Journal of Lie Theory
Volume **30** (2020) 837–849

Cuspidal Modules for Solenoidal Lie Algebras over Rational Quantum Tori

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Communicated by M. Schlichenmaier

Abstract. We classify all irreducible cuspidal modules for solenoidal Lie algebras over rational quantum tori, generalizing the results about cuspidal modules for solenoidal Lie algebras over commuting tori and the gap-p Virasoro algebras.

Mathematics Subject Classification: 17B10, 17B66.

Key Words: Solenoidal Lie algebra, cuspidal module, quantum torus, gap-p Virasoro algebra.

1. Introduction

The classification of irreducible Harish-Chandra modules for infinite dimensional Lie algebras is a classical problem in the representation theory, and was solved for the Virasoro algebra Vir [8] and its various generalizations, such as the twisted Heisenberg-Virasoro algebra[7], the gap-p Virasoro algebras Vir_p [15], the Witt algebras W_d [2], the higher rank Virasoro algebras(or solenoidal Lie algebras) W_μ [6], and so on.

In order to achieve the classification result for W_d in [2], the authors applied a new technique, called cover method. Using this cover method, they also classified all irreducible cuspidal modules for W_μ in [3], which was originally done in [13], but in a much more computational way. Let $A = \mathbb{C}[x_1^{\pm 1}, \cdots, x_d^{\pm 1}]$ be the Laurent polynomial ring with d variables, and let $\mathcal{G}$ stand for W_d or W_μ in the following. An $A\mathcal{G}$-module is defined to be a module over the Lie algebra $\mathcal{G} \ltimes A$ with an associative A-action. The A-cover $\hat{M}$ of a $\mathcal{G}$-module M is a particular $A\mathcal{G}$-quotient of the module $\mathcal{G} \otimes M$. Here A plays the role of a coordinate algebra. It turns out that $\hat{M}$ is cuspidal if M is, and M becomes a $\mathcal{G}$-quotient of $\hat{M}$ provided that M is irreducible. This reduces the classification of irreducible cuspidal $\mathcal{G}$-modules to the classification of irreducible cuspidal $A\mathcal{G}$-modules. It is well known that irreducible cuspidal AW_d-modules are the so-called modules of tensor fields, originally constructed by Shen[12], Larsson[4], and further studied in [10, 11]. While for W_μ, it was proved in [3] that cuspidal AW_μ-modules with the support being one coset of $\mathbb{Z}^d$ in $\mathbb{C}^d$ are in a one-to-one correspondence to finite dimensional modules over an subalgebra of $\mathrm{Der}\,(\mathbb{C}[x_1, \cdots, x_d])$. These finite dimensional modules must be one dimensional if the corresponding AW_μ-module is irreducible, which makes the irreducible cuspidal AW_μ-module has weight multiplicities no more than one.

ISSN 0949–5932 / \$2.50

In the present paper we classify all irreducible cuspidal modules for a solenoidal Lie algebra $\mathfrak{g}$ over a rational quantum torus $\mathbb{C}_Q$, which is a subalgebra of the algebra $\mathrm{Der}(\mathbb{C}_{\mathbf{Q}})$ of derivations over $\mathbb{C}_Q$. The universal central extension of $\mathfrak{g}$ was computed in [14]. When $d = 1$, the algebra $\mathfrak{g}$ is exactly the gap-p Virasoro algebra studied in [15]. When constructing the cover for a cuspidal module M for $\mathfrak{g}$, we choose the centre $\mathcal{Z}$ of $\mathbb{C}_Q$, instead of the whole $\mathbb{C}_Q$, as the coordinate algebra and define the $\mathcal{Z}$-cover of M to be some $\mathcal{Z}\mathfrak{g}$-quotient of the module $\mathfrak{g}'_R \otimes M$, where $\mathfrak{g}'_R$ is a proper ideal of $\mathfrak{g}$. This construction is a bit different with the W_μ or the W_d case.

We establish a one-to-one correspondence between cuspidal modules over the Lie algebra $\mathfrak{g}$ with support lying in one coset of $\mathbb{Z}^d$ in $\mathbb{C}^d$ and finite dimensional Γ-graded modules over a subquotient algebra $\mathcal{L}$ of the Lie algebra $\mathrm{Der}\,(\mathbb{C}[x_1, ..., x_d] \oplus (\mathbb{C}_Q/\mathcal{Z}))$, where Γ is a finite group closely related to $\mathbb{C}_Q$. Through this correspondence we show that any irreducible cuspidal $\mathcal{Z}\mathfrak{g}$-module is isomorphic to the module of tensor fields $\mathcal{V}(\alpha, \beta, W)$, for some $\alpha \in \mathbb{C}^d, \beta \in \mathbb{C}$ and finite dimensional Γ-graded irreducible $\mathfrak{gl}_N$-module W, where N is the order of Γ. These modules $\mathcal{V}(\alpha, \beta, W)$ may have weight multiplicities larger than one. The Lie algebra $\mathcal{L}$ has a quotient isomorphic to the direct sum of $\mathfrak{gl}_N$ and a solvable subalgebra of $\mathfrak{gl}_d$. This is how the seemingly peculiar algebra $\mathfrak{gl}_N$ comes into the picture. Furthermore, using the modified cover method we prove that any irreducible cuspidal $\mathfrak{g}$-module is isomorphic to the unique irreducible subquotient of some $\mathcal{V}(\alpha, \beta, W)$.

Since the relation between the algebra $\mathfrak{g}$ and $\mathrm{Der}(\mathbb{C}_{\mathbf{Q}})$ is similar to that between the solenoidal Lie algebra W_μ and the Witt algebra W_d, we believe our result may play a role in classification of irreducible Harish-Chandra $\mathrm{Der}(\mathbb{C}_{\mathbf{Q}})$-modules. We mention that in [5] irreducible cuspidal $\mathrm{Der}(\mathbb{C}_{\mathbf{Q}})$-modules were classified, also using a cover method.

The paper is arranged as follows. In Section 2 we recall some results about the gap-p Virasoro algebra Vir_p, the solenoidal Lie algebra W_μ, $\mathbb{C}_Q$ and $\mathfrak{gl}_N$. In section 3 we give the construction of modules of tensor fields $\mathcal{V}(\alpha, \beta, W)$ over $\mathfrak{g}$, and their irreducibility criterion. Section 4 is devoted to finite dimensional Γ-graded $\mathcal{L}$-modules. In Section 5 we classify all irreducible cuspidal $\mathcal{Z}\mathfrak{g}$-modules, and in the last section all irreducible cuspidal $\mathfrak{g}$-modules are classified using the cover method.

Throughout this paper, $\mathbb{C}, \mathbb{Z}, \mathbb{N}, \mathbb{Z}_+$ refer to the set of complex numbers, integers, nonnegative integers and positive integers respectively. For a Lie algebra $\mathcal{G}$, we denote by $\mathcal{U}(\mathcal{G})$ the universal enveloping algebra of $\mathcal{G}$. Let $d \in \mathbb{Z}_+$ and we fix a standard basis $\epsilon_1, ..., \epsilon_d$ for the space $\mathbb{C}^d$. Denote by $(\cdot \mid \cdot)$ the inner product on $\mathbb{C}^d$.

2. Notations and preliminaries

For a Lie algebra, a weight module is called a Harish-Chandra module if all weight spaces are finite dimensional, and called a *cuspidal* module if all weight spaces are uniformly bounded. The *support* of a weight module is defined to be the set of all weights.

2.1. The gap-p Virasoro algebra Vir_p

Let p be a positive integer and $\mathbb{C}[x^{\pm 1}]$ denote the Laurent polynomial ring in one variable x.

The gap-p Virasoro algebra Vir_p is a Lie algebra with a basis

$$\{x^{m+1}\frac{\partial}{\partial x}, x^s, C_i \mid m \in p\mathbb{Z}, s \notin p\mathbb{Z}, 0 \le i \le p-1\},$$

and Lie brackets

$$[x^{m+1}\frac{\partial}{\partial x}, x^{n+1}\frac{\partial}{\partial x}] = (n-m)x^{m+n+1}\frac{\partial}{\partial x} + \delta_{m+n,0}\frac{1}{12}\left((\frac{m}{p})^3 - (\frac{m}{p})\right) C_0;$$
$$[x^{m+1}\frac{\partial}{\partial x}, x^r] = rx^{m+r}; \quad [x^r, x^s] = \delta_{r+s,0} r C_{\overline{r}},$$

where $m, n \in p\mathbb{Z}, r, s \notin p\mathbb{Z}$ and we use $\overline{r}$ to represent the residue of r by p.
We recall from [15] the module of intermediate series over Vir_p. Let $F = (F_{i,j})$ be a $(p-1) \times p$ complex matrix, with index $1 \le i \le p-1,\ 0 \le j \le p-1$, satisfying the following three conditions

(I) $0 \in o(F) = \{j \mid F_{i,j} \neq 0 \text{ for some } i\}$;

(II) if $F_{i,j} \neq 0$ then $F_{s,\overline{i+j}} \neq 0$ for some $1 \le s \le p-1$;

(III) $F_{r,\overline{i+s}} F_{s,i} = F_{s,\overline{i+r}} F_{r,i}$ for any $0 \le i \le p-1$ and $1 \le r, s \le p-1$.

For any $j \in o(F)$ denote the space $V_{(j)} = \mathrm{span}_{\mathbb{C}} \{v_{j+pk} \mid k \in \mathbb{Z}\}$. Let $a, b \in \mathbb{C}$ and define the Vir_p-module structure on $\bigoplus_{j \in o(F)} V_{(j)}$ by

$$\left(x^{m+1}\frac{\partial}{\partial x}\right) \cdot v_{j+n} = (a + j + n + mb) v_{j+n+m} \text{ and}$$
$$x^s \cdot v_{j+n} = F_{\overline{s},j} v_{j+n+s}; \quad C_i v_{j+n} = 0,$$

where $m, n \in p\mathbb{Z},\ s \notin p\mathbb{Z}, j \in o(F)$ and $i = 0, 1, \cdots, p-1$. We denote this module by $V(a, b, F)$, and call it a module of intermediate series over Vir_p. Roughly speaking, the module $V(a, b, F)$ is a sum of several modules of intermediate series over the subalgebra $\mathrm{span}_{\mathbb{C}} \left\{x^{m+1}\frac{\partial}{\partial x}, C_0 \mid m \in p\mathbb{Z}\right\}$, which is isomorphic to the normal Virasoro algebra. Let $P = \mathrm{span}_{\mathbb{C}} \{x^m \mid m \in p\mathbb{Z}\}$.

Theorem 2.1 ([15])**.** (1) *Any irreducible cuspidal modules over $Vir_p \ltimes P$ with an associative P-action must be of the form $V(a, b, F)$.*

(2) *Any irreducible cuspidal modules over Vir_p is the irreducible quotient of some $V(a, b, F)$.*

2.2. The solenoidal Lie algebra W_μ

Let $d \in \mathbb{Z}_+$ and $\mu = (\mu_1, \cdots, \mu_d)^T \in \mathbb{C}^d$ is called *generic* if $\mu_1, \cdots, \mu_d$ are linearly independent over the field of rational numbers. Let $A = \mathbb{C}[x_1^{\pm 1}, \cdots, x_d^{\pm 1}]$ be the Laurent polynomial ring with d variables. For any $\mathbf{m} = (m_1, \cdots, m_d)^T \in \mathbb{Z}^d$ denote $x^{\mathbf{m}} = x_1^{m_1} \cdots x_d^{m_d}$. Set $\partial_x = \sum_{i=1}^d \mu_i x_i \frac{\partial}{\partial x_i}$. Then the solenoidal Lie algebra W_μ over

A has a basis $\{x^{\mathbf{m}}\partial_x \mid \mathbf{m} \in \mathbb{Z}^d\}$

and Lie bracket $[x^{\mathbf{m}}\partial_x, x^{\mathbf{n}}\partial_x] = (\mu \mid \mathbf{n} - \mathbf{m}) x^{\mathbf{m}+\mathbf{n}}\partial_x.$

Let $\alpha \in \mathbb{C}^d, \beta \in \mathbb{C}$. The modules $T(\alpha,\beta)$ of tensor fields over W_μ have bases $\{v_{\mathbf{s}} \mid \mathbf{s} \in \mathbb{Z}^d\}$ and the W_μ-action

$$(x^{\mathbf{m}}\partial_x) \cdot v_{\mathbf{s}} = (\mu \mid \alpha + \mathbf{s} + \beta\mathbf{m}) v_{\mathbf{m}+\mathbf{s}}.$$

It is well known that the module $T(\alpha,\beta)$ is reducible if and only if $\alpha \in \mathbb{Z}^d$ and $\beta \in \{0,1\}$. For $\alpha \in \mathbb{Z}^d$, $T(\alpha,0)$ has a unique irreducible quotient $T(\alpha,0)/\mathbb{C}v_{-\alpha}$, and $T(\alpha,1)$ has a unique submodule $\text{span}_{\mathbb{C}}\{v_{\mathbf{s}} \mid \mathbf{s} \neq -\alpha\}$ of codimension one. And, the module $T(\alpha,\beta)$ may be equipped with an additional associative A-action $x^{\mathbf{m}} v_{\mathbf{s}} = v_{\mathbf{m}+\mathbf{s}}$.

Theorem 2.2 ([3, 13])**.** *Any irreducible cuspidal module over $W_\mu \ltimes A$ with an associative A-action must be of the form $T(\alpha,\beta)$ for some $\alpha \in \mathbb{C}^d$ and $\beta \in \mathbb{C}$.*

2.3. The quantum torus $\mathbb{C}_Q$ and the Lie algebra $\mathfrak{gl}_N$

Let $Q = (q_{ij})$ be a $d \times d$ complex matrix with all q_{ij} being roots of unity and satisfying

$$q_{ii} = 1, \quad q_{ij}q_{ji} = 1 \text{ for all } 1 \leq i,j \leq d.$$

The rational quantum torus relative to Q is the unital associative algebra $\mathbb{C}_Q = \mathbb{C}_Q[t_1^{\pm 1}, \cdots, t_d^{\pm 1}]$ with commuting relation

$$t_i t_j = q_{ij} t_j t_i \text{ for all } 1 \leq i,j \leq d.$$

For $\mathbf{m} = (m_1, \cdots, m_d)^T \in \mathbb{Z}^d$ we define $t^{\mathbf{m}} = t_1^{m_1} \cdots t_d^{m_d}$ and for $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$

$$\sigma(\mathbf{m},\mathbf{n}) = \prod_{1 \leq i < j \leq d} q_{ji}^{n_j m_i}, \quad R = \{\mathbf{m} \in \mathbb{Z}^d \mid \sigma(\mathbf{m},\mathbf{n}) = \sigma(\mathbf{n},\mathbf{m}) \text{ for } \forall \mathbf{n} \in \mathbb{Z}^d\}.$$

Clearly, the center $\mathcal{Z}$ of $\mathbb{C}_Q$ is spanned by $\{t^{\mathbf{m}} \mid \mathbf{m} \in R\}$. From Theorem 4.5 in [9], up to an isomorphism of $\mathbb{C}_Q$, we may always assume that $q_{2i,2i-1} = q_i, q_{2i-,2i} = q_i^{-1}$ for $1 \leq i \leq z$, and other entries of Q are all 1, where $z \in \mathbb{Z}_+$ with $2z \leq d$ and the orders k_i of q_i as roots of unity satisfy $k_{i+1} \mid k_i$ for $1 \leq i \leq z$. Then the subgroup R of $\mathbb{Z}^d$ has a simple form

$$R = \bigoplus_{i=1}^{z} (\mathbb{Z}k_i\epsilon_{2i-1} \oplus \mathbb{Z}k_i\epsilon_{2i}) \oplus \bigoplus_{l > 2z} \mathbb{Z}\epsilon_l.$$

Moreover, we have $\sigma(\mathbf{m},\mathbf{r}) = \sigma(\mathbf{r},\mathbf{m}), \quad t^{\mathbf{m}}t^{\mathbf{r}} = t^{\mathbf{m}+\mathbf{r}}$ for all $\mathbf{m} \in R, \mathbf{r} \in \mathbb{Z}^d$.

Let $\mathcal{I} = \text{span}_{\mathbb{C}}\left\{t^{\mathbf{n}+\mathbf{r}} - t^{\mathbf{r}} \mid n \in R, r \in \mathbb{Z}^d\right\}$, which is an ideal of the associative algebra $\mathbb{C}_Q$. By [9] and [16], we have $\mathbb{C}_Q/\mathcal{I} \cong \bigotimes_{i=1}^{z} M_{k_i}(\mathbb{C}) \cong M_N(\mathbb{C})$, where $N = \prod_{i=1}^{z} k_i$ and $M_n(\mathbb{C})$ denotes the associative algebra of all $n \times n$ complex matrices. It is well known that $M_{k_i}(\mathbb{C})$ can be generated by

$$X_{2i-1} = E_{1,1} + q_i E_{2,2} + \cdots + q_i^{k_i - 1} E_{k_i,k_i},$$
$$X_{2i} = E_{1,2} + E_{2,3} + \cdots + E_{k_i-1,k_i} + E_{k_i,1},$$

where E_{kl} represents the $k_i \times k_i$ matrix with 1 in the (k,l)-entry and 0 elsewhere. Let E denote the identity matrix of suitable order. It is easy to see that $X_{2i}^{k_i} = X_{2i-1}^{k_i} = E$ and $X_{2i}X_{2i-1} = q_i X_{2i-1} X_{2i}$. Define $X^{\mathbf{n}} = \bigotimes_{i=1}^{z} X_{2i-1}^{n_{2i-1}} X_{2i}^{n_{2i}}$ for $\mathbf{n} \in \mathbb{Z}^d$. Notice that for any $\mathbf{n} \in R$, $X^{\mathbf{n}}$ is the identity matrix in $M_N(\mathbb{C})$, and $X^{\mathbf{r}}X^{\mathbf{s}} = \sigma(\mathbf{r},\mathbf{s})X^{\mathbf{r}+\mathbf{s}}$.

Define the Lie bracket $[a,b]=ab-ba$ on $M_N(\mathbb{C})$, and write $\mathfrak{gl}_N$ for $M_N(\mathbb{C})$ as a Lie algebra. We have the Lie bracket for $\mathfrak{gl}_N$

$$[X^{\mathbf{m}},X^{\mathbf{n}}]=(\sigma(\mathbf{m},\mathbf{n})-\sigma(\mathbf{n},\mathbf{m}))X^{\mathbf{m}+\mathbf{n}}.$$

Set $\Gamma=\mathbb{Z}^d/R$ and let $\overline{\mathbf{n}}$ denote the image of $\mathbf{n}$. Clearly, Γ has order N, and $\mathfrak{gl}_N$ is a Γ-graded Lie algebra with homogeneous spaces $(\mathfrak{gl}_N)_{\overline{\mathbf{n}}}=\mathrm{span}_{\mathbb{C}}\{X^{\mathbf{n}}\}$. In this paper by a Γ-gradation on $\mathfrak{gl}_N$ we always mean this gradation. Set

$$\Gamma_0=\{\mathbf{n}\in\mathbb{Z}^d\mid 0<n_{2i-1}\le k_i, 0<n_{2i}\le k_i, 1\le i\le z, \text{ and } n_l=0, 2z<l\le d\},$$

which is a complete set of representatives for Γ. When a representative of $\overline{\mathbf{n}}$ is needed we always choose $\mathbf{n}\in\Gamma_0$.

3. Modules of tensor fields for $\mathfrak{g}$

In this section we construct modules of tensor fields $\mathcal{V}(\alpha,\beta,W)$ for the solenoidal Lie algebra $\mathfrak{g}$, and give an irreducibility criterion for $\mathcal{V}(\alpha,\beta,W)$.
Let $\gamma=(\gamma_1,\cdots,\gamma_d)^T\in\mathbb{C}^d$ be generic and set

$$L_{\mathbf{m}}=\begin{cases} t^{\mathbf{m}}\sum_{i=1}^d\gamma_i t_i\frac{\partial}{\partial t_i} & \text{if } m\in R;\\ t^{\mathbf{m}} & \text{if } m\notin R,\end{cases}$$

where $t^{\mathbf{m}}$ is the inner derivation on $\mathbb{C}_Q$ defined by $t^{\mathbf{m}}(t^{\mathbf{r}})=\sigma(\mathbf{m},\mathbf{r})t^{\mathbf{m}+\mathbf{r}}$. The Lie algebra $\mathfrak{g}$ we consider in this paper has a basis $\{L_{\mathbf{m}}\mid\mathbf{m}\in\mathbb{Z}^d\}$, subject to the Lie brackets

$$[L_{\mathbf{m}},L_{\mathbf{n}}]=(\gamma\mid\mathbf{n}-\mathbf{m})L_{\mathbf{m}+\mathbf{n}};$$

$$[L_{\mathbf{m}},L_{\mathbf{s}}]=(\gamma\mid\mathbf{s})L_{\mathbf{m}+\mathbf{s}};\quad [L_{\mathbf{r}},L_{\mathbf{s}}]=(\sigma(\mathbf{r},\mathbf{s})-\sigma(\mathbf{s},\mathbf{r}))L_{\mathbf{r}+\mathbf{s}},$$

for $\mathbf{m},\mathbf{n}\in R,\mathbf{r},\mathbf{s}\notin R$. The subalgebra $\mathfrak{g}_R$ of $\mathfrak{g}$ spanned by $\{L_{\mathbf{m}}\mid\mathbf{m}\in R\}$ is isomorphic to the solenoidal Lie algebra $W_{B\gamma}$, where B is the $d\times d$-matrix

$$\mathrm{diag}\{k_1,k_1,k_2,k_2,\cdots,k_z,k_z,1,\cdots,1\}.$$

This isomorphism is given by $L_{\mathbf{m}}\mapsto x^{\mathbf{n}}\sum_{i=1}^d\mu_i k_i x_i\frac{\partial}{\partial x_i}$, where $\mathbf{n}=B^{-1}\mathbf{m}$ and $k_i=1$ for $i>2z$. Hence the Lie algebra $\mathfrak{g}$ may be considered as a quantum version of W_μ. Now we give the construction of the module of tensor fields for $\mathfrak{g}$. Let $\alpha\in\mathbb{C}^d,\beta\in\mathbb{C}$ and $W=\bigoplus_{\overline{\mathbf{s}}\in\Gamma}W_{\overline{\mathbf{s}}}$ be a Γ-graded $\mathfrak{gl}_N$-module. Define a $\mathfrak{g}$-module structure on the space

$$\mathcal{V}(\alpha,\beta,W)=\bigoplus_{\mathbf{s}\in\mathbb{Z}^d}W_{\overline{\mathbf{s}}}\otimes t^{\mathbf{s}}$$

by
$$L_{\mathbf{m}}(w_{\overline{\mathbf{s}}}\otimes t^{\mathbf{n}+\mathbf{s}})=(\gamma\mid\alpha+\mathbf{n}+\mathbf{s}+\beta\mathbf{m})w_{\overline{\mathbf{s}}}\otimes t^{\mathbf{m}+\mathbf{n}+\mathbf{s}}$$

and
$$L_{\mathbf{r}}(w_{\overline{\mathbf{s}}}\otimes t^{\mathbf{n}+\mathbf{s}})=(X^{\mathbf{r}}w_{\overline{\mathbf{s}}})\otimes t^{\mathbf{r}+\mathbf{n}+\mathbf{s}},$$

where $\mathbf{m},\mathbf{n}\in R,\mathbf{r}\notin R,\mathbf{s}\in\Gamma_0$ and $w_{\overline{\mathbf{s}}}\in W_{\overline{\mathbf{s}}}$. We call $\mathcal{V}(\alpha,\beta,W)$ a module of tensor fields over $\mathfrak{g}$.

Lemma 3.1. *If the Γ-graded $\mathfrak{gl}_N$-module is irreducible and* $\dim W>1$*, then the $\mathfrak{g}$-module $\mathcal{V}(\alpha,\beta,W)$ is irreducible for any $\alpha\in\mathbb{C}^d,\beta\in\mathbb{C}$.*

Proof. Let U be a nonzero $\mathfrak{g}$-submodule of $\mathcal{V}(\alpha,\beta,W)$ and $0\neq w_{\bar{\mathbf{s}}}\otimes t^{\mathbf{s}}\in U$ for some $w_{\bar{\mathbf{s}}}\in W_{\bar{\mathbf{s}}}, \mathbf{s}\in\mathbb{Z}^d$. We claim that

$$w_{\bar{\mathbf{k}}}\otimes t^{\mathbf{k}+\mathbf{m}}\in U \text{ for any } \mathbf{s}\neq\mathbf{k}\in\mathbb{Z}^d \text{ and } \mathbf{m}\in R.$$

Since W is a Γ-graded irreducible $\mathfrak{gl}_N$-module, we see that $w_{\bar{\mathbf{k}}}=aw_{\bar{\mathbf{s}}}$ for some homogeneous element a in $\mathcal{U}(\mathfrak{gl}_N)$ of the form

$$a=\sum c_{\mathbf{s}_1\cdots\mathbf{s}_p}X^{\mathbf{s}_1}\cdots X^{\mathbf{s}_p},$$

where $c_{\mathbf{s}_1\cdots\mathbf{s}_p}\in\mathbb{C}$ and the sum takes over finitely many $(\mathbf{s}_1,\cdots,\mathbf{s}_p)$ such that $\mathbf{s}_1+\cdots\mathbf{s}_p=\mathbf{k}-\mathbf{s}$. Denote $b=\sum c_{\mathbf{s}_1\cdots\mathbf{s}_p}L_{\mathbf{s}_1}\cdots L_{\mathbf{s}_1}\in\mathcal{U}(\mathfrak{g})$. Then we have

$$w_{\bar{\mathbf{k}}}\otimes t^{\mathbf{k}}=aw_{\bar{\mathbf{s}}}\otimes t^{\mathbf{k}}=b(w_{\bar{\mathbf{s}}}\otimes t^{\mathbf{s}}).$$

Since $L_{\mathbf{m}}(w_{\bar{\mathbf{k}}}\otimes t^{\mathbf{k}})=(\gamma\mid\alpha+\mathbf{k}+\beta\mathbf{m})(w_{\bar{\mathbf{k}}}\otimes t^{\mathbf{m}+\mathbf{k}})\in U$, we get that

$$w_{\bar{\mathbf{k}}}\otimes t^{\mathbf{m}+\mathbf{k}}\in U \text{ if } \alpha+\mathbf{k}+\beta\mathbf{m}\neq\mathbf{0}.$$

Assume $\alpha+\mathbf{k}+\beta\mathbf{m}=\mathbf{0}$. Since

$$L_{\mathbf{m}}(w_{\bar{\mathbf{s}}}\otimes t^{\mathbf{s}})=(\gamma\mid\alpha+\mathbf{s}+\beta\mathbf{m})(w_{\bar{\mathbf{s}}}\otimes t^{\mathbf{m}+\mathbf{s}})=(\gamma\mid\mathbf{s}-\mathbf{k})(w_{\bar{\mathbf{s}}}\otimes t^{\mathbf{m}+\mathbf{s}})\in U$$

we obtain $w_{\bar{\mathbf{s}}}\otimes t^{\mathbf{m}+\mathbf{s}}\in U$. Then we still have $w_{\bar{\mathbf{k}}}\otimes t^{\mathbf{m}+\mathbf{k}}=b(w_{\bar{\mathbf{s}}}\otimes t^{\mathbf{m}+\mathbf{s}})\in U$.
We still need to show $w'_{\bar{\mathbf{s}}}\otimes t^{\mathbf{m}+\mathbf{s}}\in U$ for any $\mathbf{m}\in R$ and $w'_{\bar{\mathbf{s}}}\in W_{\bar{\mathbf{s}}}$. But this is just a replicate of the proof of the above claim starting with a vector $w_{\bar{\mathbf{k}}}\otimes t^{\mathbf{k}}\in U$ with $\mathbf{k}\neq\mathbf{s}$. So $U=\mathcal{V}(\alpha,\beta,W)$ and the lemma stands. □
Furthermore, we have the following

Theorem 3.2. *Let $\alpha\in\mathbb{C}^d,\beta\in\mathbb{C}$ and W be a Γ-graded irreducible $\mathfrak{gl}_N$-module. The $\mathfrak{g}$-module $\mathcal{V}(\alpha,\beta,W)$ is reducible if and only if $\dim W=1,\alpha\in\mathbb{Z}^d$ and $\beta\in\{0,1\}$.*

Proof. By Lemma 3.1 we only need consider when $\dim W=1$. In this case $L_{\mathbf{s}}\mathcal{V}(\alpha,\beta,W)=0$ for any $\mathbf{s}\notin R$, and hence $\mathcal{V}(\alpha,\beta,W)$ reduces to a module over the subalgebra $\mathfrak{g}_R$ of $\mathfrak{g}$, which is isomorphic to the solenoidal Lie algebra $W_{B\gamma}$. Then the theorem follows from the irreducibility criterion of the $W_{B\gamma}$-module $T(\alpha,\beta)$. □

4. The algebra $\mathcal{L}$ and finite dimensional modules

In this section we introduce the related algebra $\mathcal{L}$ and study finite dimensional modules over $\mathcal{L}$.

Consider the associative algebra $AC=\mathbb{C}[x_1,\cdots,x_d]\oplus(\mathbb{C}_Q/\mathcal{Z})$ with $x_it_j=t_jx_i$ for all $1\leq i,j\leq d$. We simply write $t^{\bar{\mathbf{s}}}$ for the image of $t^{\mathbf{s}}$ in $\mathbb{C}_Q/\mathcal{Z}$. Denote by $\mathcal{L}$ the subalgebra of $\mathrm{Der}(AC)$ spanned by

$$\{x^{\mathbf{m}}d_\gamma, x^{\mathbf{n}}t^{\bar{\mathbf{s}}}\mid\mathbf{m}\in\mathbb{N}^d\backslash\{\mathbf{0}\},\mathbf{n}\in\mathbb{N}^d,\bar{\mathbf{s}}\in\Gamma\},$$

where $d_\gamma=\sum_{i=1}^d\gamma_i\frac{\partial}{\partial x_i}+\sum_{i=1}^d\gamma_it_i\frac{\partial}{\partial t_i}$.

The Lie bracket of $\mathcal{L}$ is

$$
\begin{aligned}
[x^{\mathbf{m}}d_\gamma, x^{\mathbf{n}}d_\gamma] &= \sum_{i=1}^{d} \gamma_i(n_i - m_i)x^{\mathbf{m}+\mathbf{n}-\epsilon_i}d_\gamma;\\
[x^{\mathbf{m}}d_\gamma, x^{\mathbf{l}}t^{\bar{\mathbf{s}}}] &= \sum_{i=1}^{d} \gamma_i l_i x^{\mathbf{m}+\mathbf{l}-\epsilon_i}t^{\bar{\mathbf{s}}} + (\gamma \mid \mathbf{s})x^{\mathbf{m}+\mathbf{l}}t^{\bar{\mathbf{s}}};\\
[x^{\mathbf{p}}t^{\bar{\mathbf{r}}}, x^{\mathbf{l}}t^{\bar{\mathbf{s}}}] &= (\sigma(\mathbf{r},\mathbf{s}) - \sigma(\mathbf{s},\mathbf{r}))x^{\mathbf{p}+\mathbf{l}}t^{\bar{\mathbf{r}}+\bar{\mathbf{s}}},
\end{aligned} \tag{4.1}
$$

where $\mathbf{m},\mathbf{n} \in \mathbb{N}^d\backslash\{\mathbf{0}\}, \mathbf{p},\mathbf{l} \in \mathbb{N}^d$ and $\bar{\mathbf{r}},\bar{\mathbf{s}} \in \Gamma$. Here we have chosen the representative for $\bar{\mathbf{s}} \in \Gamma$ in Γ_0 as usual. The algebra $\mathcal{L}$ has a Γ-gradation $\mathcal{L} = \bigoplus_{\bar{\mathbf{s}}\in\Gamma} \mathcal{L}(\bar{\mathbf{s}})$ where

$$
\begin{aligned}
\mathcal{L}(\bar{\mathbf{s}}) &= \operatorname{span}_{\mathbb{C}} \left\{x^{\mathbf{n}}t^{\bar{\mathbf{s}}} \mid \mathbf{n} \in \mathbb{N}^d\right\};\\
\mathcal{L}(\bar{\mathbf{0}}) &= \operatorname{span}_{\mathbb{C}} \left\{x^{\mathbf{m}}d_\gamma, x^{\mathbf{n}}t^{\bar{\mathbf{0}}} \mid \mathbf{m} \in \mathbb{N}^d\backslash\{\mathbf{0}\}, \mathbf{n} \in \mathbb{N}^d\right\}.
\end{aligned}
$$

Define two subalgebras of $\mathcal{L}$

$$
\mathcal{L}^{(x)} = \operatorname{span}_{\mathbb{C}} \left\{x^{\mathbf{m}}d_\gamma \mid \mathbf{m} \in \mathbb{N}^d\backslash\{\mathbf{0}\}\right\}, \quad \mathcal{L}^{(t)} = \operatorname{span}_{\mathbb{C}} \left\{x^{\mathbf{n}}t^{\bar{\mathbf{s}}} \mid \mathbf{n} \in \mathbb{N}^d, \bar{\mathbf{s}} \in \Gamma\right\}.
$$

Set $\deg(x_i) = 1, \deg(d_\gamma) = -1$ and $|\mathbf{m}| = \sum_{j=1}^{d} m_j$ for all $1 \le i \le d$ and $\mathbf{m} \in \mathbb{N}^d$. We get $\mathbb{Z}$-gradations for the algebras $\mathcal{L}^{(x)} = \bigoplus_{i\ge 0} \mathcal{L}_i^{(x)}$ and $\mathcal{L}^{(t)} = \bigoplus_{i\ge 0} \mathcal{L}_i^{(t)}$ with

$$
\mathcal{L}_i^{(x)} = \operatorname{span}_{\mathbb{C}} \{x^{\mathbf{m}}d_\gamma \mid |\mathbf{m}| = i+1\}, \quad \mathcal{L}_i^{(t)} = \operatorname{span}_{\mathbb{C}} \left\{x^{\mathbf{n}}t^{\bar{\mathbf{s}}} \mid |\mathbf{n}| = i, \bar{\mathbf{s}} \in \Gamma\right\}.
$$

Set $\mathcal{L}_i = \mathcal{L}_i^{(x)} \oplus \mathcal{L}_i^{(t)}$. Notice that the algebra $\mathcal{L}$ is not $\mathbb{Z}$-graded. But the subspace $\mathcal{L}_+ = \bigoplus_{i\ge 1} \mathcal{L}_i$ still makes an Γ-graded ideal of $\mathcal{L}$. The main result in this section is the following

Theorem 4.1. (1) $[\mathcal{L}^{(x)}, \mathcal{L}^{(x)}] = [\mathcal{L}_0^{(x)}, \mathcal{L}_0^{(x)}] \oplus \left(\bigoplus_{j\ge 1} \mathcal{L}_j^{(x)}\right)$;

(2) *Every finite dimensional representation* (U,ρ) *of* $\mathcal{L}^{(x)}$ *satisfies* $\rho(\mathcal{L}_p^{(x)}) = 0$ *for* $p \gg 0$;

(3) *Every finite dimensional irreducible module over* $\mathcal{L}^{(x)}$ *is one dimensional, and parametrized by some* $\beta \in \mathbb{C}$ *such that* $x_i\frac{\partial}{\partial x_i}$ *acts as a constant* $\beta\gamma_i$, *and* $\mathcal{L}_j^{(x)}$ *trivially for any* $1 \le i \le d, j \ge 1$;

(4) *Every finite dimensional representation* (U,ρ) *for* $\mathcal{L}$ *satisfies* $\rho(\mathcal{L}_p) = 0$ *for* $p \gg 0$;

(5) *The ideal* $\mathcal{L}_+$ *annihilates every finite dimensional irreducible* $\mathcal{L}$*-module.*

Proof. The first three statements are exactly the Theorem 3.1 from [3]. For (4), consider U as a $\mathcal{L}^{(x)}$-module. By (3) we have $\rho(\mathcal{L}_p^{(x)}) = 0$ for $p \gg 0$. Then it follows from $[x^{\mathbf{m}}d_\gamma, t^{\bar{\mathbf{s}}}] = (\gamma \mid \mathbf{s})x^{\mathbf{m}}t^{\bar{\mathbf{s}}}$ that $\rho(x^{\mathbf{m}}t^{\bar{\mathbf{s}}}) = 0$ if $|\mathbf{m}| > p$. This proves (4).

Let (V,ρ) be a finite dimensional irreducible representation of $\mathcal{L}$. By (4) and equation (4.1) we see that $\rho(\mathcal{L}_+)$ is a finite dimensional nilpotent Lie algebra. Let $V' = \{v \in V \mid \mathcal{L}_+ v = 0\}$, which is a $\mathcal{L}$-submodule of V. Moreover, by Lie's Theorem, there exists a common eigenvector $v \in V$ for $\rho(\mathcal{L}_+)$ such that $\rho(a)v = 0$ for all $a \in \mathcal{L}_+$. Then $V' \ne 0$. So $V' = V$ by the irreducibility of V. Hence $\mathcal{L}_+ V = 0$. □

For later use we mention that there is an isomorphism from the quotient algebra $\mathcal{L}/\mathcal{L}_+$ to $\mathfrak{gl}_d(\gamma) \oplus \mathfrak{gl}_N$, where $\mathfrak{gl}_d(\gamma) = \mathrm{span}_{\mathbb{C}}\left\{\epsilon_i\gamma^T \in \mathfrak{gl}_d \mid 1 \le i \le d\right\}$, defined by

$$x_i d_\gamma + \mathcal{L}_+ \mapsto \epsilon_i \gamma^T, \quad t^{\overline{\mathbf{s}}} + \mathcal{L}_+ \mapsto X^{\mathbf{s}}. \tag{4.2}$$

Notice that $\mathfrak{gl}_d(\gamma)$ is solvable since $[\mathfrak{gl}_d(\gamma), \mathfrak{gl}_d(\gamma)]$ is abelian.

5. Cuspidal $\mathcal{Z}\mathfrak{g}$-modules

In this section we study a specific class of cuspidal modules over $\mathfrak{g}$, and prove that the irreducible ones are exactly modules of tensor fields.

Consider the centre $\mathcal{Z}$ of $\mathbb{C}_Q$. We may form an extended Lie algebra $\mathfrak{g} \ltimes \mathcal{Z}$. We call a module V for $\mathfrak{g} \ltimes \mathcal{Z}$ a $\mathcal{Z}\mathfrak{g}$-module provided that the $\mathcal{Z}$-action on V is associative.

The following are two examples of $\mathcal{Z}\mathfrak{g}$-modules. Set $\mathfrak{g}'_R = \mathrm{span}_{\mathbb{C}}\{L_{\mathbf{s}} \mid \mathbf{s} \notin R\}$, which is an ideal of $\mathfrak{g}$. Then $\mathfrak{g}'_R$ makes a $\mathcal{Z}\mathfrak{g}$-module if we define

$$L_{\mathbf{m}} \cdot L_{\mathbf{s}} = [L_{\mathbf{m}}, L_{\mathbf{s}}], \quad t^{\mathbf{n}} \cdot L_{\mathbf{s}} = L_{\mathbf{n}+\mathbf{s}} \quad \text{for } \mathbf{m} \in \mathbb{Z}^d, \mathbf{n} \in R \text{ and } \mathbf{s} \notin R.$$

The second example is the module $\mathcal{V}(\alpha, \beta, W) = \bigoplus_{\overline{\mathbf{s}} \in \Gamma} W_{\overline{\mathbf{s}}} \otimes t^{\mathbf{s}}\mathcal{Z}$ of tensor fields for $\mathfrak{g}$ with a $\mathcal{Z}$-action given by

$$t^{\mathbf{m}}(w_{\overline{\mathbf{s}}} \otimes t^{\mathbf{n}+\mathbf{s}}) = w_{\overline{\mathbf{s}}} \otimes t^{\mathbf{m}+\mathbf{n}+\mathbf{s}},$$

for $\mathbf{m}, \mathbf{n} \in R, \mathbf{s} \notin R$ and $w_{\overline{\mathbf{s}}} \in W_{\overline{\mathbf{s}}}$. Clearly, if W is a Γ-graded irreducible $\mathfrak{gl}_N$-module, then $\mathcal{V}(\alpha, \beta, W)$ is irreducible as a $\mathcal{Z}\mathfrak{g}$-module for any $\alpha \in \mathbb{C}^d, \beta \in \mathbb{C}$.

Now let M be a cuspidal $\mathcal{Z}\mathfrak{g}$-module with support lying in $\alpha + \mathbb{Z}^d$ (may not equal). Since the $\mathcal{Z}$-action is associative, M can be represented as

$$M \cong \bigoplus_{\overline{\mathbf{s}} \in \Gamma} U_{\overline{\mathbf{s}}} \otimes \mathcal{Z},$$

where $U_{\overline{\mathbf{s}}} = M_{\alpha+\mathbf{s}}$ for $\mathbf{s} \in \Gamma_0$. From this we see that the support of M equals to $\alpha + I + R$, where I is a subset of Γ_0. Set $U = \bigoplus_{\overline{\mathbf{s}} \in \Gamma} U_{\overline{\mathbf{s}}}$.

For later use we need some operators on M. For $\mathbf{m} \in R, \mathbf{r} \notin R$, consider the following elements in $\mathcal{U}(\mathfrak{g} \ltimes \mathcal{Z})$,

$$D(\mathbf{m}) = t^{-\mathbf{m}} L_{\mathbf{m}}, \quad D(\mathbf{m}, \mathbf{r}) = t^{-\mathbf{m}} L_{\mathbf{m}+\mathbf{r}}.$$

They all lie in $\mathrm{End}(U)$. Moreover, the operator $D(\mathbf{m})$ may be restricted to $U_{\overline{\mathbf{s}}}$, and $D(\mathbf{m}, \mathbf{r})$ may be restricted to an operator $U_{\overline{\mathbf{s}}} \longrightarrow U_{\overline{\mathbf{r}+\mathbf{s}}}$ for each $\overline{\mathbf{s}} \in \Gamma$. The operators $D(\mathbf{m}), D(\mathbf{m}, \mathbf{r})$ completely determine the $\mathfrak{g}$-action on M, since

$$L_{\mathbf{m}}(t^{\mathbf{n}} v_{\overline{\mathbf{s}}}) = (\gamma \mid \mathbf{n}) t^{\mathbf{m}+\mathbf{n}} v_{\overline{\mathbf{s}}} + t^{\mathbf{m}+\mathbf{n}} D(\mathbf{m}) v_{\overline{\mathbf{s}}}$$
$$L_{\mathbf{m}+\mathbf{r}}(t^{\mathbf{n}} v_{\overline{\mathbf{s}}}) = t^{\mathbf{n}}(L_{\mathbf{m}+\mathbf{r}} v_{\overline{\mathbf{s}}}) = t^{\mathbf{m}+\mathbf{n}}(D(\mathbf{m}, \mathbf{r}) v_{\overline{\mathbf{s}}}).$$

The following lemma is easy to check.

Lemma 5.1. *For $\mathbf{m}, \mathbf{n} \in R$ and $\mathbf{r}, \mathbf{s} \notin R$, we have*

$$[D(\mathbf{m}), D(\mathbf{n})] = (\gamma \mid \mathbf{m})(D(\mathbf{m}+\mathbf{n}) - D(\mathbf{m})) - (\gamma \mid \mathbf{n})(D(\mathbf{m}+\mathbf{n}) - D(\mathbf{n}));$$
$$[D(\mathbf{m}), D(\mathbf{n}, \mathbf{s})] = (\gamma \mid \mathbf{n}+\mathbf{s}) D(\mathbf{m}+\mathbf{n}, \mathbf{s}) - (\gamma \mid \mathbf{n}) D(\mathbf{n}, \mathbf{s});$$
$$[D(\mathbf{m}, \mathbf{r}), D(\mathbf{n}, \mathbf{s})] = (\sigma(\mathbf{r}, \mathbf{s}) - \sigma(\mathbf{s}, \mathbf{r})) D(\mathbf{m}+\mathbf{n}, \mathbf{r}+\mathbf{s}).$$

Proposition 5.2. *Let M be a cuspidal $\mathcal{Z}\mathfrak{g}$-module with support lying in $\alpha+\mathbb{Z}^d$. We may write $M \cong \bigoplus_{\overline{\mathbf{s}}\in\Gamma} U_{\overline{\mathbf{s}}} \otimes \mathcal{Z}$, where $U_{\overline{\mathbf{s}}} = M_{\alpha+\mathbf{s}}$ for $\mathbf{s} \in \Gamma_0$. Then the action of $\mathfrak{g}$ on M is given by*

$$L_{\mathbf{m}}(v_{\overline{\mathbf{s}}} \otimes t^{\mathbf{n}}) = ((\gamma \mid \mathbf{n}) + D(\mathbf{m}))v_{\overline{\mathbf{s}}} \otimes t^{\mathbf{m}+\mathbf{n}};$$
$$L_{\mathbf{m}+\mathbf{r}}(v_{\overline{\mathbf{s}}} \otimes t^{\mathbf{n}}) = D(\mathbf{m},\mathbf{r}) \cdot v_{\overline{\mathbf{s}}} \otimes t^{\mathbf{m}+\mathbf{n}},$$

for $\mathbf{m},\mathbf{n} \in R, \mathbf{r} \in \Gamma_0\backslash\{\mathbf{0}\}$ and $v_{\overline{\mathbf{s}}} \in U_{\overline{\mathbf{s}}}$. Here the operators $D(\mathbf{m}) : U_{\overline{\mathbf{s}}} \longrightarrow U_{\overline{\mathbf{s}}}$ for each $\overline{\mathbf{s}} \in \Gamma$, can be expressed as $\mathrm{End}(U_{\overline{\mathbf{s}}})$*-valued polynomial in $\mathbf{m}$ with constant term $D(\mathbf{0}) = (\gamma \mid \alpha + \mathbf{s})Id$, and the operators $D(\mathbf{m},\mathbf{r}) : U_{\overline{\mathbf{s}}} \longrightarrow U_{\overline{\mathbf{r}+\mathbf{s}}}$ for each $\overline{\mathbf{s}} \in \Gamma$, can be expressed as* $\mathrm{Hom}(U_{\overline{\mathbf{s}}}, U_{\overline{\mathbf{r}+\mathbf{s}}})$*-valued polynomial in $\mathbf{m}$ whose constant term $D(\mathbf{0},\mathbf{r})$ has the form of an upper triangular matrix relative to suitable bases of $U_{\overline{\mathbf{s}}}$ and $U_{\overline{\mathbf{r}+\mathbf{s}}}$.*

Proof. Notice that $\mathfrak{g}_R$ is isomorphic to the solenoidal Lie algebra $W_{B\gamma}$. The part concerning the action of $\mathfrak{g}_R$ and $D(\mathbf{m})$ is just what was stated in Theorem 4.5 in [2]. Next we prove the $\mathfrak{g}'_R$-action by induction on d. If $d = 1$, then the algebra $\mathfrak{g}$ shrinks either to a gap-p Virasoro algebra (for a suitable p), in which case the result is clear since all $L_{\mathbf{m}+\mathbf{r}}$ act as a same constant by the construction of the module of intermediate series in Subsection 2.1, or to a Virasoro algebra, in which case $\mathfrak{g}'_R = 0$ and there is nothing to prove.

For the induction step, we assume that the operators $D(\mathbf{m} - m_i\epsilon_i, \mathbf{r} - r_i\epsilon_i)$ have polynomial dependence on $m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_d$, and $D(m_i\epsilon_i, r_i\epsilon_i)$ is a polynomial on m_i, for all $i = 1, 2, \cdots, d$ and $\mathbf{r} \notin R, r_i\epsilon_i \notin R$ and $\mathbf{r} - r_i\epsilon_i \notin R$. Note that $\mathbf{m}, m_i\epsilon_i \in R$. From

$$[D(\mathbf{0}, r_i\epsilon_i), D(\mathbf{m} - m_i\epsilon_i, \mathbf{r} - r_i\epsilon_i)] = (\sigma(r_i\epsilon_i, \mathbf{r} - r_i\epsilon_i) - \sigma(\mathbf{r} - r_i\epsilon_i, r_i\epsilon_i))D(\mathbf{m} - m_i\epsilon_i, \mathbf{r})$$

we see that $D(\mathbf{m} - m_i\epsilon_i, \mathbf{r})$ is a polynomial on $m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_d$. Then consider

$$[D(m_i\epsilon_i), D(\mathbf{m}-m_i\epsilon_i, \mathbf{r})] = (\gamma \mid \mathbf{m}-m_i\epsilon_i+\mathbf{r})D(\mathbf{m},\mathbf{r}) - (\gamma \mid \mathbf{m}-m_i\epsilon_i)D(\mathbf{m}-m_i\epsilon_i, \mathbf{r}).$$

We see that $D(\mathbf{m},\mathbf{r})$ may be expressed as rational functions $\frac{P_i(\mathbf{m})}{F_i(\mathbf{m})}, 1 \le i \le d$, where

$$F_i(\mathbf{m}) = (\gamma \mid \mathbf{m} - m_i\epsilon_i + \mathbf{r}) = \gamma_i r_i + \sum_{j\neq i} \gamma_j(m_j + r_j) \neq 0$$

since γ is generic and $m_j + r_j \neq 0$ for all $j \neq i$. In particular, we have $\frac{P_1(\mathbf{m})}{F_1(\mathbf{m})} = \frac{P_2(\mathbf{m})}{F_2(\mathbf{m})}$ in the space $\mathrm{Hom}(U_{\overline{\mathbf{s}}}, U_{\overline{\mathbf{r}+\mathbf{s}}}) \otimes \mathbb{C}(m_1, \cdots, m_d)$. Hence

$$P_1(\mathbf{m})F_2(\mathbf{m}) = P_2(\mathbf{m})F_1(\mathbf{m}).$$

Notice that $F_1(\mathbf{m}), F_2(\mathbf{m})$ are coprime to each other. It follows that $F_1(\mathbf{m})$ divides $P_1(\mathbf{m})$. So $D(\mathbf{m},\mathbf{r})$ is a polynomial.

It remains to show that the value of the polynomial $D(\mathbf{m},\mathbf{r})$ at $\mathbf{m} = \mathbf{0}$ coincides with $D(\mathbf{0},\mathbf{r})$ on $U_{\overline{\mathbf{s}}}$. Let p be the order of the image $\overline{\mathbf{r}} \in \Gamma$ of $\mathbf{r}$, and we have $p\mathbf{r} \in R$. Denote by $\mathfrak{g}(\mathbf{r})$ the Lie subalgebra of $\mathfrak{g}$ generated by $\{L_{k\mathbf{r}} \mid k \in \mathbb{Z}\}$, which is a gap-$p$ Virasoro algebra. Consider M as a $\mathfrak{g}(\mathbf{r})$-module with composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M.$$

Each quotient M_i/M_{i-1} is an irreducible cuspidal $\mathfrak{g}(\mathbf{r})$-module, and by Theorem 2.1 has weight multiplicities no more than one. It is clear that l is the maximal dimension of the weight spaces of M. Then, according to this composition series, one can choose a basis of M, with respect to which the operator $D(\mathbf{0},\mathbf{r}) = L_{\mathbf{r}}$ has the form of upper triangular matrices of order l on all weight spaces of M, hence on $U_{\overline{\mathbf{s}}}$. Moreover, from Theorem 2.1 and the construction of module of intermediate series over Vir_p in Subsection 2.1, we see that all $D(kp\mathbf{r},\mathbf{r}), k\in\mathbb{Z}$, have the same action as $D(\mathbf{0},\mathbf{r})$ on all M_i/M_{i-1}, hence on $U_{\overline{\mathbf{s}}}$. Notice that the constant term $D(\mathbf{0})$ of $D(\mathbf{m}-k\mathbf{r})$ acts as $(\gamma\mid\alpha+\mathbf{s})\mathrm{Id}$ on $U_{\overline{\mathbf{s}}}$, and as $(\gamma\mid\alpha+\mathbf{r}+\mathbf{s})\mathrm{Id}$ on $U_{\overline{\mathbf{r}+\mathbf{s}}}$. By considering the constant term in both sides of

$$[D(\mathbf{m}-k\mathbf{r}),D(k\mathbf{r},\mathbf{r})] = (k+1)(\gamma\mid\mathbf{r})D(\mathbf{m},\mathbf{r}) - k(\gamma\mid\mathbf{r})D(k\mathbf{r},\mathbf{r}),$$

we see that the value of $D(\mathbf{m},\mathbf{r})$ at $\mathbf{m}=\mathbf{0}$ equals to $D(\mathbf{0},\mathbf{r})$ on $U_{\overline{\mathbf{s}}}$. $\square$

Define operators $\partial_{\mathbf{0}}^{\mathbf{p}} \in \mathrm{End}(U_{\overline{\mathbf{s}}})$ and $\partial_{\mathbf{r}}^{\mathbf{p}} \in \mathrm{Hom}(U_{\overline{\mathbf{s}}},U_{\overline{\mathbf{r}+\mathbf{s}}})$ for $\mathbf{r}\in\Gamma_0\setminus R$, by expansions of the polynomials $D(\mathbf{m})$ and $D(\mathbf{m},\mathbf{r})$ in $\mathbf{m}$:

$$D(\mathbf{m}) = \sum_{\mathbf{p}\in\mathbb{N}^d}\frac{\mathbf{m}^{\mathbf{p}}}{\mathbf{p}!}\partial_{\mathbf{0}}^{\mathbf{p}};\qquad D(\mathbf{m},\mathbf{r}) = \sum_{\mathbf{p}\in\mathbb{N}^d}\frac{\mathbf{m}^{\mathbf{p}}}{\mathbf{p}!}\partial_{\mathbf{r}}^{\mathbf{p}},$$

with only finite number of the operators $\partial_{\mathbf{0}}^{\mathbf{p}},\partial_{\mathbf{r}}^{\mathbf{p}}$ being nonzero. Here $\mathbf{p}! = p_1!\cdots p_d!$ and $\mathbf{m}^{\mathbf{p}} = m_1^{p_1}\cdots m_d^{p_d}$. Now expand the commutator $[D(\mathbf{m}),D(\mathbf{n})]$ in $\mathbf{m},\mathbf{n}$,

$$\sum_{\mathbf{p},\mathbf{l}\in\mathbb{N}^d}\frac{\mathbf{m}^{\mathbf{p}}}{\mathbf{p}!}\frac{\mathbf{n}^{\mathbf{l}}}{\mathbf{l}!}[\partial_{\mathbf{0}}^{\mathbf{p}},\partial_{\mathbf{0}}^{\mathbf{l}}] = (\gamma\mid\mathbf{m})\sum_{\mathbf{q}\in\mathbb{N}^d}\frac{(\mathbf{m}+\mathbf{n})^{\mathbf{q}}-\mathbf{m}^{\mathbf{q}}}{\mathbf{q}!}\partial_{\mathbf{0}}^{\mathbf{q}} - (\gamma\mid\mathbf{n})\sum_{\mathbf{q}\in\mathbb{N}^d}\frac{(\mathbf{m}+\mathbf{n})^{\mathbf{q}}-\mathbf{n}^{\mathbf{q}}}{\mathbf{q}!}\partial_{\mathbf{0}}^{\mathbf{q}}.$$

Comparing the coefficients at $\frac{\mathbf{m}^{\mathbf{p}}}{\mathbf{p}!}\frac{\mathbf{n}^{\mathbf{l}}}{\mathbf{l}!}$ in both sides, we get the Lie bracket of $\partial_{\mathbf{0}}^{\mathbf{p}}$

$$[\partial_{\mathbf{0}}^{\mathbf{p}},\partial_{\mathbf{0}}^{\mathbf{l}}] = \begin{cases}\sum_{i=1}^d\gamma_i(l_i-p_i)\partial_{\mathbf{0}}^{\mathbf{p}+\mathbf{l}-\epsilon_i} & \text{if } \mathbf{p},\mathbf{l}\neq\mathbf{0};\\ 0 & \text{if } \mathbf{p}=\mathbf{0} \text{ or } \mathbf{l}=\mathbf{0}.\end{cases} \tag{5.1}$$

Similarly by expanding the commutators $[D(\mathbf{m}),D(\mathbf{n},\mathbf{s})]$ and $[D(\mathbf{m},\mathbf{r}),D(\mathbf{n},\mathbf{s})]$, then comparing coefficients at $\frac{\mathbf{m}^{\mathbf{p}}}{\mathbf{p}!}\frac{\mathbf{n}^{\mathbf{l}}}{\mathbf{l}!}$ in both sides, we get the Lie bracket of $[\partial_{\mathbf{0}}^{\mathbf{p}},\partial_{\mathbf{s}}^{\mathbf{l}}]$

$$[\partial_{\mathbf{0}}^{\mathbf{p}},\partial_{\mathbf{s}}^{\mathbf{l}}] = \begin{cases}(\gamma\mid\mathbf{s})\partial_{\mathbf{s}}^{\mathbf{p}+\mathbf{l}} + \sum_{i=1}^d\gamma_i l_i\partial_{\mathbf{s}}^{\mathbf{p}+\mathbf{l}-\epsilon_i} & \text{if } \mathbf{p}\neq\mathbf{0};\\ (\gamma\mid\mathbf{s})\partial_{\mathbf{s}}^{\mathbf{l}} & \text{if } \mathbf{p}=\mathbf{0},\end{cases} \tag{5.2}$$

and the Lie bracket of $[\partial_{\mathbf{r}}^{\mathbf{p}},\partial_{\mathbf{s}}^{\mathbf{l}}]$

$$[\partial_{\mathbf{r}}^{\mathbf{p}},\partial_{\mathbf{s}}^{\mathbf{l}}] = (\sigma(\mathbf{r},\mathbf{s})-\sigma(\mathbf{s},\mathbf{r}))\partial_{\mathbf{r}+\mathbf{s}}^{\mathbf{p}+\mathbf{l}}. \tag{5.3}$$

Recall the Lie bracket of the algebra $\mathcal{L}$. The equations (5.1), (5.2) and (5.3) imply that the operators $\{\partial_{\mathbf{0}}^{\mathbf{p}},\partial_{\mathbf{s}}^{\mathbf{l}}\mid\mathbf{p}\in\mathbb{N}^d\backslash\{\mathbf{0}\},\mathbf{s}\in\Gamma_0\backslash\{\mathbf{0}\},\mathbf{l}\in\mathbb{N}^d\}$ yield a finite dimensional Γ-graded representation of the algebra $\mathcal{L}$ on the space U. Denote by ρ this representation map. Combining with Proposition 5.2 we get the following

Theorem 5.3. *There exists an equivalence between the category of finite dimensional Γ-graded $\mathcal{L}$-modules and the category of cuspidal $\mathcal{Z}\mathfrak{g}$-modules with support lying in some coset $\alpha+\mathbb{Z}^d$. The equivalence functor associates to a finite dimensional Γ-graded $\mathcal{L}$-module $U=\bigoplus_{\overline{\mathbf{s}}\in\Gamma}U_{\overline{\mathbf{s}}}$ a $\mathcal{Z}\mathfrak{g}$-module*

$$M=\bigoplus_{\overline{\mathbf{s}}\in\Gamma}U_{\overline{\mathbf{s}}}\otimes t^{\mathbf{s}}\mathcal{Z}\quad \text{with the } \mathfrak{g}\text{-action}$$

$$\begin{aligned} L_{\mathbf{m}}(v_{\overline{\mathbf{s}}}\otimes t^{\mathbf{s}+\mathbf{n}}) &= \Big((\gamma\mid\alpha+\mathbf{n}+\mathbf{s})\mathrm{Id}+\sum_{\mathbf{p}\in\mathbb{N}^d\setminus\{\mathbf{0}\}}\frac{\mathbf{m}^{\mathbf{p}}}{\mathbf{p}!}\rho(x^{\mathbf{p}}d_\gamma)\Big)v_{\overline{\mathbf{s}}}\otimes t^{\mathbf{s}+\mathbf{n}+\mathbf{m}};\\ L_{\mathbf{m}+\mathbf{r}}(v_{\overline{\mathbf{s}}}\otimes t^{\mathbf{s}+\mathbf{n}}) &= \sum_{\mathbf{p}\in\mathbb{N}^d}\frac{\mathbf{m}^{\mathbf{p}}}{\mathbf{p}!}\rho(x^{\mathbf{p}}t^{\overline{\mathbf{r}}})v_{\overline{\mathbf{s}}}\otimes t^{\mathbf{s}+\mathbf{n}+\mathbf{m}+\mathbf{r}}, \end{aligned} \tag{5.4}$$

where $\mathbf{m},\mathbf{n}\in R,\mathbf{r}\notin R,\mathbf{s}\in\Gamma_0$ and $v_{\overline{\mathbf{s}}}\in U_{\overline{\mathbf{s}}}$.

Clearly, the subset of $\mathbf{s}\in\Gamma_0$ with $U_{\overline{\mathbf{s}}}\neq 0$ equals to the subset of $\mathbf{s}\in\Gamma_0$ with $M_{\alpha+\mathbf{s}}\neq 0$. Furthermore, we have

Theorem 5.4. *Every irreducible cuspidal $\mathcal{Z}\mathfrak{g}$-module is isomorphic to some $\mathcal{V}(\alpha,\beta,W)$, where $\alpha\in\mathbb{C}^d,\beta\in\mathbb{C}$ and W is a finite dimensional Γ-graded irreducible $\mathfrak{gl}_N$-module.*

Proof. By Theorem 5.3, $U=\bigoplus_{\overline{\mathbf{s}}\in\Gamma}U_{\overline{\mathbf{s}}}$ is a finite dimensional Γ-graded irreducible $\mathcal{L}$-module. Hence $\mathcal{L}_+U=0$ by Theorem 4.1(5). Notice that $\mathcal{L}_+$ is also Γ-graded. It reduces U to a Γ-graded irreducible module over $\mathcal{L}/\mathcal{L}_+$, which is isomorphic to $\mathfrak{gl}_d(\gamma)\oplus\mathfrak{gl}_N$.

On the other hand, since $\mathcal{L}^{(t)}$ is an ideal of $\mathcal{L}$, we obtain $\mathcal{L}^{(t)}U=0$ or U by the irreducibility of U. If $\mathcal{L}^{(t)}U=0$, then $L_{\mathbf{s}}M=0$ for all $\mathbf{s}\notin R$. Hence M is a $\mathfrak{g}_R$-module, and the result follows from [3].

Suppose $\mathcal{L}^{(t)}U=U$. Notice that the ideal $\mathcal{L}_+$ is Γ-graded and annihilates U. Therefore, U becomes a Γ-graded module over $\mathcal{L}^{(t)}/\mathcal{L}_+$, which is isomorphic to $\mathfrak{gl}_N$. Then the equation (5.4) implies $L_{\mathbf{m}+\mathbf{r}}(v_{\overline{\mathbf{s}}}\otimes t^{\mathbf{s}+\mathbf{n}})=X^{\mathbf{r}}v_{\overline{\mathbf{s}}}\otimes t^{\mathbf{s}+\mathbf{n}+\mathbf{m}+\mathbf{r}}$.

We identify elements of $\mathfrak{gl}_d(\gamma)$ and those of $\mathcal{L}^{(x)}/\mathcal{L}_+$ through the isomorphism given in (4.2). Notice that $\mathfrak{gl}_d(\gamma)$ is solvable. By the Lie's Theorem, there exists a common eigenvector $v\in U$ for $\mathfrak{gl}_d(\gamma)$ such that

$$(\epsilon_i\gamma^T)v=(x_id_\gamma)v=\beta_iv,\ \beta_i\in\mathbb{C},\quad \text{for all } 1\leq i\leq d.$$

Set $\beta=\frac{\beta_1}{\gamma_1}$. Since $0=[x_id_\gamma,x_1d_\gamma]v=(\beta_i\gamma_1-\beta_1\gamma_i)v$, we obtain $\beta_i=\gamma_i\beta$ for all $1\leq i\leq d$. Since U is irreducible as an $\mathcal{L}/\mathcal{L}_+$-module, v can generate any vector in U only by applying $t^{\overline{\mathbf{s}}}$ and $\epsilon_i\gamma^T$. For any $\overline{\mathbf{s}}\in\Gamma$, we have

$$(\epsilon_i\gamma^T)(t^{\overline{\mathbf{s}}}v)=(x_id_\gamma)(t^{\overline{\mathbf{s}}}v)=[x_id_\gamma,t^{\overline{\mathbf{s}}}]v+t^{\overline{\mathbf{s}}}(x_id_\gamma)v=(\gamma\mid\mathbf{s})(x_it^{\overline{\mathbf{s}}})v+\beta\gamma_it^{\overline{\mathbf{s}}}v=\beta\gamma_it^{\overline{\mathbf{s}}}v.$$

This implies that all vectors in U are common eigenvectors for $\mathfrak{gl}_d(\gamma)$, and that

$$(\mathbf{m}\gamma^T)w=(\sum_{i=1}^d m_ix_id_\gamma)w=\beta(\gamma\mid\mathbf{m})w,\quad \text{for any } w\in U,\mathbf{m}\in\mathbb{Z}^d.$$

It remains only to show that U is irreducible as a Γ-graded $\mathfrak{gl}_N$-module. Let V be a nonzero Γ-graded $\mathfrak{gl}_N$-submodule. Since all vectors in V are common eigenvectors for $\mathfrak{gl}_d(\gamma)$, V becomes a $\mathfrak{gl}_d(\gamma)\oplus\mathfrak{gl}_N$-module, hence equals to U by the irreducibility of U. This completes the proof. $\square$

6. Classification of irreducible cuspidal $\mathfrak{g}$-modules

In this last section we use a modified cover method to classify all irreducible cuspidal $\mathfrak{g}$-modules. First we introduce the cover of a cuspidal $\mathfrak{g}$-module M (not necessarily irreducible). The proof of the following Lemma is elementary and we omit it.

Lemma 6.1. (1) *The tensor product $\mathfrak{g}'_R\otimes M$ of $\mathfrak{g}$-modules admits a $\mathcal{Z}\mathfrak{g}$-module structure if we define the $\mathcal{Z}$-action by*

$$t^{\mathbf{n}}(L_{\mathbf{s}}\otimes v)=L_{\mathbf{n}+\mathbf{s}}\otimes v \text{ for } \mathbf{n}\in R, \mathbf{s}\notin R, v\in M.$$

(2) *The map $\pi:\mathfrak{g}'_R\otimes M\longrightarrow M$ defined by $y\otimes v\mapsto y\cdot v$ is a $\mathfrak{g}$-module homomorphism, and it is surjective if $\mathfrak{g}'_R M=M$.*

(3) *The subspace J of $\mathfrak{g}'_R\otimes M$ spanned by vectors with the form of a finite sum $\sum L_{\mathbf{s}}\otimes v_{\mathbf{s}}$, where $\mathbf{s}\notin R, v_{\mathbf{s}}\in M$, satisfying that $\sum L_{\mathbf{n}+\mathbf{s}}v_{\mathbf{s}}=0$ for all $\mathbf{n}\in R$, is a $\mathcal{Z}\mathfrak{g}$-submodule of $\mathfrak{g}'_R\otimes M$, and lies in $\ker\pi$.*

Define $\hat{M}=(\mathfrak{g}'_R\otimes M)/J$. We call this quotient $\mathcal{Z}\mathfrak{g}$-module the *$\mathcal{Z}$-cover* of M. Consider M as a cuspidal $\mathfrak{g}_R$-module and an essentially same argument as Theorem 4.7 in [2] shows that $\hat{M}$ is cuspidal as a $\mathcal{Z}\mathfrak{g}_R$-module. Hence $\hat{M}$ is also cuspidal as a $\mathcal{Z}\mathfrak{g}$-module. Now we are ready to state our main

Theorem 6.2. *Let M be an irreducible cuspidal $\mathfrak{g}$-module. Then M is isomorphic to an irreducible subquotient of some $\mathcal{V}(\alpha,\beta,W)$, where $\alpha\in\mathbb{C}^d,\beta\in\mathbb{C}$ and W is a finite dimensional Γ-graded irreducible $\mathfrak{gl}_N$-module.*

Proof. If $\mathfrak{g}'_R M=0$, then M is an irreducible $\mathfrak{g}_R$-module and the result follows from [3] or [13].

Assume $\mathfrak{g}'_R M\neq 0$. Then $\mathfrak{g}'_R M=M$ by the irreducibility of M. Take J,π as in Lemma 6.1. Since $J\subseteq\ker\pi$, we get a surjective $\mathfrak{g}$-module homomorphism $\hat{\pi}:\hat{M}\longrightarrow M$. Consider the composition series for the $\mathcal{Z}\mathfrak{g}$-module $\hat{M}$,

$$0=\hat{M}_0\subset\hat{M}_1\subset\cdots\subset\hat{M}_l=\hat{M},$$

where all $\hat{M}_i/\hat{M}_{i-1}$ are irreducible cuspidal $\mathcal{Z}\mathfrak{g}$-modules. Let k be the smallest integer such that $\hat{\pi}(\hat{M}_k)\neq 0$. Then we have $\hat{\pi}(\hat{M}_k)=M$ and $\hat{\pi}(\hat{M}_{k-1})=0$, which reduces $\hat{\pi}$ to a $\mathfrak{g}$-module epimorphism from $\hat{M}_k/\hat{M}_{k-1}$ to M. Then Theorem 6.2 follows from Theorem 5.4. $\square$

Acknowledgement: The author was supported by the National Natural Science Foundation of China (No. 11626157, 11801375).

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Received May 7, 2019
and in final form February 25, 2020