii}(u)$ from Lemma 6.1 gives

$$a' = a \left(-2a^2 - \frac{1}{2}(b+c)^2 \right), \quad b' = b \left(c^2 - b^2 - 2a^2 - \frac{1}{2}(b+c)^2 \right) \tag{17}$$

$$c' = c \left(b^2 - c^2 - 2a^2 - \frac{1}{2}(b+c)^2 \right). \tag{18}$$

This system determines the Ricci flow on $\mathfrak{M}_2 = \mathfrak{S}_2$.

To obtain the flow on $\mathfrak{N}_2 = \mathfrak{T}_2$, assume first that $a(t) \neq 0$ and consider $r(t) = \frac{b(t)}{a(t)}$, $s(t) = \frac{c(t)}{a(t)}$. Then

$$\begin{aligned} r' &= r (\log(r))' = r (\log(b)' - \log(a)') = r \left(\frac{b'}{b} - \frac{a'}{a} \right) \\ &= r (c^2 - b^2) = a^2 r (s^2 - r^2), \end{aligned}$$

and similarly $s' = a^2 s (r^2 - s^2)$. So the flow on \mathfrak{T}_2 is determined by

$$r' = a^2 r (s^2 - r^2) \quad \text{and} \quad s' = a^2 s (r^2 - s^2). \tag{19}$$

Notice that $(rs)' = rs' + r's = 0$, so $rs = K$, a constant, as expected. The trajectories of this flow are the portions of hyperbolas $rs = K$ in the region $r \geq |s|$. As in the unimodular case, the positive factor a^2 just determines the speed along the trajectory. There is a fixed point at $r = |s| = \sqrt{|K|}$ for each K . When $r > |s|$ we have $r' < 0$ and points move to the left in the rs plane along the hyperbolas as t increases. Each trajectory $(r(t), s(t))$ is traced out for t in some open interval (t_-, t_+) where $-\infty \leq t_- < 0$ and $0 < t_+ \leq +\infty$. The exact values of t_-, t_+ , as well as the speed along the trajectory, depend on the ‘‘starting point’’ $p = (a(0), b(0), c(0))$ of the flow in \mathfrak{M}_2 . We have

$$\lim_{t \rightarrow t_+} (r(t), s(t)) = \left(\sqrt{|K|}, \text{sign}(K)\sqrt{|K|} \right).$$

Also $\lim_{t \rightarrow t_-} (r(t), s(t)) = (0, 1, 0)$, the ‘‘point at infinity’’ corresponding to the ray $(0, b, 0)$ which we saw corresponds to the family of metrics on the Heisenberg algebra.

When $a(t) = 0$ we are dealing with rays $[0, b(t), c(t)]$ in the projective plane with $b(t) \geq |c(t)|$. These can be parameterized by $m(t) = \frac{c(t)}{b(t)}$ where $-1 \leq m(t) \leq +1$. Then differentiating and using equations (18) gives

$$m' = m (\log(m))' = m \left(\frac{c'}{c} - \frac{b'}{b} \right) = 2m (b^2 - c^2) = 2b^2 m (1 - m^2) \tag{20}$$

as the equation of flow. There are fixed points at $m = 0, \pm 1$, that is, when $c(t) = 0, \pm b(t)$.

Figure 4 below illustrates the Ricci flow on $\mathfrak{T}_2 = \mathfrak{N}_2$. The left diagram shows the hyperbolic trajectories in the rs -plane flowing towards fixed points on the rays $s = \pm r, r \geq 0$. The right diagram includes ‘‘points at infinity’’ in the projective plane and shows how the point $B(m = 0)$, corresponding to the Heisenberg algebra, is a ‘‘universally repelling’’ fixed point. As discussed previously, the fixed points of the Ricci flow on $\mathfrak{T}_2 = \mathfrak{N}_2$ correspond to Ricci solitons on the corresponding simply-connected Lie groups.

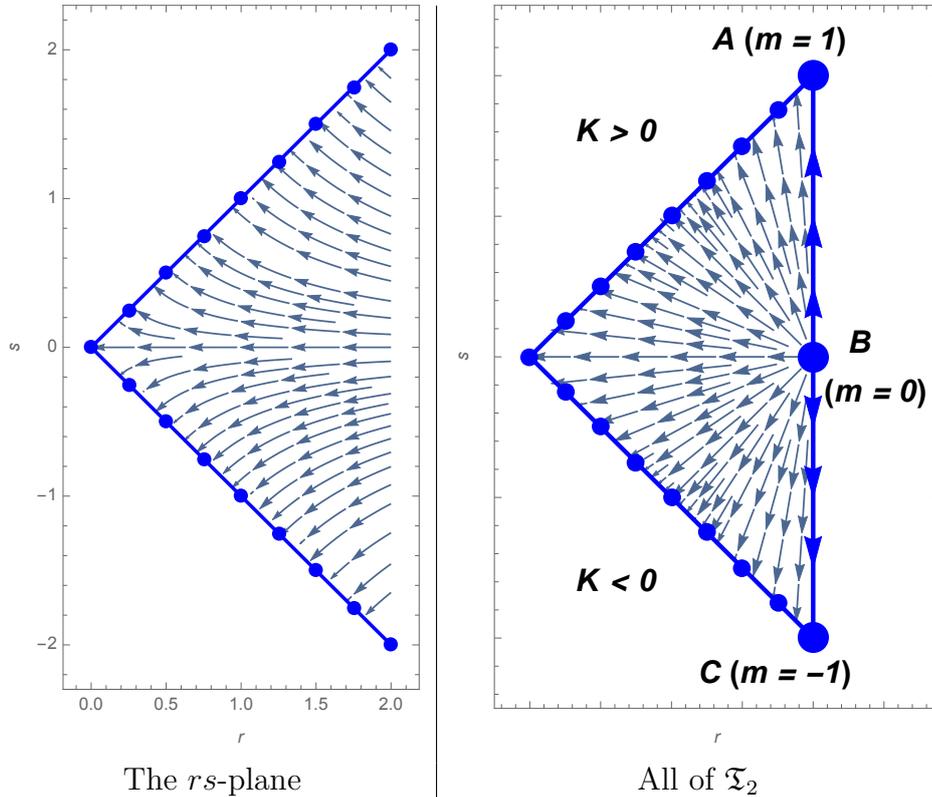


Figure 4: Ricci flow on \mathfrak{N}_2

To picture the Ricci flow on all of $\mathfrak{T} = \mathfrak{N}$ one should identify the points A, B, C and rays AB, BC in the right diagram of Figure 4 with the corresponding points and rays in Figure 3.

Finally, notice that when $a(t) = 0$ we have a unimodular Lie algebra, so the Ricci flow on \mathfrak{S}_2 or \mathfrak{T}_2 should agree with that found earlier on \mathfrak{S}_u or \mathfrak{T}_u . When $a(t) = 0$, the equations (18) become

$$b' = b\left(c^2 - b^2 - \frac{1}{2}(b+c)^2\right) \quad \text{and} \quad c' = c\left(b^2 - c^2 - \frac{1}{2}(b+c)^2\right). \quad (21)$$

But if $\{e_1, e_2, e_3\}$ is a standard basis defining a triple $(0, b, c) \in \mathfrak{S}_2$, then we saw above that considered as a unimodular Lie algebra $\{e_3, e_2, e_1\}$ is a Milnor basis defining the triple $(a_1, a_2, a_3) = (b, -c, 0) \in \mathfrak{S}_u$ when $c < 0$ or the triple $(a_1, a_2, a_3) = (b, 0, -c) \in \mathfrak{S}_u$ when $c \geq 0$. Using these values for (a_1, a_2, a_3) in equations (9) and simplifying, we do obtain the same equations (21) for b', c' . Similarly, the equation (20) for m' is equivalent to the equations (10) for m'_2, m'_3 in the cases where one of m_2, m_3 is 0.

Notice that in Figures 3 and 4 all trajectories in \mathfrak{N} through points corresponding to a given Lie algebra \mathfrak{g} converge to a common fixed point. This is somewhat analogous to results of Böhm and Lafuente [3] who study the Ricci flow on Lie bracket spaces for a large class of solvable Lie algebras (those of *real type*) and find that all flow lines through points for a given Lie algebra tend to converge (in an appropriate sense) to a common fixed point, a solvsoliton independent of the starting point of the flow.

7. Isomorphisms of metric Lie algebras and solitons of the Ricci flow

The natural definition of an isomorphism $f : M_1 \rightarrow M_2$ between two manifolds with Riemannian metrics is that f be an isometry. If both manifolds are Lie groups with

left invariant metrics, then we can require that the isometry f maps the identity to the identity (by composing with a left translation if necessary). The differential of f at the identity will then give a linear map between the associated metric Lie algebras which preserves the metric. This suggests defining an isomorphism of metric Lie algebras to be a linear map $df : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ which is the differential at the identity of an isometry $f : G_1 \rightarrow G_2$ taking the identity to the identity, where G_i is the simply-connected Lie group with left invariant metric corresponding to \mathfrak{g}_i . This is the definition used in [7] and we will call this a weak isomorphism between the metric Lie algebras.

Notice that if $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a strong isomorphism of metric Lie algebras, that is, a Lie algebra isomorphism that preserves the metric, then it lifts (uniquely) to a Lie group isomorphism $F : G_1 \rightarrow G_2$ which will be an isometry between the associated simply-connected Lie groups with the induced left invariant metrics. So strong isomorphism implies weak isomorphism. Each weak equivalence class is then a union of one or more of our strong equivalence classes. The sets $\mathfrak{M}_W, \mathfrak{N}_W$ defined using weak isomorphism would then be quotients of our $\mathfrak{M}, \mathfrak{N}$ where we identify any strong equivalence classes in the same weak class. We will state results showing that for most metric Lie algebras the weak class and strong class coincide. Proofs of all results appear at the end of this section.

We will say that two points in \mathfrak{M} are weakly isomorphic if the corresponding metric Lie algebras are weakly isomorphic. We begin stating results for \mathfrak{M}_u .

Proposition 7.1. *Let $p = (a_1, a_2, a_3)$ and $\bar{p} = (\bar{a}_1, \bar{a}_2, \bar{a}_3)$ be two points in \mathfrak{M}_u . If $a_1 = a_2, a_3 = 0$, then p is weakly isomorphic to \bar{p} if and only if $\bar{a}_1 = \bar{a}_2, \bar{a}_3 = 0$. For any other p, \bar{p} is weakly isomorphic to \bar{p} if and only if $p = \bar{p}$.*

Corollary 7.2. *Let $p = (a_1, a_2, a_3)$ and $\bar{p} = (\bar{a}_1, \bar{a}_2, \bar{a}_3)$ be two points (not equal to $(0, 0, 0)$) in \mathfrak{M}_u . Then p is equivalent to \bar{p} up to weak isomorphism and scaling if and only if p is equivalent to \bar{p} up to strong isomorphism and scaling.*

By Proposition 7.1, the points $(a, a, 0) \in \mathfrak{M}_u$ (corresponding to metrics on $\mathfrak{e}(2)$) are all weakly isomorphic to the point $(0, 0, 0)$ (the flat metric on the abelian Lie algebra \mathbb{R}^3), and there are no other cases of weak isomorphism differing from strong isomorphism. This means that projection $\pi_W : \mathfrak{M}_u \rightarrow \mathfrak{M}_{u,W}$ is one-to-one except that the points $(a, a, 0) \in \mathfrak{M}_u$ are all collapsed to a single point in $\mathfrak{M}_{u,W}$.

By Corollary 7.2, the projection $\pi_W : \mathfrak{N}_u \rightarrow \mathfrak{N}_{u,W}$ is a bijection, that is, no points are collapsed in passing from strong to weak isomorphism. That is why our results are essentially identical to those in [7] for the unimodular case. Let $\pi_u : \mathfrak{M}_u \rightarrow \mathfrak{N}_u$ be the projection. Another useful corollary is

Corollary 7.3. *If $p \in \mathfrak{M}_u$ is a Ricci soliton, then $\pi_u(p) \in \mathfrak{N}_u$ is a fixed point of the Ricci flow on \mathfrak{N}_u .*

This means that any Ricci soliton metric on a Lie algebra \mathfrak{g} in \mathfrak{M}_u will be a special Ricci soliton (and an algebraic soliton) and that any (left invariant) Ricci soliton on the corresponding simply-connected Lie group will be a special Ricci soliton and an algebraic soliton. We now consider results for \mathfrak{M}_2 .

Proposition 7.4. *Let $p = (a, b, c)$ and $\bar{p} = (\bar{a}, \bar{b}, \bar{c})$ be two points in \mathfrak{M}_2 . If $c = -b$, then p is weakly isomorphic to \bar{p} if and only if $\bar{a} = a, \bar{c} = -\bar{b}$. For any other p, \bar{p} is weakly isomorphic to \bar{p} if and only if $p = \bar{p}$.*

Corollary 7.5. *Let $p = (a, b, c)$ and $\bar{p} = (\bar{a}, \bar{b}, \bar{c})$ be two points (not equal to $(0, 0, 0)$) in \mathfrak{M}_2 . If $c = -b$, then p is equivalent to \bar{p} up to weak isomorphism and scaling if and only if $\bar{c} = -\bar{b}$. For any other case, p is equivalent to \bar{p} up to weak isomorphism and scaling if and only if p is equivalent to \bar{p} up to strong isomorphism and scaling.*

By Proposition 7.4, all points $(a, b, -b) \in \mathfrak{M}_2$ for a fixed $a \geq 0$ are weakly isomorphic to the point $(a, 0, 0)$, and there are no other cases of weak isomorphism differing from strong isomorphism. This means that projection $\pi_W : \mathfrak{M}_2 \rightarrow \mathfrak{M}_{2,W}$ is one-to-one except that for each a , the points $(a, b, -b) \in \mathfrak{M}_2$ are all collapsed to a single point in $\mathfrak{M}_{2,W}$.

By Corollary 7.5, the projection $\pi_W : \mathfrak{N}_2 \rightarrow \mathfrak{N}_{2,W}$ will be one-to-one except that all points $\pi_2(a, b, -b) \in \mathfrak{N}_2$ are identified to a single point in $\mathfrak{N}_{2,W}$. Similar to Corollary 7.3 we have

Corollary 7.6. *If $p \in \mathfrak{M}_2$ is a Ricci soliton, then $\pi_2(p) \in \mathfrak{N}_2$ is a fixed point of the Ricci flow on \mathfrak{N}_2 .*

This means that any Ricci soliton in \mathfrak{M}_2 will be a special Ricci soliton (and an algebraic soliton) and that any Ricci soliton on the corresponding simply-connected Lie group G will be a special Ricci soliton and algebraic. (Since the Lie algebra is solvable, G is a solvmanifold and the corollary implies that any Ricci soliton on G will be a solvsoliton. As remarked in Section 4, this fact can also be proved in certain cases using results in Böhm and Lafuente [3].) Combining Corollaries 7.3 and 7.6 we get

Proposition 7.7. *Any (left invariant) Ricci soliton on a simply-connected, three-dimensional Lie group is a special Ricci soliton and an algebraic Ricci soliton.*

Remark 7.8. It can easily be shown that in the projections $\pi_W : \mathfrak{M} \rightarrow \mathfrak{M}_W$ and $\pi_W : \mathfrak{N} \rightarrow \mathfrak{N}_W$ there are no additional collapsing of points other than those given above for the \mathfrak{M}_u and \mathfrak{M}_2 cases. This involves showing, however, that no point $x \in \mathfrak{M}_2 - \mathfrak{M}_2 \cap \mathfrak{M}_u$ can be weakly isomorphic to a point $\bar{x} \in \mathfrak{M}_u - \mathfrak{M}_2 \cap \mathfrak{M}_u$. Since we don't need this result, we omit the rather involved computational proof. ■

We now sketch the proofs of the results stated above. Notice that if two points p, \bar{p} in \mathfrak{M} are weakly isomorphic then the isometry between corresponding simply-connected Lie groups allows us to identify the Ricci operators R, \bar{R} and any associated quantities for the two points. If we diagonalize the (symmetric) operator R with an orthogonal transformation giving $R = \text{diag}(R_1, R_2, R_3)$, then the principal curvatures R_i are determined (up to order) as the eigenvalues of R . We define the *Ricci triple* $T(p)$ for a point p to be this (unordered) set $\{R_1, R_2, R_3\}$ of principal curvatures. We define the signature of p , $\text{Sig}(p)$, to be the signature of the form R , that is, the number of positive, negative, and zero terms among the R_i . We then have

Lemma 7.9. *The Ricci operator R , the Ricci triple $T(p)$, the signature $\text{Sig}(p)$, and $|DR|^2$ where DR is the covariant derivative of R are all weak isomorphism invariants.*

We will also need the following lemma.

Lemma 7.10. *Let G be a three-dimensional, simply-connected Lie group with a left invariant metric. Suppose all principal curvatures are constant and equal, $R_i = c$, so the Ricci tensor is a constant scalar $R = cI$ for some constant c . Then if $c < 0$, G is isometric to three-dimensional hyperbolic space (with standard metric scaled by a factor depending only on c), if $c > 0$, G is isometric to the three sphere (with standard metric scaled by a factor depending only on c), while if $c = 0$ then G is isometric to Euclidean three-space (with the standard metric).*

Proof. A useful formula of Milnor [21] for three-dimensional Lie groups gives the sectional curvature $k(u, v)$ for the plane spanned by u, v as

$$k(u, v) = \|u \times v\| \rho / 2 - r(u \times v)$$

where $\rho = R_1 + R_2 + R_3$ is the scalar curvature and r is the Ricci quadratic form given by $r(x) = R(x \otimes x)$. Taking any pair of orthogonal unit vectors in the plane for u, v , we find in $k(u, v) = 3c/2 - c = c/2$. But when all sectional curvatures are constant and equal (to $c/2$), the Killing-Hopf theorem (see for example Wolf [24]) yields the desired result. ■

Corollary 7.11. *Let p, \bar{p} be two points in \mathfrak{M} with equal Ricci triples $T(p) = T(\bar{p}) = \{c, c, c\}$ for some constant c . Then p and \bar{p} are weakly isomorphic.*

Proof. By the lemma, the corresponding simply-connected Lie groups will be isometric, so the metric Lie algebras will be weakly isomorphic. ■

We can now prove Proposition 7.1

Proof. For a \mathfrak{g} corresponding to a point $(a_1, a_2, a_3) \in \mathfrak{M}_u$ the diagonal elements in the diagonalized Ricci matrix of Section 3 are $R_i = (a_i^2 - (a_j - a_k)^2)/2$ where i, j, k is any permutation of 1, 2, 3. This can be rewritten to give

$$T(\mathfrak{g}) = \{R_1, R_2, R_3\} = \{2u_2u_3, 2u_1u_3, 2u_1u_2\}$$

where $u_i = (a_1 + a_2 + a_3)/2 - a_i$ for $i = 1, 2, 3$. Notice that $a_i = u_j + u_k$.

For a general point $p = (a_1, a_2, a_3) \in \mathfrak{M}_u$, if none of the R_i are zero, we can solve for the u_i getting $u_i = \pm \sqrt{\frac{R_j R_k}{2R_i}}$ where i, j, k is any permutation of 1, 2, 3. Permuting the R_i just permutes the u_i . Changing the sign of one u_i in the solution requires changing the sign of all. So a solution triple $\{u_1, u_2, u_3\}$ is determined up to permutation and change of all signs. Then since $a_i = u_j + u_k$, the triple (a_1, a_2, a_3) is determined up to permutation and change of all signs. But there is only one such choice in \mathfrak{M}_u , that is, with $a_1 \geq a_2 \geq 0, a_2 \geq a_3$. So there is at most one $p = (a_1, a_2, a_3) \in \mathfrak{M}_u$ corresponding to a given triple $\{R_1, R_2, R_3\}$ with no R_i equal to zero. But if p, \bar{p} are weakly isomorphic, then $T(p) = T(\bar{p})$ by Lemma 7.9, so we must have $p = \bar{p}$.

If one R_i is zero, then either $u_j = 0$ or $u_k = 0$. But then either $R_k = 0$ or $R_j = 0$. So if one entry in the Ricci triple is 0, at least two will be. So assume $R_i = R_j = 0$ with $u_k = 0$. Assume the triple is not the trivial $(0, 0, 0)$, so $R_k = 2u_i u_j \neq 0$. The covariant derivative DR of the Ricci matrix R satisfies

$$|DR|^2 = 8 \{u_2^4 (u_1 - u_3)^2 + u_1^4 (u_3 - u_2)^2 + u_3^4 (u_2 - u_1)^2\}$$

or, when $u_k = 0$, just $|DR|^2 = 8u_i^2 u_j^2 (u_i^2 + u_j^2) = 2R_k^2 (u_i^2 + u_j^2)$.

The intersection of the hyperbola $R_k = 2u_i u_j$ with the circle $u_i^2 + u_j^2 = |DR|^2 / 2R_k^2$ can give at most four points for (u_i, u_j) . If (s, t) is one solution, then by symmetry the remaining solutions are (t, s) , $(-s, -t)$, $(-t, -s)$. Since $a_k = u_i + u_j$, $a_i = u_j$, $a_j = u_i$, the triple (a_1, a_2, a_3) is again determined up to sign and permutation by $T(p)$ and $|DR(p)|^2$ and there is only one possible triple $p = (a_1, a_2, a_3) \in \mathfrak{M}_u$. Since $T(p)$ and $|DR(p)|^2$ are weak isomorphism invariants by Lemma 7.9, if p, \bar{p} are weakly isomorphic we must have $p = \bar{p}$.

There remains the case where $T(p) = \{0, 0, 0\}$, so the metric is flat. Any two such points with $T(p) = T(\bar{p}) = \{0, 0, 0\}$ will be weakly isomorphic by corollary 7.11. This happens if and only if at least two of the u_i are zero. So assume $u_i = u_j = 0$. Then $a_k = 0$, $a_i = a_j = u_k$. For this to give a point in \mathfrak{M}_u we must have $a_1 = a_2 \geq 0$, $a_3 = 0$ as stated in the proposition. This completes the proof of Proposition 7.1. ■

Corollary 7.2 follows at once from the proposition since any distinct points of \mathfrak{M}_u which are weakly isomorphic are seen to be equivalent up to scaling. Then for Corollary 7.3 we argue as follows: If a point $p \in \mathfrak{M}_u$ corresponds to a left invariant Ricci soliton, then all points $p(t)$ in the Ricci flow starting with p are equivalent to p up to weak isomorphism and scaling. Then by Corollary 7.2 they are equivalent to p up to (strong) isomorphism and scaling. Then $\pi_u(p)$ is a fixed point of the flow on \mathfrak{N}_u as claimed.

Turning now to the results for \mathfrak{M}_2 , one can check that for a point $p = (a, b, c) \in \mathfrak{M}_2$ the Ricci triple is $T(p) = \{R_1, R_2, R_3\}$ where

$$\begin{aligned} R_1 &= -2a^2 - \frac{(b+c)^2}{2}, & R_2 &= -2a^2 - \frac{(b+c)}{2} \sqrt{(b-c)^2 + 4a^2}, \\ R_3 &= -2a^2 + \frac{(b+c)}{2} \sqrt{(b-c)^2 + 4a^2} \end{aligned} \tag{22}$$

The R_i are obtained as the eigenvalues of the Ricci matrix given in Lemma 6.1 which appear when we diagonalize the matrix. If $a > 0$ we can write this as

$$\begin{aligned} R_1 &= -2a^2(1 + \frac{1}{4}(r+s)^2), & R_2 &= -2a^2(1 + \frac{1}{4}(r-s)\sqrt{(r-s)^2 + 4}), \\ R_3 &= -2a^2(1 - \frac{1}{4}(r-s)\sqrt{(r-s)^2 + 4}), \end{aligned} \tag{23}$$

where $r = b/a$ and $s = c/a$. If $a = 0$, then

$$R_1 = -(b+c)^2/2, \quad R_2 = (c^2 - b^2)/2, \quad R_3 = (b^2 - c^2)/2. \tag{24}$$

For the proof of Proposition 7.4 we need two additional lemmas.

Lemma 7.12. *Let $p = (a, b, c)$ and $\bar{p} = (\bar{a}, \bar{b}, \bar{c})$ be two points in \mathfrak{M}_2 . If $c = -b$, then p is weakly isomorphic to \bar{p} if and only if $\bar{a} = a$, $\bar{c} = -\bar{b}$.*

Proof. By inspection of equations (22), we have $c = -b \Leftrightarrow R_1 = R_2 = R_3 = -2a^2$. So unless $\bar{a} = a$, $\bar{c} = -\bar{b}$ we will have $T(p) \neq T(\bar{p})$ and then p is not weakly isomorphic to \bar{p} by Lemma 7.9.

On the other hand, if $\bar{a} = a$, $\bar{c} = -\bar{b}$, then we have $T(p) = T(\bar{p}) = \{-2a^2, -2a^2, -2a^2\}$, so p is weakly isomorphic to \bar{p} by Corollary 7.11. ■

If we scale a point p by a factor of $k > 0$, each entry a, b, c is scaled by k .

Then by inspection of equations (22) each entry R_1, R_2, R_3 in $T(p)$ is scaled by a (positive) factor k^2 , so the signature will remain unchanged under scaling.

Definition 7.13. For a point $x \in \mathfrak{N}_2$ the *signature*, $\text{Sig}(x)$, is the signature of any point $p \in \mathfrak{M}_2$ such that $\pi(p) = x$. The signature is *non-degenerate* if the corresponding Ricci matrix is non-degenerate, that is, if none of the Ricci triple entries are 0. ■

Observe that since the entries in $T(p)$ vary continuously with $p = (a, b, c) \in \mathfrak{M}_2$, the set of points $x \in \mathfrak{N}_2$ having a particular non-degenerate signature will be open.

Given a point $p = (a, b, c) \in \mathfrak{M}_2$, define the trajectory, $\text{traj}(p)$, to be the set of points $\{p(t) = (a(t), b(t), c(t)) : t \in (t_-, t_+)\} \subset \mathfrak{M}_2$ on the solution curve for the Ricci flow on \mathfrak{M}_2 with initial value $p(0) = p$. If p is weakly isomorphic to \bar{p} , then the associated Lie group isometry $F : G \rightarrow \bar{G}$ will pull back the entire Ricci flow starting from \bar{p} to the Ricci flow starting from p . But then each point $\bar{p}(t) \in \text{traj}(\bar{p})$ is weakly isomorphic to the corresponding point $p(t) \in \text{traj}(p)$. So $p(t), \bar{p}(t)$ will have the same Ricci triple and the same signature. Then considering the projections $\pi(\text{traj}(p)), \pi(\text{traj}(\bar{p})) \subset \mathfrak{N}_2$, we have $\text{Sig}(\pi(p(t))) = \text{Sig}(\pi(\bar{p}(t)))$ for each t .

Examining the trajectories in figure (4) for the Ricci flow on \mathfrak{N}_2 through each $\pi(p)$, we see that there will be a unique fixed point $x(p, t_+) \in \mathfrak{N}_2$ such that $\lim_{t \rightarrow t_+} (\pi(p(t))) = x(p, t_+)$. Evidently, if $\pi(p)$ is a fixed point of the flow on \mathfrak{N}_2 , then $x(p, t_+) = \pi(p)$. If $\pi(p)$ is on the hyperbola $rs = K$, then $x(p, t_+)$ is the ‘‘endpoint’’ $r = \sqrt{|K|}, s = \text{sign}(K)\sqrt{|K|}$. If $a = 0$ and $c > 0$, then $x(p, t_+) = A = [0, 1, 1]$, while if $a = 0$ and $c < 0$, $x(p, t_+) = C = [0, 1, -1]$.

Lemma 7.14. Let $p = (a, b, c)$ and $\bar{p} = (\bar{a}, \bar{b}, \bar{c})$ be two points in \mathfrak{M}_2 and assume both signatures $\text{Sig}(x(p, t_+))$ and $\text{Sig}(x(\bar{p}, t_+))$ are non-degenerate. Then p and \bar{p} cannot be weakly isomorphic unless $\text{Sig}(x(\bar{p}, t_+)) = \text{Sig}(x(p, t_+))$.

Proof. If $\text{Sig}(x(p, t_+))$ is non-degenerate, then as observed above the set of points with this signature is open. In particular, all points $\pi(p(t))$ must have this signature for sufficiently large t . But then if p and \bar{p} are weakly isomorphic, all points $\pi(\bar{p}(t))$ will also have this signature for sufficiently large t . But if $\text{Sig}(x(\bar{p}, t_+))$ is also non-degenerate, then the signature for all points $\pi(\bar{p}(t))$ with t sufficiently large will be $\text{Sig}(x(\bar{p}, t_+))$. So we must have $\text{Sig}(x(\bar{p}, t_+)) = \text{Sig}(x(p, t_+))$ as claimed. ■

We can now prove the following lemma.

Lemma 7.15. Two points $p = (a, b, c), \bar{p} = (\bar{a}, \bar{b}, \bar{c}) \in \mathfrak{M}_2$ cannot be weakly isomorphic if $a^2 < bc$ but $\bar{a}^2 > \bar{b}\bar{c}$.

Proof. If $a^2 < bc, a > 0$, then $K = (bc)/a^2 > 1$. Inspecting the trajectory along a hyperbola with $K > 1$ in \mathfrak{N}_2 we find the fixed point $x(p, t_+) = (\sqrt{K}, \sqrt{K})$. Then using equations (23) with $r = s = \sqrt{K} > 1$, one finds $\text{Sig}(x(p, t_+)) = (-, -, +)$.

If $a = 0, b > c > 0$, then p is on the ‘‘line at infinity’’ in \mathfrak{N}_2 and we have $a(t) = 0, b(t) > c(t) > 0$ for each point $p(t) \in \text{traj}(p)$. Then inspection of equations (22) gives $\text{Sig}(p(t)) = (-, -, +)$ for all t .

Finally, if $a = 0, b = c > 0$, then $\pi(0, b, b) = A$ is a fixed point and equations (24) give $\text{Sig}(p(t)) = \text{Sig}(p) = (-, 0, 0)$ for all t .

On the other hand, if $\bar{a}^2 > \bar{b}\bar{c}$, $\bar{a} > 0$, then $K = (\bar{b}\bar{c})/\bar{a}^2 < 1$. Inspection of the hyperbolic trajectories in \mathfrak{M}_2 gives $x(\bar{p}, t_+) = (\sqrt{|K|}, \text{sign}(K)\sqrt{|K|})$. Then using equations (23) with $r = \sqrt{|K|}$, $s = \text{sign}(K)\sqrt{|K|}$, $K < 1$ one finds $\text{Sig}(x(\bar{p}, t_+)) = (-, -, -)$ in all cases.

So if $\bar{a} > 0$ the signatures corresponding to $p(t)$ and $\bar{p}(t)$ differ for all large t . It follows that we cannot have p and \bar{p} weakly isomorphic.

If $\bar{a} = 0$ and $a = 0$ we are dealing with two unimodular algebras and we cannot have p and \bar{p} weakly isomorphic by Proposition 7.1.

There remains the case when $\bar{a} = 0$ but $a > 0$. We cannot have p and \bar{p} weakly isomorphic unless their Ricci triples coincide. But by equations (24) for \bar{p} we have $\bar{R}_2 + \bar{R}_3 = 0$. For p , since $a > 0$, equations (23) show that $R_2 + R_3 < 0$ and $R_2 + R_1 < 0$. A computation shows that when $K > 1$, $R_1 + R_3 < 0$ also. So the Ricci triples cannot coincide and we cannot have p and \bar{p} weakly isomorphic in this case either. This completes the proof of the lemma. ■

We can now prove Proposition 7.4.

Proof. Let $p = (a, b, c)$ and $\bar{p} = (\bar{a}, \bar{b}, \bar{c})$ be two points in \mathfrak{M}_2 . If $c = -b$, then by Lemma 7.12, p is weakly isomorphic to \bar{p} if and only if $\bar{a} = a$, $\bar{c} = -\bar{b}$. We must show that if $c \neq -b$, then p cannot be weakly isomorphic to \bar{p} unless $p = \bar{p}$. By Lemma 7.15, p cannot be weakly isomorphic to \bar{p} unless either $a^2 \leq bc$ and $\bar{a}^2 \leq \bar{b}\bar{c}$ or $a^2 \geq bc$ and $\bar{a}^2 \geq \bar{b}\bar{c}$, so we need only prove p weakly isomorphic to \bar{p} implies $p = \bar{p}$ in such cases. Notice that $R_1 \leq -2a^2 \leq R_3$, $R_2 \leq -2a^2 \leq R_3$ and a computation shows the following:

$$R_1 \leq R_2 \Leftrightarrow a^2 \leq bc \text{ and } R_1 \geq R_2 \Leftrightarrow a^2 \geq bc.$$

Assume the two points $p = (a, b, c)$, $\bar{p} = (\bar{a}, \bar{b}, \bar{c}) \in \mathfrak{M}_2$ are weakly isomorphic, so their Ricci triples are equal, $T(p) = \{R_1, R_2, R_3\} = T(\bar{p}) = \{\bar{R}_1, \bar{R}_2, \bar{R}_3\}$ by Lemma 7.9. Then if either both $a^2 \leq bc$ and $\bar{a}^2 \leq \bar{b}\bar{c}$ or both $a^2 \geq bc$ and $\bar{a}^2 \geq \bar{b}\bar{c}$ we must have $R_i = \bar{R}_i$, $i = 1, 2, 3$. Since we are assuming $c \neq -b$, we can solve for a, b, c in terms of R_1, R_2, R_3 :

$$a = \frac{1}{2}\sqrt{-(R_2+R_3)}, \quad b = \frac{1}{2}\left\{ \sqrt{R_2+R_3-2R_1} + \sqrt{\frac{2R_2^2+2R_3^2-2R_1R_2-2R_1R_3}{R_2+R_3-2R_1}} \right\}$$

$$c = \frac{1}{2}\left\{ \sqrt{R_2+R_3-2R_1} - \sqrt{\frac{2R_2^2+2R_3^2-2R_1R_2-2R_1R_3}{R_2+R_3-2R_1}} \right\}$$

Notice that if $c \neq -b$, then $R_2 + R_3 - 2R_1 \neq 0$ so these expressions are well-defined. Then if $c \neq -b$ and either both $a^2 \leq bc$ and $\bar{a}^2 \leq \bar{b}\bar{c}$ or both $a^2 \geq bc$ and $\bar{a}^2 \geq \bar{b}\bar{c}$ we must have $(a, b, c) = (\bar{a}, \bar{b}, \bar{c})$, that is, $p = \bar{p}$ as desired. This completes the proof of the proposition. ■

Corollaries 7.5 and 7.6 follow easily from Proposition 7.4 in the same way that Corollaries 7.2 and 7.3 followed from Proposition 7.1.

Finally, for Proposition 7.7, notice that for any left invariant Ricci soliton on a simply-connected, three-dimensional Lie group the corresponding metric Lie algebra must lie in either \mathfrak{M}_u or \mathfrak{M}_2 . So by Corollaries 7.3 and 7.6 the soliton must be a special Ricci soliton and an algebraic Ricci soliton as claimed.

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