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Geometric Cycles in Compact Riemannian Locally Symmetric Spaces of Type IV and Automorphic Representations of Complex Simple Lie Groups

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Abstract. Let G be a connected complex simple Lie group with maximal compact subgroup U. Let $\mathfrak{g}$ be the Lie algebra of G, and $X = G/U$ be the associated Riemannian globally symmetric space of type IV. We have constructed three types of arithmetic uniform lattices in G, say of type 1, type 2, and type 3 respectively. If $\mathfrak{g} \neq \mathfrak{b}_n$ $(n \geq 1)$, then for each $1 \leq i \leq 3$, there is an arithmetic uniform torsion-free lattice Γ in G which is commensurable with a lattice of type i such that the corresponding locally symmetric space $\Gamma\backslash X$ has some non-vanishing (in the cohomology level) geometric cycles, and the Poincaré duals of fundamental classes of such cycles are not represented by G-invariant differential forms on X. As a consequence, we are able to detect some automorphic representations of G, when $\mathfrak{g} = \delta_n$ $(n > 4)$, $\mathfrak{c}_n$ $(n \geq 6)$, or $\mathfrak{f}_4$. To prove these, we have simplified Kač's description of finite order automorphisms of $\mathfrak{g}$ with respect to a Chevalley basis of $\mathfrak{g}$. Also we have determined some orientation preserving group action on some subsymmetric spaces of X.

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1. Introduction

Let G be a non-compact semisimple Lie group with finite centre and K be a maximal compact subgroup of G with θ, the corresponding Cartan involution of G. Let Γ be a torsion-free uniform lattice in G. Then Γ acts freely on the Riemannian globally symmetric space $X := G/K$ and the canonical projection $\pi : X \longrightarrow \Gamma\backslash X$ is a covering projection. Let B be a reductive subgroup of G such that $K_B = B \cap K$ is a maximal compact subgroup of B. Set $X_B = B/K_B$ and $\Gamma_B = B \cap \Gamma$. Note that X_B is a connected totally geodesic submanifold of X. Assume that the natural map $j : \Gamma_B\backslash X_B \longrightarrow \Gamma\backslash X$ is an embedding. Then the image $C_B := j(\Gamma_B\backslash X_B)$ is called a geometric cycle. In literature, these are also known as modular symbols. Under certain conditions, the fundamental class $[C_B] \in H_d(\Gamma\backslash X; \mathbb{C})$ $(d = \dim(\Gamma_B\backslash X_B))$ is non-trivial. So the Poincaré dual of $[C_B]$ contributes nontrivially to $H^*(\Gamma\backslash X; \mathbb{C})$. See [24, Th. 2.1], [30, Th. 4.11]. These theorems are restated here as Theorem 4.1, Theorem 4.2 respectively.

If $\Gamma \subset G$ be any lattice, the Hilbert space $L^2(\Gamma\backslash G)$ of square integrable functions on $\Gamma\backslash G$ with respect to a G-invariant measure, is a unitary representation of G. Here

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the group action on $L^2(\Gamma\backslash G)$ is given by the right translation of G on $\Gamma\backslash G$. When Γ is a uniform lattice, we have

$$L^2(\Gamma\backslash G) \cong \widehat{\bigoplus}_{\pi\in\hat{G}} m(\pi,\Gamma)H_\pi,$$

due to Gelfand and Pyatetskii-Shapiro [9], [10]; where H_π is the representation space of $\pi \in \hat{G}$; $m(\pi,\Gamma) \in \mathbb{N}\cup\{0\}$, the multiplicity of π in $L^2(\Gamma\backslash G)$. If $(\tau,\mathbb{C})$ is the trivial representation of G, then $m(\tau,\Gamma) = 1$. A unitary representation $\pi \in \hat{G}$ such that $m(\pi,\Gamma) > 0$ for some uniform lattice Γ, is called an automorphic representation of G with respect to Γ.

The connection between the geometric cycles and automorphic representations has been made by the Matsushima's isomorphism. Assume now that Γ is a torsion-free uniform lattice in G. Then the isomorphism $L^2(\Gamma\backslash G) \cong \widehat{\bigoplus}_{\pi\in\hat{G}} m(\pi,\Gamma)H_\pi$ implies

$$\bigoplus_{\pi\in\hat{G}} m_\pi H_{\pi,K} \hookrightarrow C^\infty(\Gamma\backslash G)_K.$$

Matsushima's formula [23] says that the above inclusion induces an isomorphism

$$\bigoplus_{\pi\in\hat{G}} m_\pi H^p(\mathfrak{g}^{\mathbb{C}},K;H_{\pi,K}) \cong H^p(\mathfrak{g}^{\mathbb{C}},K;C^\infty(\Gamma\backslash G)_K) \cong H^p(\Gamma\backslash X;\mathbb{C}), \tag{1}$$

where $\mathfrak{g}$ is the Lie algebra of G.

Hence a non-vanishing (in the cohomology level) geometric cycle will contribute to the RHS of (1) and it may help to detect occurrence of some $\pi \in \hat{G}$ with non-zero $(\mathfrak{g}^{\mathbb{C}},K)$-cohomology. In fact, Theorem 2.1 in [24] states that under certain conditions, we have a pair of geometric cycles such that the corresponding cohomology classes are not only non-zero, but also these have non-zero $H^*(\mathfrak{g}^{\mathbb{C}},K;H_{\pi,K})$-components for some non-trivial $\pi \in \hat{G}$.

Based on Theorem 2.1 in [24], this technique was used by Millson and Raghunathan [24] when $G = SU(p,q), SO_0(p,q), Sp(p,q)$. Based on Theorem 2.1 in [24] and Theorem 4.11 in [30], Schwermer and Waldner [33] have done the case for $G = SU^*(2n)$, Waldner [37] has done the case when G is the non-compact real form of the exceptional complex Lie group G_2. The cases $G = SL(n,\mathbb{R}), SL(n,\mathbb{C})$ were considered by Schimpf [32], the case $G = SO^*(2n)$ and more generally the case when G/K is a Hermitian symmetric space were considered by Mondal and Sankaran [26], [27]. Here we consider the case when G is a connected complex simple Lie group.

The main results are stated as Theorem 1.1 and Theorem 1.2. In obtaining our results, we have first considered three types of arithmetic uniform lattices of a connected complex semisimple Lie group of adjoint type. The three types of lattices depend on how one views the Lie algebra $\mathfrak{g}$. Type 1 corresponds to viewing it as a real Lie algebra. Type 2 views it as the complexification of the compact real form of $\mathfrak{g}$. Type 3 involves a choice of a non-compact real form of $\mathfrak{g}$. Actually type 3 is a union of a family of types, one for each non-compact real form of $\mathfrak{g}$. Any lattice of type i $(i = 1, 2, \text{or } 3)$ is θ-stable. See Section 3 for details.

Let $\mathcal{L}_i(G)$ be the collection of θ-stable torsion-free lattices of G which are commensurable to $\mathrm{Ad}^{-1}(\Gamma)$ for some Γ of type i $(i = 1, 2, 3)$.

Theorem 1.1. *Let G be a connected complex simple Lie group with maximal compact subgroup U, and $X = G/U$. For each $i = 1, 2$; there exists $\Gamma \in \mathcal{L}_i(G)$ such that $H^k(\Gamma\backslash X; \mathbb{C})$ contains a non-zero cohomology class which is not represented by G-invariant differential forms on X for all k of the form $dim(X(\bar{\sigma}))$, $dim(X(\bar{\sigma}\bar{\theta}))$ given in the Table 2. Also depending on each pair $X(\bar{\sigma})$, $X(\bar{\sigma}\bar{\theta})$ in the Table 2, there exists $\Gamma \in \mathcal{L}_3(G)$ such that $H^k(\Gamma\backslash X; \mathbb{C})$ contains a non-zero cohomology class which is not represented by G-invariant differential forms on X for k of the form $dim(X(\bar{\sigma}))$, $dim(X(\bar{\sigma}\bar{\theta}))$ given in the Table 2.*

The proof of the above theorem is given in Section 4.4. To prove the theorem we have used Kač's classification of finite order automorphisms of a complex simple Lie algebra $\mathfrak{g}$ ([14]). Actually we have given a simple description of a finite order automorphism with respect to a Chevalley basis of $\mathfrak{g}$ (see Section 4.1), which is a new addition, as far as we know. Also we have described all finite order automorphisms of $\mathfrak{g}^{\mathbb{R}}$ up to conjugation (see Remark 4.4(iii)). Remark 4.4(i), (ii) might be interesting from representation theoretic point of view. We also need to determine some orientation preserving group action on a subsymmetric space of a Riemannian globally symmetric space of type IV. The work has been done in Section 4.2 and the result is summarised in Table 1. These are important in topology and other areas of mathematics also.

If G is a connected complex simple Lie group, Theorem 1.1 gives us some non-vanishing (in the cohomology level) geometric cycles in the RHS of (1). To detect some automorphic representation of G, it is important to know the irreducible unitary representations of G with non-zero relative Lie algebra cohomology, which appear in the LHS of (1). Let G be a connected semisimple Lie group with finite centre, $\mathfrak{g}$ =Lie(G), K be a maximal compact subgroup of G with Cartan involution θ. The irreducible unitary representations of G with non-zero $(\mathfrak{g}^{\mathbb{C}}, K)$-cohomology are classified in terms of the θ-stable parabolic subalgebras $\mathfrak{q} \subset \mathfrak{g}^{\mathbb{C}}$.

A θ-stable parabolic subalgebra of $\mathfrak{g}$ is by definition, a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}^{\mathbb{C}}$ such that $\theta(\mathfrak{q}) = \mathfrak{q}$ and $\bar{\mathfrak{q}} \cap \mathfrak{q}$ is a Levi subalgebra of $\mathfrak{q}$, where $^-$ denotes the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$. Associated with a θ-stable parabolic subalgebra $\mathfrak{q}$, we have an irreducible unitary representation $A_{\mathfrak{q}}$ of G with trivial infinitesimal character and non-zero $(\mathfrak{g}^{\mathbb{C}}, K)$-cohomology. If $\mathfrak{q}$ is a θ-stable parabolic subalgebra, then so is $\mathrm{Ad}(k)(\mathfrak{q})$ $(k \in K)$; and $A_{\mathfrak{q}}$, $A_{\mathrm{Ad}(k)(\mathfrak{q})}$ are unitarily equivalent. If $\mathfrak{q} = \mathfrak{g}$, then $A_{\mathfrak{q}} = \mathbb{C}$, the trivial representation of G. If $\mathrm{rank}(G) = \mathrm{rank}(K)$ and $\mathfrak{q}$ is a θ-stable Borel subalgebra, then $A_{\mathfrak{q}}$ is a discrete series representation of G with trivial infinitesimal character. See Section 5.1 for more details.

Now let G be complex, $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$, and θ be the corresponding Cartan involution of $\mathfrak{g}$. Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$. Choose a system of positive roots Δ^+ in the set of all non-zero roots $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{b}$ be the corresponding Borel subalgebra of $\mathfrak{g}$ and Φ be the set of all simple roots in Δ^+. We can deduce that the θ-stable parabolic subalgebras of $\mathfrak{g}$ are of the form $\mathfrak{q} \times \mathfrak{q}$, where $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ containing a θ-stable Cartan subalgebra of $\mathfrak{g}$ (see Section 5.2). Also it is sufficient to consider the θ-stable parabolic subalgebras of $\mathfrak{g}$ of the form $\mathfrak{q} \times \mathfrak{q}$, where $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{b}$ (see Section 5.2 again). The parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{b}$ are in one-one correspondence with the power set of Φ. That is, $\mathfrak{q}$ is of the form

$\mathfrak{q}_{\Phi'} = \mathfrak{l}_{\Phi'} \oplus \mathfrak{u}_{\Phi'}$, where

$$\mathfrak{l}_{\Phi'} = \mathfrak{h} \oplus \sum_{\substack{n_\psi(\alpha)=0 \\ \forall \psi \in \Phi'}} \mathfrak{g}_\alpha, \mathfrak{u}_{\Phi'} = \sum_{\substack{n_\psi(\alpha)>0 \\ \text{for some } \psi \in \Phi'}} \mathfrak{g}_\alpha;$$

and $\alpha = \sum_{\psi \in \Phi} n_\psi(\alpha)\psi \in \Delta$, for some $\Phi' \subset \Phi$. Let $A_{\Phi'}$ be the irreducible unitary representation of G with non-zero $(\mathfrak{g} \times \mathfrak{g}, U)$-cohomology corresponding to the θ-stable parabolic subalgebra $\mathfrak{q}_{\Phi'} \times \mathfrak{q}_{\Phi'}$, where U is the connected Lie subgroup of G with Lie algebra $\mathfrak{u}$. Then the Poincaré polynomial of $H^*(\mathfrak{g} \times \mathfrak{g}, U; A_{\Phi',U})$ is given by

$$P(\Phi', t) = t^{\dim(\mathfrak{u}_{\Phi'})}(1+t)^{|\Phi'|} P(\mathfrak{l}_1, t) P(\mathfrak{l}_2, t) \cdots P(\mathfrak{l}_k, t),$$

where $\mathfrak{l}_1, \mathfrak{l}_2, \dots \mathfrak{l}_k$ are the simple factors of the semisimple part $[\mathfrak{l}_{\Phi'}, \mathfrak{l}_{\Phi'}]$ of $\mathfrak{l}_{\Phi'}$ and each $P(\mathfrak{l}_i, t)$ is given by the formula below. If $\mathfrak{s}$ is a finite dimensional complex simple Lie algebra, the Poincaré polynomial $P(\mathfrak{s}, t)$ is given by

$$P(\mathfrak{s}, t) = (1 + t^{2d_1+1})(1 + t^{2d_2+1}) \cdots (1 + t^{2d_l+1}),$$

where $l = \text{rank}(\mathfrak{s})$ and $d_1, d_2, \dots d_l$ are the exponents of $\mathfrak{s}$. We have deduced the formula for $P(\Phi', t)$ from a more general result in [15]. Also for Φ', $\Phi'' \subset \Phi$, $A_{\Phi'}$ is unitarily equivalent to $A_{\Phi''}$ if and only if $\Phi' = \Phi''$. See Section 5.2 for details.

Now combining these with Theorem 1.1, we get

Theorem 1.2. *Let G be a connected complex simple Lie group. For each $i = 1, 2, 3$, there exists a uniform lattice $\Gamma \in \mathcal{L}_i(G)$ of G such that $L^2(\Gamma \backslash G)$ has an irreducible component $A_{\Phi'}$, where*

(i) $\Phi' = \{\psi_1\}$*, or* $\{\psi_2\}$*, or* $\{\psi_1, \psi_2\} \subset \Phi$*, if G is of C_n-type ($n \geq 6$). If* $n = 6$, 8*, or* 10*, we can discard* $\{\psi_1\}$ *among these.*

(ii) $\Phi' = \{\psi_1\} \subset \Phi$*, if G is of D_n-type ($n > 4$).*

(iii) $\Phi' = \{\psi_1\}$ *or* $\{\psi_4\} \subset \Phi$*, if G is of F_4-type.*

The proof of the above theorem is given in Section 5.3. In literature, there are non-vanishing results of the multiplicity of automorphic representations in $L^2(\Gamma \backslash G)$, for example see [1], [5], [6], [22], [3, Ch. VIII], [29, § 6], [27]. In all those cases, G is an equi-rank group, that is rank of G is equal to the rank of a maximal compact subgroup. But in our case, G is complex, so can not be an equi-rank group. Schimpf [32] has identified some automorphic representation, when $G = SL(n, \mathbb{C})$ $(n = 2,\ 3)$. We also get Schimpf's result for $n = 3$, see Remark 5.5(i). The problem in identifying automorphic representation using this technique is that if a geometric cycle gives non-zero cohomology class in $H^k(\Gamma \backslash X; \mathbb{C})$, then most of the times it happens that there are more than one $A_\mathfrak{q}$ with $H^k(\mathfrak{g}^\mathbb{C}, K; A_{\mathfrak{q},K}) \neq 0$. Theorem 4.1 in [25], or Theorem 1.2 in [18] might be a way to solve this problem. See Remark 5.5(ii).

2. Cartan involution of a real semisimple Lie algebra with complex structure

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$. Let $\mathfrak{g}^\mathbb{R}$ denote the Lie algebra $\mathfrak{g}$ considered as a real Lie algebra and let J denote the complex structure of $\mathfrak{g}^\mathbb{R}$ corresponding to the multiplication by i of $\mathfrak{g}$.

Then $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} \oplus J\mathfrak{u}$ is a Cartan decomposition of $\mathfrak{g}^{\mathbb{R}}$ with the corresponding Cartan involution θ (say). The complex linear extension of θ to the complexification $(\mathfrak{g}^{\mathbb{R}})^{\mathbb{C}}$ is denoted by the same notation θ.

Let s denote the the involution $(X,Y) \mapsto (Y,X)$ of the product algebra $\mathfrak{l} = \mathfrak{u} \times \mathfrak{u}$. Then $(\mathfrak{l}, s)$ is an orthogonal symmetric algebra of the compact type and $\mathfrak{l} = \mathfrak{u}_* + \mathfrak{e}_*$ is the decomposition of $\mathfrak{l}$ into eigenspaces of s, where $\mathfrak{u}_* = \{(X,X) : X \in \mathfrak{u}\}$ and $\mathfrak{e}_* = \{(X,-X) : X \in \mathfrak{u}\}$. Let $(\mathfrak{l}^*, s^*)$ denote the dual of $(\mathfrak{l}, s)$, where $\mathfrak{l}^*$ is the subset $\mathfrak{u}_* + i\mathfrak{e}_*$ of the complexification $\mathfrak{l}^{\mathbb{C}}$ of $\mathfrak{l}$ and s^* is the map $T + iX \mapsto T - iX (T \in \mathfrak{u}_*, X \in \mathfrak{e}_*)$. Now $\mathfrak{g}$ is isomorphic to $\mathfrak{l}^*$ (as a real Lie algebra) via the map $\phi : X + JY \mapsto (X,X) + i(Y,-Y)$, where $X, Y \in \mathfrak{u}$. Also we have $\phi \circ \theta = s^* \circ \phi$. Hence the complexification $(\mathfrak{g}^{\mathbb{R}})^{\mathbb{C}}$ is isomorphic to $(\mathfrak{l}^*)^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}} \cong \mathfrak{g} \times \mathfrak{g}$ in such a way that θ corresponds to the complex linear extension of s, that is θ corresponds to the map $(Z_1, Z_2) \mapsto (Z_2, Z_1)$ of $\mathfrak{g} \times \mathfrak{g}$.

3. Arithmetic uniform lattices of a connected complex semisimple Lie group

Let G be a connected semisimple Lie group. The natural way to construct arithmetic uniform discrete subgroups of G is Weil's restriction of scalars described below:

Let F be an algebraic number field of degree > 1 and G' be a linear connected semisimple Lie group defined over F such that G is isogenous with G'. Then It is sufficient to consider arithmetic uniform discrete subgroups of G'. Let S be the set of all infinite places of F. For each $s \in S$, define $F_s = \mathbb{R}$, if $s(F) \subset \mathbb{R}$; and $F_s = \mathbb{C}$, if $s(F) \not\subset \mathbb{R}$. We can identify G' with a subgroup of $SL(N, F_{\text{id}})$ defined over F that is, there exists a finite subset P of $F[x_{11}, ..., x_{NN}]$ such that G' is the identity component of the group $\{g \in SL(N, F_{\text{id}}) : p(g) = 0 \text{ for all } p \in P\}$. For each $s \in S$ let G'^s be the identity component of the group $\{g \in SL(N, F_s) : s(p)(g) = 0 \text{ for all } p \in P\}$. Let $\mathcal{O}$ be the ring of integers of F, and $G'_{\mathcal{O}} = G' \cap \mathrm{GL}_N(\mathcal{O})$. Then $G'_{\mathcal{O}}$ is an arithmetic uniform lattice of G' if G'^s is compact for all $s \in S \setminus \{\text{id}\}$.

We shall follow the construction of Borel [2] to construct some arithmetic uniform lattices in a connected complex semisimple Lie group. Let G be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. As before, let $\mathfrak{g}^{\mathbb{R}}$ denote the Lie algebra $\mathfrak{g}$ considered as a real Lie algebra and let J denote the complex structure of $\mathfrak{g}^{\mathbb{R}}$ corresponding to the multiplication by i of $\mathfrak{g}$. Note that $G/Z \cong \mathrm{Ad}(G)$, where Z denotes the centre of G. As G is a connected complex semisimple Lie group, Z is finite. So it is sufficient to determine arithmetic uniform lattices of $\mathrm{Ad}(G)$, which is the identity component of $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$. As the Lie group $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$ has finitely many components, it is sufficient to determine uniform arithmetic lattices of $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$. We shall construct three types of arithmetic uniform lattices in $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$. But before proceeding further, we need some facts about algebraic number fields.

3.1. Algebraic number fields. Let F be an algebraic number field and S be the set of all real places of F. By the Theorem of the Primitive Element, we may write $F = \mathbb{Q}(u)$ for some $u \in F$.

Proposition 3.1. *For any $k, l \in \mathbb{N} \cup \{0\}$ with $k + l = |S|$, We may choose a primitive element $u \in F$ such that the number of positive real conjugates of u is k and the number of negative real conjugates of u is l.*

Proof. Let $S = \{s_1, s_2, \ldots, s_n\}$, where $n = |S|$. Let $u_i = s_i(u)$ for all $1 \le i \le n$. Assume that $u_1 < u_2 < \cdots < u_l < u_{l+1} < \cdots < u_{l+k}$ (here $k + l = n$). Choose $r \in \mathbb{Q}$ such that $u_l < r < u_{l+1}$. Then $u_i - r < 0$ for $1 \le i \le l$, and $u_{l+j} - r > 0$ for $1 \le j \le k$. Clearly $F = \mathbb{Q}(u) = \mathbb{Q}(u - r)$, and $u_1 - r, u_2 - r, \ldots, u_l - r, u_{l+1} - r, \ldots, u_{l+k} - r$ are all real conjugates of u. So this $u - r$ is a primitive element with the required property. ■

Remark 3.2. (i) If F is a totally real number field, F has a primitive element u such that u has exactly one positive conjugate (by Proposition 3.1). Replacing F by a conjugate of F (if necessary), we may assume that $F = \mathbb{Q}(u)$ with $u > 0$ and $s(u) < 0$ for all $s \in S - \{\mathrm{id}\}$, S is the set of all infinite places of F.

(ii) Let F be an algebraic number field such that $F \not\subset \mathbb{R}$ and all other conjugates of F are real. Then again by Proposition 3.1, we may write $F = \mathbb{Q}(u)$, where $u \in \mathbb{C}$ with $s(u) < 0$ for all $s \in S - \{\mathrm{id}\}$, S is the set of all infinite places of F. ■

Examples 3.3. (1) If m is a positive square-free integer, the quadratic number field $\mathbb{Q}(\sqrt{m})$ is a totally real number field. More generally, if $f \in \mathbb{Q}[x]$ is irreducible and all roots of f are real, then $\mathbb{Q}(\alpha)$ is a totally real number field, where α is a root of f.

(2) Let $h \in \mathbb{Q}[x]$ be an irreducible polynomial such that h has exactly two non-real roots. For each $n \in \mathbb{N}$ with $n \ge 2$, there exists such a polynomial of degree n. For example, start with $f(x) = (x^2 + k)(x - k_1)\cdots(x - k_{n-2})$, where $k, k_1, \cdots, k_{n-2}$ are positive even integers and $k_1, k_2, \ldots, k_{n-2}$ are distinct. Let then $x_1, x_2, \ldots, x_m$ $(n-3 \le m \le n-1)$ be the real roots of $f'(x) = 0$. Since the real roots of f are all distinct, $f(x_i) \ne 0$ for all $1 \le i \le m$. Let $\epsilon = \min\{|f(x_i)| : 1 \le i \le m\}$. For any $a \in \mathbb{R}$ with $|a| < \epsilon$, let $g_a(x) = f(x) + a$. Then $g_a(x) = 0$ has exactly $n - 2$ real roots. For if f has a local optimum value above (respectively, below) the x-axis, the corresponding local optimum value of g_a is above (respectively, below) the x-axis; and vice versa. Let q be an odd integer such that $\frac{2}{q} < \epsilon$. Then $f(x) + \frac{2}{q} = 0$ has exactly $n - 2$ real roots. Hence if $h(x) = qf(x) + 2$, then $h(x) = 0$ also has exactly $n - 2$ real roots. If $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $a_0, a_1, \cdots, a_{n-1}$ are all even integers. Also $h(x) = qx^n + (qa_{n-1})x^{n-1} + \cdots + (qa_1)x + (qa_0 + 2)$. So $h \in \mathbb{Z}[x]$ is irreducible, by Eisenstein's Criterion (see [13, Ch. 4]). ■

The algebraic number field $\mathbb{Q}(\alpha)$ has exactly one complex place, where α is a root of h.

3.2. Construction of some arithmetic uniform lattices in $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$. Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{h} = (\mathfrak{u} \cap \mathfrak{h}) \oplus (J\mathfrak{u} \cap \mathfrak{h})$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of all non-zero roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$, Δ^+ be the set of positive roots in Δ with respect to some chosen ordering and Φ the set of all simple roots in Δ^+. Let B denote the Killing form of $\mathfrak{g}$. For each $\alpha \in \Delta$, there exists unique $H_\alpha \in \mathfrak{h}$ such that

$$\alpha(H) = B(H, H_\alpha) \text{ for all } H \in \mathfrak{h}.$$

Let $H_\alpha^* = 2H_\alpha/\alpha(H_\alpha)$ for all $\alpha \in \Delta$. For each $\alpha \in \Delta$ there exists $E_\alpha \in \mathfrak{g}$ such that

$$[H, E_\alpha] = \alpha(H)E_\alpha \text{ for all } H \in \mathfrak{h},$$

$$\begin{aligned}
[E_\alpha, E_{-\alpha}] &= H^*_\alpha \ \text{ for all } \alpha \in \Delta, \\
[E_\alpha, E_\beta] &= 0 \ \text{ if } \alpha, \beta \in \Delta, \alpha+\beta \notin \Delta, \alpha+\beta \neq 0, \\
[E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta} \ \text{ if } \alpha, \beta, \alpha+\beta \in \Delta, \text{ where} \\
N_{\alpha,\beta} &= -N_{-\alpha,-\beta} = \pm(1-p),
\end{aligned} \tag{2}$$

and $\beta + n\alpha (p \leq n \leq q)$ is the α-series containing β. Also we can choose E_α $(\alpha \in \Delta)$ in such a way that

$$E_\alpha - E_{-\alpha},\ i(E_\alpha + E_{-\alpha}) \in \mathfrak{u} \text{ for all } \alpha \in \Delta.$$

Then $\{H^*_\phi, E_\alpha : \phi \in \Phi, \alpha \in \Delta\}$ is a Chevalley basis of $\mathfrak{g}$ such that

$$\mathfrak{u} = \sum_{\phi \in \Phi} \mathbb{R}(iH^*_\phi) \oplus \sum_{\alpha \in \Delta^+} \mathbb{R}(E_\alpha - E_{-\alpha}) \oplus \sum_{\alpha \in \Delta^+} \mathbb{R}(i(E_\alpha + E_{-\alpha})). \tag{3}$$

Let $X_\alpha = E_\alpha - E_{-\alpha}, Y_\alpha = i(E_\alpha + E_{-\alpha})$ for all $\alpha \in \Delta^+$.

Let F be an algebraic number field of degree > 1, $\mathcal{O}$ be the ring of integers of F and S be the set of all infinite places of F. Assume that $s(F) \subset \mathbb{R}$ for all $s \in S \setminus \{\mathrm{id}\}$ (see Examples in Section 3.1). If G is real, we assume that $F \subset \mathbb{R}$. If G is complex, we assume that $F \not\subset \mathbb{R}$. In any case, we may write $F = \mathbb{Q}(u)$, where $s(u) < 0$ for all $s \in S \setminus \{\mathrm{id}\}$ (Remark 3.2). If $F \subset \mathbb{R}$, then we may choose $u > 0$. Otherwise $u \in \mathbb{C}$. Let $v = \sqrt{u}$ and $v_s = \sqrt{-s(u)}$ for all $s \in S \setminus \{\mathrm{id}\}$. Note that $v_s > 0$ for all $s \in S \setminus \{\mathrm{id}\}$. Also if $u > 0$, then $v > 0$.

Now we shall construct some arithmetic uniform lattices of $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$ as follows:

1. First view $\mathfrak{g}$ as a real Lie algebra $\mathfrak{g}^{\mathbb{R}}$. Let F be an algebraic number field as above with $F \subset \mathbb{R}$. Recall that $\mathfrak{g}$ is isomorphic to the non-compact real form $\mathfrak{l}^*$ of $\mathfrak{g} \times \mathfrak{g}$ in such a way that the Cartan decomposition $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} \oplus J\mathfrak{u}$ corresponds to the Cartan decomposition $\mathfrak{l}^* = \mathfrak{u}_* \oplus \mathfrak{e}$ of $\mathfrak{l}^*$, where $\mathfrak{u}_* = \{(X, X) : X \in \mathfrak{u}\}$ and $\mathfrak{e} = \{(iX, -iX) : X \in \mathfrak{u}\}$ (see Section 2). Then

$$\{(iH^*_\phi, iH^*_\phi), (X_\alpha, X_\alpha), (Y_\alpha, Y_\alpha) : \phi \in \Phi, \alpha \in \Delta^+\}$$
$$\cup \{H^*_\phi, -H^*_\phi), (iX_\alpha, -iX_\alpha), (iY_\alpha, -iY_\alpha) : \phi \in \Phi, \alpha \in \Delta^+\}$$

is a basis of $\mathfrak{g}^{\mathbb{R}}$ (via this identification) consisting of vectors belonging to either $\mathfrak{u}_*$ or to $\mathfrak{e}$, with respect to which the structural constants are integers.

Let $\mathfrak{m}$ be the vector space over F spanned by the set

$$\{(iH^*_\phi, iH^*_\phi), (X_\alpha, X_\alpha), (Y_\alpha, Y_\alpha) : \phi \in \Phi, \alpha \in \Delta^+\}$$
$$\cup \{vH^*_\phi, -vH^*_\phi), (ivX_\alpha, -ivX_\alpha), (ivY_\alpha, -ivY_\alpha) : \phi \in \Phi, \alpha \in \Delta^+\}$$

and $\mathfrak{m}^s$ be the vector space over $F^s = s(F)$ spanned by the set

$$\{(iH^*_\phi, iH^*_\phi), (X_\alpha, X_\alpha), (Y_\alpha, Y_\alpha) : \phi \in \Phi, \alpha \in \Delta^+\}$$
$$\cup \{iv_sH^*_\phi, -iv_sH^*_\phi), (-v_sX_\alpha, v_sX_\alpha),\ (-v_sY_\alpha, v_sY_\alpha) : \phi \in \Phi, \alpha \in \Delta^+\}$$

for all $s \in S - \{\mathrm{id}\}$. Then $\mathfrak{m}$ is a Lie algebra over F, $\mathfrak{m}^s$ is a Lie algebra over F^s, and the structural constants of $\mathfrak{m}^s$ are the conjugates by s of the structural constants of $\mathfrak{m}$ with respect to the given bases for all $s \in S - \{\mathrm{id}\}$. Thus $\mathfrak{m}^s$ is the conjugate of $\mathfrak{m}$ by s. We also have $\mathfrak{m} \otimes \mathbb{R} = \mathfrak{g}^{\mathbb{R}}$, $\mathfrak{m}^s \otimes \mathbb{R} = \mathfrak{u} \times \mathfrak{u}$ for all $s \in S - \{\mathrm{id}\}$.

Take a basis of $\mathfrak{g}^{\mathbb{R}}$ contained in $\mathfrak{m}$ and identify $\mathrm{Aut}((\mathfrak{g}^{\mathbb{R}})^{\mathbb{C}})$ with an algebraic subgroup G' of $GL(2N,\mathbb{C})$ ($2N = \dim(\mathfrak{g}^{\mathbb{R}})$) defined over F, via this basis. Then $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$ is identified with $G'_{\mathbb{R}}$, the group of real matrices in G'. The group $(G'_{\mathbb{R}})^s$ is then $\mathrm{Aut}(\mathfrak{u}\times\mathfrak{u})$, hence compact, for all $s \in S - \{\mathrm{id}\}$. Let $\mathcal{O}$ be the ring of algebraic integers of F and $\Gamma = G'_{\mathcal{O}} = G' \cap GL(2N,\mathcal{O})$. As $(G'_{\mathbb{R}})^s$ is compact for all $s \in S - \{\mathrm{id}\}$, Γ is a cocompact arithmetic lattice in $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$. An arithmetic uniform lattice of $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$ constructed in this way, is called a lattice of type 1.

2. Now view $\mathfrak{g}$ as a complex Lie algebra, and F be an algebraic number field $F \not\subset \mathbb{R}$ with $s(F) \subset \mathbb{R}$ for all $s \in S \setminus \{\mathrm{id}\}$.

(i) Let $B = \{iH^*_{\phi}, X_{\alpha}, Y_{\alpha} : \phi \in \Phi, \alpha \in \Delta^+\}$. Let $\mathfrak{m}$ be the vector space over F spanned by the set B and $\mathfrak{m}^s$ be the vector space over F^s spanned by the set B for all $s \in S - \{\mathrm{id}\}$. Then $\mathfrak{m}$ is a Lie algebra over F, $\mathfrak{m}^s$ is a Lie algebra over F^s, and the structural constants of $\mathfrak{m}$ and $\mathfrak{m}^s$ are integers with respect to the basis B for all $s \in S - \{\mathrm{id}\}$. Thus $\mathfrak{m}^s$ is the conjugate of $\mathfrak{m}$ by s. We also have $\mathfrak{m}\otimes\mathbb{C} = \mathfrak{g}$, $\mathfrak{m}^s \otimes \mathbb{R} = \mathfrak{u}$ for all $s \in S - \{\mathrm{id}\}$. Here note that the real span of B is the compact real form of $\mathfrak{g}$.

(ii) Let $\mathfrak{g}_0$ be a non-compact real form of $\mathfrak{g}$ and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$ such that $\mathfrak{u} = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$. Let $\{e_{\lambda}\}$ be a basis of $\mathfrak{g}_0$ consisting of vectors belonging either to $\mathfrak{k}_0$ or to $\mathfrak{p}_0$, with respect to which the structural constants are all rational numbers [2, Prop. 3.7]. Let k and p stand for indices of the subbases for $\mathfrak{k}_0$ and $\mathfrak{p}_0$ respectively.

Let $\mathfrak{m}$ be the vector space over F spanned by the elements e_k and ve_p, and $\mathfrak{m}^s$ be the vector space over F^s spanned by the elements e_k and $iv_s e_p$, for all $s \in S - \{\mathrm{id}\}$. Then $\mathfrak{m}$ is a Lie algebra over F, $\mathfrak{m}^s$ is a Lie algebra over F^s, $\mathfrak{m}^s$ is the conjugate of $\mathfrak{m}$ by s and $\mathfrak{m}\otimes\mathbb{C} = \mathfrak{g}$, $\mathfrak{m}^s\otimes\mathbb{R} = \mathfrak{u}$ for all $s \in S - \{\mathrm{id}\}$. Let B' be the set consisting of vectors e_k and ve_p.

Identify $\mathrm{Aut}(\mathfrak{g})$ with an algebraic subgroup G' of $GL(N,\mathbb{C})$ ($N = \dim_{\mathbb{C}}(\mathfrak{g})$) defined over F, via the basis B (respectively, B') in case (i) (respectively case (ii)). The group $(G')^s$ is then $\mathrm{Aut}(\mathfrak{u})$, hence compact, for all $s \in S - \{\mathrm{id}\}$. Let $\mathcal{O}$ be the ring of algebraic integers of F and $\Gamma = G'_{\mathcal{O}} = G' \cap GL(N,\mathcal{O})$. As $(G')^s$ is compact for all $s \in S - \{\mathrm{id}\}$, Γ is a cocompact arithmetic lattice in $\mathrm{Aut}(\mathfrak{g})$. In case (i), $G'_{\mathbb{R}}$ is $\mathrm{Aut}(\mathfrak{u})$, which is compact. And in case (ii), $G'_{\mathbb{R}}$ is $\mathrm{Aut}(\mathfrak{g}_0)$, which is non-compact. An arithmetic uniform lattice of $\mathrm{Aut}(\mathfrak{g})$ constructed as in 2.(i), is called a lattice of type 2; and an arithmetic uniform lattice of $\mathrm{Aut}(\mathfrak{g})$ constructed in 2.(ii), is called a lattice of type 3.

Note that $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} \oplus J\mathfrak{u}$ is a Cartan decomposition of $\mathfrak{g}^{\mathbb{R}}$. Let θ be the corresponding Cartan involution. Let Γ be a cocompact arithmetic lattice of $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$ constructed as in 1 or 2. Note that,

(i) if Γ is as in 1, then $\theta \in \Gamma$; and (ii) if Γ is as in 2, then $\theta \notin \Gamma$, as $\theta \notin \mathrm{Aut}(\mathfrak{g})$.

But $\theta\Gamma\theta^{-1} = \Gamma$, in both cases 1 and 2. Also $\mathfrak{g} \cong \mathrm{ad}(\mathfrak{g})$ and the real Lie algebra isomorphism of $\mathrm{ad}(\mathfrak{g})$ corresponding to the Cartan involution θ of $\mathfrak{g}^{\mathbb{R}}$ is given by $\mathrm{ad}(X) \mapsto \mathrm{ad}(\theta X) = \theta\mathrm{ad}(X)\theta^{-1}$, which is denoted by the same notation θ. Then θ is the differential at identity of the Lie group isomorphism $\tilde{\theta}$ of $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$ given by $\tilde{\theta}(\sigma) = \theta\sigma\theta^{-1}$. The Lie group isomorphism of G whose differential at identity is θ, is also denoted by the same notation θ. Then we have $\mathrm{Ad}\circ\theta = \tilde{\theta}\circ\mathrm{Ad}$. So if Γ is a cocompact arithmetic lattice of $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$ constructed as in 1 or 2, then $\theta(\mathrm{Ad}^{-1}(\Gamma)) = \mathrm{Ad}^{-1}(\Gamma)$.

4. Special cycles in Riemannian globally symmetric space of type IV

Let G be a real semisimple Lie group with finite centre and K be a maximal compact subgroup of G. Let Γ be a torsion-free uniform discrete subgroup of G. Then Γ acts freely on the Riemannian globally symmetric space $X := G/K$ and the canonical projection $\pi : X \longrightarrow \Gamma\backslash X$ is a covering projection. One can identify the group cohomology $H^*(\Gamma;\mathbb{C})$ with the cohomology $H^*(\Gamma\backslash X;\mathbb{C})$ of the locally symmetric space $\Gamma\backslash X$.

Let B be a reductive subgroup of G such that $K_B = B \cap K$ is a maximal compact subgroup of B. Set $X_B = B/K_B$ and $\Gamma_B = B \cap \Gamma$. Note that X_B is a connected totally geodesic submanifold of X. Assume that the natural map $j\colon \Gamma_B\backslash X_B \longrightarrow \Gamma\backslash X$ is an embedding. Then the image $C_B := j(\Gamma_B\backslash X_B)$ is called a geometric cycle. Under certain conditions, the fundamental class $[C_B] \in H_d(\Gamma\backslash X;\mathbb{C})$ $(d = \dim(\Gamma_B\backslash X_B))$ is non-trivial. So the Poincaré dual of $[C_B]$ contributes nontrivially to $H^*(\Gamma\backslash X;\mathbb{C})$.

If the reductive subgroup B is the fixed point set of a finite order automorphism μ of G such that $\mu(K) = K$ and $\mu(\Gamma) = \Gamma$, then we denote B by $G(\mu)$, K_B by $K(\mu)$, X_B by $X(\mu)$ and Γ_B by $\Gamma(\mu)$. In this case, the natural map $j : \Gamma(\mu)\backslash X(\mu) \longrightarrow \Gamma\backslash X$ is an embedding and the image $C(\mu,\Gamma) := j(\Gamma(\mu)\backslash X(\mu))$ is called a special cycle.

Let X_u denote the compact dual of X. We can identify the cohomology $H^*(X_u;\mathbb{C})$ of X_u with the cohomology $H^*(\Omega(X;\mathbb{C})^G)$ of the complex $\Omega(X;\mathbb{C})^G$ of G-invariant complex valued differential forms on X. Since Γ is a cocompact discrete subgroup of G, the inclusion $j_\Gamma : \Omega(X;\mathbb{C})^G \hookrightarrow \Omega(X;\mathbb{C})^\Gamma$ induces an injective map

$$j_\Gamma^*\colon H^*(\Omega(X;\mathbb{C})^G) \hookrightarrow H^*(\Omega(X;\mathbb{C})^\Gamma)$$

(the so called Matsushima map). Now we can identify the cohomology $H^*(\Gamma\backslash X;\mathbb{C})$ of $\Gamma\backslash X$ with the cohomology $H^*(\Omega(X;\mathbb{C})^\Gamma)$ of the complex $\Omega(X;\mathbb{C})^\Gamma$. In this way we have an injective map

$$k_\Gamma\colon H^*(X_u;\mathbb{C}) \longrightarrow H^*(\Gamma\backslash X;\mathbb{C}).$$

So the elements in the image $k_\Gamma(H^*(X_u;\mathbb{C}))$ are represented by the G-invariant differential forms on X.

The following results state some conditions under which fundamental class of a special cycle is non-zero and the corresponding cohomology class does not lie in the image of the Matsushima map that is, it is not represented by a G-invariant differential form on X.

Theorem 4.1. (Th. 2.1, [24]) *Let F be an algebraic number field of degree > 1 with ring of integers $\mathcal{O}$. Let G be a linear connected semisimple Lie group defined over F, θ be a Cartan involution of G defined over F and $K = \{g \in G : \theta(g) = g\}$. Let σ be an involutive automorphism of G defined over F with $\sigma\theta = \theta\sigma$ and $\Gamma \subset G_{\mathcal{O}}$ be a torsion-free, $\langle\sigma,\theta\rangle$-stable, arithmetic uniform lattice of G such that the Lie groups $G, G(\sigma), G(\sigma\theta)$ act orientation preservingly on $X, X(\sigma)$ and $X(\sigma\theta)$ respectively.*
Then there exists a $\langle\sigma,\theta\rangle$-stable subgroup Γ' of Γ of finite index such that the cohomology classes defined by $[C(\sigma,\Gamma')], [C(\sigma\theta,\Gamma')]$ via Poincaré duality are non-zero and are not represented by G-invariant differential forms on X.

Theorem 4.2. (Th. 4.11, [30]) *Let $F, \mathcal{O}, G, \theta, K$ be as in the above theorem. Let σ and τ be finite order automorphisms of G defined over F with $\sigma\theta = \theta\sigma, \tau\theta = \theta\tau$ and $\sigma\tau = \tau\sigma$. Let $\Gamma \subset G_{\mathcal{O}}$ be a torsion-free, $\langle\sigma,\tau\rangle$-stable, arithmetic uniform lattice of G such that $\Gamma\backslash X, C(\sigma,\Gamma), C(\tau,\Gamma)$ and all connected components of their intersection are orientable. Assume that*

(i) $dim(C(\sigma,\Gamma)) + dim(C(\tau,\Gamma)) = dim(\Gamma\backslash X)$,

(ii) *the Lie groups $G, G(\sigma), G(\tau)$ act orientation preservingly on $X, X(\sigma)$ and $X(\tau)$ respectively, and*

(iii) *the group $G(\langle\sigma,\tau\rangle)$ is compact. Then there exists a $\langle\sigma,\tau\rangle$-stable normal subgroup Γ'' of Γ of finite index such that $[C(\sigma,\Gamma'')][C(\tau,\Gamma'')] \neq 0$.*

Remark 4.3. (i) If σ is an involution with $\sigma\theta = \theta\sigma$, and $\tau = \sigma\theta$, then obviously $\dim(C(\sigma,\Gamma)) + \dim(C(\tau,\Gamma)) = \dim(\Gamma\backslash X)$, and the group $G(\langle\sigma,\tau\rangle)$ is a closed subgroup of K, hence compact. Also in this case, the cycles $C(\sigma,\Gamma), C(\sigma\theta,\Gamma)$ intersect transversely, and so the connected components of their intersection are points. Hence if the Lie groups $G(\sigma), G(\sigma\theta)$ act orientation preservingly on $X(\sigma)$ and $X(\sigma\theta)$ respectively, then in particular, $C(\sigma,\Gamma), C(\sigma\theta,\Gamma)$ and all connected components of their intersection are orientable.

(ii) Originally, Th. 2.1 in [24] has been stated under the assumption that $C(\sigma,\Gamma)$, $C(\sigma\theta,\Gamma)$ are orientable, and all intersections of $C(\sigma,\Gamma), C(\sigma\theta,\Gamma)$ are of positive multiplicity. Now the assumption in Th. 4.1 implies that there is a $\langle\sigma,\theta\rangle$-stable subgroup Γ'' of Γ of finite index such that $[C(\sigma,\Gamma'')][C(\tau,\Gamma'')] \neq 0$, by Th. 4.2. Now the Th. 4.1 follows from the proof of Th. 2.1 in [24].

(iii) If G is a connected complex semisimple Lie group, then since the simply connected cover of G is a linear Lie group, without loss of generality we may assume that $\Gamma\backslash X, C(\sigma,\Gamma), C(\sigma\theta,\Gamma)$ and all connected components of their intersection are orientable [24][Prop. 2.3 and its Cor.]. In general, Rohlfs and Schwermer [30] proved that by passing to a suitable subgroup of finite index in Γ if necessary, we may assume that $\Gamma\backslash X, C(\sigma,\Gamma), C(\sigma\theta,\Gamma)$ and all connected components of their intersection are orientable.

(iv) We say that the *condition Or* (as in [30]) is satisfied for G, σ, τ if the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \sigma, \tau$. ■

Now the hypotheses of the above theorems have been checked in the following subsections for a connected complex simple Lie group G so that we can apply the above theorems.

4.1. Automorphisms of finite order of a complex simple Lie algebra. Here we describe Victor Kač's Classification [14] of finite order automorphisms of a complex simple Lie algebra. We follow [12, § 5, Ch. X] for this purpose.

Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$. As before, $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} \oplus J\mathfrak{u}$ is a Cartan decomposition of $\mathfrak{g}^{\mathbb{R}}$, where $\mathfrak{g}^{\mathbb{R}}$ is the underlying real Lie algebra of $\mathfrak{g}$ and J is the complex structure of $\mathfrak{g}^{\mathbb{R}}$ corresponding to the multiplication by i of $\mathfrak{g}$. Let θ be the corresponding Cartan involution. Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$. Choose a system of positive roots Δ^+ in the set of all non-zero roots $\Delta = \Delta(\mathfrak{g},\mathfrak{h})$. Let Φ be the set of all simple roots in Δ^+. Let $\{H^*_\phi, E_\alpha : \phi \in \Phi, \alpha \in \Delta\}$ be a Chevalley basis for $\mathfrak{g}$ as in (2). Then the Lie algebra $\mathfrak{g}$ is generated by the vectors $H^*_\phi,\ E_\phi,\ E_{-\phi}\ (\phi \in \Phi)$.

For any $\sigma \in \mathrm{Aut}(\mathfrak{g})$ with $\sigma(\mathfrak{h}) = \mathfrak{h}$, define $\sigma(\alpha)(H) = \alpha(\sigma H)$ for all $H \in \mathfrak{h}$, where $\alpha \in \mathfrak{h}^*$. Then $\sigma(\Delta) = \Delta$. Now assume that σ is a finite order automorphism of $\mathfrak{g}$ with $\sigma\theta = \theta\sigma$, $\sigma(\mathfrak{h}) = \mathfrak{h}$ and $\sigma(\Delta^+) = \Delta^+$. Then σ induces an automorphism of the Dynkin diagram of $\mathfrak{g}$. As the order of any automorphism of a Dynkin diagram is $1, 2$, or 3; $\sigma|_{\mathfrak{h}}$ has order $1, 2$, or 3 respectively.

Conversely, let $\bar{\nu}$ be an automorphism of the Dynkin diagram of $\mathfrak{g}$ of order k ($k = 1, 2, \text{or } 3$). As $\mathfrak{g}$ is generated by H^*_ϕ, E_ϕ, $E_{-\phi}$ ($\phi \in \Phi$), there exists a unique $\nu \in \mathrm{Aut}(\mathfrak{g})$ with

$$\nu(H^*_\phi) = H^*_{\bar{\nu}(\phi)}, \quad \nu(E_\phi) = E_{\bar{\nu}(\phi)}, \quad \nu(E_{-\phi}) = E_{-\bar{\nu}(\phi)} \quad (\phi \in \Phi).$$

Note that ν is of order k, and $\nu\theta = \theta\nu$. We call ν, an automorphism of $\mathfrak{g}$ induced by an automorphism of the Dynkin diagram of $\mathfrak{g}$. Let $\epsilon_0 = e^{\frac{2\pi i}{k}}$ be a primitive k-th root of unity. As ν has order k, any eigenvalue of ν has the form ϵ_0^i ($i \in \mathbb{Z}_k$) and $\mathfrak{g} = \oplus_{i \in \mathbb{Z}_k} \mathfrak{g}^\nu_i$ such that $[\mathfrak{g}^\nu_i, \mathfrak{g}^\nu_j] \subset \mathfrak{g}^\nu_{i+j}$, where $\mathfrak{g}^\nu_i$ is the eigenspace of ν corresponding to the eigenvalue ϵ_0^i. Since $k = 1, 2$, or 3, $\mathfrak{g}^\nu_{\bar{0}}$, $\mathfrak{g}^\nu_{\bar{1}}$, $\mathfrak{g}^\nu_{\bar{2}} \neq 0$, where $\bar{a} = a + k\mathbb{Z} \in \mathbb{Z}_k$ for all $a \in \mathbb{Z}$. The Lie algebra $\mathfrak{g}^\nu_{\bar{0}}$ is reductive (in fact, it is simple [12, the proof of Lemma 5.11, Ch. X]) and $\mathfrak{h}^\nu = \mathfrak{h} \cap \mathfrak{g}^\nu_{\bar{0}}$ is a Cartan subalgebra of $\mathfrak{g}^\nu_{\bar{0}}$. Define a root of $\mathfrak{g}$ with respect to $\mathfrak{h}^\nu$ as a pair (α, i) ($\alpha \in (\mathfrak{h}^\nu)^*, i \in \mathbb{Z}_k$), if the joint eigenspace $\mathfrak{g}_{(\alpha,i)} = \{X \in \mathfrak{g}^\nu_i : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}^\nu\} \neq 0$. Note that a root of $\mathfrak{g}$ with respect to $\mathfrak{h}^\nu$ is just a weight of the $\mathfrak{g}^\nu_{\bar{0}}$-module $\mathfrak{g}^\nu_i$. We may add pairs by $(\alpha, i) + (\beta, j) = (\alpha + \beta, i + j)$. Let $\bar{\Delta}$ denote the set of all non-zero roots and $\bar{\Delta}_0$ the set of roots of the form $(0, i), i \in \mathbb{Z}_k$. Then we have [12, § 5, Ch. X]

$$\mathfrak{g} = \mathfrak{h}^\nu \oplus \sum_{(\alpha,i) \in \bar{\Delta}} \mathfrak{g}_{(\alpha,i)}, \ \mathfrak{h} = \sum_{(\alpha,i) \in \bar{\Delta}_0} \mathfrak{g}_{(\alpha,i)}, \ \mathfrak{h}^\nu = \mathfrak{g}_{(0,\bar{0})}, \tag{4}$$

$$[\mathfrak{g}_{(\alpha,i)}, \mathfrak{g}_{(\beta,j)}] \subset \mathfrak{g}_{(\alpha,i)+(\beta,j)}, \tag{5}$$

$$\dim \mathfrak{g}_{(\alpha,i)} = 1 \text{ for all } (\alpha, i) \in \bar{\Delta} \setminus \bar{\Delta}_0, \tag{6}$$

$$[\mathfrak{g}_{(\alpha,i)}, \mathfrak{g}_{(\beta,j)}] \neq 0, \text{ if } (\alpha, i) \in \bar{\Delta} \setminus \bar{\Delta}_0 \,; (\beta, j), (\alpha, i) + (\beta, j) \in \bar{\Delta}. \tag{7}$$

Let $\Delta_0 = \Delta(\mathfrak{g}^\nu_{\bar{0}}, \mathfrak{h}^\nu)$ be the set of all non-zero roots of the simple Lie algebra $\mathfrak{g}^\nu_{\bar{0}}$ with respect to the Cartan subalgebra $\mathfrak{h}^\nu$. Then $\Delta_0 = \{(\alpha, \bar{0}) \in \bar{\Delta} : \alpha \neq 0\}$. Define

$$\bar{H}^*_\phi = \sum_{i=0}^{k-1} H^*_{\bar{\nu}^i(\phi)}, \ \bar{E}_\phi = \sum_{i=0}^{k-1} E_{\bar{\nu}^i(\phi)}, \text{ and } \bar{E}_{-\phi} = \sum_{i=0}^{k-1} E_{-\bar{\nu}^i(\phi)} \ (\phi \in \Phi).$$

Note that $\mathfrak{h}^\nu = \sum_{\phi \in \Phi} \mathbb{C}\bar{H}^*_\phi$, and the vectors $\bar{E}_\phi$, $\bar{E}_{-\phi} \in \mathfrak{g}^\nu_{\bar{0}}$ for all $\phi \in \Phi$.

Also the vectors $\sum_{i=0}^{k-1} \epsilon_0^i E_{\bar{\nu}^i(\phi)}$, $\sum_{i=0}^{k-1} \epsilon_0^i E_{-\bar{\nu}^i(\phi)} \in \mathfrak{g}^\nu_{\overline{k-1}}$, and for $k = 3$, the vectors $E_\phi + \epsilon_0^2 E_{\bar{\nu}(\phi)} + \epsilon_0 E_{\bar{\nu}^2(\phi)}$, $E_{-\phi} + \epsilon_0^2 E_{-\bar{\nu}(\phi)} + \epsilon_0 E_{-\bar{\nu}^2(\phi)} \in \mathfrak{g}^\nu_{\bar{1}}$ for all $\phi \in \Phi$ with $\phi \neq \bar{\nu}(\phi)$. Let $a_{\phi\psi} = \phi(H^*_\psi)$ for all $\phi, \psi \in \Phi$. Then we have

$$\begin{aligned}[\bar{H}^*_\psi, \bar{E}_\phi] &= \sum_{i,j=0}^{k-1} [H^*_{\bar{\nu}^i(\psi)}, E_{\bar{\nu}^j(\phi)}] = \sum_{i,j=0}^{k-1} a_{\bar{\nu}^j(\phi)\bar{\nu}^i(\psi)} E_{\bar{\nu}^j(\phi)} = \sum_{j=0}^{k-1} (\sum_{i=0}^{k-1} a_{\bar{\nu}^j(\phi)\bar{\nu}^i(\psi)}) E_{\bar{\nu}^j(\phi)} \\ &= \sum_{j=0}^{k-1} (\sum_{i=0}^{k-1} a_{\phi\bar{\nu}^{i-j}(\psi)}) E_{\bar{\nu}^j(\phi)} \text{ (as } a_{\phi\psi} = a_{\bar{\nu}(\phi)\bar{\nu}(\psi)})\end{aligned}$$

$$= (\sum_{i=0}^{k-1} a_{\phi\bar{\nu}^i(\psi)}) \sum_{j=0}^{k-1} E_{\bar{\nu}^j(\phi)} \text{ (as } \bar{\nu}^k = \text{id}) = (\sum_{i=0}^{k-1} a_{\phi\bar{\nu}^i(\psi)})\bar{E}_\phi.$$

Similarly
$$[\bar{H}^*_\psi, \bar{E}_{-\phi}] = -(\sum_{i=0}^{k-1} a_{\phi\bar{\nu}^i(\psi)})\bar{E}_{-\phi},$$

for all $\phi, \psi \in \Phi$. Thus $\bar{E}_\phi$ is a root vector corresponding to some root $\psi \in \Delta_0$, $\bar{E}_{-\phi}$ is a root vector corresponding to $-\psi \in \Delta_0$. Also note that

$$[\bar{H}^*_\psi, \sum_{j=0}^{k-1} \epsilon_0^j E_{\bar{\nu}^j(\phi)}] = (\sum_{i=0}^{k-1} a_{\phi\bar{\nu}^i(\psi)}) \sum_{j=0}^{k-1} \epsilon_0^j E_{\bar{\nu}^j(\phi)}, \quad \text{and for } k = 3,$$
$$[\bar{H}^*_\psi, E_\phi + \epsilon_0^2 E_{\bar{\nu}(\phi)} + \epsilon_0 E_{\bar{\nu}^2(\phi)}] = (\sum_{i=0}^{k-1} a_{\phi\bar{\nu}^i(\psi)})(E_\phi + \epsilon_0^2 E_{\bar{\nu}(\phi)} + \epsilon_0 E_{\bar{\nu}^2(\phi)})$$

for all $\phi, \psi \in \Phi$. So if $\phi \in \Phi$ with $\phi \neq \bar{\nu}(\phi)$, and $\bar{E}_\phi$ is a root vector corresponding to the root $\psi \in \Delta_0$, then $\sum_{j=0}^{k-1} \epsilon_0^j E_{\bar{\nu}^j(\phi)}$ is a weight vector corresponding to the weight $\psi \in (\mathfrak{h}^\nu)^*$ of the $\mathfrak{g}_0^\nu$-module $\mathfrak{g}^\nu_{\overline{k-1}}$, and for $k = 3$, $E_\phi + \epsilon_0^2 E_{\bar{\nu}(\phi)} + \epsilon_0 E_{\bar{\nu}^2(\phi)}$ is weight vector corresponding to the weight ψ of $\mathfrak{g}^\nu_{\bar{1}}$.

Similarly $\sum_{j=0}^{k-1} \epsilon_0^j E_{-\bar{\nu}^j(\phi)}$ is a weight vector corresponding to the weight $-\psi \in (\mathfrak{h}^\nu)^*$ of $\mathfrak{g}^\nu_{\overline{k-1}}$, and for $k = 3$, $E_{-\phi} + \epsilon_0^2 E_{-\bar{\nu}(\phi)} + \epsilon_0 E_{-\bar{\nu}^2(\phi)}$ is weight vector corresponding to the weight $-\psi$ of $\mathfrak{g}^\nu_{\bar{1}}$.

Actually there exists a basis $\Psi = \{\psi_1, \psi_2, \ldots, \psi_n\}$ of the root system $\Delta_0 = \Delta(\mathfrak{g}_0^\nu, \mathfrak{h}^\nu)$ such that $\bar{E}_\phi$ is a root vector corresponding to some root $\psi_i \in \Psi$, $\bar{E}_{-\phi}$ is a root vector corresponding to $-\psi_i$, and $\{\bar{H}^*_\phi,\ \bar{E}_\phi,\ \bar{E}_{-\phi} : \phi \in \Phi\}$ generates $\mathfrak{g}_0^\nu$ [12, the proof of Lemma 5.11, Ch. X]. Let Δ_0^+ be the system of positive roots in Δ_0 generated by the basis Ψ. Let α_0 be the lowest weight (with respect to Δ_0^+) of the $\mathfrak{g}_0^\nu$-module $\mathfrak{g}^\nu_{\bar{1}}$. Then $\alpha_0 \neq 0$, the set $B := \{\alpha_0, \psi_1, \ldots, \psi_n\}$ is linearly dependent and B generates $\bar{\Delta}$ in the sense that each $(\alpha, i) \in \bar{\Delta}$ can be written in the form

$$(\alpha, i) = \pm(n_0(\alpha_0, \bar{1}) + \sum_{j=1}^{n} n_j(\psi_j, 0)) \ (n_j \in \mathbb{N} \cup \{0\} \text{ for all } 0 \leq j \leq n), \tag{8}$$

[12, Lemma 5.7, Ch. X]. Note that if $\alpha \in (\mathfrak{h}^\nu)^*$ is a weight of the $\mathfrak{g}_0^\nu$-module $\mathfrak{g}^\nu_{\bar{a}}$ $(0 \leq a \leq k-1)$, then we may take $n_0 = a$ in the above decomposition. Also if $(\alpha, i) \in \bar{\Delta}$ with $i \neq 0$ or $\alpha \in \Delta_0^+$ for $i = 0$, then α can be written as

$$\alpha = \beta_1 + \cdots + \beta_k, \tag{9}$$

where all $\beta_i \in B$, not necessarily distinct, such that each partial sum $\beta_1 + \cdots + \beta_j$ is the first component of some root in $\bar{\Delta}$ [12, follows from (v) of Lemma 5.7, Ch. X]. For $X_1, X_2, \ldots, X_r \in \mathfrak{g}$ we denote the element $\text{ad}(X_1) \cdots \text{ad}(X_{r-1})(X_r) \in \mathfrak{g}$ by $[X_1, \ldots, X_{r-1}, X_r]$. If $(\alpha, \bar{a}) \in \bar{\Delta} \setminus \bar{\Delta}_0$ $(a \in \mathbb{N})$ with

$$(\alpha, \bar{a}) = a(\alpha_0, \bar{1}) + \textstyle\sum_{i=1}^n n_i(\psi_i, 0) \ \ (n_i \in \mathbb{N} \cup \{0\} \text{ for all } 1 \leq i \leq n),$$

then by (6), (7) and (9) we have

$$\mathfrak{g}_{(\alpha, \bar{a})} = \mathbb{C}[X_1, \ldots, X_r], \tag{10}$$

for suitable vectors $X_1,\dots,X_r$ lie in the eigenspaces of the roots $(\alpha_0,\bar 1)$, $(\psi_i,0)$ $(1\le i\le n)$ such that the sum of the corresponding roots is $a(\alpha_0,\bar 1)+\sum_{i=1}^n n_i(\psi_i,0)$. Choose $E_0(\neq 0)\in\mathfrak{g}_{(\alpha_0,\bar 1)}$. Then the vectors E_0, $\bar E_\phi$ $(\phi\in\Phi)$ generate the Lie algebra $\mathfrak{g}$ [12, Th. 5.15(i), Ch. X]. Let $\alpha_0+\sum_{i=1}^n a_i\psi_i=0$ ($a_i\in\mathbb{N}$ for all $1\le i\le n$) [12, Tables of Diagrams $S(A)$, §5, Ch. X]. Let $s_0,s_1,\dots,s_n$ be non-negative integers without non-trivial common factor and put $m=k(s_0+\sum_{i=1}^n a_is_i)$. Let ϵ be a primitive m-th root of unity and $s_\phi:=s_i$, if $\bar E_\phi$ is a root vector corresponding to the simple root $\psi_i\in\Psi$. Note that $s_{\bar\nu^j(\phi)}=s_\phi$ for all $\phi\in\Phi$. There exists a unique automorphism σ of $\mathfrak{g}$ of order m with

$$\sigma(E_0)=\epsilon^{s_0}E_0,\ \sigma(\bar E_\phi)=\epsilon^{s_\phi}\bar E_\phi\ (\phi\in\Phi) \tag{11}$$

[12, Th. 5.15(i), Ch. X]. The automorphism σ is called an automorphism of type $(s_0,s_1,\dots,s_n;k)$. Note that the automorphism ν induced by the Dynkin diagram automorphism $\bar\nu$ is of type $(1,0,\dots,0;k)$. The automorphism σ is inner if and only if $k=1$ [12, Th. 5.16(i), Ch. X].

For $1\le i\le n$, if $\mathfrak{g}_{(\psi_i,\bar 1)}\neq 0$, the decomposition (8) for $(\psi_i,\bar 1)$ is given by

$$(\psi_i,\bar 1)=(\alpha_0,\bar 1)+\sum_{\substack{j=1\\ j\neq i}}^n a_j(\psi_j,0)+(a_i+1)(\psi_i,0),\ \text{as } \alpha_0+\sum_{j=0}^n a_j\psi_j=0.$$

Similarly

$$\mathfrak{g}_{(\psi_i,\bar 2)}\neq 0 \text{ implies } (\psi_i,\bar 2)=2(\alpha_0,\bar 1)+\sum_{\substack{j=1\\ j\neq i}}^n 2a_j(\psi_j,0)+(2a_i+1)(\psi_i,0),$$

$$\mathfrak{g}_{(-\psi_i,\bar 1)}\neq 0 \text{ implies } (-\psi_i,\bar 1)=(\alpha_0,\bar 1)+\sum_{\substack{j=1\\ j\neq i}}^n a_j(\psi_j,0)+(a_i-1)(\psi_i,0),\ \text{and}$$

$$\mathfrak{g}_{(-\psi_i,\bar 2)}\neq 0 \text{ implies } (-\psi_i,\bar 2)=2(\alpha_0,\bar 1)+\sum_{\substack{j=1\\ j\neq i}}^n 2a_j(\psi_j,0)+(2a_i-1)(\psi_i,0),$$

as $\mathfrak{g}^\nu_{\bar k}=\mathfrak{g}^\nu_{\bar 0}$, $\mathfrak{g}_{(-\psi_i,\bar k)}\neq 0$ and

$$(-\psi_i,\bar k)=k(\alpha_0,\bar 1)+\sum_{\substack{j=1\\ j\neq i}}^n ka_j(\psi_j,0)+(ka_i-1)(\psi_i,0).$$

By (10) for $\phi\in\Phi$ with $\phi\neq\bar\nu(\phi)$, and $k\neq 3$, we have

$$\sigma(\sum_{j=0}^{k-1}\epsilon_0^jE_{\bar\nu^j(\phi)})=\epsilon^{s_i+\frac{m}{k}}\sum_{j=0}^{k-1}\epsilon_0^jE_{\bar\nu^j(\phi)}=\epsilon_0\epsilon^{s_i}\sum_{j=0}^{k-1}\epsilon_0^jE_{\bar\nu^j(\phi)},$$

$$\sigma(\sum_{j=0}^{k-1}\epsilon_0^jE_{-\bar\nu^j(\phi)})=\epsilon^{-s_i+\frac{m}{k}}\sum_{j=0}^{k-1}\epsilon_0^jE_{-\bar\nu^j(\phi)}=\epsilon_0\epsilon^{-s_i}\sum_{j=0}^{k-1}\epsilon_0^jE_{-\bar\nu^j(\phi)},$$

as $m=k(s_0+\sum\limits_{j=1}^n a_js_j)$ implies $s_0+\sum\limits_{\substack{j=1\\ j\neq i}}^n a_js_j+(a_i+1)s_i=s_i+\frac{m}{k}$,

$s_0+\sum\limits_{\substack{j=1\\ j\neq i}}^n a_js_j+(a_i-1)s_i=-s_i+\frac{m}{k}$, and $\epsilon^{\frac{m}{k}}=\epsilon_0$ for $k=1,2$.

Similarly for $\phi \in \Phi$ with $\phi \neq \bar{\nu}(\phi)$, and $k = 3$, we have

$$\sigma(\sum_{j=0}^{k-1} \epsilon_0^j E_{\bar{\nu}^j(\phi)}) = \epsilon^{s_i + \frac{2m}{3}} \sum_{j=0}^{k-1} \epsilon_0^j E_{\bar{\nu}^j(\phi)} = \begin{cases} \epsilon_0^2 \epsilon^{s_i} \sum_{j=0}^{k-1} \epsilon_0^j E_{\bar{\nu}^j(\phi)}, & \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0 \\ \epsilon_0 \epsilon^{s_i} \sum_{j=0}^{k-1} \epsilon_0^j E_{\bar{\nu}^j(\phi)}, & \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0^2. \end{cases}$$

$$\begin{aligned} \sigma(E_\phi + \epsilon_0^2 E_{\bar{\nu}(\phi)} + \epsilon_0 E_{\bar{\nu}^2(\phi)}) &= \epsilon^{s_i + \frac{m}{3}} (E_\phi + \epsilon_0^2 E_{\bar{\nu}(\phi)} + \epsilon_0 E_{\bar{\nu}^2(\phi)}) \\ &= \begin{cases} \epsilon_0 \epsilon^{s_i} (E_\phi + \epsilon_0^2 E_{\bar{\nu}(\phi)} + \epsilon_0 E_{\bar{\nu}^2(\phi)}), & \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0 \\ \epsilon_0^2 \epsilon^{s_i} (E_\phi + \epsilon_0^2 E_{\bar{\nu}(\phi)} + \epsilon_0 E_{\bar{\nu}^2(\phi)}), & \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0^2. \end{cases} \end{aligned}$$

$$\sigma(\sum_{j=0}^{k-1} \epsilon_0^j E_{-\bar{\nu}^j(\phi)}) = \epsilon^{-s_i + \frac{2m}{3}} \sum_{j=0}^{k-1} \epsilon_0^j E_{-\bar{\nu}^j(\phi)} = \begin{cases} \epsilon_0^2 \epsilon^{-s_i} \sum_{j=0}^{k-1} \epsilon_0^j E_{-\bar{\nu}^j(\phi)}, & \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0 \\ \epsilon_0 \epsilon^{-s_i} \sum_{j=0}^{k-1} \epsilon_0^j E_{-\bar{\nu}^j(\phi)}, & \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0^2. \end{cases}$$

$$\begin{aligned} \sigma(E_{-\phi} + \epsilon_0^2 E_{-\bar{\nu}(\phi)} + \epsilon_0 E_{-\bar{\nu}^2(\phi)}) &= \epsilon^{-s_i + \frac{m}{3}} (E_{-\phi} + \epsilon_0^2 E_{-\bar{\nu}(\phi)} + \epsilon_0 E_{-\bar{\nu}^2(\phi)}) \\ &= \begin{cases} \epsilon_0 \epsilon^{-s_i} (E_{-\phi} + \epsilon_0^2 E_{-\bar{\nu}(\phi)} + \epsilon_0 E_{-\bar{\nu}^2(\phi)}), & \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0 \\ \epsilon_0^2 \epsilon^{-s_i} (E_{-\phi} + \epsilon_0^2 E_{-\bar{\nu}(\phi)} + \epsilon_0 E_{-\bar{\nu}^2(\phi)}), & \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0^2. \end{cases} \end{aligned}$$

Also for any k, $\sigma(\bar{E}_{-\phi}) = \epsilon^{-s_i + m} \bar{E}_{-\phi} = \epsilon^{-s_i} \bar{E}_{-\phi}$ for all $\phi \in \Phi$, as $\bar{E}_{-\phi}$ is a root vector of $\mathfrak{g}_0^\nu$ corresponding to the root $-\psi_i$ and

$$-\psi_i = k\alpha_0 + \sum_{\substack{j=1 \\ j \neq i}}^{n} k a_j \psi_j + (k a_i - 1)\psi_i$$

via the identification $\mathfrak{g}_0^\nu = \mathfrak{g}_k^\nu$. Note that $s_i = s_\phi$. Obviously for all $\phi \in \Phi$ with $\phi = \bar{\nu}(\phi)$,

$$\sigma(E_\phi) = \epsilon^{s_\phi} E_\phi, \ \sigma(E_{-\phi}) = \epsilon^{-s_\phi} E_{-\phi}, \text{ and}$$

$$\sigma(H_\phi^*) = \sigma([E_\phi, E_{-\phi}]) = [\sigma(E_\phi), \sigma(E_{-\phi})] = [\epsilon^{s_\phi} E_\phi, \epsilon^{-s_\phi} E_{-\phi}] = H_\phi^*.$$

For $\phi \in \Phi$ with $\phi \neq \bar{\nu}(\phi)$, and $k = 2$,

$$\begin{aligned} \sigma(E_\phi) &= \sigma\left(\frac{E_\phi + E_{\bar{\nu}(\phi)}}{2}\right) + \sigma\left(\frac{E_\phi - E_{\bar{\nu}(\phi)}}{2}\right) \\ &= \epsilon^{s_\phi} \frac{E_\phi + E_{\bar{\nu}(\phi)}}{2} - \epsilon^{s_\phi} \frac{E_\phi - E_{\bar{\nu}(\phi)}}{2} = \epsilon^{s_\phi} E_{\bar{\nu}(\phi)}, \\ \sigma(E_{-\phi}) &= \sigma\left(\frac{E_{-\phi} + E_{-\bar{\nu}(\phi)}}{2}\right) + \sigma\left(\frac{E_{-\phi} - E_{-\bar{\nu}(\phi)}}{2}\right) \\ &= \epsilon^{-s_\phi} \frac{E_{-\phi} + E_{-\bar{\nu}(\phi)}}{2} - \epsilon^{-s_\phi} \frac{E_{-\phi} - E_{-\bar{\nu}(\phi)}}{2} = \epsilon^{-s_\phi} E_{-\bar{\nu}(\phi)}, \end{aligned}$$

as $\epsilon_0 = -1$ here. That is for all $\phi \in \Phi$ with $\phi \neq \bar{\nu}(\phi)$,

$$\sigma(E_\phi) = \epsilon^{s_\phi} E_{\bar{\nu}(\phi)}, \ \sigma(E_{-\phi}) = \epsilon^{-s_\phi} E_{-\bar{\nu}(\phi)}, \text{ and } \sigma(H_\phi^*) = H_{\bar{\nu}(\phi)}^* \text{ for } k = 2.$$

For $\phi \in \Phi$ with $\phi \neq \bar{\nu}(\phi)$, and $k = 3$,

$$\sigma(E_\phi) = \sigma\left(\frac{E_\phi + E_{\bar{\nu}(\phi)} + E_{\bar{\nu}^2(\phi)}}{3}\right) + \sigma\left(\frac{E_\phi + \epsilon_0 E_{\bar{\nu}(\phi)} + \epsilon_0^2 E_{\bar{\nu}^2(\phi)}}{3}\right)$$

$$+\sigma\left(\frac{E_\phi + \epsilon_0^2 E_{\bar{\nu}(\phi)} + \epsilon_0 E_{\bar{\nu}^2(\phi)}}{3}\right)$$

$$= \begin{cases} \epsilon^{s_\phi} \frac{E_\phi + E_{\bar{\nu}(\phi)} + E_{\bar{\nu}^2(\phi)}}{3} + \epsilon_0^2 \epsilon^{s_\phi} \frac{E_\phi + \epsilon_0 E_{\bar{\nu}(\phi)} + \epsilon_0^2 E_{\bar{\nu}^2(\phi)}}{3} + \epsilon_0 \epsilon^{s_\phi} \frac{E_\phi + \epsilon_0^2 E_{\bar{\nu}(\phi)} + \epsilon_0 E_{\bar{\nu}^2(\phi)}}{3}, & \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0 \\ \epsilon^{s_\phi} \frac{E_\phi + E_{\bar{\nu}(\phi)} + E_{\bar{\nu}^2(\phi)}}{3} + \epsilon_0 \epsilon^{s_\phi} \frac{E_\phi + \epsilon_0 E_{\bar{\nu}(\phi)} + \epsilon_0^2 E_{\bar{\nu}^2(\phi)}}{3} + \epsilon_0^2 \epsilon^{s_\phi} \frac{E_\phi + \epsilon_0^2 E_{\bar{\nu}(\phi)} + \epsilon_0 E_{\bar{\nu}^2(\phi)}}{3}, & \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0^2 \end{cases}$$

$$= \begin{cases} \epsilon^{s_\phi} E_{\bar{\nu}(\phi)}, & \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0 \\ \epsilon^{s_\phi} E_{\bar{\nu}^2(\phi)}, & \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0^2. \end{cases}$$

$$\sigma(E_{-\phi}) = \sigma\left(\frac{E_{-\phi} + E_{-\bar{\nu}(\phi)} + E_{-\bar{\nu}^2(\phi)}}{3}\right) + \sigma\left(\frac{E_{-\phi} + \epsilon_0 E_{-\bar{\nu}(\phi)} + \epsilon_0^2 E_{-\bar{\nu}^2(\phi)}}{3}\right)$$

$$+\sigma\left(\frac{E_{-\phi} + \epsilon_0^2 E_{-\bar{\nu}(\phi)} + \epsilon_0 E_{-\bar{\nu}^2(\phi)}}{3}\right).$$

This is equal to

$$\epsilon^{-s_\phi} \frac{E_{-\phi} + E_{-\bar{\nu}(\phi)} + E_{-\bar{\nu}^2(\phi)}}{3} + \epsilon_0^2 \epsilon^{-s_\phi} \frac{E_{-\phi} + \epsilon_0 E_{-\bar{\nu}(\phi)} + \epsilon_0^2 E_{-\bar{\nu}^2(\phi)}}{3}$$
$$+ \epsilon_0 \epsilon^{-s_\phi} \frac{E_{-\phi} + \epsilon_0^2 E_{-\bar{\nu}(\phi)} + \epsilon_0 E_{-\bar{\nu}^2(\phi)}}{3} = \epsilon^{-s_\phi} E_{-\bar{\nu}(\phi)},$$

if $\epsilon^{\frac{m}{3}} = \epsilon_0$, and equal to

$$\epsilon^{-s_\phi} \frac{E_{-\phi} + E_{-\bar{\nu}(\phi)} + E_{-\bar{\nu}^2(\phi)}}{3} + \epsilon_0 \epsilon^{-s_\phi} \frac{E_{-\phi} + \epsilon_0 E_{-\bar{\nu}(\phi)} + \epsilon_0^2 E_{-\bar{\nu}^2(\phi)}}{3}$$
$$+ \epsilon_0^2 \epsilon^{-s_\phi} \frac{E_{-\phi} + \epsilon_0^2 E_{-\bar{\nu}(\phi)} + \epsilon_0 E_{-\bar{\nu}^2(\phi)}}{3} = \epsilon^{-s_\phi} E_{-\bar{\nu}^2(\phi)},$$

if $\epsilon^{\frac{m}{3}} = \epsilon_0^2$, as $\epsilon_0 = \omega$ here. Hence for all $\phi \in \Phi$ with $\phi \neq \bar{\nu}(\phi)$,

$$\sigma(E_\phi) = \epsilon^{s_\phi} E_{\bar{\nu}(\phi)},\ \sigma(E_{-\phi}) = \epsilon^{-s_\phi} E_{-\bar{\nu}(\phi)},\ \text{and } \sigma(H_\phi^*) = H_{\bar{\nu}(\phi)}^* \text{ for } k = 3,\ \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0,$$

or

$$\sigma(E_\phi) = \epsilon^{s_\phi} E_{\bar{\nu}^2(\phi)},\ \sigma(E_{-\phi}) = \epsilon^{-s_\phi} E_{-\bar{\nu}^2(\phi)},\ \text{and } \sigma(H_\phi^*) = H_{\bar{\nu}^2(\phi)}^* \text{ for } k = 3,\ \text{if } \epsilon^{\frac{m}{3}} = \epsilon_0^2.$$

So if we want the Dynkin diagram automorphism induced by σ to be $\bar{\nu}$, we must take ϵ to be a primitive m-th root of unity with $\epsilon^{\frac{m}{k}} = \epsilon_0$. In this case, we have

$$\sigma(E_\alpha) = q_\alpha \epsilon^{n_\alpha} E_{\nu(\alpha)},\ \sigma(E_{-\alpha}) = q_\alpha \epsilon^{-n_\alpha} E_{-\nu(\alpha)},\ \text{and } \sigma(H_\phi^*) = H_{\bar{\nu}(\phi)}^*,$$

for all $\alpha \in \Delta^+, \phi \in \Phi$; where $\{H_\phi^*, E_\alpha : \phi \in \Phi, \alpha \in \Delta\}$ is a Chevalley basis for $\mathfrak{g}$ as in (2), $q_\alpha = \pm 1$, and $n_\alpha = \sum_{\phi \in \Phi} n_\phi(\alpha) s_\phi$ (if $\alpha = \sum_{\phi \in \Phi} n_\phi(\alpha)\phi$, $n_\phi(\alpha) \in \mathbb{N} \cup \{0\}$) for all $\alpha \in \Delta^+$. Recall that ν is the unique automorphism of $\mathfrak{g}$ with

$$\nu(H_\phi^*) = H_{\bar{\nu}(\phi)}^*, \quad \nu(E_\phi) = E_{\bar{\nu}(\phi)}, \quad \nu(E_{-\phi}) = E_{-\bar{\nu}(\phi)} \quad (\phi \in \Phi).$$

So $\nu(\mathfrak{h}) = \mathfrak{h}$ and hence $\nu(\alpha)$ $(\alpha \in \mathfrak{h}^*)$ makes sense, where $\nu(\alpha)(H) = \alpha(\nu H)$ for all $H \in \mathfrak{h}$. Note that $\nu(\phi) = \bar{\nu}(\phi)$ for all $\phi \in \Phi$.

Also $q_\alpha = \pm 1$ by (2), for if $\beta + n\alpha$ $(p \le n \le q)$ is the α-string containing β, then $\nu(\beta) + n\nu(\alpha)$ $(p \le n \le q)$ is the $\nu(\alpha)$-string containing $\nu(\beta)$, where $\alpha, \beta \in \Delta$.

If $X_\alpha = E_\alpha - E_{-\alpha}, Y_\alpha = i(E_\alpha + E_{-\alpha})$ $(\alpha \in \Delta^+)$, then the compact real form $\mathfrak{u}$ is given by

$$\mathfrak{u} = \sum_{\phi \in \Phi} \mathbb{R}(iH_\phi^*) \oplus \sum_{\alpha \in \Delta^+} \mathbb{R}X_\alpha \oplus \sum_{\alpha \in \Delta^+} \mathbb{R}Y_\alpha.$$

Now
$$\sigma(X_\alpha) = q_\alpha(\epsilon^{n_\alpha} E_{\nu(\alpha)} - \epsilon^{-n_\alpha} E_{-\nu(\alpha)})$$
$$= q_\alpha\left(\cos\frac{2bn_\alpha\pi}{m} E_{\nu(\alpha)} + i\sin\frac{2bn_\alpha\pi}{m} E_{\nu(\alpha)} - \cos\frac{2bn_\alpha\pi}{m} E_{-\nu(\alpha)} + i\sin\frac{2bn_\alpha\pi}{m} E_{-\nu(\alpha)}\right)$$
$$= q_\alpha \cos\frac{2bn_\alpha\pi}{m}(E_{\nu(\alpha)} - E_{-\nu(\alpha)}) + iq_\alpha \sin\frac{2bn_\alpha\pi}{m}(E_{\nu(\alpha)} + E_{-\nu(\alpha)})$$
$$= q_\alpha \cos\frac{2bn_\alpha\pi}{m} X_{\nu(\alpha)} + q_\alpha \sin\frac{2bn_\alpha\pi}{m} Y_{\nu(\alpha)},$$
$$\sigma(Y_\alpha) = iq_\alpha(\epsilon^{n_\alpha} E_{\nu(\alpha)} + \epsilon^{-n_\alpha} E_{-\nu(\alpha)})$$
$$= q_\alpha\left(i\cos\frac{2bn_\alpha\pi}{m} E_{\nu(\alpha)} - \sin\frac{2bn_\alpha\pi}{m} E_{\nu(\alpha)} + i\cos\frac{2bn_\alpha\pi}{m} E_{-\nu(\alpha)} + \sin\frac{2bn_\alpha\pi}{m} E_{-\nu(\alpha)}\right)$$
$$= iq_\alpha \cos\frac{2bn_\alpha\pi}{m}(E_{\nu(\alpha)} + E_{-\nu(\alpha)}) - q_\alpha \sin\frac{2bn_\alpha\pi}{m}(E_{\nu(\alpha)} - E_{-\nu(\alpha)})$$
$$= q_\alpha \cos\frac{2bn_\alpha\pi}{m} Y_{\nu(\alpha)} - q_\alpha \sin\frac{2bn_\alpha\pi}{m} X_{\nu(\alpha)},$$

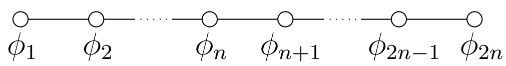
for all $\alpha \in \Delta^+$, where $\epsilon = e^{\frac{2b\pi i}{m}}$ with $\gcd(b, m) = 1$, is a primitive m-th root of unity. Obviously $\sigma(iH_\phi^*) = iH_{\bar{\nu}(\phi)}^*$ for all $\phi \in \Phi$. Hence σ is an automorphism of $\mathfrak{g}$ of order m such that $\sigma\theta = \theta\sigma$, $\sigma(\mathfrak{h}) = \mathfrak{h}$, $\sigma(\Delta^+) = \Delta^+$, and the Dynkin diagram automorphism induced by σ is $\bar{\nu}$. Let $i_1, \ldots, i_t$ be all the indices with $s_{i_1} = \cdots = s_{i_t} = 0$. Then the Lie algebra $\mathfrak{g}_0^\sigma = \{X \in \mathfrak{g} : \sigma(X) = X\}$ is the direct sum of an $(n-t)$-dimensional centre and a semisimple Lie algebra whose Dynkin diagram is the subdiagram of the diagram $\mathfrak{g}^{(k)}$ in Table 4.1 consisting of the vertices consisting of the vertices $\psi_{i_1}, \ldots, \psi_{i_t}$ [12, Th. 5.15(ii), Ch. X]. Except for conjugation, these are all automorphisms of $\mathfrak{g}$ of order m [12, Th. 5.15(iii), Ch. X].

Remark 4.4. (i) Let $\delta \in \Delta^+$ be the highest root of $\mathfrak{g}$. Then $\mathfrak{g}_\delta$, $\mathfrak{g}_{-\delta} \subset \mathfrak{g}_0^\nu$, except for $\mathfrak{g} = \mathfrak{a}_{2n}$ $(n \ge 1)$ $(k = 2)$. For $\mathfrak{g} = \mathfrak{a}_{2n}$ $(n \ge 1)$ with $k = 2$, $\mathfrak{g}_\delta, \mathfrak{g}_{-\delta} \subset \mathfrak{g}_{\bar{1}}^\nu$. Consequently $\alpha_0 = -\delta\big|_{\mathfrak{h}^\nu}$ for $\mathfrak{g} = \mathfrak{a}_{2n}$ $(n \ge 1)$ with $k = 2$: For $k = 1$, $\mathfrak{g}_0^\nu = \mathfrak{g}_{\bar{1}}^\nu = \mathfrak{g}$. Then obviously, $\mathfrak{g}_\delta, \mathfrak{g}_{-\delta} \subset \mathfrak{g}_0^\nu$. For $k = 2$, or 3, we prove it via case by case consideration.

Note that $\nu(\delta) = \delta$ and hence for any $E(\neq 0) \in \mathfrak{g}_\delta$, $\nu(E) = E$ or $\nu(E) = -E$, by the definition of ν. Thus if ν is an automorphism of order 3, then $\mathfrak{g}_\delta \subset \mathfrak{g}_0^\nu$. Similarly $\mathfrak{g}_{-\delta} \subset \mathfrak{g}_0^\nu$.

Now assume that $k = 2$. Let $\mathfrak{g} = \mathfrak{a}_{2n}$ $(n \ge 1)$.

$\mathfrak{a}_{2n}$: $(n \ge 1)$



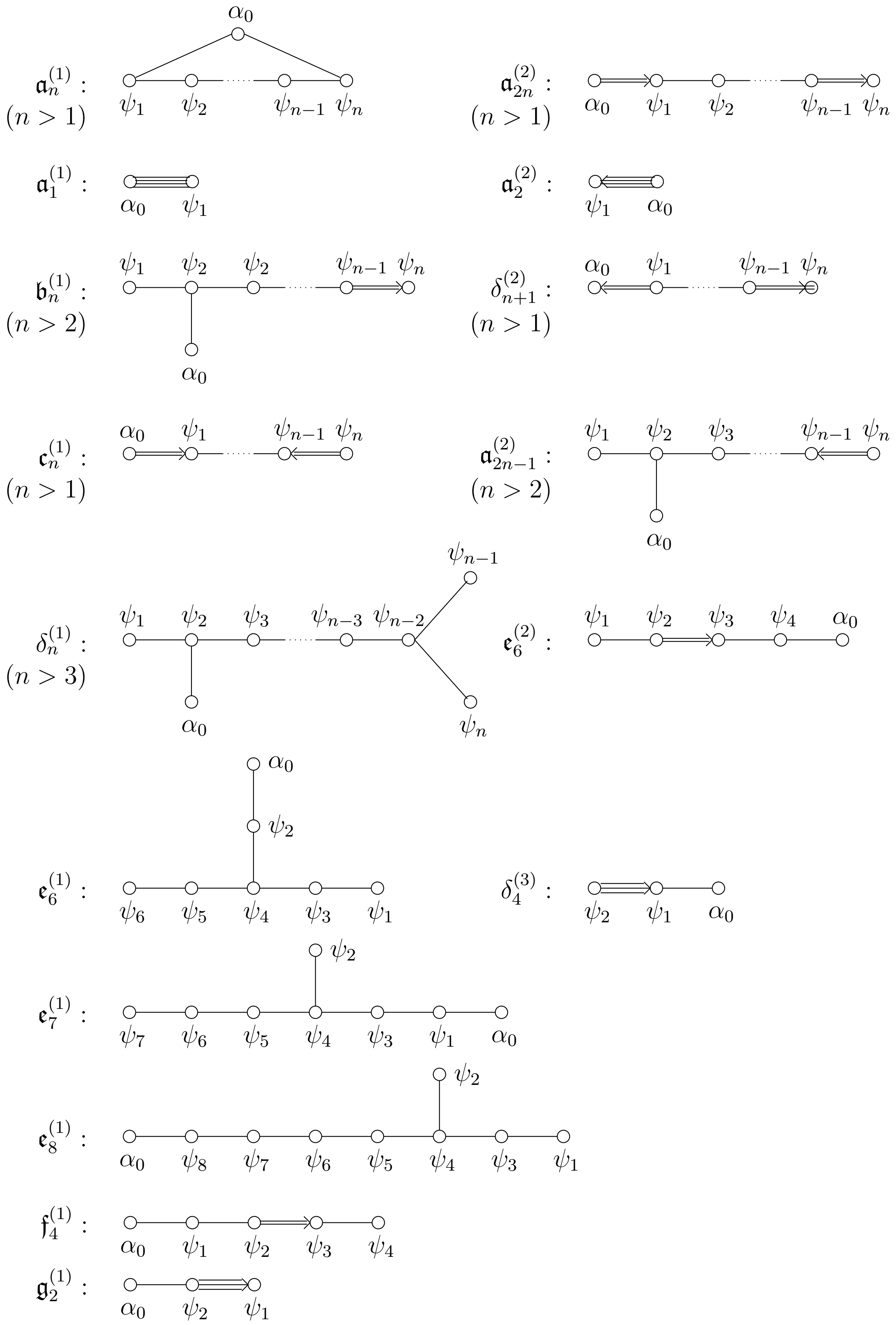


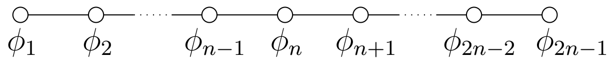
Table 4.1: Diagrams $\mathfrak{g}^k$ of a complex simple Lie algebra $\mathfrak{g}$.

The highest root $\delta = \phi_1 + \cdots + \phi_{2n}$. Note that $\bar{\nu}$ is given by $\bar{\nu}(\phi_j) = \phi_{2n-j+1}$ for all $1 \leq j \leq 2n$. Let E_j, E_{-j} be non-zero root vectors corresponding to the roots ϕ_j, $-\phi_j$ respectively, for all $1 \leq j \leq 2n$. Then $[E_n, E_{n-1}, \dots, E_1]$, $[E_{n+1}, E_{n+2}, \dots, E_{2n}] \neq 0$, as $\phi_i + \cdots + \phi_j$ is a root for all $1 \leq i < j \leq 2n$. Let

$$E = \big[[E_n, E_{n-1}, \dots, E_1], [E_{n+1}, E_{n+2}, \dots, E_{2n}]\big].$$

Then $E \neq 0$, $E \in \mathfrak{g}_\delta$, and $\nu(E) = -E$. Hence $\mathfrak{g}_\delta \subset \mathfrak{g}^\nu_{\bar{1}}$. Similarly $\mathfrak{g}_{-\delta} \subset \mathfrak{g}^\nu_{\bar{1}}$.
Let $\mathfrak{g} = \mathfrak{a}_{2n-1}$ $(n \geq 2)$.

$\mathfrak{a}_{2n-1}$:
$(n \geq 2)$

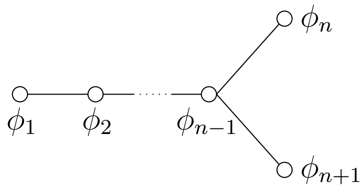


The highest root $\delta = \phi_1 + \cdots + \phi_{2n-1}$. Note that $\bar{\nu}$ is given by $\bar{\nu}(\phi_j) = \phi_{2n-j}$ for all $1 \leq j \leq 2n-1$. Let E_j, E_{-j} be non-zero root vectors corresponding to the roots ϕ_j, $-\phi_j$ respectively, for all $1 \leq j \leq 2n-1$. Then $[E_{n-1}, \dots, E_1]$, $[E_{n+1}, \dots, E_{2n-1}] \neq 0$. Let

$$E = \big[[E_{n-1}, \dots, E_1], [E_n, [E_{n+1}, \dots, E_{2n-1}]]\big].$$

Then $E \neq 0$, $E \in \mathfrak{g}_\delta$, and $\nu(E) = E$. Hence $\mathfrak{g}_\delta \subset \mathfrak{g}^\nu_{\bar{0}}$. Similarly $\mathfrak{g}_{-\delta} \subset \mathfrak{g}^\nu_{\bar{0}}$.
Let $\mathfrak{g} = \delta_{n+1}$ $(n \geq 3)$.

δ_{n+1} :
$(n \geq 3)$

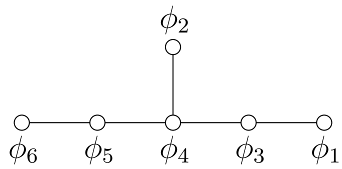


The highest root $\delta = \phi_1 + 2\phi_2 + \cdots + 2\phi_{n-1} + \phi_n + \phi_{n+1}$. Note that $\bar{\nu}$ is given by $\bar{\nu}(\phi_j) = \phi_j$ for all $1 \leq j \leq n-1$, $\bar{\nu}(\phi_n) = \phi_{n+1}$. Let E_j, E_{-j} be non-zero root vectors corresponding to the roots ϕ_j, $-\phi_j$ respectively, for all $1 \leq j \leq n+1$. Then $[E_2, \dots, E_{n-1}, E_n]$, $[E_2, \dots, E_{n-1}, E_{n+1}] \neq 0$. Let

$$E = \big[[E_2, \dots, E_{n-1}, E_n], [E_1, [E_2, \dots, E_{n-1}, E_{n+1}]]\big].$$

Then $E \neq 0$, $E \in \mathfrak{g}_\delta$, and $\nu(E) = E$. Hence $\mathfrak{g}_\delta \subset \mathfrak{g}^\nu_{\bar{0}}$. Similarly $\mathfrak{g}_{-\delta} \subset \mathfrak{g}^\nu_{\bar{0}}$.
Let $\mathfrak{g} = \mathfrak{e}_6$.

$\mathfrak{e}_6$:



The highest root $\delta = \phi_1 + 2\phi_2 + 2\phi_3 + 3\phi_4 + +2\phi_5 + \phi_6$. Note that $\bar{\nu}$ is given by $\bar{\nu}(\phi_1) = \phi_6$, $\bar{\nu}(\phi_3) = \phi_5$, $\bar{\nu}(\phi_2) = \phi_2$, $\bar{\nu}(\phi_4) = \phi_4$. Let E_j, E_{-j} be non-zero root vectors corresponding to the roots ϕ_j, $-\phi_j$ respectively, for all $1 \leq j \leq 6$. Let $E'_1 = \big[[E_1, E_3, E_4], [E_2, [E_6, E_5, E_4]]\big]$. Then $E'_1 \neq 0$ and $\nu(E'_1) = E'_1$. Let $E'_2 = [E_5, [E_3, E'_1]]$. Then $E'_2 \neq 0$ and $\nu(E'_2) = E'_2$. Let

$$E = [E_2, E_4, E'_2].$$

Then $E \neq 0$, $E \in \mathfrak{g}_\delta$, and $\nu(E) = E$. Hence $\mathfrak{g}_\delta \subset \mathfrak{g}_{\bar{0}}^\nu$. Similarly $\mathfrak{g}_{-\delta} \subset \mathfrak{g}_{\bar{0}}^\nu$.

(ii) The module $\mathfrak{g}_{\bar{a}}^\nu$ is an irreducible $\mathfrak{g}_{\bar{0}}^\nu$-module for all $0 \leq a \leq k-1$: It remain to prove that the $\mathfrak{g}_{\bar{0}}^\nu$-modules $\mathfrak{g}_{\bar{1}}^\nu$ (for $k = 2, 3$) and $\mathfrak{g}_{\bar{2}}^\nu$ (for $k = 3$) are irreducible. Let $\{H_\phi^*, E_\alpha : \phi \in \Phi, \alpha \in \Delta\}$ be a Chevalley basis for $\mathfrak{g}$ as in (2). As $\mathfrak{g}_{\bar{a}}^\nu$ are finite dimensional, $\mathfrak{g}_{\bar{a}}^\nu$ are direct sums of irreducible $\mathfrak{g}_{\bar{0}}^\nu$-modules. First we show that the module $\mathfrak{g}_{\bar{1}}^\nu$ is irreducible.

Recall that E_0 $(\neq 0) \in \mathfrak{g}_{(\alpha_0,1)} \subset \mathfrak{g}_{\bar{1}}^\nu$ and $\dim(\mathfrak{g}_{(\alpha_0,1)}) = 1$. By (10), any weight space of the $\mathfrak{g}_{\bar{0}}^\nu$-module $\mathfrak{g}_{\bar{1}}^\nu$ corresponding to a non-zero weight is generated by E_0. So if V is the irreducible submodule of $\mathfrak{g}_{\bar{1}}^\nu$ containing $\mathfrak{g}_{(\alpha_0,1)}$, then any weight space of $\mathfrak{g}_{\bar{1}}^\nu$ corresponding to a non-zero weight is contained in V. Now we show that V also contains the weight space corresponding to the zero weight. If not, then there is a non-zero vector H_0 corresponding to the zero weight such that $[H_0, \bar{E}_\phi] = 0$ for all $\phi \in \Phi$. Here recall that $\bar{E}_\phi = \sum_{i=0}^{k-1} E_{\bar{\nu}^i(\phi)}$ is a root vector corresponding to a simple root of $\mathfrak{g}_{\bar{0}}^\nu$.

Assume that $k = 2$. The weight space of $\mathfrak{g}_{\bar{1}}^\nu$ corresponding to the zero weight is given by $\mathfrak{h}_{\bar{1}}^\nu = \sum\limits_{\phi\in\Phi,\ \phi\neq\bar{\nu}(\phi)} \mathbb{C}(H_\phi^* - H_{\bar{\nu}(\phi)}^*)$. Now

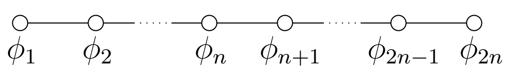
$$[H_\phi^* - H_{\bar{\nu}(\phi)}^*,\ \bar{E}_\psi] = [H_\phi^* - H_{\bar{\nu}(\phi)}^*,\ E_\psi + E_{\bar{\nu}(\psi)}] = (a_{\psi\phi} - a_{\psi\bar{\nu}(\phi)})(E_\psi - E_{\bar{\nu}(\psi)}),$$

where $a_{\psi\phi} = \psi(H_\phi^*)$ for all $\phi, \psi \in \Phi$. So for $\psi \in \Phi$, if $\psi = \bar{\nu}(\psi)$, then $[H,\ \bar{E}_\psi] = 0$, for all $H \in \mathfrak{h}_{\bar{1}}^\nu$. Note that $H_\phi^* - H_{\bar{\nu}(\phi)}^* = -(H_{\bar{\nu}(\phi)}^* - H_{\bar{\nu}^2(\phi)}^*)$, $(\phi \in \Phi,\ \phi \neq \bar{\nu}(\phi))$. So the vectors $H_\phi^* - H_{\bar{\nu}(\phi)}^*$ $(\phi \in \Phi,\ \phi \neq \bar{\nu}(\phi))$ are linearly dependent. Choose a maximal linearly independent subset $\{H_{\phi_i}^* - H_{\bar{\nu}(\phi_i)}^* : 1 \leq i \leq p\}$ in the linearly dependent set $\{H_\phi^* - H_{\bar{\nu}(\phi)}^* : \phi \in \Phi,\ \phi \neq \bar{\nu}(\phi)\}$ and define $a_{ij} = a_{\phi_i\phi_j}$, $a_{i\bar{\nu}(j)} = a_{\phi_i\bar{\nu}(\phi_j)}$ for all $1 \leq i, j \leq p$. Note that $p \leq n$, where $n = \mathrm{rank}(\mathfrak{g}_{\bar{0}}^\nu)$. Let $H_0 = \sum_{i=1}^p c_i(H_{\phi_i}^* - H_{\bar{\nu}(\phi_i)}^*)$. Now $[H_0, \bar{E}_{\phi_i}] = 0$ for all $1 \leq i \leq p$ implies

$$\sum_{j=1}^p (a_{ij} - a_{i\bar{\nu}(j)})c_j = 0 \ \text{ for all } 1 \leq i \leq p.$$

So if the $(p \times p)$ matrix $A = (a_{ij} - a_{i\bar{\nu}(j)})$ is non-singular, then H_0 must be zero, which contradicts our assumption. So we show that the matrix A is non-singular, via case by case consideration.

Let $\mathfrak{g} = \mathfrak{a}_{2n}$ $(n \geq 1)$. $\mathfrak{a}_{2n}$: $(n \geq 1)$



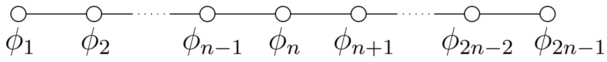
Here $p = n$, $\mathfrak{h}_{\bar{1}}^\nu = \sum_{i=1}^n \mathbb{C}(H_{\phi_i}^* - H_{\phi_{2n-i+1}}^*)$, and for all $1 \leq i, j \leq n$, we have

$$a_{ij} = \begin{cases} 2, & \text{if } i = j \\ -1, & \text{if } |i-j| = 1 \\ 0, & \text{otherwise;} \end{cases} \qquad a_{i\bar{\nu}(j)} = \begin{cases} -1, & \text{if } i = j = n, \\ 0, & \text{otherwise.} \end{cases}$$

Hence the matrix $A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 3 \end{pmatrix}$, which is non-singular.

Let $\mathfrak{g} = \mathfrak{a}_{2n-1}$ $(n \geq 2)$.

$\mathfrak{a}_{2n-1}$: $(n \geq 2)$



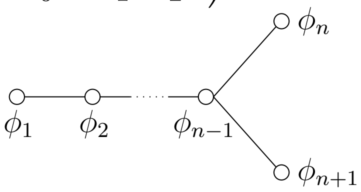
Here $p = n-1$, $\mathfrak{h}_{\bar{1}}^{\nu} = \sum_{i=1}^{n-1} \mathbb{C}(H^*_{\phi_i} - H^*_{\phi_{2n-i}})$, and for all $1 \leq i, j \leq n-1$, we have

$$a_{ij} = \begin{cases} 2, & \text{if } i = j \\ -1, & \text{if } |i-j| = 1 \\ 0, & \text{otherwise;} \end{cases} \qquad \text{and } a_{i\bar{\nu}(j)} = 0 \text{ always.}$$

Hence the matrix $A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$, which is non-singular.

Let $\mathfrak{g} = \delta_{n+1}$ $(n \geq 3)$:

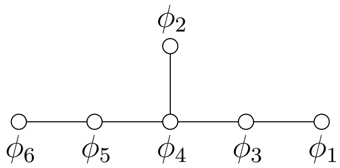
δ_{n+1} : $(n \geq 3)$



Here $p = 1$, $\mathfrak{h}_{\bar{1}}^{\nu} = \mathbb{C}(H^*_{\phi_n} - H^*_{\phi_{n+1}})$, and the matrix $A = (2)$, obviously non-singular.

Let $\mathfrak{g} = \mathfrak{e}_6$.

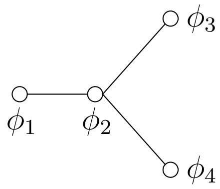
$\mathfrak{e}_6$:



Here $p = 2$, $\mathfrak{h}_{\bar{1}}^{\nu} = \mathbb{C}(H^*_{\phi_1} - H^*_{\phi_6}) \oplus \mathbb{C}(H^*_{\phi_3} - H^*_{\phi_5})$, and the matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, which is non-singular.

Now let $k = 3$. Then $\mathfrak{g} = \delta_4$.

δ_4 :



The zero weight space of $\mathfrak{g}_{\bar{1}}^{\nu}$ is given by $\mathfrak{h}_{\bar{1}}^{\nu} = \mathbb{C}(H^*_{\phi_1} + \epsilon_0^2 H^*_{\bar{\nu}(\phi_1)} + \epsilon_0 H^*_{\bar{\nu}^2(\phi_1)})$, $\epsilon_0 = e^{\frac{2\pi i}{3}}$. Now $\bar{E}_{\phi_1} = E_{\phi_1} + E_{\bar{\nu}(\phi_1)} + E_{\bar{\nu}^2(\phi_1)}$ is a non-zero root vector of $\mathfrak{g}_0^{\nu}$, and

$$[(H^*_{\phi_1} + \epsilon_0^2 H^*_{\bar{\nu}(\phi_1)} + \epsilon_0 H^*_{\bar{\nu}^2(\phi_1)}, E_{\phi_1} + E_{\bar{\nu}(\phi_1)} + E_{\bar{\nu}^2(\phi_1)}] = 2(E_{\phi_1} + \epsilon_0^2 E_{\bar{\nu}(\phi_1)} + \epsilon_0 E_{\bar{\nu}^2(\phi_1)}) \neq 0.$$

Hence $\mathfrak{h}_{\bar{1}}^{\nu}$ is not invariant under $\mathfrak{g}_0^{\nu}$ and so it is contained in V. Now we show that $\mathfrak{g}_{\bar{2}}^{\nu}$ for $\mathfrak{g} = \delta_4$, is irreducible. In this case, $\mathfrak{g}_0^{\nu} = \mathfrak{g}_2$.

$\mathfrak{g}_2$:

Note that $\bar{E}_{\phi_1} = E_{\phi_1}+E_{\phi_3}+E_{\phi_4}$ and $\bar{E}_{\phi_2} = 3E_{\phi_2}$ are root vectors of $\mathfrak{g}_0^\nu$ corresponding to the roots ψ_1 and ψ_2, respectively. Therefore $E_{\phi_1} + \epsilon_0 E_{\bar{\nu}(\phi_1)} + \epsilon_0^2 E_{\bar{\nu}^2(\phi_1)}$ and $E_{-\phi_1} + \epsilon_0 E_{-\bar{\nu}(\phi_1)} + \epsilon_0^2 E_{-\bar{\nu}^2(\phi_1)}$ are weight vectors of $\mathfrak{g}_2^\nu$ corresponding to the weights ψ_1, $-\psi_1$, respectively. Clearly,

$$[\bar{E}_{-\phi_1}, E_{\phi_1} + \epsilon_0 E_{\bar{\nu}(\phi_1)} + \epsilon_0^2 E_{\bar{\nu}^2(\phi_1)}] = -(H^*_{\phi_1} + \epsilon_0 H^*_{\bar{\nu}(\phi_1)} + \epsilon_0^2 H^*_{\bar{\nu}^2(\phi_1)}),$$

$$[\bar{E}_{-\phi_1}, -(H^*_{\phi_1} + \epsilon_0 H^*_{\bar{\nu}(\phi_1)} + \epsilon_0^2 H^*_{\bar{\nu}^2(\phi_1)})] = -2(E_{-\phi_1} + \epsilon_0 E_{-\bar{\nu}(\phi_1)} + \epsilon_0^2 E_{-\bar{\nu}^2(\phi_1)}),$$

$$[\bar{E}_{\phi_2}, E_{\phi_1} + \epsilon_0 E_{\bar{\nu}(\phi_1)} + \epsilon_0^2 E_{\bar{\nu}^2(\phi_1)}] \neq 0, \quad [\bar{E}_{\phi_1}, \bar{E}_{\phi_2}, E_{\phi_1} + \epsilon_0 E_{\bar{\nu}(\phi_1)} + \epsilon_0^2 E_{\bar{\nu}^2(\phi_1)}] \neq 0,$$

$$[\bar{E}_{-\phi_2}, E_{-\phi_1} + \epsilon_0 E_{-\bar{\nu}(\phi_1)} + \epsilon_0^2 E_{-\bar{\nu}^2(\phi_1)}] \neq 0, [\bar{E}_{-\phi_1}, \bar{E}_{-\phi_2}, E_{-\phi_1} + \epsilon_0 E_{-\bar{\nu}(\phi_1)} + \epsilon_0^2 E_{-\bar{\nu}^2(\phi_1)}] \neq 0.$$

These are weight vectors of $\mathfrak{g}_2^\nu$ corresponding to the weights $\psi_1 + \psi_2$, $2\psi_1 + \psi_2$, $-\psi_1 - \psi_2$, $-2\psi_1 - \psi_2$, respectively. As $\dim(\mathfrak{g}_2^\nu) = 7$, $\mathfrak{g}_2^\nu$ is generated by the sum $E_{\phi_1} + \epsilon_0 E_{\bar{\nu}(\phi_1)} + \epsilon_0^2 E_{\bar{\nu}^2(\phi_1)}$ as a $\mathfrak{g}_0^\nu$-module. Hence $\mathfrak{g}_2^\nu$ is irreducible. Note that the lowest weight of $\mathfrak{g}_2^\nu$ is $-2\psi_1 - \psi_2 = \alpha_0$, as $\alpha_0 + 2\psi_1 + \psi_2 = 0$, [12, Tables of Diagrams $S(A)$, Ch. X]. Hence $\mathfrak{g}_2^\nu \cong \mathfrak{g}_1^\nu$, as $\mathfrak{g}_0^\nu$-modules.

(iii) *Except for conjugation,* $\{\sigma, \sigma\theta : \sigma$ *is defined as in* (11)$\}$ *are all automorphisms of* $\mathfrak{g}^{\mathbb{R}}$ *of order* m *and leave* $\mathfrak{u}$ *invariant*: Since $\mathfrak{g}$ is simple, $\mathrm{Aut}(\mathfrak{g})$ is a subgroup of $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$ of index 2. Hence $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}}) = \mathrm{Aut}(\mathfrak{g}) \cup \mathrm{Aut}(\mathfrak{g})\theta$. So it is sufficient to prove that if σ_1, σ_2 are automorphisms of $\mathfrak{g}$ of order m such that these are conjugate in $\mathrm{Aut}(\mathfrak{g})$ and leave $\mathfrak{u}$ invariant, then σ_1, σ_2 are conjugate in $\mathrm{Aut}(\mathfrak{u})$. To prove this, we follow the argument of [12, Prop. 1.4, Ch. X]. Let $g \in \mathrm{Aut}(\mathfrak{g})$ be such that $\sigma_2 = g\sigma_1 g^{-1}$. Now $g\mathfrak{u}$ is also a compact real form of $\mathfrak{g}$.

So there exists $g_0 \in \mathrm{Int}(\mathfrak{g})$ such that $g\mathfrak{u} = g_0\mathfrak{u}$, where $\mathrm{Int}(\mathfrak{g})$ is the identity component of $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$. Hence $g_0^{-1} g \in \tilde{U}$, the normaliser of $\mathfrak{u}$ in $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$. So we can write g as $g = pu$, where $p \in \exp(J\mathfrak{u})$, $u \in \tilde{U}$. Thus $\sigma_2 = pu\sigma_1 u^{-1} p^{-1}$ $(\sigma_1, \sigma_2, u \in \tilde{U})$. Let $\tilde{\theta}$ be the Lie group automorphism of $\mathrm{Aut}(\mathfrak{g}^{\mathbb{R}})$ given by $\tilde{\theta}(\sigma) = \theta\sigma\theta^{-1}$. Now applying $\tilde{\theta}$ on both sides of equation $\sigma_2 = pu\sigma_1 u^{-1} p^{-1}$, we have $\sigma_2 = p^{-1} u\sigma_1 u^{-1} p$. This implies $Ap^2A^{-1} = p^2$, where $A = u\sigma_1 u^{-1}$. Let $p = \exp(JX)$ $(X \in \mathfrak{u})$. Now $Ap^2A^{-1} = p^2$ implies $\exp(2J\mathrm{Ad}(A)(X)) = \exp(2JX)$. As exp is one-to-one on $J\mathfrak{u}$, we have $\mathrm{Ad}(A)(X) = X$ and so A commutes with $\exp(JX) = p$. Hence $\sigma_2 = pu\sigma_1 u^{-1} p^{-1} = u\sigma_1 u^{-1}$, and σ_1, σ_2 are conjugate in $\mathrm{Aut}(\mathfrak{u})$.

4.2. The condition Or: Let G be a connected complex simple Lie group and U be a maximal compact subgroup of G. Let $\mathrm{Lie}(G) = \mathfrak{g}$, $\mathrm{Lie}(U) = \mathfrak{u}$, and θ be the Cartan involution corresponding to the Cartan decomposition $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} \oplus J\mathfrak{u}$, where J denotes, as usual, the complex structure of $\mathfrak{g}^{\mathbb{R}}$ corresponding to the multiplication by i of $\mathfrak{g}$. Let $\bar{\theta}$ denote the corresponding Cartan involution of G. Let $\mathfrak{t}$ be a maximal abelian subspace of $\mathfrak{u}$ and $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Choose a system of positive roots Δ^+ in the set of all non-zero roots $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$. Let Φ be the set of simple roots in Δ^+. Let $\{H^*_\phi, E_\alpha : \phi \in \Phi, \alpha \in \Delta\}$ be a Chevalley basis for $\mathfrak{g}$ as in (2). Then

$$\mathfrak{u} = \sum_{\phi \in \Phi} \mathbb{R}(iH^*_\phi) \oplus \sum_{\alpha \in \Delta^+} \mathbb{R}X_\alpha \oplus \sum_{\alpha \in \Delta^+} \mathbb{R}Y_\alpha,$$

where $X_\alpha = E_\alpha - E_{-\alpha}$, $Y_\alpha = i(E_\alpha + E_{-\alpha})$ for all $\alpha \in \Delta^+$.

Let $\bar{\sigma}$ be an involution of G whose differential at identity is an automorphism σ of $\mathfrak{g}$ of order 2 as in (11). Recall that $\sigma(\mathfrak{u}) = \mathfrak{u}$. Let $\mathfrak{u} = \mathfrak{u}_0 \oplus \mathfrak{u}_1$, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the decompositions of $\mathfrak{u}, \mathfrak{g}$ into 1 and -1 eigenspaces of σ respectively. Note that U is invariant under $\bar{\sigma}$. Let $G(\mu) = \{g \in G : \mu(g) = g\}$ and $U(\mu) = \{u \in U : \mu(u) = u\}$, where $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$. Then $G(\mu)$ is a closed reductive subgroup of G and $U(\mu)$ is a maximal compact subgroup of $G(\mu)$. $X(\mu) = G(\mu)/U(\mu)$ is a Riemannian globally symmetric space of non-compact type. Note that $X(\bar{\sigma}\bar{\theta})$ is an irreducible Riemannian globally symmetric space of type III. For our purpose, it is important to know that when the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$.

We proceed as follows:

Note that $G(\bar{\sigma}) = U(\bar{\sigma})\exp(J\mathfrak{u}_0)$ and $G(\bar{\sigma}\bar{\theta}) = U(\bar{\sigma})\exp(J\mathfrak{u}_1)$. So it is sufficient to check whether the canonical action of $U(\bar{\sigma})$ on $X(\mu)$ is orientation preserving. If $o = U(\bar{\sigma})$ is the identity coset in $X(\mu)$, then $U(\bar{\sigma})(o) = o$ and the differential of this action is given by $\mathrm{Ad} : U(\bar{\sigma}) \longrightarrow T_o(X(\mu))$. Hence it is sufficient to check whether $\det(\mathrm{Ad}(u)|_{i\mathfrak{u}_k}) = 1$ for all $u \in U(\bar{\sigma})$, where $k = 0, 1$.

Let $\tilde{U}$ be the simply connected Lie group with Lie algebra $\mathfrak{u}$ and $p : \tilde{U} \longrightarrow U$ be the covering projection whose differential is the identity map of $\mathfrak{u}$. Let $\tilde{Z}$ denote the centre of $\tilde{U}, S = \ker(p) \subset \tilde{Z}$, and $\tilde{\sigma}$ be the automorphism of $\tilde{U}$ with $d\tilde{\sigma} = (d\tilde{\sigma})_e = \sigma|_{\mathfrak{u}}$. Then $\tilde{U}(\tilde{\sigma})$, the set of fixed points of $\tilde{\sigma}$ is connected [12, Th. 8.2, Ch. VII]. Let

$$\begin{aligned} L = p^{-1}(U(\bar{\sigma})) &= \{u \in \tilde{U} : \bar{\sigma}p(u) = p(u)\} = \{u \in \tilde{U} : p\tilde{\sigma}(u) = p(u)\} \\ &= \{u \in \tilde{U} : p(\tilde{\sigma}(u)u^{-1}) = e\} = \{u \in \tilde{U} : \tilde{\sigma}(u)u^{-1} \in S\}. \end{aligned}$$

Then L is a closed subgroup $\tilde{U}$ and $\tilde{U}(\tilde{\sigma})$ is the connected component of L. Also note that $\tilde{U}(\tilde{\sigma})S \subset L$. If $\tilde{U}(\tilde{\sigma})S = L$, then $U(\bar{\sigma}) = p(L) = p(\tilde{U}(\tilde{\sigma}))$, hence connected. But it may happen that $\tilde{U}(\tilde{\sigma})S \subset L$ but $\tilde{U}(\tilde{\sigma})S \neq L$. Since the covering projection p is orientation preserving, we need to check that $\det(\mathrm{Ad}(u)|_{i\mathfrak{u}_k}) = 1$ for all $u \in L$, where $k = 0, 1$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{u}_1$. For any $u \in \tilde{U}$, there exist $u_1, u_2 \in \tilde{U}(\tilde{\sigma})$ and $X \in \mathfrak{a}$ such that $u = \exp(\mathrm{Ad}(u_1)(X))u_2$ [12, Th. 8.6, Ch. VII]. Now, as $S \subset \tilde{Z}$,

$$\begin{aligned} u \in L &\Leftrightarrow \tilde{\sigma}(u)u^{-1} \in S \\ &\Leftrightarrow \exp(\mathrm{Ad}(u_1)(-X))u_2u_2^{-1}\exp(\mathrm{Ad}(u_1)(-X)) = \exp(\mathrm{Ad}(u_1)(-2X)) \in S \\ &\Leftrightarrow u_1\exp(-2X)u_1^{-1} \in S \Leftrightarrow \exp(-2X) \in S. \end{aligned}$$

To check whether $\det(\mathrm{Ad}(u)|_{i\mathfrak{u}_k}) = 1$ for all $u \in L$, where $k = 0, 1$; it is sufficient to check whether $\det(\mathrm{Ad}(\exp(X))|_{i\mathfrak{u}_k}) = 1$ for all $X \in \mathfrak{a}$ with $\exp(-2X) \in S$, where $k = 0, 1$. Now $\det(\mathrm{Ad}(u)|_{i\mathfrak{u}_0})\ \det(\mathrm{Ad}(u)|_{i\mathfrak{u}_1}) = \det(\mathrm{Ad}(u)|_{i\mathfrak{u}}) = 1$ for all $u \in L$. So it is sufficient to check whether $\det(\mathrm{Ad}(\exp(X))|_{\mathfrak{u}_0}) = 1$ for all $X \in \mathfrak{a}$ with $\exp(-2X) \in S$.

Let the Dynkin diagram automorphism induced by σ be $\bar{\nu}$. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ be a maximal set of strongly orthogonal roots in $\{\alpha \in \Delta^+ : \sigma(H^*_\alpha) = H^*_\alpha,\ \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$.

Then $\mathfrak{a} = \sum\limits_{\substack{\phi \in \Phi \\ \bar{\nu}(\phi) \neq \phi}} \mathbb{R}i(H^*_\phi - H^*_{\bar{\nu}(\phi)}) \oplus \sum\limits_{j=1}^{r} \mathbb{R}Y_{\gamma_j}$ is a maximal abelian subspace of $\mathfrak{u}_1$.

Let $c = \mathrm{Ad}(\exp(\frac{\pi}{4}\sum_{j=1}^{r} X_{\gamma_j}))$. Then $c(Y_{\gamma_j}) = iH^*_{\gamma_j}$ for all $1 \le j \le r$ and we have $c(H^*_\phi - H^*_{\bar\nu(\phi)}) = H^*_\phi - H^*_{\bar\nu(\phi)}$ for all $\phi \in \Phi$. So

$$c(\mathfrak{a}) = \sum_{\phi\in\Phi,\ \bar\nu(\phi)\neq\phi} \mathbb{R}i(H^*_\phi - H^*_{\bar\nu(\phi)}) \oplus \sum_{j=1}^{r} \mathbb{R}(iH^*_{\gamma_j}).$$

Let $\mathfrak{a}^\perp$ be the orthogonal complement of $c(\mathfrak{a})$ in $\mathfrak{t}$ with respect to the positive definite symmetric bilinear form $-B(H,H')(H,H' \in \mathfrak{t})$. Since $\gamma_j(H) = 0$ for all $H \in \mathfrak{a}^\perp$ and for all $1 \le j \le r$, we have $c(H) = H$ for all $H \in \mathfrak{a}^\perp$. Hence if $\mathfrak{t}' = \mathfrak{a}^\perp \oplus \mathfrak{a}$, then $c(\mathfrak{t}') = \mathfrak{a}^\perp \oplus c(\mathfrak{a}) = \mathfrak{t}$. For $X \in \mathfrak{a}, \exp(-2X) \in \tilde Z \Leftrightarrow \alpha'(-2X) \in 2\pi i\mathbb{Z}$ for all $\alpha' \in \Delta(\mathfrak{g}, \mathfrak{t}'^{\mathbb{C}})$ [12, Th. 6.7, Ch.VII] $\Leftrightarrow \alpha(c(X)) \in \pi i\mathbb{Z}$ for all $\alpha \in \Delta$. So if

$$X = iH + \sum_{j=1}^{r} c_j Y_{\gamma_j} \quad (H \in \sum_{\phi\in\Phi,\ \bar\nu(\phi)\neq\phi} \mathbb{R}(H^*_\phi - H^*_{\bar\nu(\phi)}),\ c_j \in \mathbb{R}),$$

then
$$\exp(-2X) \in \tilde Z \ \Leftrightarrow\ \alpha(H) + \sum_{j=1}^{r} c_j \alpha(H^*_{\gamma_j}) \in \pi\mathbb{Z} \ \text{ for all } \alpha \in \Delta$$
$$\Leftrightarrow\ \phi(H) + \sum_{j=1}^{r} c_j \phi(H^*_{\gamma_j}) \in \pi\mathbb{Z} \ \text{ for all } \phi \in \Phi.$$

In particular, we have for all $1 \le j \le r$ that $\exp(-2X) \in \tilde Z$ implies $c_j \in \frac{\pi}{2}\mathbb{Z}$, taking $\alpha = \gamma_j$.

For $X \in \mathfrak{a}$ with $\exp(-2X) \in S$, let $\sigma_X = \exp(X)$. Then $\sigma_X^{-2} \in \tilde Z$ and so $\mathrm{Ad}(\sigma_X)|_{\mathfrak{u}_0} : \mathfrak{u}_0 \longrightarrow \mathfrak{u}_0$ is an involution. Note that $\mathfrak{t}_0 = \sum_{\phi\in\Phi} \mathbb{R}i(H^*_\phi + H^*_{\bar\nu(\phi)})$ is a maximal abelian subspace of $\mathfrak{u}_0$.

Now $\mathfrak{t}^- = \sum_{j=1}^{r} \mathbb{R}(iH^*_{\gamma_j}) \subset \mathfrak{t}_0$. Let $\mathfrak{t}^+ = \{H \in \mathfrak{t}_0 : \gamma_j(H) = 0 \text{ for all } 1 \le j \le r\}$. So $\mathrm{Ad}(\sigma_X)(H) = H$ for all $H \in \mathfrak{t}^+$.

Now $[X, iH^*_{\gamma_j}] = c_j[Y_{\gamma_j}, iH^*_{\gamma_j}] = 2c_j X_{\gamma_j},\ [X, X_{\gamma_j}] = c_j[Y_{\gamma_j}, X_{\gamma_j}] = -2c_j iH^*_{\gamma_j}$. Hence

$$\mathrm{Ad}(\sigma_X)(iH^*_{\gamma_j}) = (\cos 2c_j) iH^*_{\gamma_j} + (\sin 2c_j) X_{\gamma_j} = (\cos 2c_j) iH^*_{\gamma_j} \ \text{ for all } 1 \le j \le r.$$

So $\mathrm{Ad}(\sigma_X)(\mathfrak{t}_0) = \mathfrak{t}_0$. Let $\Delta_0 = \Delta(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ and Δ_0^+ be a system of positive roots in Δ_0. For $X \in \mathfrak{a}$ with $\exp(-2X) \in S$, let $s'_X \in W(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ be such that $\mathrm{Ad}(\sigma_X) \circ s'_X(\Delta_0^+) = \Delta_0^+$ and $s_X = \mathrm{Ad}(\sigma_X) \circ s'_X$. For $\alpha \in \Delta_0$, choose $\bar E_\alpha \in (\mathfrak{g}_0)_\alpha$ such that

$$\mathfrak{u}_0 = \mathfrak{t}_0 \oplus \sum_{\alpha\in\Delta_0^+} \mathbb{R}(\bar E_\alpha - \bar E_{-\alpha}) \oplus \sum_{\alpha\in\Delta_0^+} \mathbb{R}i(\bar E_\alpha + \bar E_{-\alpha})$$

and $s_X(\bar E_\alpha) = a_\alpha \bar E_{\alpha'} (s_X(\alpha') = \alpha,\ a_\alpha \in \mathbb{C})$ with $a_\alpha a_{-\alpha} = 1$ and $|a_\alpha| = 1$ [12, Cor. 5.2, Ch. IX]. For $\alpha \in \Delta_0^+$, if $\bar X_\alpha = \bar E_\alpha - \bar E_{-\alpha}$ and $\bar Y_\alpha = i(\bar E_\alpha + \bar E_{-\alpha})$, then

$$s_X(\bar X_\alpha) = x_\alpha \bar X_{\alpha'} + y_\alpha \bar Y_{\alpha'} \ \text{ and } \ s_X(\bar Y_\alpha) = -y_\alpha \bar X_{\alpha'} + x_\alpha \bar Y_{\alpha'},$$

where $a_\alpha = x_\alpha + iy_\alpha,\ x_\alpha, y_\alpha \in \mathbb{R}$. Then $\det(s_X\big|_{\mathbb{R}\bar X_\alpha + \mathbb{R}\bar Y_\alpha}) = x_\alpha^2 + y_\alpha^2 = |a_\alpha|^2 = 1$ for all $\alpha \in \Delta_0^+$. So it is sufficient to check whether $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$ for all $X \in \mathfrak{a}$ with $\exp(-2X) \in S$.

Recall that the Dynkin diagram automorphism $\bar\nu$ of $\mathfrak{g}$ induced by σ has order k ($k = 1$ or 2). Let ν be an automorphism of $\mathfrak{g}$ induced by $\bar\nu$, as in Section 4.1. Then ν has order k and we have an $\mathbb{Z}_k$-gradation of $\mathfrak{g}$ as $\mathfrak{g} = \oplus_{i\in\mathbb{Z}_k} \mathfrak{g}_i^\nu$ into the eigenspaces of ν. Recall that $\mathfrak{g}_0^\nu$ is a simple Lie algebra and $\mathfrak{h}^\nu = \mathfrak{h} \cap \mathfrak{g}_0^\nu$ is a Cartan subalgebra of $\mathfrak{g}_0^\nu$. Also the set Φ of simple roots in Δ^+ induces a basis $\Psi = \{\psi_1, \psi_2, \ldots, \psi_n\}$ (determined by the map $\bar{\nu}\colon \Phi \longrightarrow \Phi$) of the root system $\Delta(\mathfrak{g}_0^\nu, \mathfrak{h}^\nu)$ [12, the proof of Lemma 5.11, Ch. X]. Let α_0 be the lowest weight of the $\mathfrak{g}_0^\nu$-module $\mathfrak{g}_{\bar{1}}^\nu$. Note that if $k = 1$ that is, if $\bar{\nu}$ is the identity map, then $\alpha_0 = -\delta$, where δ is the highest root of Δ^+. Let $\alpha_0 + \sum i = 1^n a_i\psi_i = 0$ ($a_i \in \mathbb{N}$ for all $1 \leq i \leq n$). Then as in (11), there are non-zero integers $s_0, s_1, \ldots, s_n$ without non-trivial common factor such that $2 = k(s_0 + \sum_{i=1}^n a_i s_i)$ and σ is an involution of $\mathfrak{g}$ of type $(s_0, s_1, \ldots, s_n; k)$.

Let $i_1, \ldots, i_t$ be all the indices with $s_{i_1} = \cdots = s_{i_t} = 0$. Then the Lie algebra $\mathfrak{g}_0 = \{X \in \mathfrak{g} : \sigma(X) = X\}$ is the direct sum of an $(n-t)$-dimensional centre and a semisimple Lie algebra whose Dynkin diagram is the subdiagram of the diagram $\mathfrak{g}^{(k)}$ (given in Section 4.1) consisting of the vertices $\psi_{i_1}, \ldots, \psi_{i_t}$. From now on we assume that Δ_0^+ is the system of positive roots in $\Delta_0 = \Delta(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ corresponding to the basis $\{\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_t}\}$.

Remark 4.5. (i) We may choose $\bar{E}_\alpha \in (\mathfrak{g}_0)_\alpha$ such that

$$\mathfrak{u}_0 = \mathfrak{t}_0 \oplus \sum_{\alpha\in\Delta_0^+} \mathbb{R}(\bar{E}_\alpha - \bar{E}_{-\alpha}) \oplus \sum_{\alpha\in\Delta_0^+} \mathbb{R}i(\bar{E}_\alpha + \bar{E}_{-\alpha})$$

and $s_X(\bar{E}_\alpha) = a_\alpha \bar{E}_{\alpha'} (s_X(\alpha') = \alpha,\ a_\alpha \in \mathbb{C})$ with $a_\alpha a_{-\alpha} = 1$ and $a_\alpha = \pm 1$: For $X \in \mathfrak{a}$ with $\exp(-2X) \in S$, $\mathrm{Ad}(\sigma_X)|_{\mathfrak{u}_0}$ is an involution and $s'_X \in W(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ be such that $\mathrm{Ad}(\sigma_X) \circ s'_X(\Delta_0^+) = \Delta_0^+$. So s'_X is also an involution. Hence $s_X = \mathrm{Ad}(\sigma_X) \circ s'_X : \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0$ is an involution with $s_X(\Delta_0^+) = \Delta_0^+$. Now the result follows from [2, Lemma 3.5].

(ii) If $\mathfrak{g} = \mathfrak{e}_8, \mathfrak{f}_4$, or $\mathfrak{g}_2$, then the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$: In these cases, the simply connected Lie group $\tilde{U}$ has trivial centre, that is $\tilde{Z} = \{e\}$ [12, Cor. 7.8, Ch. VII, Lemma 3.30 and Th. 3.32, Ch. X]. So the result follows.

(iii) If $\mathfrak{a} = \sum_{\phi\in\Phi,\ \bar{\nu}(\phi)\neq\phi} \mathbb{R}i(H_\phi^* - H_{\bar{\nu}(\phi)}^*)$ is a maximal abelian subspace of $\mathfrak{u}_1$ (this happens exactly when the Dynkin diagram automorphism $\bar{\nu}$ has order 2 and $\sigma = \nu$, the automorphism of $\mathfrak{g}$ induced by $\bar{\nu}$), then the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$: For $\mathrm{Ad}(\sigma_X)|_{\mathfrak{t}_0}$ is the identity map, in this case.

(iv) If $X(\bar{\sigma}\bar{\theta})$ is a Hermitian symmetric space and is not of tube type, then the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$: If $X(\bar{\sigma}\bar{\theta})$ is a Hermitian symmetric space, then $\sigma|_{\mathfrak{h}}$ is the identity map and there is a maximal set $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ of strongly orthogonal roots in $\{\alpha \in \Delta^+ : \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$ such that if $\psi \in \Delta_0^+$ is a simple root of $\mathfrak{g}_0$, then $\psi\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \frac{1}{2}(\gamma_{i+1} - \gamma_i)$ or 0 or $-\frac{1}{2}\gamma_i$ for some i [11, Lemma.13]. Now $X(\bar{\sigma}\bar{\theta})$ is not of tube type iff there is a simple root $\psi \in \Delta_0^+$ with $\psi\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \frac{1}{2}\gamma_i$ for some i [21, Prop. 4.4 and its Remark]. Then for $X = \sum_{j=1}^r c_j Y_{\gamma_j} \in \mathfrak{a}$, $\sum_{j=1}^r c_j \phi(H_{\gamma_j}^*) \in \pi\mathbb{Z}$ for all $\phi \in \Phi$ $\Leftrightarrow 2c_1, c_2 - c_1, \ldots, c_r - c_{r-1}, -c_i$ (for some i) $\in \pi\mathbb{Z} \Leftrightarrow c_j \in \pi\mathbb{Z}$ for all $1 \leq j \leq r$. So $\mathrm{Ad}(\sigma_X)|_{\mathfrak{t}_0}$ is the identity map, in this case.

(v) If $X(\bar{\sigma}\bar{\theta})$ is a Hermitian symmetric space and is of tube type, then the canonical action of $G(\mu)$ on $X(\mu)$ may not be orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$: Since $X(\bar{\sigma}\bar{\theta})$ is a Hermitian symmetric space, $\mathfrak{g}_0$ has one dimensional centre. Since $X(\bar{\sigma}\bar{\theta})$

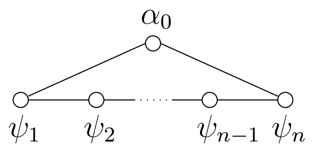
is of tube type, the element $Z = \sum_{j=1}^{r} iH^*_{\gamma_j}$ lies in the centre of $\mathfrak{g}_0$ [21, Prop. 3.12]. Again if $\psi \in \Delta_0^+$ is a simple root of $\mathfrak{g}_0$, then $\psi\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \frac{1}{2}(\gamma_{i+1} - \gamma_i)$ or 0 for some i. So for $X = \sum_{j=1}^{r} c_j Y_{\gamma_j} \in \mathfrak{a}$, $\sum_{j=1}^{r} c_j \phi(H^*_{\gamma_j}) \in \pi\mathbb{Z}$ for all $\phi \in \Phi$ $\Leftrightarrow 2c_1, c_2 - c_1, \ldots, c_r - c_{r-1} \in \pi\mathbb{Z} \Leftrightarrow \cos 2c_j = \cos 2c_1$ for all $1 \le j \le r$, where $2c_1 \in \pi\mathbb{Z}$. Hence $s_X(Z) = \mathrm{Ad}(\sigma_X)(Z) = \pm Z$. So if $s_X(Z) = Z$, then $\mathrm{Ad}(\sigma_X|_{\mathfrak{t}_0})$ is the identity map and hence $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$. If $s_X(Z) = -Z$, then $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$ if the Dynkin diagram automorphism of $[\mathfrak{g}_0, \mathfrak{g}_0]$ induced by s_X is an odd permutation.

(vi) If the Riemannian globally symmetric space $X(\bar\sigma\bar\theta)$ is not a Hermitian symmetric space, then the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar\sigma, \bar\sigma\bar\theta$ iff the Dynkin diagram automorphism of $\mathfrak{g}_0$ induced by s_X is an even permutation for any $X \in \mathfrak{a}$ with $\exp(-2X) \in S$: In this case, $\mathfrak{g}_0$ is semisimple and for all $X \in \mathfrak{a}$ with $\exp(-2X) \in S$, $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$ iff the Dynkin diagram automorphism of $\mathfrak{g}_0$ induced by s_X is an even permutation. ■

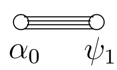
Now we check whether $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$ for all $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$, via case by case consideration.

1.

$\mathfrak{a}_n^{(1)}$: $(n > 1)$



$\mathfrak{a}_1^{(1)}$:



Here $\mathfrak{g} = \mathfrak{a}_n$ and $\alpha_0 + \psi_1 + \psi_2 + \cdots + \psi_n = 0$. Without loss of generality, we may assume that σ is an involution of $\mathfrak{g}$ of type $(s_0, 0, \ldots, 0, s_p, 0, \ldots, 0; 1)$ $(1 \le p \le n,\ p \le n+1-p)$ with $s_0 = 1 = s_p$. Then $\mathfrak{u}_0 = \mathfrak{su}(p) \oplus \mathfrak{su}(n+1-p) \oplus i\mathbb{R}$ and $X(\bar\sigma\bar\theta)$ is a Hermitian symmetric space. Let

$$\gamma_1 = \psi_p, \gamma_2 = \psi_{p-1} + \psi_p + \psi_{p+1}, \ldots, \gamma_p = \psi_1 + \cdots + \psi_p + \cdots \psi_{2p-1}.$$

Then $\{\gamma_1, \gamma_2, \ldots, \gamma_p\}$ is a maximal set of strongly orthogonal roots in the set $\{\alpha \in \Delta^+ : \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$. The Hermitian symmetric space $X(\bar\sigma\bar\theta)$ is of tube type iff $n = 2p-1$ that is, $p = n+1-p$. So if $p < n+1-p$, then the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar\sigma, \bar\sigma\bar\theta$, by Remark 4.5(iv).

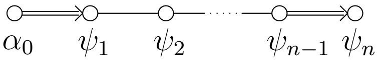
Now assume that $p = n+1-p$. Then we get $\psi_{p-1}\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \frac{1}{2}(\gamma_2 - \gamma_1) = \psi_{p+1}\big|_{(\mathfrak{t}^-)^{\mathbb{C}}}$ and $\psi_{p-2}\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \frac{1}{2}(\gamma_3 - \gamma_2) = \psi_{p+2}\big|_{(\mathfrak{t}^-)^{\mathbb{C}}}, \ldots, \psi_1\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \frac{1}{2}(\gamma_p - \gamma_{p-1}) = \psi_{2p-1}\Big|_{(\mathfrak{t}^-)^{\mathbb{C}}}$, where $n = 2p-1$. Let $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$ and $\mathrm{Ad}(\sigma_X)(Z) = -Z$, where $Z = \sum_{j=1}^{r} iH^*_{\gamma_j}$. Then

$$\begin{aligned}\mathrm{Ad}(\sigma_X)(\psi_{p\pm j}) &= \mathrm{Ad}(\sigma_X)(\frac{1}{2}(\gamma_{j+1} - \gamma_j) + \psi_{p\pm j} - \frac{1}{2}(\gamma_{j+1} - \gamma_j)) \\ &= -\frac{1}{2}(\gamma_{j+1} - \gamma_j) + \psi_{p\pm j} - \frac{1}{2}(\gamma_{j+1} - \gamma_j) = \psi_{p\pm j} - (\gamma_{j+1} - \gamma_j) \\ &= -\psi_{p\mp j}, \quad \text{for all } 1 \le j \le p-1.\end{aligned}$$

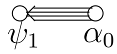
Let $w^0_{\mathfrak{g}_0} \in W(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ be the longest element, that is, we have $w^0_{\mathfrak{g}_0}(\psi_j) = -\psi_{p-j}$ and $w^0_{\mathfrak{g}_0}(\psi_{p+j}) = -\psi_{2p-j}$ for all $1 \le j \le p-1$. Then $s_X = \mathrm{Ad}(\sigma_X) \circ w^0_{\mathfrak{g}_0}$ with $s_X(\Delta_0^+) = \Delta_0^+$. Now $s_X(\psi_{p-j}) = \psi_{2p-j}$ and $s_X(\psi_{p+j}) = \psi_j$ for all $1 \le j \le p-1$.

So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = -(-1)^{p-1} = (-1)^p$. So if p is even, then the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$.

2. $\mathfrak{a}_{2n}^{(2)}$: $(n > 1)$

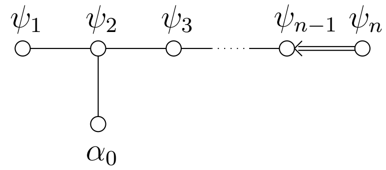


$\mathfrak{a}_{2}^{(2)}$:



Here $\mathfrak{g} = \mathfrak{a}_{2n}$, $\alpha_0 + 2\psi_1 + 2\psi_2 + \cdots + 2\psi_n = 0$, and σ is an involution of $\mathfrak{g}$ of type $(1, 0, \ldots, 0; 2)$. Then $\mathfrak{u}_0 = \mathfrak{so}(2n+1)$ and $\mathfrak{g}_0 = \mathfrak{b}_n$, which does not have any non-trivial Dynkin diagram automorphism. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$ for all $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$, by Remark 4.5(vi). Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$.

3. $\mathfrak{a}_{2n-1}^{(2)}$: $(n > 2)$

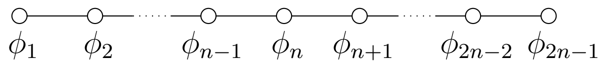


Now $\mathfrak{g} = \mathfrak{a}_{2n-1}(n > 2)$. In this case, $\alpha_0 + \psi_1 + 2\psi_2 + \cdots + 2\psi_{n-1} + \psi_n = 0$.

(i) First assume that σ is an involution of $\mathfrak{g}$ of type $(1, 0, \ldots, 0; 2)$ (similarly for type $(0, 1, 0, \ldots, 0; 2)$). Then $\mathfrak{u}_0 = \mathfrak{sp}(n)$ and $\mathfrak{g}_0 = \mathfrak{c}_n(n > 2)$, which does not have any non-trivial Dynkin diagram automorphism. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$ for all $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$, by Remark 4.5(vi). Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$.

(ii) Next assume that σ is an involution of type $(0, 0, \ldots, 0, 1; 2)$. Then $\mathfrak{u}_0 = \mathfrak{so}(2n)$ and $\mathfrak{g}_0 = \delta_n(n > 2)$. The diagram $\mathfrak{a}_{2n-1}^{(2)}$ is corresponding to the Dynkin diagram automorphism $\bar{\nu}$ of $\mathfrak{a}_{2n-1}$ given by $\bar{\nu}(\phi_j) = \phi_{2n-j}$ for all $1 \leq j \leq 2n-1$.

$\mathfrak{a}_{2n-1}$: $(n > 2)$



Now let $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ be a maximal set of strongly orthogonal roots in the set $\{\alpha \in \Delta^+ : \sigma(H_\alpha^*) = H_\alpha^*,\ \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$, where $\gamma_1 = \phi_n$. Then

$$\mathfrak{a} = \sum_{j=1}^{n-1} \mathbb{R}i(H_{\phi_j}^* - H_{\phi_{2n-j}}^*) \oplus \sum_{j=1}^{r} \mathbb{R}Y_{\gamma_j}$$

is a maximal abelian subspace of $\mathfrak{u}_1$. Let $X = iH + \frac{\pi}{2}Y_{\gamma_1}$, where

$$H = \tfrac{\pi}{2n} \sum_{j=1}^{n-1} j(H_{\phi_j}^* - H_{\phi_{2n-j}}^*).$$

Now $\phi_{n+j}(H) = -\phi_{n-j}(H)$ for all $1 \leq j \leq n-1$, $\phi_j(H) = 0$ for $1 \leq j \leq n-2$ and $j = n$, and $\phi_{n-1}(H) = \frac{\pi}{2}$. Hence $\phi_j(H) + \frac{\pi}{2}\phi_j(H_{\gamma_1}^*) = 0$ for $1 \leq j \leq 2n-1$, $j \neq n, n+1$, $\phi_n(H) + \frac{\pi}{2}\phi_n(H_{\gamma_1}^*) = \pi$, and $\phi_{n+1}(H) + \frac{\pi}{2}\phi_{n+1}(H_{\gamma_1}^*) = -\pi$. Hence $\phi_j(H) + \frac{\pi}{2}\phi(H_{\gamma_1}^*) \in \pi\mathbb{Z}$ for all $1 \leq j \leq 2n-1$ that is, $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$.

Now $\mathrm{Ad}(\sigma_X)(H_{\gamma_1}^*) = -H_{\gamma_1}^*$ and $\mathrm{Ad}(\sigma_X)(H) = H$ for all $\{H \in \mathfrak{h}^\nu = \mathfrak{t}_0^{\mathbb{C}} : \gamma_1(H) = 0\}$. Hence $\mathrm{Ad}(\sigma_X)(\psi_j) = \psi_j$ for all $1 \leq j \leq n-2$, $\mathrm{Ad}(\sigma_X)(\alpha_0) = \alpha_0$ (as $n > 2$), $\mathrm{Ad}(\sigma_X)(\psi_n) = -\psi_n$, and

$$\begin{aligned}\operatorname{Ad}(\sigma_X)(\psi_{n-1}) &= \operatorname{Ad}(\sigma_X)(-\tfrac{1}{2}\psi_n + \psi_{n-1} + \tfrac{1}{2}\psi_n) = \tfrac{1}{2}\psi_n + \psi_{n-1} + \tfrac{1}{2}\psi_n \\ &= \psi_{n-1} + \psi_n = \psi_{n-1} - (\alpha_0 + \psi_1 + 2\psi_2 + \cdots + 2\psi_{n-1}) = -\mu,\end{aligned}$$

where μ is the highest root in Δ_0^+. Therefore

$$\operatorname{Ad}(\sigma_X)(\{\alpha_0, \psi_1, \ldots, \psi_{n-2}, \psi_{n-1}\}) = \{\alpha_0, \psi_1, \ldots, \psi_{n-2}, -\mu\}.$$

Lemma 4.6. *Let $\mathfrak{l}_0$ be a real simple Lie algebra, $\mathfrak{l}_0 = \mathfrak{k}_0 \oplus \mathfrak{e}_0$ be a Cartan decomposition of $\mathfrak{l}_0$, and $\mathfrak{k}_0$ has one dimensional centre (that is, the corresponding Riemannian globally symmetric space is Hermitian symmetric space). Let $\mathfrak{b}_0$ be a maximal abelian subspace of $\mathfrak{k}_0$, $\mathfrak{l} = \mathfrak{l}_0^{\mathbb{C}}$, $\mathfrak{k} = \mathfrak{k}_0^{\mathbb{C}}$, and $\mathfrak{b} = \mathfrak{b}_0^{\mathbb{C}}$. Then $\mathfrak{b} \subset \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{l}$. Let $\Delta = \Delta(\mathfrak{l}, \mathfrak{b})$, and $\Delta_0 = \Delta(\mathfrak{k}, \mathfrak{b}) =$ the set of all compact roots in Δ. Let Δ^+ be a system of positive roots in Δ such that the corresponding simple system contains exactly one non-compact root ν and the coefficient $n_\nu(\mu)$ of ν in the highest root μ when expressed as a sum of simple roots is 1. Let $\Delta_0^+ = \Delta_0 \cap \Delta^+$, $\Delta_{\pm 1} = \{\alpha \in \Delta : n_\nu(\alpha) = \pm 1\}$, and $w_{\mathfrak{l}}^0$ (respectively, $w_{\mathfrak{k}}^0$) denote the longest element of the Weyl group $W(\mathfrak{g}, \mathfrak{b})$ (respectively, $W(\mathfrak{k}, \mathfrak{b})$) with respect to the positive system Δ^+ (respectively, Δ_0^+). Then $\Delta^+ = \Delta_0^+ \cup \Delta_1$, and $w_0(\Delta^+) = \Delta_0^+ \cup \Delta_{-1}$, where $w_0 = w_{\mathfrak{k}}^0 w_{\mathfrak{l}}^0 \in W(\mathfrak{l}, \mathfrak{b})$. If $w_{\mathfrak{l}}^0(\nu) = -\nu$ (that is, the Hermitian symmetric space is of tube type), then $w_0(\Delta_0^+) = \Delta_0^+$ and $w_0(\nu) = -\mu$.*

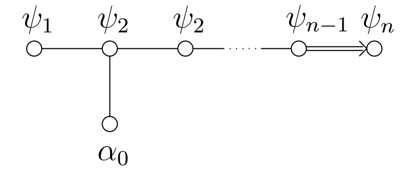
Proof. Let $\mathfrak{e} = \mathfrak{e}_0^{\mathbb{C}}$. Then $\mathfrak{k} = \mathfrak{b} \oplus \sum_{\substack{\alpha \in \Delta \\ n_\nu(\alpha) = 0}} \mathfrak{l}_\alpha$, $\mathfrak{e} = \mathfrak{e}_+ \oplus \mathfrak{e}_-$, $\mathfrak{e}_\pm = \sum_{\substack{\alpha \in \Delta \\ n_\nu(\alpha) = \pm 1}} \mathfrak{l}_\alpha$.
Also $[\mathfrak{k}, \mathfrak{e}_+] \subset \mathfrak{e}_+$, $[\mathfrak{k}, \mathfrak{e}_-] \subset \mathfrak{e}_-$. Since $\mathfrak{l}$ is simple, the $\mathfrak{k}$-modules $\mathfrak{e}_+, \mathfrak{e}_-$ are irreducible with highest weight μ, $-\nu$ respectively. Also Δ_1 (respectively, Δ_{-1}) is the set of all weights of the $\mathfrak{k}$-module $\mathfrak{e}_+$ (respectively, $\mathfrak{e}_-$). Hence $w_{\mathfrak{k}}^0(\Delta_1) = \Delta_1, w_{\mathfrak{k}}^0(\Delta_{-1}) = \Delta_{-1}$, and $w_{\mathfrak{k}}^0(\mu)$ (respectively, $w_{\mathfrak{k}}^0(-\nu)$) is the lowest weight of $\mathfrak{e}_+$ (respectively, $\mathfrak{e}_-$). Hence $w_{\mathfrak{k}}^0(\mu) = \nu$, and $w_{\mathfrak{k}}^0(-\nu) = -\mu$.

Now $w_0(\Delta^+) = w_{\mathfrak{k}}^0 w_{\mathfrak{l}}^0(\Delta_0^+ \cup \Delta_1) = w_{\mathfrak{k}}^0(-(\Delta_0^+) \cup \Delta_{-1}) = \Delta_0^+ \cup \Delta_{-1}$, and if $w_{\mathfrak{l}}^0(\nu) = -\nu$, then $w_0(\nu) = w_{\mathfrak{k}}^0 w_{\mathfrak{l}}^0(\nu) = w_{\mathfrak{k}}^0(-\nu) = -\mu$. ■

Remark 4.7. The above remark will be useful to determine the Weyl group element s'_X (defined in Section 4.2) in case by case consideration. ■

Returning to case 3.(ii), let $w_{\mathfrak{g}_0}^0$ denote the longest element of the Weyl group of δ_n with respect to the simple system $\{\alpha_0, \psi_1, ..., \psi_{n-2}, \psi_{n-1}\}$. The hypotheses of Lemma 4.6 are satisfied for the Hermitian symmetric space $SO_0(2, 2n-2)/SO(2) \times SO(2n-2)$ and $\nu = \psi_{n-1}$. Let $w_0 \in W(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ be as in Lemma 4.6 and $s'_X = w_0$. Then $s'_X(\{\alpha_0, \psi_1, \ldots, \psi_{n-2}, \psi_{n-1}\}) = \{\alpha_0, \psi_1, \ldots, \psi_{n-2}, -\mu\}$. Since $w_{\mathfrak{g}_0}^0(\psi_{n-1}) = -\psi_{n-1}$, we have $s'_X(\psi_{n-1}) = -\mu$ and $s'_X(\{\alpha_0, \psi_1, \ldots, \psi_{n-2}\}) = \{\alpha_0, \psi_1, \ldots, \psi_{n-2}\}$.
Now $w_{\mathfrak{g}_0}^0(\alpha_0)$ is $-\alpha_0$ or $-\psi_1$ according as n is even or n is odd. In any case, we have $s'_X(\alpha_0) = \psi_1, s'_X(\psi_1) = \alpha_0, s'_X(\psi_2) = \psi_2, \ldots, s'_X(\psi_{n-2}) = \psi_{n-2}$. Let $s_X = \operatorname{Ad}(\sigma_X) \circ s'_X$. Then $s_X(\alpha_0) = \psi_1, s_X(\psi_1) = \alpha_0, s_X(\psi_j) = \psi_j$ for $2 \leq j \leq n-1$. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = -1$.

4. $\mathfrak{b}_n^{(1)}$: $(n > 2)$



Here $\mathfrak{g} = \mathfrak{b}_n$ and $\alpha_0 + \psi_1 + 2\psi_2 + \cdots + 2\psi_n = 0$.

(i) First assume that σ is an involution of $\mathfrak{g}$ of type $(1,1,0,\ldots,0;1)$. Therefore $\mathfrak{u}_0 = \mathfrak{so}(2n-1) \oplus i\mathbb{R}$ and $X(\bar{\sigma}\bar{\theta})$ is a Hermitian symmetric space of tube type. Let

$$\gamma_1 = \psi_1, \gamma_2 = \psi_1 + 2\psi_2 + \cdots + 2\psi_n.$$

Then $\{\gamma_1, \gamma_2\}$ is a maximal set of strongly orthogonal roots in $\{\alpha \in \Delta^+ : \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$ and so $\mathfrak{a} = \mathbb{R}Y_{\gamma_1} \oplus \mathbb{R}Y_{\gamma_2}$ is a maximal abelian subspace of $\mathfrak{u}_1$. Since $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{b}_{n-1}$ does not admit any non-trivial Dynkin diagram automorphism, $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = -1$ for some $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$, by Remark 4.5(v).

(ii) Next assume that σ is an involution of $\mathfrak{g}$ of type $(0,\ldots,0,s_p,0,\ldots,0;1)$ with $s_p = 1$ where $2 \leq p < n$. Then $\mathfrak{u}_0 = \mathfrak{so}(2p) \oplus \mathfrak{so}(2n+1-2p)$. Let $q = \min\{p, n-p\}$, and define

$$\begin{aligned}
&\gamma_1 = \psi_p, \gamma_1' = \psi_p + 2\psi_{p+1} + \cdots + 2\psi_n, \\
&\gamma_2 = \psi_{p-1} + \psi_p + \psi_{p+1}, \gamma_2' = \psi_{p-1} + \psi_p + \psi_{p+1} + 2\psi_{p+2} + \cdots + 2\psi_n, \ldots, \\
&\gamma_q = \psi_{p-q+1} + \cdots + \psi_p + \cdots + \psi_{p+q-1}, \gamma_q' = \psi_{p-q+1} + \cdots + \psi_p + \cdots + \psi_{p+q-1} \\
&\qquad\qquad + 2\psi_{p+q} + \cdots + 2\psi_n.
\end{aligned}$$

If $n-p < p$, define $\gamma_0 = \psi_{2p-n} + \cdots + \psi_n$. Let $\Gamma = \{\gamma_1, \gamma_1', \ldots, \gamma_q, \gamma_q'\}$ if $p \leq n-p$, and $\Gamma = \{\gamma_1, \gamma_1', \ldots, \gamma_q, \gamma_q', \gamma_0\}$ if $p > n-p$. Then Γ is a maximal set of strongly orthogonal roots in $\{\alpha \in \Delta^+ : \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$, and $\mathfrak{a} = \sum_{\gamma \in \Gamma} \mathbb{R}Y_\gamma$ is a maximal abelian subspace of $\mathfrak{u}_1$. Let $X = \frac{\pi}{2}(Y_{\gamma_1} + Y_{\gamma_1'})$. Since $\frac{\pi}{2}\psi_j(H^*_{\gamma_1} + H^*_{\gamma_1'}) = 0$ for $1 \leq j \leq n, j \neq p-1, p$, $\frac{\pi}{2}\psi_{p-1}(H^*_{\gamma_1} + H^*_{\gamma_1'}) = -\pi$, and $\frac{\pi}{2}\psi_p(H^*_{\gamma_1} + H^*_{\gamma_1'}) = \pi$; hence $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$. Now $\mathrm{Ad}(\sigma_X)(H^*_{\gamma_1}) = -H^*_{\gamma_1}$, $\mathrm{Ad}(\sigma_X)(H^*_{\gamma_1'}) = -H^*_{\gamma_1'}$, and $\mathrm{Ad}(\sigma_X)(H) = H$ for all $\{H \in \mathfrak{t}_0^{\mathbb{C}} : \gamma_1(H) = 0 = \gamma_1'(H)\}$. Hence

$$\begin{aligned}
\mathrm{Ad}(\sigma_X)(\psi_j) &= \psi_j \text{ for all } 1 \leq j \leq p-2,\ p+2 \leq j \leq n; \\
\mathrm{Ad}(\sigma_X)(\alpha_0) &= \alpha_0 \text{ (if } p > 2), \text{ and } \mathrm{Ad}(\sigma_X)(\psi_p) = -\psi_p, \text{ and} \\
\mathrm{Ad}(\sigma_X)(\psi_{p-1}) &= \mathrm{Ad}(\sigma_X)(-\tfrac{1}{2}(\gamma_1 + \gamma_1') + \tfrac{1}{2}(\gamma_1 + \gamma_1') + \psi_{p-1}) \\
&= \tfrac{1}{2}(\gamma_1 + \gamma_1') + \tfrac{1}{2}(\gamma_1 + \gamma_1') + \psi_{p-1} \\
&= \psi_{p-1} + \gamma_1 + \gamma_1' = \psi_{p-1} + 2\psi_p + \cdots + 2\psi_n.
\end{aligned}$$

So $\mathrm{Ad}(\sigma_X)(\psi_{p-1}) = -\alpha_0$, if $p = 2$. $\mathrm{Ad}(\sigma_X)(\psi_{p-1}) = -\alpha_0 - \psi_1 - \psi_2 = -\mu$, if $p = 3$; and $\mathrm{Ad}(\sigma_X)(\psi_{p-1}) = -\alpha_0 - \psi_1 - 2\psi_2 - \cdots - 2\psi_{p-2} - \psi_{p-1} = -\mu$, if $p > 3$, where μ is the highest root of δ_p with respect to the basis $\{\alpha_0, \psi_1, \psi_2, \ldots, \psi_{p-1}\}$ of the root system of δ_p.

Similarly if $p = 2$, then $\mathrm{Ad}(\sigma_X)(\alpha_0) = -\psi_1$.

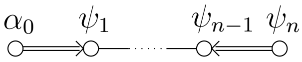
Let $s_X' \in W(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ be such that $\mathrm{Ad}(\sigma_X) \circ s_X'(\Delta_0^+) = \Delta_0^+$, and $s_X = \mathrm{Ad}(\sigma_X) \circ s_X'$. Since $\mathfrak{b}_{n-p}$ does not admit any non-trivial Dynkin diagram automorphism, we have $s_X(\alpha_0) = \psi_1, s_X(\psi_1) = \alpha_0, s_X(\psi_j) = \psi_j$ for all $2 \leq j \leq n, j \neq p$, as in case 3(ii). So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = -1$.

(iii) Finally assume that σ is an involution of $\mathfrak{g}$ of type $(0,\ldots,0,1;1)$. Then $\mathfrak{u}_0 = \mathfrak{so}(2n)$. Define $\gamma_1 = \psi_n$. Then $\{\gamma_1\}$ is a maximal set of strongly orthogonal roots in $\{\alpha \in \Delta^+ : \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$, and $\mathfrak{a} = \mathbb{R}Y_{\gamma_1}$ is a maximal abelian subspace of $\mathfrak{u}_1$. Let $X = \frac{\pi}{2}Y_{\gamma_1}$. Since $\frac{\pi}{2}\psi_j(H^*_{\gamma_1}) = 0$ for $1 \leq j \leq n-2$, $\frac{\pi}{2}\psi_{n-1}(H^*_{\gamma_1}) = -\pi$, and $\frac{\pi}{2}\psi_n(H^*_{\gamma_1}) = \pi$; hence $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$. Now $\mathrm{Ad}(\sigma_X)(H^*_{\gamma_1}) = -H^*_{\gamma_1}$, and $\mathrm{Ad}(\sigma_X)(H) = H$ for all $\{H \in \mathfrak{t}_0^{\mathbb{C}} : \gamma_1(H) = 0\}$. Hence

$$\begin{aligned}
\mathrm{Ad}(\sigma_X)(\psi_j) &= \psi_j \ \text{ for all } 1 \le j \le n-2,\\
\mathrm{Ad}(\sigma_X)(\alpha_0) &= \alpha_0 \ \text{ (as } n > 2\text{), and } \mathrm{Ad}(\sigma_X)(\psi_n) = -\psi_n, \text{ and}\\
\mathrm{Ad}(\sigma_X)(\psi_{n-1}) &= \mathrm{Ad}(\sigma_X)(-\psi_n + \psi_{n-1} + \psi_n) = \psi_n + \psi_{n-1} + \psi_n\\
&= \psi_{n-1} + 2\psi_n = \psi_{n-1} - \alpha_0 - \psi_1 - 2\psi_2 - \cdots - 2\psi_{n-1} = -\mu,
\end{aligned}$$

where μ is the highest root in Δ_0^+. Take $s'_X \in W(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ in such a way that $\mathrm{Ad}(\sigma_X) \circ s'_X(\Delta_0^+) = \Delta_0^+$, and $s_X = \mathrm{Ad}(\sigma_X) \circ s'_X$. Then $s_X(\alpha_0) = \psi_1$, $s_X(\psi_1) = \alpha_0$, $s_X(\psi_j) = \psi_j$ for all $2 \le j \le n-1$, as in case 3(ii). So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = -1$.

5. $\mathfrak{c}_n^{(1)}$: $(n > 1)$



Here $\mathfrak{g} = \mathfrak{c}_n$ and $\alpha_0 + 2\psi_1 + 2\psi_2 + \cdots + 2\psi_{n-1} + \psi_n = 0$.

(i) First assume that σ is an involution of $\mathfrak{g}$ of type $(1, 0, \ldots, 0, 1; 1)$. Then $\mathfrak{u}_0 = \mathfrak{su}(n) \oplus i\mathbb{R}$ and $X(\bar\sigma\bar\theta)$ is a Hermitian symmetric space of tube type. Let

$$\gamma_1 = \psi_n, \gamma_2 = 2\psi_{n-1} + \psi_n, \ldots, \gamma_n = 2\psi_1 + \cdots + 2\psi_{n-1} + \psi_n.$$

In conseqequence $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is a maximal set of strongly orthogonal roots in the set $\{\alpha \in \Delta^+ : \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$. Note that $\psi_{n-1} = \frac{1}{2}(\gamma_2 - \gamma_1)$, $\psi_{n-2} = \frac{1}{2}(\gamma_3 - \gamma_2), \ldots$, $\psi_1 = \frac{1}{2}(\gamma_n - \gamma_{n-1})$. Let $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde Z$ and $\mathrm{Ad}(\sigma_X)(Z) = -Z$, where $Z = \sum_{j=1}^n iH^*_{\gamma_j}$. Then $\mathrm{Ad}(\sigma_X)(\psi_{n-j}) = -\frac{1}{2}(\gamma_{j+1} - \gamma_j) = -\psi_{n-j}$ for all $1 \le j \le n-1$. Let $w^0_{\mathfrak{g}_0} \in W(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ be the longest element that is, $w^0_{\mathfrak{g}_0}(\psi_j) = -\psi_{n-j}$ for all $1 \le j \le n-1$. Then $s_X = \mathrm{Ad}(\sigma_X) \circ w^0_{\mathfrak{g}_0}$ with $s_X(\Delta_0^+) = \Delta_0^+$. Now $s_X(\psi_j) = \psi_{n-j}$ for all $1 \le j \le n-1$. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = -(-1)^{[\frac{n-1}{2}]} = 1$ iff $n \in 4\mathbb{Z}$ or $n \in 3 + 4\mathbb{Z}$.
So if $n \in 4\mathbb{Z}$ or $n \in 3+4\mathbb{Z}$, then the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar\sigma, \bar\sigma\bar\theta$.

(ii) Next assume that σ is an involution of $\mathfrak{g}$ of type $(0, 0, \ldots, 0, s_p, 0, \ldots, 0; 1)$ $(1 \le p \le n-1,\ p \ne n-p)$ with $s_p = 1$. Then $\mathfrak{u}_0 = \mathfrak{sp}(p) \oplus \mathfrak{sp}(n-p)$ and $\mathfrak{g}_0 = \mathfrak{c}_p \oplus \mathfrak{c}_{n-p}$ $(p \ne n-p)$, which does not have any non-trivial Dynkin diagram automorphism. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$ for all $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde Z$. Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar\sigma, \bar\sigma\bar\theta$.

(iii) Finally assume that σ is an involution of $\mathfrak{g}$ of type $(0, \ldots, 0, s_p, 0, \ldots, 0; 1)$ $(s_p = 1)$, where n is even and $p = \frac{n}{2}$. Then $\mathfrak{u}_0 = \mathfrak{sp}(p) \oplus \mathfrak{sp}(p)$. Define $\gamma_1 = \psi_p$, $\gamma_2 = \psi_{p-1} + \psi_p + \psi_{p+1}, \ldots, \gamma_p = \psi_1 + \cdots + \psi_{p-1} + \psi_p + \psi_{p+1} + \cdots + \psi_{n-1}$ (as $n = 2p$). Then $\{\gamma_1, \gamma_2, \ldots, \gamma_p\}$ is a maximal set of strongly orthogonal roots in $\{\alpha \in \Delta^+ : \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$, and $\mathfrak{a} = \sum_{j=1}^p \mathbb{R}Y_{\gamma_j}$ is a maximal abelian subspace of $\mathfrak{u}_1$. Let $X = \sum_{j=1}^p c_j Y_{\gamma_j}$. Then $\exp(-2X) \in \tilde Z$ iff $c_p - c_{p-1}, c_{p-1} - c_{p-2}, \ldots, c_2 - c_1$, $2c_1$, $-2c_p \in \pi\mathbb{Z}$ iff $\cos 2c_j = \cos 2c_1$ for all $1 \le j \le p$, and $\cos 2c_1 = \pm 1$.
Let $X = \sum_{j=1}^p c_j Y_{\gamma_j}$ with $\exp(-2X) \in \tilde Z$ and $\cos 2c_1 = -1$. Then

$$\mathrm{Ad}(\sigma_X)(H^*_{\gamma_j}) = -H^*_{\gamma_j}, \text{ and } \mathrm{Ad}(\sigma_X)(H) = H$$

for all $\{H \in \mathfrak{t}_0^{\mathbb{C}} : \gamma_j(H) = 0 \text{ for all } 1 \le j \le p\}$.

Recall that $\mathfrak{t}^- = \sum_{j=1}^{p} \mathbb{R}(iH^*_{\gamma_j}) \subset \mathfrak{t}_0$, and $\mathfrak{t}^+ = \{H \in \mathfrak{t}_0 : \gamma_j(H) = 0 \text{ for all } 1 \leq j \leq p\}$. Now

$$\psi_1\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \tfrac{1}{2}(\gamma_p - \gamma_{p-1}),\ \ \psi_2\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \tfrac{1}{2}(\gamma_{p-1} - \gamma_{p-2}),\ \ldots, \psi_{p-1}\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \tfrac{1}{2}(\gamma_2 - \gamma_1),$$
$$\psi_{p+1}\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \tfrac{1}{2}(\gamma_2 - \gamma_1),\ \ldots, \psi_{n-1}\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \tfrac{1}{2}(\gamma_p - \gamma_{p-1}),\ \ \psi_n\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = -\gamma_p,$$
$$\alpha_0\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = -\gamma_p.$$

Hence
$$\begin{aligned}\mathrm{Ad}(\sigma_X)(\psi_{p\pm j}) &= \mathrm{Ad}(\sigma_X)(\tfrac{1}{2}(\gamma_{j+1} - \gamma_j) + \psi_{p\pm j} - \tfrac{1}{2}(\gamma_{j+1} - \gamma_j)) \\ &= -\tfrac{1}{2}(\gamma_{j+1} - \gamma_j) + \psi_{p\pm j} - \tfrac{1}{2}(\gamma_{j+1} - \gamma_j) \\ &= \psi_{p\pm j} - (\gamma_{j+1} - \gamma_j) = -\psi_{p\mp j}, \quad \text{for all } 1 \leq j \leq p-1;\end{aligned}$$

$$\mathrm{Ad}(\sigma_X)(\psi_n) = \mathrm{Ad}(\sigma_X)(-\gamma_p + \psi_n + \gamma_p) = \gamma_p + \psi_n + \gamma_p = \psi_n + 2\gamma_p = -\alpha_0, \text{ and}$$
$$\mathrm{Ad}(\sigma_X)(\alpha_0) = \mathrm{Ad}(\sigma_X)(-\gamma_p + \alpha_0 + \gamma_p) = \gamma_p + \alpha_0 + \gamma_p = \alpha_0 + 2\gamma_p = -\psi_n.$$

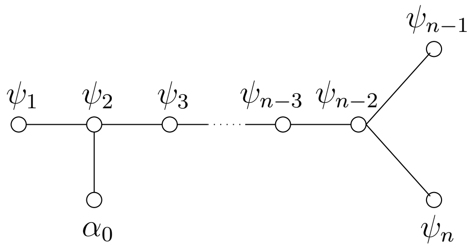
Therefore
$$\begin{aligned}&\mathrm{Ad}(\sigma_X)(\{\alpha_0, \psi_1, \ldots, \psi_{p-1}, \psi_{p+1}, \ldots, \psi_n\}) \\ &\qquad = \{-\alpha_0, -\psi_1, \ldots, -\psi_{p-1}, -\psi_{p+1}, \ldots, -\psi_n\}.\end{aligned}$$

Let $w^0_{\mathfrak{g}_0} \in W(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ be the longest element that is, $w^0_{\mathfrak{g}_0}(\psi_j) = -\psi_j$ for all $1 \leq j \leq n$, $j \neq p$; and $w^0_{\mathfrak{g}_0}(\alpha_0) = -\alpha_0$. Then $s_X = \mathrm{Ad}(\sigma_X) \circ w^0_{\mathfrak{g}_0}$ with $s_X(\Delta_0^+) = \Delta_0^+$.

Now $s_X(\psi_{p\pm j}) = \psi_{p\mp j}$ for all $1 \leq j \leq p-1$, $s_X(\psi_n) = \alpha_0$, $s_X(\alpha_0) = \psi_n$. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = (-1)^p$. Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$ if p is even that is, if $n \in 4\mathbb{Z}$ and $p = \frac{n}{2}$.

6.

$\delta_n^{(1)}$:
$(n > 3)$



Here $\mathfrak{g} = \delta_n$ and $\alpha_0 + \psi_1 + 2\psi_2 + \cdots + 2\psi_{n-2} + \psi_{n-1} + \psi_n = 0$.

(i) Assume that σ is an involution of $\mathfrak{g}$ of type $(1, 0, \ldots, 0, 1; 1)$ (similarly for types $(1, 0, \ldots, 0, 1, 0; 1)$, $(0, 1, 0, \ldots, 0, 1, 0; 1)$, or $(0, 1, 0, \ldots, 0, 1; 1)$). Therefore $\mathfrak{u}_0 = \mathfrak{su}(n) \oplus i\mathbb{R}$ and $X(\bar{\sigma}\bar{\theta})$ is a Hermitian symmetric space. This Hermitian symmetric space is of tube type iff n is even. So if n is odd, then the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$, by Remark 4.5(iv). Now assume that n is even and $r = \frac{n}{2}$. Define

$$\gamma_1 = \psi_n, \quad \gamma_2 = \psi_{n-3} + 2\psi_{n-2} + \psi_{n-1} + \psi_n,$$
$$\gamma_3 = \psi_{n-5} + 2\psi_{n-4} + 2\psi_{n-3} + 2\psi_{n-2} + \psi_{n-1} + \psi_n,\ \ldots,$$
$$\gamma_r = \psi_1 + 2\psi_2 + \cdots + 2\psi_{n-2} + \psi_{n-1} + \psi_n.$$

That is, $\gamma_j = \psi_{n-2j+1} + 2\psi_{n-2j+2} + \cdots + 2\psi_{n-2} + \psi_{n-1} + \psi_n$, for all $2 \leq j \leq r$; and $\gamma_1 = \psi_n$. Then $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ is a maximal set of strongly orthogonal roots in $\{\alpha \in \Delta^+ : \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$. Now

$$\psi_1\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = 0,\ \ \psi_2\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \tfrac{1}{2}(\gamma_r - \gamma_{r-1}),\ \ \psi_3\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = 0,$$
$$\psi_4\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \tfrac{1}{2}(\gamma_{r-1} - \gamma_{r-2}),\ \ldots,\ \psi_{n-2}\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \tfrac{1}{2}(\gamma_2 - \gamma_1),\ \ \psi_{n-1}\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = 0, \text{ i.e.}$$
$$\psi_{2j}\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \tfrac{1}{2}(\gamma_{r-j+1} - \gamma_{r-j}) \text{ for all } 1 \leq j \leq r-1,\ \ \psi_{2j-1}\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = 0 \text{ for all } 1 \leq j \leq r.$$

Let $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$ and $\mathrm{Ad}(\sigma_X)(Z) = -Z$, where $Z = \sum_{j=1}^{r} iH^*_{\gamma_j}$.

Then $\quad \mathrm{Ad}(\sigma_X)(\psi_{2j-1}) = \psi_{2j-1}$, for all $1 \le j \le r$, and

$$\begin{aligned}\mathrm{Ad}(\sigma_X)(\psi_{2j}) &= \mathrm{Ad}(\sigma_X)(\tfrac{1}{2}(\gamma_{r-j+1} - \gamma_{r-j}) + \psi_{2j} - \tfrac{1}{2}(\gamma_{r-j+1} - \gamma_{r-j})) \\ &= -\tfrac{1}{2}(\gamma_{r-j+1} - \gamma_{r-j}) + \psi_{2j} - \tfrac{1}{2}(\gamma_{r-j+1} - \gamma_{r-j}) = \psi_{2j} - (\gamma_{r-j+1} - \gamma_{r-j}) \\ &= -\psi_{2j-1} - \psi_{2j} - \psi_{2j+1}, \quad \text{for all } 1 \le j \le r-1.\end{aligned}$$

So $$\mathrm{Ad}(\sigma_X)(\{\psi_1, \psi_2, \ldots, \psi_{n-1}\}) =$$
$$\{\psi_1, -\psi_1 - \psi_2 - \psi_3, \psi_3, -\psi_3 - \psi_4 - \psi_5, \ldots, \psi_{n-3}, -\psi_{n-3} - \psi_{n-2} - \psi_{n-1}, \psi_{n-1}\}.$$

Let $w^0_{\mathfrak{g}_0} \in W(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ be the longest element and $s'_X = s_{\psi_{n-1}} s_{\psi_{n-3}} \ldots s_{\psi_3} s_{\psi_1} w^0_{\mathfrak{g}_0}$. Then $s'_X(\psi_{2j-1}) = \psi_{n-2j+1}$ for all $1 \le j \le r$, and $s'_X(\psi_{2j}) = -\psi_{n-2j-1} - \psi_{n-2j} - \psi_{n-2j+1}$ for all $1 \le j \le r-1$. Then $s_X = \mathrm{Ad}(\sigma_X) \circ s'_X$ with $s_X(\Delta_0^+) = \Delta_0^+$. Now $s_X(\psi_{2j-1}) = \psi_{n-2j+1}$ for all $1 \le j \le r$, and $s_X(\psi_{2j}) = \psi_{n-2j}$ for all $1 \le j \le r-1$. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = -(-1)^{r-1} = (-1)^r$. Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$ if r is even that is, if $n \in 4\mathbb{Z}$.

(ii) Assume that σ is an involution of $\mathfrak{g}$ of type $(1, 1, 0, \ldots, 0; 1)$ (similarly for type $(0, \ldots, 0, 1, 1; 1)$). Then $\mathfrak{u}_0 = \mathfrak{so}(2n-2) \oplus i\mathbb{R}$ and $X(\bar{\sigma}\bar{\theta})$ is a Hermitian symmetric space of tube type. Let

$$\gamma_1 = \psi_1, \gamma_2 = \psi_1 + 2\psi_2 + \cdots + 2\psi_{n-2} + \psi_{n-1} + \psi_n.$$

Then $\{\gamma_1, \gamma_2\}$ is a maximal set of strongly orthogonal roots in $\{\alpha \in \Delta^+ : \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$ and so $\mathfrak{a} = \mathbb{R}Y_{\gamma_1} \oplus \mathbb{R}Y_{\gamma_2}$ is a maximal abelian subspace of $\mathfrak{u}_1$.

Now $\psi_2\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = \frac{1}{2}(\gamma_2 - \gamma_1), \psi_j\big|_{(\mathfrak{t}^-)^{\mathbb{C}}} = 0$ for all $3 \le j \le n$.

Let $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$ and $\mathrm{Ad}(\sigma_X)(Z) = -Z$, where $Z = i(H^*_{\gamma_1} + H^*_{\gamma_2})$.

Then $\quad \mathrm{Ad}(\sigma_X)(\psi_j) = \psi_j$ for all $3 \le j \le n$, and

$$\begin{aligned}\mathrm{Ad}(\sigma_X)(\psi_2) &= \mathrm{Ad}(\sigma_X)(\tfrac{1}{2}(\gamma_2 - \gamma_1) + \psi_2 - \tfrac{1}{2}(\gamma_2 - \gamma_1)) \\ &= -\tfrac{1}{2}(\gamma_2 - \gamma_1) + \psi_2 - \tfrac{1}{2}(\gamma_2 - \gamma_1) = \psi_2 - (\gamma_2 - \gamma_1) \\ &= \psi_2 - (2\psi_2 + \cdots + 2\psi_{n-2} + \psi_{n-1} + \psi_n) = -\mu,\end{aligned}$$

where μ is the highest root in Δ_0^+.

Let $s'_X \in W(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ be such that $\mathrm{Ad}(\sigma_X) \circ s'_X(\Delta_0^+) = \Delta_0^+$, and $s_X = \mathrm{Ad}(\sigma_X) \circ s'_X$. Then $s_X(\psi_{n-1}) = \psi_n, s_X(\psi_n) = \psi_{n-1}, s_X(\psi_j) = \psi_j$ for all $2 \le j \le n-2$, as in case 3(ii). So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = -1 \times -1 = 1$. Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$.

(iii) Next assume that σ is an involution of $\mathfrak{g}$ of type $(0, 0, \ldots, 0, s_p, 0, \ldots, 0; 1)$ $(2 \le p \le n-2,\ p \le n-p)$ with $s_p = 1$. Then $\mathfrak{u}_0 = \mathfrak{so}(2p) \oplus \mathfrak{so}(2n-2p)$. Define

$$\begin{aligned}&\gamma_1 = \psi_p, \gamma'_1 = (\psi_p + \cdots + \psi_{n-2}) + (\psi_{p+1} + \cdots + \psi_n), \\ &\gamma_2 = \psi_{p-1} + \psi_p + \psi_{p+1}, \gamma'_2 = (\psi_{p-1} + \cdots + \psi_{n-2}) + (\psi_{p+2} + \cdots + \psi_n), \ldots, \\ &\gamma_p = \psi_1 + \cdots + \psi_p + \cdots + \psi_{2p-1}, \\ &\gamma'_p = (\psi_1 + \cdots + \psi_{n-2}) + (\psi_{2p} + \cdots + \psi_n).\end{aligned}$$

In consequence $\{\gamma_1, \gamma_1', \ldots, \gamma_p, \gamma_p'\}$ is a maximal set of strongly orthogonal roots in $\{\alpha \in \Delta^+ : \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$, and $\mathfrak{a} = \sum_{j=1}^{p}(\mathbb{R}Y_{\gamma_j} + \mathbb{R}Y_{\gamma_j'})$ is a maximal abelian subspace of $\mathfrak{u}_1$. Let $X = \sum_{j=1}^{p}(c_j Y_{\gamma_j} + c_j' Y_{\gamma_j'}) \in \mathfrak{a}$. Then $\exp(-2X) \in \tilde{Z}$ iff

$$c_p + c_p' - c_{p-1} - c_{p-1}',\ c_{p-1} + c_{p-1}' - c_{p-2} - c_{p-2}',\ \ldots,\ c_2 + c_2' - c_1 - c_1',$$
$$2c_1,\ -c_1 + c_1' + c_2 - c_2', -c_2 + c_2' + c_3 - c_3',\ \ldots,\ -c_{p-1} + c_{p-1}' + c_p - c_p' \in \pi\mathbb{Z}$$

and $-c_p + c_p'$ (respectively, $-c_{p-1} + c_{p-1}' - c_p + c_p') \in \pi\mathbb{Z}$, if $p < n-p$ (respectively, $p = n-p$). This is true iff $\cos 2c_j = \cos 2c_j' = \pm 1$ (respectively, $\cos 2c_j = \pm 1, \cos 2c_j' = \pm 1$) for all $1 \le j \le p$, if $p < n-p$ (respectively, $p = n-p$).

Let $\exp(-2X) \in \tilde{Z}$ and $p < n-p$. Then for $1 \le j \le p$, either $\mathrm{Ad}(\sigma_X)(\gamma_j) = \gamma_j, \mathrm{Ad}(\sigma_X)(\gamma_j') = \gamma_j'$, or $\mathrm{Ad}(\sigma_X)(\gamma_j) = -\gamma_j, \mathrm{Ad}(\sigma_X)(\gamma_j') = -\gamma_j'$. Thus for $1 \le j \le p$,

either $$\mathrm{Ad}(\sigma_X)(\frac{\gamma_j + \gamma_j'}{2}) = \frac{\gamma_j + \gamma_j'}{2},\quad \mathrm{Ad}(\sigma_X)(\frac{\gamma_j' - \gamma_j}{2}) = \frac{\gamma_j' - \gamma_j}{2},$$

or $$\mathrm{Ad}(\sigma_X)(\frac{\gamma_j + \gamma_j'}{2}) = -\frac{\gamma_j + \gamma_j'}{2},\quad \mathrm{Ad}(\sigma_X)(\frac{\gamma_j' - \gamma_j}{2}) = -\frac{\gamma_j' - \gamma_j}{2}.$$

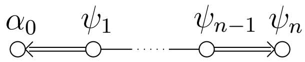
Now $\mathfrak{g}_0 = \delta_p \oplus \delta_{n-p}$ and $\mathrm{Ad}(\sigma_X)(\delta_p) = \delta_p, \mathrm{Ad}(\sigma_X)(\delta_{n-p}) = \delta_{n-p}$. So $\mathrm{Ad}(\sigma_X)$ is an inner automorphism of $\mathfrak{g}_0$ iff $|\{j : \cos 2c_j = -1\}|$ is even [4, Planche IV]. So if $s_X' \in W(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ be such that $\mathrm{Ad}(\sigma_X) \circ s_X'(\Delta_0^+) = \Delta_0^+$, and $s_X = \mathrm{Ad}(\sigma_X) \circ s_X'$, then either $s_X(\alpha_0) = \alpha_0, s_X(\psi_j) = \psi_j$ for all $1 \le j \le n, j \ne p$; or $s_X(\alpha_0) = \psi_1,\ s_X(\psi_1) = \alpha_0$, $s_X(\psi_{n-1}) = \psi_n,\ s_X(\psi_n) = \psi_{n-1},\ s_X(\psi_j) = \psi_j$ for all $2 \le j \le n-2,\ j \ne p$. In any case, $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$. Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$.

Let $p = n-p$ that is $n = 2p$. Then $\mathfrak{g}_0$ is the sum of two ideals, each is isomorphic with δ_p. Let $\delta_p^{(1)}$ be the ideal of $\mathfrak{g}_0$ whose Dynkin diagram is generated by $\{\alpha_0, \psi_1, \ldots, \psi_{p-1}\}$, and $\delta_p^{(2)}$ be the ideal of $\mathfrak{g}_0$ whose Dynkin diagram is generated by $\{\psi_{p+1}, \ldots, \psi_n\}$. Let $\exp(-2X) \in \tilde{Z}$. If $\cos 2c_j = \cos 2c_j'$ for all $1 \le j \le p$, then as before $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$. If $\cos 2c_j \ne \cos 2c_j'$ for some j, then

$$\mathrm{Ad}(\sigma_X)(\tfrac{\gamma_j + \gamma_j'}{2}) = \pm\tfrac{\gamma_j' - \gamma_j}{2},\quad \mathrm{Ad}(\sigma_X)(\tfrac{\gamma_j' - \gamma_j}{2}) = \pm\tfrac{\gamma_j + \gamma_j'}{2}.$$

Hence $\mathrm{Ad}(\sigma_X)(\delta_p^{(1)}) = \delta_p^{(2)}$, and so $\mathrm{Ad}(\sigma_X)$ is not an inner automorphism of $\mathfrak{g}_0$. Therefore s_X induces a non-trivial Dynkin diagram automorphism of $\mathfrak{g}_0$. Since $\mathrm{Ad}(\sigma_X)(\delta_p^{(1)}) = \delta_p^{(2)}$, we have $s_X(\psi_j) = \psi_{n-j}$ for all $2 \le j \le n-2,\ j \ne p,\ s_X(\alpha_0) = \psi_{n-1}$ or $\psi_n,\ s_X(\psi_1) = \psi_n$ or $\psi_{n-1},\ s_X(\psi_{n-1}) = \alpha_0$ or $\psi_1, s_X(\psi_n) = \psi_1$ or α_0. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = (-1)^p$. Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$ if p is even that is, if $n \in 4\mathbb{Z}$ and $p = \frac{n}{2}$.

7. $\delta_{n+1}^{(2)}$: $(n > 1)$

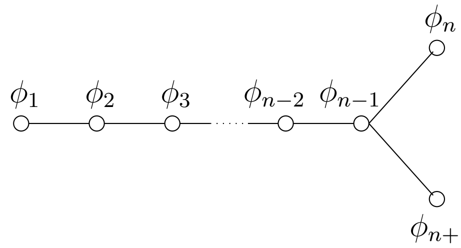


Here $\mathfrak{g} = \delta_{n+1}(n > 1)$, and $\alpha_0 + \psi_1 + \psi_2 + \cdots + \psi_{n-1} + \psi_n = 0$.

(i) First assume that σ is an involution of $\mathfrak{g}$ of type $(0, \ldots, 0, s_p, 0, \ldots, 0; 2)$ with $s_p = 1$, where $0 \le p \le n,\ p \ne n-p$. Then $\mathfrak{u}_0 = \mathfrak{so}(2p+1) \oplus \mathfrak{so}(2n-2p+1)$ and $\mathfrak{g}_0 = \mathfrak{b}_p \oplus \mathfrak{b}_{n-p}$ $(p \ne n-p)$, which does not have any non-trivial Dynkin diagram automorphism. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$ for all $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$. Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$.

(ii) Next assume that $n > 2$ is even and σ is an involution of type $(0, \ldots, 0, s_p, 0, \ldots, 0; 2)$ with $p = \frac{n}{2}$ and $s_p = 1$. Then $\mathfrak{g}_0$ is the sum of two ideals, each is isomorphic with $\mathfrak{b}_p$. Let $\mathfrak{b}_p^{(1)}$ be the ideal of $\mathfrak{g}_0$ whose Dynkin diagram is generated by $\{\alpha_0, \psi_1, \ldots, \psi_{p-1}\}$, and $\mathfrak{b}_p^{(2)}$ be the ideal of $\mathfrak{g}_0$ whose Dynkin diagram is generated by $\{\psi_{p+1}, \ldots, \psi_n\}$. The diagram $\delta_{n+1}^{(2)}(n > 2)$ is corresponding to the Dynkin diagram automorphism $\bar{\nu}$ of δ_{n+1} given by $\bar{\nu}(\phi_j) = \phi_j$ for all $1 \le j \le n-1$, $\bar{\nu}(\phi_n) = \phi_{n+1}$, $\bar{\nu}(\phi_{n+1}) = \phi_n$.

δ_{n+1} :
$(n > 2)$



Now we want to determine $\{\alpha \in \Delta^+ : \sigma(H_\alpha^*) = H_\alpha^*,\ \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$. Note that

$$\begin{aligned}&\{\alpha \in \Delta^+ : \sigma(H_\alpha^*) = H_\alpha^*\} = \{\alpha \in \Delta^+ : n_{\phi_n}(\alpha) = n_{\phi_{n+1}}(\alpha)\}\\ &= \{\phi_i + \cdots + \phi_{j-1},\ (\phi_i + \cdots \phi_{n-1}) + (\phi_j + \cdots + \phi_{n+1}) : 1 \le i < j \le n\}.\end{aligned}$$

Again since $\sigma|_{\mathfrak{h}} = \nu|_{\mathfrak{h}}$, $\{\alpha \in \Delta^+ : \sigma(H_\alpha^*) = H_\alpha^*\} \subset (\mathfrak{h}^\nu)^*$. Let E_j be a non-zero root vector corresponding to the root ϕ_j for all $1 \le j \le n+1$. Define

$$\begin{aligned}E_{ij} &= [E_i, \ldots, E_{j-1}] \text{ for all } 1 \le i < j-1 < j \le n+1,\\ E_{ij} &= E_i \text{ for all } 1 \le i = j-1 < j \le n+1; \text{ and}\\ E'_{i(n+1)} &= [E_i, \ldots, E_{n-1}, E_{n+1}] \text{ for all } 1 \le i \le n-1, \quad E'_{n(n+1)} = E_{n+1}.\end{aligned}$$

Then $E_{ij} \neq 0$, $E'_{i(n+1)} \neq 0$, $\nu(E_{ij}) = E_{ij}$ (for all $1 \le i < j \le n$), and further $\nu(E_{i(n+1)}) = E'_{i(n+1)}$, $\nu(E'_{i(n+1)}) = E_{i(n+1)}$ (for all $1 \le i \le n$). Define

$$E^{ij} = \left[E_{j(n+1)}, [E_{ij}, E'_{j(n+1)}]\right] \text{ for all } 1 \le i < j \le n.$$

Then $E^{ij} \neq 0$, $\nu(E^{ij}) = E^{ij}$ (for all $1 \le i < j \le n$). This shows that $\mathfrak{g}_\alpha \subset \mathfrak{g}_0^\nu$ for all $\alpha \in \{\alpha \in \Delta^+ : \sigma(H_\alpha^*) = H_\alpha^*\}$. Thus $\{\alpha \in \Delta^+ : \sigma(H_\alpha^*) = H_\alpha^*\} \subset \Delta(\mathfrak{g}_0^\nu, \mathfrak{h}^\nu)$.

Now $\sigma(\mathfrak{g}_0^\nu) = \mathfrak{g}_0^\nu$ and so $\mathfrak{g}_0^\nu = \mathfrak{k}^\nu \oplus \mathfrak{p}^\nu$, where $\mathfrak{k}^\nu = \mathfrak{g}_0^\nu \cap \mathfrak{g}_0$, $\mathfrak{p}^\nu = \mathfrak{g}_0^\nu \cap \mathfrak{g}_1$.

Since $\mathfrak{h}^\nu \subset \mathfrak{k}^\nu$, $[\mathfrak{k}^\nu, \mathfrak{k}^\nu] \subset \mathfrak{k}^\nu$, $[\mathfrak{k}^\nu, \mathfrak{p}^\nu] \subset \mathfrak{p}^\nu$, and $(\mathfrak{g}_0^\nu)_\alpha$ is one-dimensional for all $\alpha \in \Delta(\mathfrak{g}_0^\nu, \mathfrak{h}^\nu)$; we have $(\mathfrak{g}_0^\nu)_\alpha \subset \mathfrak{k}^\nu$ or $\mathfrak{p}^\nu$. Thus

$$\begin{aligned}\{\alpha \in \Delta^+ : \sigma(H_\alpha^*) = H_\alpha^*,\ \mathfrak{g}_\alpha \subset \mathfrak{g}_1\} &= \{\alpha \in \Delta^+ : \sigma(H_\alpha^*) = H_\alpha^*,\ (\mathfrak{g}_0^\nu)_\alpha \subset \mathfrak{p}^\nu\}\\ &= \{\alpha \in \Delta^+ : n_{\phi_n}(\alpha) = n_{\phi_{n+1}}(\alpha), \text{ and } n_{\phi_p}(\alpha) \text{ is odd}\}.\end{aligned}$$

Let
$$\begin{aligned}&\gamma_1 = \phi_p, \gamma'_1 = (\phi_p + \cdots + \phi_{n-1}) + (\phi_{p+1} + \cdots + \phi_{n+1}),\\ &\gamma_2 = \phi_{p-1} + \phi_p + \phi_{p+1}, \gamma'_2 = (\phi_{p-1} + \cdots + \phi_{n-1}) + (\phi_{p+2} + \cdots + \phi_{n+1}), \ldots,\\ &\gamma_p = \phi_1 + \cdots + \phi_p + \cdots + \phi_{n-1}, \gamma'_p = (\phi_1 + \cdots + \phi_{n-1}) + (\phi_n + \phi_{n+1}).\end{aligned}$$

Then $\{\gamma_1, \gamma'_1, \gamma_2, \gamma'_2, \ldots, \gamma_p, \gamma'_p\}$ is a maximal set of strongly orthogonal roots in $\{\alpha \in \Delta^+ : \sigma(H_\alpha^*) = H_\alpha^*,\ \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$, and $\mathfrak{a} = \mathbb{R}i(H_{\phi_n}^* - H_{\phi_{n+1}}^*) \oplus \sum_{j=1}^p (\mathbb{R}Y_{\gamma_j} + \mathbb{R}Y_{\gamma'_j})$ is a maximal abelian subspace of $\mathfrak{u}_1$.

Let $X = ic_0(H^*_{\phi_n} - H^*_{\phi_{n+1}}) + \sum_{j=1}^{p}(c_j Y_{\gamma_j} + c'_j Y_{\gamma'_j}) \in \mathfrak{a}$. Then $\exp(-2X) \in \tilde{Z}$

iff $c_p + c'_p - c_{p-1} - c'_{p-1},\ c_{p-1} + c'_{p-1} - c_{p-2} - c'_{p-2}, \ldots, c_2 + c'_2 - c_1 - c'_1,\ 2c_1,$
$-c_1 + c'_1 + c_2 - c'_2, -c_2 + c'_2 + c_3 - c'_3, \ldots, -c_{p-1} + c'_{p-1} + c_p - c'_p,\ 2c_0 - c_p + c'_p,$
$-2c_0 - c_p + c'_p \in \pi\mathbb{Z}$

iff $2c_0 \in \pi\mathbb{Z},\ \cos 2c_j = \pm 1,\ \cos 2c'_j = \pm 1$ for all $1 \leq j \leq p$.

If $\cos 2c_j = \cos 2c'_j$ for all $1 \leq j \leq p$, then either $\mathrm{Ad}(\sigma_X)(\gamma_j) = \gamma_j, \mathrm{Ad}(\sigma_X)(\gamma'_j) = \gamma'_j$, or $\mathrm{Ad}(\sigma_X)(\gamma_j) = -\gamma_j,\ \mathrm{Ad}(\sigma_X)(\gamma'_j) = -\gamma'_j$. Thus for $1 \leq j \leq p$,

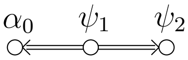
either $$\mathrm{Ad}(\sigma_X)(\frac{\gamma_j + \gamma'_j}{2}) = \frac{\gamma_j + \gamma'_j}{2},\quad \mathrm{Ad}(\sigma_X)(\frac{\gamma'_j - \gamma_j}{2}) = \frac{\gamma'_j - \gamma_j}{2},$$

or $$\mathrm{Ad}(\sigma_X)(\frac{\gamma_j + \gamma'_j}{2}) = -\frac{\gamma_j + \gamma'_j}{2},\quad \mathrm{Ad}(\sigma_X)(\frac{\gamma'_j - \gamma_j}{2}) = -\frac{\gamma'_j - \gamma_j}{2}.$$

Now $\mathfrak{g}_0 = \mathfrak{b}_p^{(1)} \oplus \mathfrak{b}_p^{(2)}$ and $\mathrm{Ad}(\sigma_X)(\mathfrak{b}_p^{(1)}) = \mathfrak{b}_p^{(1)}$ and $\mathrm{Ad}(\sigma_X)(\mathfrak{b}_p^{(2)}) = \mathfrak{b}_p^{(2)}$. Since $\mathfrak{b}_p^{(1)} \cong \mathfrak{b}_p \cong \mathfrak{b}_p^{(2)}$ does not admit any non-trivial Dynkin diagram automorphism, we have $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$.

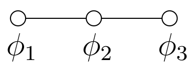
If $\cos 2c_j \neq \cos 2c'_j$ for some j, then it follows that $\mathrm{Ad}(\sigma_X)(\frac{\gamma_j+\gamma'_j}{2}) = \pm\frac{\gamma'_j-\gamma_j}{2}$ and $\mathrm{Ad}(\sigma_X)(\frac{\gamma'_j-\gamma_j}{2}) = \pm\frac{\gamma_j+\gamma'_j}{2}$. Hence $\mathrm{Ad}(\sigma_X)(\mathfrak{b}_p^{(1)}) = \mathfrak{b}_p^{(2)}$, and so $\mathrm{Ad}(\sigma_X)$ is not an inner automorphism of $\mathfrak{g}_0$. Therefore s_X induces a non-trivial Dynkin diagram automorphism of $\mathfrak{g}_0$. Since $\mathrm{Ad}(\sigma_X)(\mathfrak{b}_p^{(1)}) = \mathfrak{b}_p^{(2)}$, we have $s_X(\psi_j) = \psi_{n-j}$ for all $1 \leq j \leq n-1,\ j \neq p,\ s_X(\alpha_0) = \psi_n, s_X(\psi_n) = \alpha_0$. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = (-1)^p$. Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$ if p is even that is, if $n \in 4\mathbb{Z}$ and $p = \frac{n}{2}$.

(iii) $\delta_3^{(2)}$:



Finally assume that $n = 2$ and σ is an involution of type $(0, 1, 0; 2)$. Then $\mathfrak{g}_0$ is the sum of two ideals, each is isomorphic with $\mathfrak{a}_1$. Let $\mathfrak{a}_1^{(1)}$ be the ideal of $\mathfrak{g}_0$ whose Dynkin diagram is generated by $\{\alpha_0\}$, and $\mathfrak{a}_1^{(2)}$ be the ideal of $\mathfrak{g}_0$ whose Dynkin diagram is generated by $\{\psi_2\}$. The diagram $\delta_3^{(2)}$ is corresponding to the Dynkin diagram automorphism $\bar{\nu}$ of δ_3 given by $\bar{\nu}(\phi_j) = \phi_{4-j}$ for all $1 \leq j \leq 3$.

δ_3 :



Now $\{\alpha \in \Delta^+ : \sigma(H^*_\alpha) = H^*_\alpha,\ \mathfrak{g}_\alpha \subset \mathfrak{g}_1\} = \{\alpha \in \Delta^+ : n_{\phi_1}(\alpha) = n_{\phi_3}(\alpha)$, and $n_{\phi_2}(\alpha)$ is odd$\}$ (as in the case 7(ii)), since $\psi_1 = \phi_2|_{\mathfrak{h}^\nu}$. Let $\gamma_1 = \phi_2, \gamma_2 = \phi_1 + \phi_2 + \phi_3$. Then $\{\gamma_1, \gamma_2\}$ is a maximal set of strongly orthogonal roots in $\{\alpha \in \Delta^+ : \sigma(H^*_\alpha) = H^*_\alpha,\ \mathfrak{g}_\alpha \subset \mathfrak{g}_1\}$, and $\mathfrak{a} = \mathbb{R}i(H^*_{\phi_1} - H^*_{\phi_3}) \oplus \mathbb{R}Y_{\gamma_1} + \mathbb{R}Y_{\gamma_2}$ is a maximal abelian subspace of $\mathfrak{u}_1$.

Let $X = ic_0(H^*_{\phi_1} - H^*_{\phi_3}) + c_1 Y_{\gamma_1} + c_2 Y_{\gamma_2} \in \mathfrak{a}$. Then we have $\exp(-2X) \in \tilde{Z}$ iff $2c_1$, $2c_0 - c_1 + c_2$, $-2c_0 - c_1 + c_2 \in \pi\mathbb{Z}$ iff $2c_0 \in \pi\mathbb{Z},\ \cos 2c_1 = \pm 1,\ \cos 2c_2 = \pm 1$.

Let $\cos 2c_1 = -1$, and $\cos 2c_2 = 1$.

Then $\mathrm{Ad}(\sigma_X)(\psi_1) = -\psi_1$,

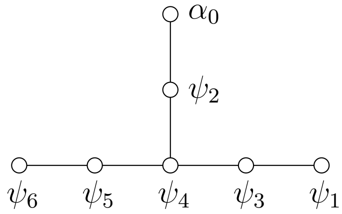
$$\mathrm{Ad}(\sigma_X)(\alpha_0) = \mathrm{Ad}(\sigma_X)(-\frac{\psi_1}{2} + \alpha_0 + \frac{\psi_1}{2}) = \frac{\psi_1}{2} + \alpha_0 + \frac{\psi_1}{2} = \alpha_0 + \psi_1 = -\psi_2,$$

and $\mathrm{Ad}(\sigma_X)(\psi_2) = -\alpha_0$ similarly.

Let $w^0_{\mathfrak{g}_0} \in W(\mathfrak{g}_0, \mathfrak{t}_0^{\mathbb{C}})$ be the longest element, that is, we have $w^0_{\mathfrak{g}_0}(\alpha_0) = -\alpha_0$ and $w^0_{\mathfrak{g}_0}(\psi_2) = -\psi_2$. Then $s_X = \mathrm{Ad}(\sigma_X) \circ w^0_{\mathfrak{g}_0}$ with $s_X(\Delta_0^+) = \Delta_0^+$. Now $s_X(\alpha_0) = \psi_2$ and $s_X(\psi_2) = \alpha_0$. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = -1$.

8.

$\mathfrak{e}_6^{(1)}$:



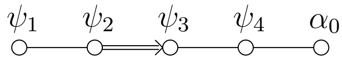
Here $\mathfrak{g} = \mathfrak{e}_6$ and $\alpha_0 + \psi_1 + 2\psi_2 + 2\psi_3 + 3\psi_4 + 2\psi_5 + \psi_6 = 0$.

(i) First assume that σ is an involution of $\mathfrak{g}$ of type $(1,1,0,0,0,0,0;1)$ (similarly for types $(1,0,0,0,0,0,1;1)$ or $(0,1,0,0,0,0,1;1)$). Then $\mathfrak{u}_0 = \mathfrak{so}(10) \oplus i\mathbb{R}$ and $X(\bar{\sigma}\bar{\theta})$ is a Hermitian symmetric space. This Hermitian symmetric space is not of tube type. So the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$, by Remark 4.5(iv).

(ii) Next assume that σ is an involution of $\mathfrak{g}$ of type $(0,0,1,0,0,0,0;1)$ (similarly for types $(0,0,0,1,0,0,0;1)$ or $(0,0,0,0,0,1,0;1)$). Then $\mathfrak{u}_0 = \mathfrak{su}(2) \oplus \mathfrak{su}(6)$ and $\mathfrak{g}_0 = \mathfrak{a}_1 \oplus \mathfrak{a}_5$, which has only one non-trivial Dynkin diagram automorphism namely, $\alpha_0 \mapsto \alpha_0, \psi_1 \mapsto \psi_6, \psi_3 \mapsto \psi_5, \psi_4 \mapsto \psi_4, \psi_5 \mapsto \psi_3, \psi_6 \mapsto \psi_1$; and this is an even permutation. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$ for all $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$. Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$.

9.

$\mathfrak{e}_6^{(2)}$:



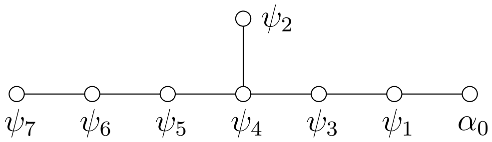
Here $\mathfrak{g} = \mathfrak{e}_6$, and $\alpha_0 + \psi_1 + 2\psi_2 + 3\psi_3 + 2\psi_4 = 0$.

(i) First assume that σ is an involution of $\mathfrak{g}$ of type $(1,0,0,0,0;2)$. Then $\mathfrak{g}_0 = \mathfrak{f}_4$, which does not have any non-trivial Dynkin diagram automorphism. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$ for all $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$. Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$.

(ii) Next assume that σ is an involution of $\mathfrak{g}$ of type $(0,1,0,0,0;2)$. Then $\mathfrak{g}_0 = \mathfrak{c}_4$, which does not have any non-trivial Dynkin diagram automorphism. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$ for all $X \in \mathfrak{a}$ with $\exp(-2X) \in \tilde{Z}$. Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$.

10.

$\mathfrak{e}_7^{(1)}$:



Here $\mathfrak{g} = \mathfrak{e}_7$ and $\alpha_0 + 2\psi_1 + 2\psi_2 + 3\psi_3 + 4\psi_4 + 3\psi_5 + 2\psi_6 + \psi_7 = 0$.

(i) First assume that σ is an involution of $\mathfrak{g}$ of type $(1,0,0,0,0,0,0,1;1)$. Then $\mathfrak{g}_0 = \mathfrak{e}_6 \oplus \mathbb{C}$ and $X(\bar{\sigma}\bar{\theta})$ is a Hermitian symmetric space of tube type. Now $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{e}_6$, which has only one non-trivial Dynkin diagram automorphism namely, $\psi_1 \mapsto \psi_6, \psi_2 \mapsto \psi_2, \psi_3 \mapsto \psi_5, \psi_4 \mapsto \psi_4, \psi_5 \mapsto \psi_3, \psi_6 \mapsto \psi_1$; and this is an even permutation.

Let $$\gamma_1=\psi_7,\quad \gamma_2=\psi_2+\psi_3+2\psi_4+2\psi_5+2\psi_6+\psi_7,$$
$$\gamma_3=2\psi_1+2\psi_2+3\psi_3+4\psi_4+3\psi_5+2\psi_6+\psi_7.$$

Then $\{\gamma_1,\gamma_2,\gamma_3\}$ is a maximal set of strongly orthogonal roots in $\{\alpha\in\Delta^+:\mathfrak{g}_\alpha\subset\mathfrak{g}_1\}$. Let $X\in\mathfrak{a}$ with $\exp(-2X)\in\tilde{Z}$ and $\mathrm{Ad}(\sigma_X)(Z)=-Z$, where $Z=\sum_{j=1}^{3} iH^*_{\gamma_j}$. Then $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}})=-1$, even if s_X induces the non-trivial Dynkin diagram automorphism of $[\mathfrak{g}_0,\mathfrak{g}_0]$.

(ii) Next assume that σ is an involution of $\mathfrak{g}$ of type $(0,0,1,0,0,0,0,0;1)$. Then $\mathfrak{u}_0=\mathfrak{su}(8)$. Define

$\gamma_1=\psi_2,\ \gamma_2=\psi_2+\psi_3+2\psi_4+\psi_5,\ \gamma_3=\psi_1+\psi_2+\psi_3+2\psi_4+\psi_5+\psi_6,$
$\gamma_4=\psi_1+\psi_2+2\psi_3+2\psi_4+2\psi_5+\psi_6,\ \gamma_5=\psi_2+\psi_3+2\psi_4+2\psi_5+2\psi_6+\psi_7,$
$\gamma_6=\psi_1+\psi_2+\psi_3+2\psi_4+2\psi_5+\psi_6+\psi_7, \gamma_7=\psi_1+\psi_2+2\psi_3+2\psi_4+\psi_5+\psi_6+\psi_7.$

In consequence $\{\gamma_1,\gamma_2,\dots,\gamma_7\}$ is a maximal set of strongly orthogonal roots in the set $\{\alpha\in\Delta^+:\mathfrak{g}_\alpha\subset\mathfrak{g}_1\}$, and $\mathfrak{a}=\sum_{j=1}^{7}\mathbb{R}Y_{\gamma_j}$ is a maximal abelian subspace of $\mathfrak{u}_1$. Also we have

$\psi_1=\frac12(-\gamma_2+\gamma_3-\gamma_5+\gamma_6),\ \psi_3=\frac12(-\gamma_3+\gamma_4-\gamma_6+\gamma_7),\ \psi_4=\frac12(-\gamma_1+\gamma_2+\gamma_3-\gamma_4),$
$\psi_5=\frac12(-\gamma_3+\gamma_4+\gamma_6-\gamma_7),\ \psi_6=\frac12(-\gamma_2+\gamma_3+\gamma_5-\gamma_6),\ \psi_7=\frac12(-\gamma_3-\gamma_4+\gamma_6+\gamma_7),$
$\alpha_0=\frac12(-\gamma_3-\gamma_4-\gamma_6-\gamma_7).$

Let $X=\frac{\pi}{2}(Y_{\gamma_5}+Y_{\gamma_6}+Y_{\gamma_7})$. Then $\frac{\pi}{2}\psi_j(H^*_{\gamma_5}+H^*_{\gamma_6}+H^*_{\gamma_7})=0$ for $1\le j\le 6$, and $\frac{\pi}{2}\psi_7(H^*_{\gamma_5}+H^*_{\gamma_6}+H^*_{\gamma_7})=\pi$; hence $X\in\mathfrak{a}$ with $\exp(-2X)\in\tilde{Z}$.

Now $\mathrm{Ad}(\sigma_X)(H^*_{\gamma_j})=H^*_{\gamma_j}$ for all $1\le j\le 4$, and $\mathrm{Ad}(\sigma_X)(H^*_{\gamma_j})=-H^*_{\gamma_j}$ for all $5\le j\le 7$. Thus

$$\mathrm{Ad}(\sigma_X)(\alpha_0)=\psi_7,\quad \mathrm{Ad}(\sigma_X)(\psi_1)=\psi_6,\quad \mathrm{Ad}(\sigma_X)(\psi_3)=\psi_5,$$
$$\mathrm{Ad}(\sigma_X)(\psi_4)=\psi_4,\ \mathrm{Ad}(\sigma_X)(\psi_5)=\psi_3,\ \mathrm{Ad}(\sigma_X)(\psi_6)=\psi_1,\ \mathrm{Ad}(\sigma_X)(\psi_7)=\alpha_0.$$

Therefore $\mathrm{Ad}(\sigma_X)(\Delta_0^+)=\Delta_0^+$, and $s_X=\mathrm{Ad}(\sigma_X)$. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}})=-1$.

(iii) Assume that σ is an involution of $\mathfrak{g}$ of type $(0,1,0,0,0,0,0,0;1)$ (similarly for type $(0,0,0,0,0,0,1,0;1)$). Then $\mathfrak{u}_0=\mathfrak{su}(2)\oplus\mathfrak{so}(12)$. Define

$\gamma_1=\psi_1,\ \gamma_2=\psi_1+\psi_2+2\psi_3+2\psi_4+\psi_5,\ \gamma_3=\psi_1+\psi_2+2\psi_3+2\psi_4+2\psi_5+2\psi_6+\psi_7,$
$\gamma_4=\psi_1+2\psi_2+2\psi_3+4\psi_4+3\psi_5+2\psi_6+\psi_7.$

In consequence $\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$ is a maximal set of strongly orthogonal roots in the set $\{\alpha\in\Delta^+:\mathfrak{g}_\alpha\subset\mathfrak{g}_1\}$, and $\mathfrak{a}=\sum_{j=1}^{4}\mathbb{R}Y_{\gamma_j}$ is a maximal abelian subspace of $\mathfrak{u}_1$.

Now $$\alpha_0\big|_{(\mathfrak{t}^-)^{\mathbb{C}}}=-\frac{1}{2}(\gamma_1+\gamma_2+\gamma_3+\gamma_4),\ \psi_2\big|_{(\mathfrak{t}^-)^{\mathbb{C}}},\ \psi_5\big|_{(\mathfrak{t}^-)^{\mathbb{C}}},\ \psi_7\big|_{(\mathfrak{t}^-)^{\mathbb{C}}}=0,$$
$$\psi_3\big|_{(\mathfrak{t}^-)^{\mathbb{C}}}=\tfrac12(-\gamma_1+\gamma_2+\gamma_3-\gamma_4),\ \psi_4\big|_{(\mathfrak{t}^-)^{\mathbb{C}}}=\tfrac12(-\gamma_3+\gamma_4),\ \psi_6\big|_{(\mathfrak{t}^-)^{\mathbb{C}}}=\tfrac12(-\gamma_2+\gamma_3).$$

Let $X=\sum_{j=1}^{4}c_jY_{\gamma_j}\in\mathfrak{a}$. Then $\exp(-2X)\in\tilde{Z}$ iff $2c_1$, $-c_1+c_2+c_3-c_4$, $-c_3+c_4$, $-c_2+c_3\in\pi\mathbb{Z}$ iff $\cos 2c_j=\cos 2c_1=\pm1$ for all $1\le j\le 4$. We assume now that $X=\sum_{j=1}^{4}c_jY_{\gamma_j}$ with $\exp(-2X)\in\tilde{Z}$ and $\cos 2c_1=-1$. Then $\mathrm{Ad}(\sigma_X)(H^*_{\gamma_j})=-H^*_{\gamma_j}$, and $\mathrm{Ad}(\sigma_X)(H)=H$ for all $\{H\in\mathfrak{t}_0^{\mathbb{C}}:\gamma_j(H)=0\text{ for all }1\le j\le 4\}$. Thus

$\mathrm{Ad}(\sigma_X)(\alpha_0)=-\alpha_0,\quad \mathrm{Ad}(\sigma_X)(\psi_j)=\psi_j$, for $j=2,5,7$;

$\mathrm{Ad}(\sigma_X)(\psi_3) = \mathrm{Ad}(\sigma_X)(\frac{1}{2}(-\gamma_1+\gamma_2+\gamma_3-\gamma_4)+\psi_3-\frac{1}{2}(-\gamma_1+\gamma_2+\gamma_3-\gamma_4))$
$= -\frac{1}{2}(-\gamma_1+\gamma_2+\gamma_3-\gamma_4)+\psi_3-\frac{1}{2}(-\gamma_1+\gamma_2+\gamma_3-\gamma_4) = \psi_3-(-\gamma_1+\gamma_2+\gamma_3-\gamma_4) = -\psi_3;$
$\mathrm{Ad}(\sigma_X)(\psi_4) = \mathrm{Ad}(\sigma_X)(\frac{1}{2}(-\gamma_3+\gamma_4)+\psi_4-\frac{1}{2}(-\gamma_3+\gamma_4))$
$= -\frac{1}{2}(-\gamma_3+\gamma_4)+\psi_4-\frac{1}{2}(-\gamma_3+\gamma_4) = \psi_4-(-\gamma_3+\gamma_4) = -\psi_2-\psi_4-\psi_5;$ and
$\mathrm{Ad}(\sigma_X)(\psi_6) = \mathrm{Ad}(\sigma_X)(\frac{1}{2}(-\gamma_2+\gamma_3)+\psi_6-\frac{1}{2}(-\gamma_2+\gamma_3))$
$= -\frac{1}{2}(-\gamma_2+\gamma_3)+\psi_6-\frac{1}{2}(-\gamma_2+\gamma_3) = \psi_6-(-\gamma_2+\gamma_3) = -\psi_5-\psi_6-\psi_7.$

So $\quad \mathrm{Ad}(\sigma_X)(\{\alpha_0,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7\})$
$$= \{-\alpha_0,\psi_2,-\psi_3,-\psi_2-\psi_4-\psi_5,\psi_5,-\psi_5-\psi_6-\psi_7,\psi_7\}.$$
Let $w^0_{\mathfrak{g}_0} \in W(\mathfrak{g}_0,\mathfrak{t}_0^{\mathbb{C}})$ be the longest element that is, $w^0_{\mathfrak{g}_0}(\alpha_0) = -\alpha_0$, $w^0_{\mathfrak{g}_0}(\psi_j) = -\psi_j$ for all $2 \le j \le 7$; and $s'_X = s_{\psi_7}s_{\psi_5}s_{\psi_2}w^0_{\mathfrak{g}_0}$. Then $s'_X(\alpha_0) = -\alpha_0$, $s'_X(\psi_j) = \psi_j$ for $j = 2,5,7$, $s'_X(\psi_3) = -\psi_3$, $s'_X(\psi_4) = -\psi_2-\psi_4-\psi_5$, and $s'_X(\psi_6) = -\psi_5-\psi_6-\psi_7$. Thus if $s_X = \mathrm{Ad}(\sigma_X)\circ s'_X$, then $s_X(\Delta_0^+) = \Delta_0^+$. Clearly $s_X(\psi_j) = \psi_j$ for all $2 \le j \le 7$, and $s_X(\alpha_0) = \alpha_0$. So $\det(s_X|_{\mathfrak{t}_0^{\mathbb{C}}}) = 1$. Hence the canonical action of $G(\mu)$ on $X(\mu)$ is orientation preserving for $\mu = \bar{\sigma}, \bar{\sigma}\bar{\theta}$.

4.3. Table for the condition Or of a connected complex simple Lie group of adjoint type on the following pages:

Let $\bar{G} = \mathrm{Int}(\mathfrak{g})$, the connected component of $\mathrm{Aut}(\mathfrak{g})$. Then $\bar{G}$ is a connected complex simple Lie group of adjoint type, $\mathrm{Lie}(\bar{G}) = \mathfrak{g}$, and $\bar{G} \cong \tilde{G}/\tilde{Z}$. The *condition Or* for $\bar{G},\bar{\sigma},\bar{\sigma}\bar{\theta}$; in each case, is given in Table 4.2. If the *condition Or* for $\bar{G},\bar{\sigma},\bar{\sigma}\bar{\theta}$ is satisfied, then the dimensions of $X(\bar{\sigma})$ and $X(\bar{\sigma}\bar{\theta})$ are given in Table 4.3. Here

$$S(GL(p,\mathbb{C})\times GL(q,\mathbb{C})) = \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : \begin{array}{l} A_1 \in GL(p,\mathbb{C}),\ A_2 \in GL(q,\mathbb{C}), \\ \text{and } \det A_1 \ \det A_2 = 1 \end{array} \right\},$$

and

$$S(U(p)\times U(q)) = \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} : A_1 \in U(p), A_2 \in U(q), \text{ and } \det A_1 \ \det A_2 = 1 \right\}.$$

We follow [12] for other notations.

4.4. Proof of Theorem 1.1: Note that $X = G/U$ is a Riemannian globally symmetric space of type IV. Let $\bar{G} = \mathrm{Ad}(G)$ be the adjoint group of G, and $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$, and θ be the Cartan involution of $\mathfrak{g}^{\mathbb{R}}$ corresponding to the Cartan decomposition $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} \oplus i\mathfrak{u}$. Let $\bar{\theta}$ denote the corresponding Cartan involution of $\bar{G}$. Let $\bar{U} = \{g \in \bar{G} : \bar{\theta}(g) = g\}$.

Then $X = \bar{G}/\bar{U}$. Let $\mathfrak{t}$ be a maximal abelian subspace of $\mathfrak{u}$, and $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Choose a system of positive roots Δ^+ in the set of all non-zero roots $\Delta = \Delta(\mathfrak{g},\mathfrak{h})$. Let Φ be the set of simple roots in Δ^+. Let $\{H^*_\phi, E_\alpha : \phi \in \Phi, \alpha \in \Delta\}$ be a Chevalley basis for $\mathfrak{g}$ as in (2). Then

$$\mathfrak{u} = \sum_{\phi\in\Phi} \mathbb{R}(iH^*_\phi) \oplus \sum_{\alpha\in\Delta^+} \mathbb{R}X_\alpha \oplus \sum_{\alpha\in\Delta^+} \mathbb{R}Y_\alpha,$$

where $X_\alpha = E_\alpha - E_{-\alpha}, Y_\alpha = i(E_\alpha + E_{-\alpha})$ for all $\alpha \in \Delta^+$.

Table 4.2: Table for the *condition Or* of a connected complex simple Lie group of adjoint type

$\mathfrak{g}$	type of σ	$X(\bar\sigma)$	$X(\bar\sigma\bar\theta)$	is *condition Or* for $\bar G, \sigma, \sigma\theta$ satisfied?
$\mathfrak{a}_{n-1}$ $(n > 1, n \in 4\mathbb{Z})$	$(s_0, 0, \dots, 0, s_p, 0, \dots, 0; 1)$ $(1 \le p \le n/2)$ with $s_0 = 1 = s_p$.	$\frac{S(GL(p,\mathbb{C})\times GL(n-p,\mathbb{C}))}{S(U(p)\times U(n-p))}$	$\frac{SU(p,n-p)}{S(U(p)\times U(n-p))}$	yes
$\mathfrak{a}_{n-1}$ $(n > 4, n \in 4\mathbb{Z})$	$(1, 0, \dots, 0; 2)$ $(0, \dots, 0, 1; 2)$	$\frac{Sp(\frac{n}{2},\mathbb{C})}{Sp(\frac{n}{2})}$ $\frac{SO(n,\mathbb{C})}{SO(n)}$	$\frac{SU^*(n)}{Sp(\frac{n}{2})}$ $\frac{SL(n,\mathbb{R})}{SO(n)}$	yes no
δ_3	$(1, 0, 0; 2)$ $(0, 1, 0; 2)$	$\frac{SO(5,\mathbb{C})}{SO(5)}$ $\frac{SO(3,\mathbb{C})\times SO(3,\mathbb{C})}{SO(3)\times SO(3)}$	$\frac{SO_0(1,5)}{SO(5)}$ $\frac{SO_0(3,3)}{SO(3)\times SO(3)}$	yes no
$\mathfrak{a}_{n-1}$ $(n > 1, n \in 2 + 4\mathbb{Z})$	$(s_0, 0, \dots, 0, s_p, 0, \dots, 0; 1)$ $(1 \le p < n/2)$ with $s_0 = 1 = s_p$ $(s_0, 0, \dots, 0, s_{\frac{n}{2}}, 0, \dots, 0; 1)$ with $s_0 = 1 = s_{\frac{n}{2}}$	$\frac{S(GL(p,\mathbb{C})\times GL(n-p,\mathbb{C}))}{S(U(p)\times U(n-p))}$ $\frac{S(GL(\frac{n}{2},\mathbb{C})\times GL(\frac{n}{2},\mathbb{C}))}{S(U(\frac{n}{2})\times U(\frac{n}{2}))}$	$\frac{SU(p,n-p)}{S(U(p)\times U(n-p))}$ $\frac{SU(\frac{n}{2},\frac{n}{2})}{S(U(\frac{n}{2})\times U(\frac{n}{2}))}$	yes no
$\mathfrak{a}_{n-1}$ $(n > 2, n \in 2 + 4\mathbb{Z})$	$(1, 0, \dots, 0; 2)$ $(0, \dots, 0, 1; 2)$	$\frac{Sp(\frac{n}{2},\mathbb{C})}{Sp(\frac{n}{2})}$ $\frac{SO(n,\mathbb{C})}{SO(n)}$	$\frac{SU^*(n)}{Sp(\frac{n}{2})}$ $\frac{SL(n,\mathbb{R})}{SO(n)}$	yes no
$\mathfrak{a}_{n-1}$ $(n > 1, n \in 1 + 2\mathbb{Z})$	$(s_0, 0, \dots, 0, s_p, 0, \dots, 0; 1)$ $(1 \le p \le \frac{n-1}{2})$ with $s_0 = 1 = s_p$ $(0, \dots, 0, 1; 2)$	$\frac{S(GL(p,\mathbb{C})\times GL(n-p,\mathbb{C}))}{S(U(p)\times U(n-p))}$ $\frac{SO(n,\mathbb{C})}{SO(n)}$	$\frac{SU(p,n-p)}{S(U(p)\times U(n-p))}$ $\frac{SL(n,\mathbb{R})}{SO(n)}$	yes yes

$\mathfrak{g}$	type of σ	$X(\bar{\sigma})$	$X(\bar{\sigma}\bar{\theta})$	is *condition Or* for $\bar{G}, \sigma, \sigma\theta$ satisfied?
$\mathfrak{c}_n$ $(n \geq 2, n \in 4\mathbb{Z},$ or $n \in 3+4\mathbb{Z})$	$(1,0,\ldots,0,1;1)$	$\frac{GL(n,\mathbb{C})}{U(n)}$	$\frac{Sp(n,\mathbb{R})}{U(n)}$	yes
	$(0,\ldots,0,s_p,0,\ldots,0;1)$ $(1 \leq p \leq n-1)$ with $s_p = 1$	$\frac{Sp(p,\mathbb{C})\times Sp(n-p,\mathbb{C})}{Sp(p)\times Sp(n-p)}$	$\frac{Sp(p,n-p)}{Sp(p)\times Sp(n-p)}$	yes
$\mathfrak{c}_n$ $(n \geq 2, n \in 1+4\mathbb{Z},$ or $n \in 2+4\mathbb{Z})$	$(1,0,\ldots,0,1;1)$	$\frac{GL(n,\mathbb{C})}{U(n)}$	$\frac{Sp(n,\mathbb{R})}{U(n)}$	no
	$(0,\ldots,0,s_p,0,\ldots,0;1)$ $(1 \leq p \leq n-1, p \neq \frac{n}{2})$ with $s_p = 1$	$\frac{Sp(p,\mathbb{C})\times Sp(n-p,\mathbb{C})}{Sp(p)\times Sp(n-p)}$	$\frac{Sp(p,n-p)}{Sp(p)\times Sp(n-p)}$	yes
$\mathfrak{c}_n$ $(n \geq 2, n \in 2+4\mathbb{Z})$	$(0,\ldots,0,s_{\frac{n}{2}},0,\ldots,0;1)$ with $s_{\frac{n}{2}} = 1$	$\frac{Sp(\frac{n}{2},\mathbb{C})\times Sp(\frac{n}{2},\mathbb{C})}{Sp(\frac{n}{2})\times Sp(\frac{n}{2})}$	$\frac{Sp(\frac{n}{2},\frac{n}{2})}{Sp(\frac{n}{2})\times Sp(\frac{n}{2})}$	no
$\mathfrak{b}_n$ $(n \geq 3)$	$(1,1,0,\ldots,0;1)$	$\frac{SO(2n-1,\mathbb{C})\times SO(2,\mathbb{C})}{SO(2n-1)\times SO(2)}$	$\frac{SO_0(2n-1,2)}{SO(2n-1)\times SO(2)}$	no
	$(0,\ldots,0,s_p,0,\ldots,0;1)$ $(2 \leq p \leq n)$ with $s_p = 1$	$\frac{SO(2p,\mathbb{C})\times SO(2n-2p+1,\mathbb{C})}{SO(2p)\times SO(2n-2p+1)}$	$\frac{SO_0(2p,2n-2p+1)}{SO(2p)\times SO(2n-2p+1)}$	no
δ_n $(n \geq 4, n \in 4\mathbb{Z},$ or $n \in 1+4\mathbb{Z})$	$(1,0,\ldots,0,1;1)$	$\frac{GL(n,\mathbb{C})}{U(n)}$	$\frac{SO^*(2n)}{U(n)}$	yes
	$(1,1,0,\ldots,0;1)$	$\frac{SO(2n-2,\mathbb{C})\times SO(2,\mathbb{C})}{SO(2n-2)\times SO(2)}$	$\frac{SO_0(2n-2,2)}{SO(2n-2)\times SO(2)}$	yes
	$(0,\ldots,0,s_p,0,\ldots,0;1)$ $(2 \leq p \leq n-2, 2p \leq n)$ with $s_p = 1$	$\frac{SO(2p,\mathbb{C})\times SO(2n-2p,\mathbb{C})}{SO(2p)\times SO(2n-2p)}$	$\frac{SO_0(2p,2n-2p)}{SO(2p)\times SO(2n-2p)}$	yes
	$(0,\ldots,0,s_p,0,\ldots,0;2)$ $(0 \leq p \leq n-1)$ with $s_p = 1$	$\frac{SO(2p+1,\mathbb{C})\times SO(2n-2p-1,\mathbb{C})}{SO(2p+1)\times SO(2n-2p-1)}$	$\frac{SO_0(2p+1,2n-2p-1)}{SO(2p+1)\times SO(2n-2p-1)}$	yes

$\mathfrak{g}$	type of σ	$X(\bar{\sigma})$	$X(\bar{\sigma}\bar{\theta})$	is *condition* *Or* for $\bar{G}, \sigma$, $\sigma\theta$ satisfied?
δ_n $(n \geq 4, n \in 2 + 4\mathbb{Z})$	$(1, 0, \ldots, 0, 1; 1)$	$\frac{GL(n,\mathbb{C})}{U(n)}$	$\frac{SO^*(2n)}{U(n)}$	no
	$(1, 1, 0, \ldots, 0; 1)$	$\frac{SO(2n-2,\mathbb{C}) \times SO(2,\mathbb{C})}{SO(2n-2) \times SO(2)}$	$\frac{SO_0(2n-2,2)}{SO(2n-2) \times SO(2)}$	yes
	$(0, \ldots, 0, s_p, 0, \ldots, 0; 1)$ $(2 \leq p \leq n-2, 2p < n)$ with $s_p = 1$	$\frac{SO(2p,\mathbb{C}) \times SO(2n-2p,\mathbb{C})}{SO(2p) \times SO(2n-2p)}$	$\frac{SO_0(2p,2n-2p)}{SO(2p) \times SO(2n-2p)}$	yes
	$(0, \ldots, 0, s_{\frac{n}{2}}, 0, \ldots, 0; 1)$ with $s_{\frac{n}{2}} = 1$	$\frac{SO(n,\mathbb{C}) \times SO(n,\mathbb{C})}{SO(n) \times SO(n)}$	$\frac{SO_0(n,n)}{SO(n) \times SO(n)}$	no
	$(0, \ldots, 0, s_p, 0, \ldots, 0; 2)$ $(0 \leq p \leq n-1)$ with $s_p = 1$	$\frac{SO(2p+1,\mathbb{C}) \times SO(2n-2p-1,\mathbb{C})}{SO(2p+1) \times SO(2n-2p-1)}$	$\frac{SO_0(2p+1,2n-2p-1)}{SO(2p+1) \times SO(2n-2p-1)}$	yes
δ_n $(n \geq 4, n \in 3 + 4\mathbb{Z})$	$(1, 0, \ldots, 0, 1; 1)$	$\frac{GL(n,\mathbb{C})}{U(n)}$	$\frac{SO^*(2n)}{U(n)}$	yes
	$(1, 1, 0, \ldots, 0; 1)$	$\frac{SO(2n-2,\mathbb{C}) \times SO(2,\mathbb{C})}{SO(2n-2) \times SO(2)}$	$\frac{SO_0(2n-2,2)}{SO(2n-2) \times SO(2)}$	yes
	$(0, \ldots, 0, s_p, 0, \ldots, 0; 1)$ $(2 \leq p \leq n-2, 2p \leq n)$ with $s_p = 1$	$\frac{SO(2p,\mathbb{C}) \times SO(2n-2p,\mathbb{C})}{SO(2p) \times SO(2n-2p)}$	$\frac{SO_0(2p,2n-2p)}{SO(2p) \times SO(2n-2p)}$	yes
	$(0, \ldots, 0, s_p, 0, \ldots, 0; 2)$ $(0 \leq p \leq n-1, p \neq \frac{n-1}{2})$ with $s_p = 1$	$\frac{SO(2p+1,\mathbb{C}) \times SO(2n-2p-1,\mathbb{C})}{SO(2p+1) \times SO(2n-2p-1)}$	$\frac{SO_0(2p+1,2n-2p-1)}{SO(2p+1) \times SO(2n-2p-1)}$	yes
	$(0, \ldots, 0, s_{\frac{n-1}{2}}, 0, \ldots, 0; 2)$ with $s_p = 1$	$\frac{SO(n,\mathbb{C}) \times SO(n,\mathbb{C})}{SO(n) \times SO(n)}$	$\frac{SO_0(n,n)}{SO(n) \times SO(n)}$	no

$\mathfrak{g}$	type of σ	$X(\bar{\sigma})$	$X(\bar{\sigma}\bar{\theta})$	is *condition Or* for $\bar{G}, \sigma$, $\sigma\theta$ satisfied?
$\mathfrak{e}_6$	$(1,1,0,0,0,0,0;1)$	$\frac{SO(10,\mathbb{C})\times SO(2,\mathbb{C})}{SO(10)\times SO(2)}$	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10)+\mathbb{R})$	yes
	$(0,0,1,0,0,0,0;1)$	$\frac{SL(2,\mathbb{C})\times SL(6,\mathbb{C})}{SU(2)\times SU(6)}$	$(\mathfrak{e}_{6(2)}, \mathfrak{su}(2)+\mathfrak{su}(6))$	yes
	$(1,0,0,0,0;2)$	$\frac{F_4^{\mathbb{C}}}{F_4}$	$(\mathfrak{e}_{6(-26)}, \mathfrak{f}_4)$	yes
	$(0,1,0,0,0;2)$	$\frac{Sp(4,\mathbb{C})}{Sp(4)}$	$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4))$	yes
$\mathfrak{e}_7$	$(1,0,0,0,0,0,0,1;1)$	$\frac{E_6^{\mathbb{C}}\times SO(2,\mathbb{C})}{E_6\times SO(2)}$	$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_6+\mathbb{R})$	no
	$(0,0,1,0,0,0,0,0;1)$	$\frac{SL(8,\mathbb{C})}{SU(8)}$	$(\mathfrak{e}_{7(7)}, \mathfrak{su}(8))$	no
	$(0,1,0,0,0,0,0,0;1)$	$\frac{SL(2,\mathbb{C})\times SO(12,\mathbb{C})}{SU(2)\times SO(12)}$	$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(2)+\mathfrak{so}(12))$	yes
$\mathfrak{e}_8$	$(0,1,0,0,0,0,0,0,0;1)$	$\frac{SO(16,\mathbb{C})}{SO(16)}$	$(\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$	yes
	$(0,0,0,0,0,0,0,0,1;1)$	$\frac{SL(2,\mathbb{C})\times E_7^{\mathbb{C}}}{SU(2)\times E_7}$	$(\mathfrak{e}_{8(-24)}, \mathfrak{su}(2)+\mathfrak{e}_7)$	yes
$\mathfrak{f}_4$	$(0,1,0,0,0;1)$	$\frac{SL(2,\mathbb{C})\times Sp(3,\mathbb{C})}{SU(2)\times Sp(3)}$	$(\mathfrak{f}_{4(4)}, \mathfrak{su}(2)+\mathfrak{sp}(3))$	yes
	$(0,0,0,0,1;1)$	$\frac{SO(9,\mathbb{C})}{SO(9)}$	$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$	yes
$\mathfrak{g}_2$	$(0,0,1;1)$	$\frac{SL(2,\mathbb{C})\times SL(2,\mathbb{C})}{SU(2)\times SU(2)}$	$(\mathfrak{g}_{2(2)}, \mathfrak{su}(2)+\mathfrak{su}(2))$	yes

Table 4.3: Table for the dimensions of $X(\bar{\sigma})$ and $X(\bar{\sigma}\bar{\theta})$ when the *condition Or* has been satisfied for $\bar{G}$, $\bar{\sigma}$, $\bar{\sigma}\bar{\theta}$

$\mathfrak{g}$	$X(\bar{\sigma})$	$X(\bar{\sigma}\bar{\theta})$	$\dim(X(\bar{\sigma}))$	$\dim(X(\bar{\sigma}\bar{\theta}))$
$\mathfrak{a}_{n-1}$ $(n > 1, n \in 4\mathbb{Z})$	$\frac{S(GL(p,\mathbb{C})\times GL(n-p,\mathbb{C}))}{S(U(p)\times U(n-p))}$ $(1 \leq p \leq \frac{n}{2})$ $\frac{Sp(\frac{n}{2},\mathbb{C})}{Sp(\frac{n}{2})}$	$\frac{SU(p,n-p)}{S(U(p)\times U(n-p))}$ $(1 \leq p \leq \frac{n}{2})$ $\frac{SU^*(n)}{Sp(\frac{n}{2})}$	$p^2 + (n-p)^2 - 1$ $\frac{n(n+1)}{2}$	$2p(n-p)$ $\frac{(n-2)(n+1)}{2}$
$\mathfrak{a}_{n-1}$ $(n > 1, n \in 2 + 4\mathbb{Z})$	$\frac{S(GL(p,\mathbb{C})\times GL(n-p,\mathbb{C}))}{S(U(p)\times U(n-p))}$ $(1 \leq p < \frac{n}{2})$ $\frac{Sp(\frac{n}{2},\mathbb{C})}{Sp(\frac{n}{2})}$ $(n > 2)$	$\frac{SU(p,n-p)}{S(U(p)\times U(n-p))}$ $(1 \leq p < \frac{n}{2})$ $\frac{SU^*(n)}{Sp(\frac{n}{2})}$ $(n > 2)$	$p^2 + (n-p)^2 - 1$ $\frac{n(n+1)}{2}$	$2p(n-p)$ $\frac{(n-2)(n+1)}{2}$
$\mathfrak{a}_{n-1}$ $(n > 1, n \in 1 + 2\mathbb{Z})$	$\frac{S(GL(p,\mathbb{C})\times GL(n-p,\mathbb{C}))}{S(U(p)\times U(n-p))}$ $(1 \leq p \leq \frac{n-1}{2})$ $\frac{SO(n,\mathbb{C})}{SO(n)}$	$\frac{SU(p,n-p)}{S(U(p)\times U(n-p))}$ $(1 \leq p \leq \frac{n-1}{2})$ $\frac{SL(n,\mathbb{R})}{SO(n)}$	$p^2 + (n-p)^2 - 1$ $\frac{n(n-1)}{2}$	$2p(n-p)$ $\frac{(n-1)(n+2)}{2}$
δ_n $(n \geq 4)$	$\frac{GL(n,\mathbb{C})}{U(n)}$ $(n \notin 2 + 4\mathbb{Z})$ $\frac{SO(p,\mathbb{C})\times SO(2n-p,\mathbb{C})}{SO(p)\times SO(2n-p)}$ $(1 \leq p < n)$ $\frac{SO(n,\mathbb{C})\times SO(n,\mathbb{C})}{SO(n)\times SO(n)}$ $(n \in 4\mathbb{Z}, \text{or } 1 + 4\mathbb{Z})$	$\frac{SO^*(2n)}{U(n)}$ $(n \notin 2 + 4\mathbb{Z})$ $\frac{SO_0(p,2n-p)}{SO(p)\times SO(2n-p)}$ $(1 \leq p < n)$ $\frac{SO_0(n,n)}{SO(n)\times SO(n)}$ $(n \in 4\mathbb{Z}, \text{or } 1 + 4\mathbb{Z})$	n^2 $\frac{p(p-1)+(2n-p)(2n-p-1)}{2}$ $n(n-1)$	$n(n-1)$ $p(2n-p)$ n^2

$\mathfrak{g}$	$X(\bar\sigma)$	$X(\bar\sigma\bar\theta)$	$\dim(X(\bar\sigma))$	$\dim(X(\bar\sigma\bar\theta))$
$\mathfrak{c}_n$ $(n \geq 2, n \in 4\mathbb{Z},$ or $n \in 3 + 4\mathbb{Z})$	$\frac{GL(n,\mathbb{C})}{U(n)}$ $\frac{Sp(p,\mathbb{C})\times Sp(n-p,\mathbb{C})}{Sp(p)\times Sp(n-p)}$ $(1 \leq p \leq n-1)$	$\frac{Sp(n,\mathbb{R})}{U(n)}$ $\frac{Sp(p,n-p)}{Sp(p)\times Sp(n-p)}$ $(1 \leq p \leq n-1)$	n^2 $p(2p+1)+$ $(n-p)(2n-2p+1)$	$n(n+1)$ $4p(n-p)$
$\mathfrak{c}_n$ $(n \geq 2, n \in 1 + 4\mathbb{Z},$ or $n \in 2 + 4\mathbb{Z})$	$\frac{Sp(p,\mathbb{C})\times Sp(n-p,\mathbb{C})}{Sp(p)\times Sp(n-p)}$ $(1 \leq p \leq n-1, p \neq \frac{n}{2})$	$\frac{Sp(p,n-p)}{Sp(p)\times Sp(n-p)}$ $(1 \leq p \leq n-1, p \neq \frac{n}{2})$	$p(2p+1)+$ $(n-p)(2n-2p+1)$	$4p(n-p)$
$\mathfrak{e}_6$	$\frac{SO(10,\mathbb{C})\times SO(2,\mathbb{C})}{SO(10)\times SO(2)}$ $\frac{SL(2,\mathbb{C})\times SL(6,\mathbb{C})}{SU(2)\times SU(6)}$ $\frac{F_4^{\mathbb{C}}}{F_4}$ $\frac{Sp(4,\mathbb{C})}{Sp(4)}$	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) + \mathbb{R})$ $(\mathfrak{e}_{6(2)}, \mathfrak{su}(2) + \mathfrak{su}(6))$ $(\mathfrak{e}_{6(-26)}, \mathfrak{f}_4)$ $(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4))$	46 38 52 36	32 40 26 42
$\mathfrak{e}_7$	$\frac{SL(2,\mathbb{C})\times SO(12,\mathbb{C})}{SU(2)\times SO(12)}$	$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(2) + \mathfrak{so}(12))$	69	64
$\mathfrak{e}_8$	$\frac{SO(16,\mathbb{C})}{SO(16)}$ $\frac{SL(2,\mathbb{C})\times E_7^{\mathbb{C}}}{SU(2)\times E_7}$	$(\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$ $(\mathfrak{e}_{8(-24)}, \mathfrak{su}(2) + \mathfrak{e}_7)$	120 136	128 112
$\mathfrak{f}_4$	$\frac{SL(2,\mathbb{C})\times Sp(3,\mathbb{C})}{SU(2)\times Sp(3)}$ $\frac{SO(9,\mathbb{C})}{SO(9)}$	$(\mathfrak{f}_{4(4)}, \mathfrak{su}(2) + \mathfrak{sp}(3))$ $(\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$	24 36	28 16
$\mathfrak{g}_2$	$\frac{SL(2,\mathbb{C})\times SL(2,\mathbb{C})}{SU(2)\times SU(2)}$	$(\mathfrak{g}_{2(2)}, \mathfrak{su}(2) + \mathfrak{su}(2))$	6	8

Let σ of an involution of $\mathfrak{g}$ as in (11) and $\bar{\sigma} : \bar{G} \longrightarrow \bar{G}$ be the involution with $d\bar{\sigma} = \sigma$. Then $\sigma\theta = \theta\sigma$. Let $\bar{\nu}$ be the Dynkin diagram automorphism induced by σ and ν be the linear extension of $\bar{\nu}$ on the dual space of $i\mathfrak{t}$. Recall that $\sigma(iH_\phi^*) = iH_{\nu(\phi)}^*$ for all $\phi \in \Phi$, and $\sigma(X_\alpha) = q_\alpha X_{\nu(\alpha)}$, $\sigma(Y_\alpha) = q_\alpha Y_{\nu(\alpha)}$ $(q_\alpha = \pm 1)$; for all $\alpha \in \Delta^+$. Note that $q_\alpha = q_{\nu(\alpha)}$ for all $\alpha \in \Delta^+$.

Let $\mathfrak{u} = \mathfrak{u}_0 \oplus \mathfrak{u}_1$ be the decomposition of $\mathfrak{u}$ in to 1 and -1 eigenspaces of σ. Then $\mathfrak{g}^\sigma = \mathfrak{u}_0 \oplus i\mathfrak{u}_1$ is a non-compact real form of $\mathfrak{g}$, and $\sigma\big|_{\mathfrak{g}^\sigma}$ is a Cartan involution of $\mathfrak{g}^\sigma$. Note that

$$\begin{aligned} \mathfrak{u}_0 = \sum_{\phi\in\Phi} \mathbb{R}i(H_\phi^* + H_{\nu(\phi)}^*) &\oplus \sum_{\alpha\in\Delta^+,\ q_\alpha=1} (\mathbb{R}(X_\alpha + X_{\nu(\alpha)}) \oplus \mathbb{R}(Y_\alpha + Y_{\nu(\alpha)})) \\ &\oplus \sum_{\alpha\in\Delta^+,\ q_\alpha=-1} (\mathbb{R}(X_\alpha - X_{\nu(\alpha)}) \oplus \mathbb{R}(Y_\alpha - Y_{\nu(\alpha)})), \quad \text{and} \\ i\mathfrak{u}_1 = \sum_{\phi\in\Phi} \mathbb{R}(H_\phi^* - H_{\nu(\phi)}^*) &\oplus \sum_{\alpha\in\Delta^+.\ q_\alpha=1} (\mathbb{R}i(X_\alpha - X_{\nu(\alpha)}) \oplus \mathbb{R}i(Y_\alpha - Y_{\nu(\alpha)})) \\ &\oplus \sum_{\alpha\in\Delta^+,\ q_\alpha=-1} (\mathbb{R}i(X_\alpha + X_{\nu(\alpha)}) \oplus \mathbb{R}i(Y_\alpha + Y_{\nu(\alpha)})). \end{aligned}$$

Let

$$\begin{aligned} B' \subset \{ & i(H_\phi^* + H_{\nu(\phi)}^*),\ (H_\phi^* - H_{\nu(\phi)}^*) : \phi \in \Phi\} \\ & \cup \{X_\alpha + X_{\nu(\alpha)},\ Y_\alpha + Y_{\nu(\alpha)},\ i(X_\alpha - X_{\nu(\alpha)}),\ i(Y_\alpha - Y_{\nu(\alpha)}) : \alpha \in \Delta^+,\ q_\alpha = 1\} \\ & \cup \{X_\alpha - X_{\nu(\alpha)},\ Y_\alpha - Y_{\nu(\alpha)},\ i(X_\alpha + X_{\nu(\alpha)}),\ i(Y_\alpha + Y_{\nu(\alpha)}) : \alpha \in \Delta^+,\ q_\alpha = -1\} \end{aligned}$$

be a basis of $\mathfrak{g}^\sigma$. Then B' is a basis of $\mathfrak{g}^\sigma$ consisting of eigenvectors of the Cartan involution $\sigma\big|_{\mathfrak{g}^\sigma}$, with respect to which the structural constants are all integers. Let Γ be an arithmetic uniform lattice of $\mathrm{Aut}(\mathfrak{g})$ of type 3 with respect to the non-compact real form $\mathfrak{g}^\sigma$ and the basis B' of $\mathfrak{g}^\sigma$. Then $\sigma \in \Gamma$. Also $\sigma \in \Gamma$ for any arithmetic uniform lattice of $\mathrm{Aut}(\mathfrak{g}^\mathbb{R})$ of type i, $i = 1$, or 2.

Now assume that Γ' be an arithmetic uniform lattice of $\mathrm{Aut}(\mathfrak{g}^\mathbb{R})$ of type $i (i = 1, 2$, or $3)$, and F be the corresponding algebraic number field with ring of integers $\mathcal{O}$. Arithmetic uniform lattices of $\mathrm{Aut}(\mathfrak{g}^\mathbb{R})$ of type 3 considered here are defined with respect to the non-compact real form $\mathfrak{g}^\sigma$ and the basis B' of $\mathfrak{g}^\sigma$. Let Γ be the set of all torsion-free elements of $\Gamma' \cap \bar{G}$. Then $\bar{G}$ is defined over F, $\bar{\theta}$, $\bar{\sigma}$ are defined over F, and $\Gamma \subset \bar{G}_\mathcal{O}$ is a torsion-free, $\langle\bar{\sigma}, \bar{\theta}\rangle$-stable, arithmetic uniform lattice of $\bar{G}$. Then if the *condition Or* is satisfied for $\bar{G}, \bar{\sigma}, \bar{\sigma}\bar{\theta}$; there exists a $\langle\bar{\sigma}, \bar{\theta}\rangle$-stable subgroup Γ'' of Γ of finite index such that the cohomology classes defined by $[C(\bar{\sigma}, \Gamma'')], [C(\bar{\sigma}\bar{\theta}, \Gamma'')]$ via Poincaré duality are non-zero and are not represented by $\bar{G}$-invariant differential forms on X, by Theorem 4.1. Since G is a covering group of $\bar{G}$, the cohomology classes defined by $[C(\bar{\sigma}, \Gamma'')], [C(\bar{\sigma}\bar{\theta}, \Gamma'')]$ via Poincaré duality are also not represented by G-invariant differential forms on X. This completes the proof.

5. Automorphic representations of a connected complex simple Lie group

Let G be a non-compact semisimple Lie group with finite centre and $\Gamma \subset G$ be a lattice. Consider the Hilbert space $L^2(\Gamma\backslash G)$ of square integrable functions on $\Gamma\backslash G$ with respect to a finite G-invariant measure. The group G acts unitarily on the Hilbert space $L^2(\Gamma\backslash G)$ via the right translation action of G on $\Gamma\backslash G$.

When Γ is a uniform lattice, we have

$$L^2(\Gamma\backslash G) \cong \widehat{\bigoplus_{\pi\in\hat{G}}} m(\pi,\Gamma)H_\pi,$$

due to Gelfand and Pyatetskii-Shapiro [9], [10]; where $\hat{G}$ denotes the unitary dual of G; H_π is the representation space of $\pi \in \hat{G}$; and $m(\pi,\Gamma) \in \mathbb{N}\cup\{0\}$, the multiplicity of π in $L^2(\Gamma\backslash G)$. If $(\tau,\mathbb{C})$ is the trivial representation of G, then $m(\tau,\Gamma) = 1$.

A unitary representation $\pi \in \hat{G}$ such that $m(\pi,\Gamma) > 0$ for some uniform lattice Γ, is called an automorphic representation with respect to Γ. The connection between geometric cycles and automorphic representations has been made by the Matsushima's isomorphism.

Let G be a connected semisimple Lie group with finite centre and K be a maximal compact subgroup of G with Cartan involution θ. Let $X = G/K$ be the associated Riemannian globally symmetric space, $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. If π be an admissible unitary representation of G on a Hilbert space H_π, we denote by $H_{\pi,K}$ the space of all K-finite vectors of H_π. The space $H_{\pi,K}$ is the associated $(\mathfrak{g}^{\mathbb{C}},K)$-module.

Let $\Gamma \subset G$ be a torsion-free uniform lattice. Then the isomorphism $L^2(\Gamma\backslash G) \cong \widehat{\bigoplus}_{\pi\in\hat{G}} m(\pi,\Gamma)H_\pi$ implies

$$\bigoplus_{\pi\in\hat{G}} m_\pi H_{\pi,K} \hookrightarrow C^\infty(\Gamma\backslash G)_K.$$

Matsushima's formula [23] says that the above inclusion induces an isomorphism

$$\bigoplus_{\pi\in\hat{G}} m_\pi H^p(\mathfrak{g}^{\mathbb{C}},K;H_{\pi,K}) \cong H^p(\mathfrak{g}^{\mathbb{C}},K;C^\infty(\Gamma\backslash G)_K).$$

Also we have the well-known isomorphism

$$H^p(\mathfrak{g}^{\mathbb{C}},K;C^\infty(\Gamma\backslash G)_K) \cong H^p(\Gamma\backslash X;\mathbb{C}).$$

See [3, Cor. 2.7, Ch. VII]. Hence

$$H^p(\Gamma\backslash X;\mathbb{C}) \cong \bigoplus_{\pi\in\hat{G}} m_\pi H^p(\mathfrak{g}^{\mathbb{C}},K;H_{\pi,K}).$$

Hence a non-vanishing (in the cohomology level) geometric cycle will contribute to the LHS and it may help to detect occurrence of some $\pi \in \hat{G}$ with non-zero $(\mathfrak{g}^{\mathbb{C}},K)$-cohomology. If X_u denotes the compact dual of X, then the image of the Matsushima map $k_\Gamma : H^*(X_u;\mathbb{C}) \longrightarrow H^*(\Gamma\backslash X;\mathbb{C})$ corresponds to the trivial representation $(\tau,\mathbb{C})$ of G. So if the cohomology class of a geometric cycle does not lie in the image $k_\Gamma(H^*(X_u;\mathbb{C}))$, then it may help to detect occurrence of some non-trivial $\pi \in \hat{G}$ with non-zero $(\mathfrak{g}^{\mathbb{C}},K)$-cohomology. For this purpose, it is important to know the irreducible unitary representations of G with non-zero $(\mathfrak{g}^{\mathbb{C}},K)$-cohomology. The details are given in the following subsections.

5.1. Irreducible unitary representations with non-zero $(\mathfrak{g}^{\mathbb{C}},K)$-cohomology. Let G be a connected semisimple Lie group with finite centre and $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition and θ be the corresponding Cartan involution. Let K be the connected Lie subgroup of G with $\mathrm{Lie}(K) = \mathfrak{k}$. Then K is a maximal compact subgroup of G. Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$

and $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{p}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ be the complexifications of $\mathfrak{k}$, $\mathfrak{p}$ respectively. The complex linear extension of θ to $\mathfrak{g}^{\mathbb{C}}$ is denoted by the same notation θ. If π be an admissible unitary representation of G on a Hilbert space H_π, recall that $H_{\pi,K}$ is the space of all K-finite vectors of H_π. By a theorem of D. Wigner, if $\pi \in \hat{G}$, then $H^*(\mathfrak{g}^{\mathbb{C}}, K; H_{\pi,K}) \neq 0$ implies the infinitesimal character χ_π of π is trivial that is, $\chi_\pi = \chi_0$, the infinitesimal character of the trivial representation of G. Hence there are only finitely irreducible unitary representations with non-zero $(\mathfrak{g}^{\mathbb{C}}, K)$-cohomology. In fact, the irreducible unitary representations with non-zero relative Lie algebra cohomology have been classified in terms of the θ-stable parabolic subalgebras $\mathfrak{q} \subset \mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$.

A θ-stable parabolic subalgebra of $\mathfrak{g}$ is by definition, a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}^{\mathbb{C}}$ such that (a) $\theta(\mathfrak{q}) = \mathfrak{q}$, and (b) $\bar{\mathfrak{q}} \cap \mathfrak{q} = \mathfrak{l}^{\mathbb{C}}$ is a Levi subalgebra of $\mathfrak{q}$, where $^-$ denotes the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$. By (b), $\mathfrak{l}^{\mathbb{C}}$ is the complexification of a real subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. Also $\theta(\mathfrak{l}) = \mathfrak{l}$ and and $\mathfrak{l}$ contains a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. Then $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t})$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{q}$. Let $\mathfrak{u}_{\mathfrak{q}}$ be the nilradical of $\mathfrak{q}$ so that $\mathfrak{q} = \mathfrak{l}^{\mathbb{C}} \oplus \mathfrak{u}_{\mathfrak{q}}$. Then $\mathfrak{u}_{\mathfrak{q}}$ is θ-stable and so $\mathfrak{u}_{\mathfrak{q}} = (\mathfrak{u}_{\mathfrak{q}} \cap \mathfrak{k}^{\mathbb{C}}) \oplus (\mathfrak{u}_{\mathfrak{q}} \cap \mathfrak{p}^{\mathbb{C}})$.

If V is finite dimensional complex L-module, where L is an abelian Lie algebra; we denote by $\Delta(V)$ (or by $\Delta(V, L)$) , the set of all non-zero weights of V and by $\delta(V)$ (or by $\delta(V, L)$), 1/2 of the sum of elements in $\Delta(V)$ counted with their respective multiplicities.

Fix systems of positive roots $\Delta^+((\mathfrak{l} \cap \mathfrak{k})^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and $\Delta^+(\mathfrak{l}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, compatible with the system $\Delta^+((\mathfrak{l} \cap \mathfrak{k})^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Then $\Delta^+_{\mathfrak{k}} = \Delta^+((\mathfrak{l} \cap \mathfrak{k})^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) \cup \Delta(\mathfrak{u}_{\mathfrak{q}} \cap \mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and also $\Delta^+ = \Delta^+(\mathfrak{l}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}) \cup \Delta(\mathfrak{u}_{\mathfrak{q}}, \mathfrak{h}^{\mathbb{C}})$ are systems of positive roots in $\Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ respectively.

Now associated with a θ-stable parabolic subalgebra $\mathfrak{q}$, we have an irreducible unitary representation $\mathcal{R}^S_{\mathfrak{q}}(\mathbb{C}) = A_{\mathfrak{q}}$ of G with trivial infinitesimal character, where $S = \dim(\mathfrak{u}_{\mathfrak{q}} \cap \mathfrak{k}^{\mathbb{C}})$. The associated $(\mathfrak{g}^{\mathbb{C}}, K)$-module $A_{\mathfrak{q},K}$ contains an irreducible K-submodule V of highest weight (with respect to $\Delta^+_{\mathfrak{k}}$)

$$2\delta(\mathfrak{u}_{\mathfrak{q}} \cap \mathfrak{p}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) = \sum_{\alpha \in \Delta(\mathfrak{u}_{\mathfrak{q}} \cap \mathfrak{p}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})} \alpha$$

and it occurs with multiplicity one in $A_{\mathfrak{q},K}$. Any other irreducible K-module that occurs in $A_{\mathfrak{q},K}$ has highest weight of the form

$$2\delta(\mathfrak{u}_{\mathfrak{q}} \cap \mathfrak{p}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) + \sum_{\gamma \in \Delta(\mathfrak{u}_{\mathfrak{q}} \cap \mathfrak{p}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})} n_\gamma \gamma,$$

with n_γ a non-negative integer [36, Th. 2.5].

If $\mathfrak{q}$ is a θ-stable parabolic subalgebra, then so is $\mathrm{Ad}(k)(\mathfrak{q})$ $(k \in K)$; and $A_{\mathfrak{q}}$, $A_{\mathrm{Ad}(k)(\mathfrak{q})}$ are unitarily equivalent. So it is sufficient to consider θ-stable parabolic subalgebras of $\mathfrak{g}$ which contain $\mathfrak{t}$, and $\Delta^+_{\mathfrak{k}}$ is contained in the corresponding system of positive roots Δ^+. It is known that [31, Prop. 4.5], for two such parabolic subalgebras $\mathfrak{q}$ and $\mathfrak{q}'$, $A_{\mathfrak{q}}$ is unitarily equivalent to $A_{\mathfrak{q}'}$ if and only if $\mathfrak{u}_{\mathfrak{q}} \cap \mathfrak{p}^{\mathbb{C}} = \mathfrak{u}_{\mathfrak{q}'} \cap \mathfrak{p}^{\mathbb{C}}$.

Actually $A_{\mathfrak{q}}$ is unitarily equivalent to $A_{\mathfrak{q}'}$ if and only if $\delta(\mathfrak{u}_{\mathfrak{q}} \cap \mathfrak{p}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) = \delta(\mathfrak{u}_{\mathfrak{q}'} \cap \mathfrak{p}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. The proof can be deduced from [31, Lemma 4.6 and Lemma 4.8] just noting the fact that if $\mathfrak{q}, \tilde{\mathfrak{q}}$ are two θ-stable parabolic subalgebras with $\mathfrak{q} \subset \tilde{\mathfrak{q}}$, then as they contain the same Borel subalgebra of $\mathfrak{g}^{\mathbb{C}}$, we have $\mathfrak{u}_{\mathfrak{q}} \cap \mathfrak{p}^{\mathbb{C}} = \mathfrak{u}_{\tilde{\mathfrak{q}}} \cap \mathfrak{p}^{\mathbb{C}}$ if and only if $\delta(\mathfrak{u}_{\mathfrak{q}} \cap \mathfrak{p}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) = \delta(\mathfrak{u}_{\tilde{\mathfrak{q}}} \cap \mathfrak{p}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$.

If $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$, then, in consequence, the Levi subgroup $L = \{g \in G : \mathrm{Ad}(g)(\mathfrak{q}) = \mathfrak{q}\}$ is a connected reductive Lie subgroup of G with Lie algebra $\mathfrak{l}$. As $\theta(\mathfrak{l}) = \mathfrak{l}$, $L \cap K$ is a maximal compact subgroup of L. One has

$$H^p(\mathfrak{g}^{\mathbb{C}}, K; A_{\mathfrak{q},K}) \cong H^{p-R(\mathfrak{q})}(\mathfrak{l}^{\mathbb{C}}, L \cap K; \mathbb{C}),$$

where $R(\mathfrak{q}) := \dim(\mathfrak{u}_{\mathfrak{q}} \cap \mathfrak{p}^{\mathbb{C}})$. Let $Y_{\mathfrak{q}}$ denote the compact dual of the Riemannian globally symmetric space $L/L \cap K$. Then $H^p(\mathfrak{l}^{\mathbb{C}}, L \cap K; \mathbb{C}) \cong H^p(Y_{\mathfrak{q}}; \mathbb{C})$. And hence

$$H^p(\mathfrak{g}^{\mathbb{C}}, K; A_{\mathfrak{q},K}) \cong H^{p-R(\mathfrak{q})}(Y_{\mathfrak{q}}; \mathbb{C}).$$

If $P(\mathfrak{q}, t)$ denotes the Poincaré polynomial of $H^*(\mathfrak{g}^{\mathbb{C}}, K; A_{\mathfrak{q},K})$. Then by the above result, we have

$$P(\mathfrak{q}, t) = t^{R(\mathfrak{q})} P(Y_{\mathfrak{q}}, t).$$

If $\mathrm{rank}(G) = \mathrm{rank}(K)$ and $\mathfrak{q}$ is a θ-stable Borel subalgebra that is, $\mathfrak{q}$ is a Borel subalgebra of $\mathfrak{g}^{\mathbb{C}}$ containing a Cartan subalgebra of $\mathfrak{k}^{\mathbb{C}}$, then $A_{\mathfrak{q}}$ is a discrete series representation of G with trivial infinitesimal character. In this case, $R(\mathfrak{q}) = \frac{1}{2}\dim(G/K)$, L is a maximal torus in K and hence

$$H^p(\mathfrak{g}^{\mathbb{C}}, K; A_{\mathfrak{q},K}) = \begin{cases} 0 & \text{if } p \neq R(\mathfrak{q}), \\ \mathbb{C} & \text{if } p = R(\mathfrak{q}). \end{cases}$$

If we take $\mathfrak{q} = \mathfrak{g}^{\mathbb{C}}$, then $L = G$ and $A_{\mathfrak{q}} = \mathbb{C}$, the trivial representation of G.
Conversely, if $\pi \in \hat{G}$ with $H^*(\mathfrak{g}^{\mathbb{C}}, K; H_{\pi,K}) \neq 0$, then H_π is unitarily equivalent to $A_{\mathfrak{q}}$ for some θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ [36, Th. 4.1].
The $(\mathfrak{g}^{\mathbb{C}}, K)$-modules $A_{\mathfrak{q},K}$ were first constructed, in general, by Parthasarathy [28]. Delorme [7] and Enright [8] gave a construction of those for complex Lie groups. Vogan and Zuckerman [36] gave a construction of the $(\mathfrak{g}^{\mathbb{C}}, K)$-modules $A_{\mathfrak{q},K}$ via cohomological induction and Vogan [34] proved that these are unitarizable. See [35] for a beautiful description of Matsushima isomorphism and the theory of $(\mathfrak{g}^{\mathbb{C}}, K)$-modules $A_{\mathfrak{q},K}$.

5.2. Irreducible unitary representations with non-zero $(\mathfrak{g}^{\mathbb{C}}, K)$-cohomology of a connected complex semisimple Lie group: Now assume that $\mathfrak{g}$ is a complex semisimple Lie algebra and θ is a Cartan involution on $\mathfrak{g}^{\mathbb{R}}$. Let $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} \oplus i\mathfrak{u}$ be the corresponding Cartan decomposition, for some compact real form $\mathfrak{u}$ of $\mathfrak{g}$. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and U be a Lie subgroup of G corresponding to the subalgebra $\mathfrak{u}$ of $\mathfrak{g}^{\mathbb{R}}$. Then U is a maximal compact subgroup of G. Recall that we shall identify $\mathfrak{g}$ with the subalgebra $\{(X, X) + i(Y, -Y) : X, Y \in \mathfrak{u}\}$ of $(\mathfrak{g}^{\mathbb{R}})^{\mathbb{C}} \cong \mathfrak{g} \times \mathfrak{g}$ and via this identification the complex linear extension of θ (denoted by the same notation) on $\mathfrak{g} \times \mathfrak{g}$ is given by $(Z_1, Z_2) \mapsto (Z_2, Z_1)$, where $Z_1, Z_2 \in \mathfrak{g}$. Then $\mathfrak{k} = \{(Z, Z) : Z \in \mathfrak{g}\}$ and $\mathfrak{p} = \{(Z, -Z) : Z \in \mathfrak{g}\}$ are the eigenspaces of θ corresponding to the eigenvalues 1 and -1 respectively.

A parabolic subalgebra of $(\mathfrak{g}^{\mathbb{R}})^{\mathbb{C}} \cong \mathfrak{g} \times \mathfrak{g}$ is of the form $\mathfrak{q}_1 \times \mathfrak{q}_2$, for some parabolic subalgebras $\mathfrak{q}_1, \mathfrak{q}_2$ of $\mathfrak{g}$. Hence $\theta(\mathfrak{q}_1 \times \mathfrak{q}_2) = \mathfrak{q}_1 \times \mathfrak{q}_2$ if and only if $\mathfrak{q}_1 = \mathfrak{q}_2$. If $\mathfrak{q} \times \mathfrak{q}$ is θ-stable, then $\mathfrak{q}$ contains a θ-stable Cartan subalgebra of $\mathfrak{g}$ (see Section 5.1). Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{u}$. Then $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$ is θ-stable Cartan subalgebra of $\mathfrak{g}$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$. Since $\mathfrak{h}$ is θ-stable, define $\theta(\alpha)(H) = \overline{\alpha(\theta H)}$

for all $H \in \mathfrak{h}$, where $\alpha \in \mathfrak{h}^*$. Note that $\theta(\alpha) = -\alpha$ for all $\alpha \in \Delta$. So if $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ containing the Cartan subalgebra $\mathfrak{h}$, then $\mathfrak{q} \cap \theta(\mathfrak{q}) = \mathfrak{l}$, the Levi factor of $\mathfrak{q}$ relative to the Cartan subalgebra $\mathfrak{h}$. Let $^-$ denote the conjugation of $\mathfrak{g} \times \mathfrak{g}$ with respect to the real form $\mathfrak{g} \cong \{(X, X) + i(Y, -Y) : X, Y \in \mathfrak{u}\}$. The map $^- : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \times \mathfrak{g}$ is given by $(Z_1, Z_2) \mapsto (\theta(Z_2), \theta(Z_1))$. Hence $\overline{\mathfrak{q} \times \mathfrak{q}} = \theta(\mathfrak{q}) \times \theta(\mathfrak{q})$ and so $\overline{(\mathfrak{q} \times \mathfrak{q})} \cap (\mathfrak{q} \times \mathfrak{q}) = (\theta(\mathfrak{q}) \cap \mathfrak{q}) \times (\theta(\mathfrak{q}) \cap \mathfrak{q}) = \mathfrak{l} \times \mathfrak{l}$, the Levi factor of the parabolic subalgebra $\mathfrak{q} \times \mathfrak{q}$ of $\mathfrak{g} \times \mathfrak{g}$ relative to the Cartan subalgebra $\mathfrak{h} \times \mathfrak{h}$. Consequently, we have a parabolic subalgebra of $\mathfrak{g} \times \mathfrak{g}$ is θ-stable if and only if it is of the form $\mathfrak{q} \times \mathfrak{q}$, for some parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ containing a θ-stable Cartan subalgebra of $\mathfrak{g}$.

Fix a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{u}$ and a system of positive roots Δ^+ in $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$, a θ-stable Cartan subalgebra of $\mathfrak{g}$. By Section 5.1, it is sufficient to consider θ-stable parabolic subalgebras of $\mathfrak{g}$ which contain $\mathfrak{t}$, and Δ^+ is contained in the corresponding system of positive roots in $\Delta(\mathfrak{g} \times \mathfrak{g}, \mathfrak{h} \times \mathfrak{h})$. Now the θ-stable parabolic subalgebras of $\mathfrak{g} \times \mathfrak{g}$, which contain $\mathfrak{t}$ and Δ^+ is contained in the corresponding system of positive roots in $\Delta(\mathfrak{g} \times \mathfrak{g}, \mathfrak{h} \times \mathfrak{h})$, are of the form $\mathfrak{q} \times \mathfrak{q}$, where $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$.

Let Φ be the set of simple roots in Δ^+. The parabolic subalgebras of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{b}$ are in one-one correspondence with $p(\Phi)$, the power set of Φ. Namely, for $\Phi' \subset \Phi$, the parabolic subalgebra $\mathfrak{q}_{\Phi'}$ corresponding to Φ' is given by

$$\mathfrak{q}_{\Phi'} = \mathfrak{l}_{\Phi'} \oplus \mathfrak{u}_{\Phi'}, \quad \text{where}$$

$$\mathfrak{l}_{\Phi'} = \mathfrak{h} \oplus \sum_{\substack{n_\psi(\alpha)=0 \\ \forall \psi \in \Phi'}} \mathfrak{g}_\alpha, \quad \mathfrak{u}_{\Phi'} = \sum_{\substack{n_\psi(\alpha)>0 \\ \text{for some } \psi \in \Phi'}} \mathfrak{g}_\alpha \quad \text{and} \quad \alpha = \sum_{\psi \in \Phi} n_\psi(\alpha)\psi \in \Delta.$$

So the θ-stable parabolic subalgebras of $\mathfrak{g} \times \mathfrak{g}$, which contain $\mathfrak{t}$ and Δ^+ is contained in the corresponding system of positive roots in $\Delta(\mathfrak{g} \times \mathfrak{g}, \mathfrak{h} \times \mathfrak{h})$, are in one-one correspondence with $p(\Phi)$. The one corresponding to $\Phi' \subset \Phi$ is given by $\mathfrak{q}_{\Phi'} \times \mathfrak{q}_{\Phi'}$, where $\mathfrak{q}_{\Phi'}$ is given above.

Note that the root space decomposition of $\mathfrak{g} \times \mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h} \times \mathfrak{h}$ is given by

$$\mathfrak{g} \times \mathfrak{g} = \mathfrak{h} \times \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_{(\alpha,0)} \oplus \mathfrak{g}_{(0,\alpha)},$$

where $\mathfrak{g}_{(\alpha,0)} = \{(Z, 0) : Z \in \mathfrak{g}_\alpha\}$, $\mathfrak{g}_{(0,\alpha)} = \{(0, Z) : Z \in \mathfrak{g}_\alpha\}$ for all $\alpha \in \Delta$. So for $\Phi' \subset \Phi$, if $\tilde{\mathfrak{u}}_{\Phi'}$ denotes the nilradical of the θ-stable parabolic subalgebra $\mathfrak{q}_{\Phi'} \times \mathfrak{q}_{\Phi'}$, then

$$\tilde{\mathfrak{u}}_{\Phi'} = \sum_{\substack{\alpha \in \Delta,\ n_\psi(\alpha)>0 \\ \text{for some } \psi \in \Phi'}} \mathfrak{g}_{(\alpha,0)} \oplus \mathfrak{g}_{(0,\alpha)}.$$

Again $\theta(\tilde{\mathfrak{u}}_{\Phi'}) = \tilde{\mathfrak{u}}_{\Phi'}$ implies $\tilde{\mathfrak{u}}_{\Phi'} = (\tilde{\mathfrak{u}}_{\Phi'} \cap \mathfrak{k}) \oplus (\tilde{\mathfrak{u}}_{\Phi'} \cap \mathfrak{p})$. Hence

$$\tilde{\mathfrak{u}}_{\Phi'} \cap \mathfrak{p} = \sum_{\substack{\alpha \in \Delta,\ n_\psi(\alpha)>0 \\ \text{for some } \psi \in \Phi'}} \{(Z, -Z) : Z \in \mathfrak{g}_\alpha\}, \quad \text{and so} \quad \dim((\tilde{\mathfrak{u}}_{\Phi'} \cap \mathfrak{p}) = \dim(\mathfrak{u}_{\Phi'}).$$

The Levi subgroup $L = \{g \in G : \mathrm{Ad}(g)(\mathfrak{q}_{\Phi'} \times \mathfrak{q}_{\Phi'}) = \mathfrak{q}_{\Phi'} \times \mathfrak{q}_{\Phi'}\}$ is a connected reductive Lie subgroup of G with Lie algebra $\mathfrak{l}_{\Phi'}$. As $\theta(\mathfrak{l}_{\Phi'}) = \mathfrak{l}_{\Phi'}$, $\mathfrak{l}_{\Phi'} \cap \mathfrak{u}$ is compact real form of $\mathfrak{l}_{\Phi'}$ and $L \cap U$ is a maximal compact subgroup of L. Also the centre of the reductive Lie algebra $\mathfrak{l}_{\Phi'}$ is $|\Phi'|$-dimensional, where $|\Phi'|$ denotes the cardinality of the set Φ'. Let $Y_{\Phi'}$ denote the compact dual of the Riemannian globally symmetric space $L/L\cap U$. Then $Y_{\Phi'} = L\cap U$, a connected compact Lie group. Hence

$$H^p(Y_{\Phi'};\mathbb{C}) = H^p((\mathfrak{l}_{\Phi'}\cap\mathfrak{u})^{\mathbb{C}};\mathbb{C}) = H^p(\mathfrak{l}_{\Phi'};\mathbb{C}).$$

If $\mathfrak{s}$ is a finite dimensional complex Lie algebra, we denote by $P(\mathfrak{s},t)$, the Poincaré polynomial of $H^*(\mathfrak{s};\mathbb{C})$. So

$$P(Y_{\Phi'},t) = (1+t)^{|\Phi'|}P(\mathfrak{l}_1,t)P(\mathfrak{l}_2,t)\cdots P(\mathfrak{l}_k,t), \text{ by the Künneth formula ;}$$

where $\mathfrak{l}_1,\mathfrak{l}_2,\dots\mathfrak{l}_k$ are the simple factors of the semisimple part $[\mathfrak{l}_{\Phi'},\mathfrak{l}_{\Phi'}]$ of $\mathfrak{l}_{\Phi'}$. If $\mathfrak{s}$ is a finite dimensional complex simple Lie algebra, the Poincaré polynomial $P(\mathfrak{s},t)$ is given by

$$P(\mathfrak{s},t) = (1+t^{2d_1+1})(1+t^{2d_2+1})\cdots(1+t^{2d_l+1}), \qquad (*)$$

where $l = \text{rank}(\mathfrak{s})$ and $d_1,d_2,\dots d_l$ are the exponents of $\mathfrak{s}$ (see the table given below). If $A_{\Phi'}$ is the irreducible unitary representation of G associated with the θ-stable parabolic subalgebra $\mathfrak{q}_{\Phi'}\times\mathfrak{q}_{\Phi'}$, then the Poincaré polynomial of $H^*(\mathfrak{g}\times\mathfrak{g},U;A_{\Phi',U})$ is given by

$$P(\Phi',t) = t^{\dim(\mathfrak{u}_{\Phi'})}(1+t)^{|\Phi'|}P(\mathfrak{l}_1,t)P(\mathfrak{l}_2,t)\cdots P(\mathfrak{l}_k,t),$$

where each $P(\mathfrak{l}_i,t)$ is given by the formula $(*)$. Also for Φ', $\Phi''\subset\Phi$, $A_{\Phi'}$ is unitarily equivalent to $A_{\Phi''}$ if and only if $\tilde{\mathfrak{u}}_{\Phi'}\cap\mathfrak{p} = \tilde{\mathfrak{u}}_{\Phi''}\cap\mathfrak{p}$ if and only if $\mathfrak{u}_{\Phi'} = \mathfrak{u}_{\Phi''}$ if and only if $\Phi' = \Phi''$.
The exponents of complex simple Lie algebras are given below :

Table 5.1: Table for the exponents of a complex simple Lie algebra $\mathfrak{s}$ of rank l

$\mathfrak{s}$	$d_1,d_2,\dots,d_l$
$\mathfrak{a}_l$	$1,2,\dots,l$
$\mathfrak{b}_l$	$1,3,\dots,2l-1$
$\mathfrak{c}_l$	$1,3,\dots,2l-1$
δ_l	$1,3,\dots,2l-3,l-1$
$\mathfrak{e}_6$	$1,4,5,7,8,11$
$\mathfrak{e}_7$	$1,5,7,9,11,13,17$
$\mathfrak{e}_8$	$1,7,11,13,17,19,23,29$
$\mathfrak{f}_4$	$1,5,7,11$
$\mathfrak{g}_2$	$1,5$

5.3. Proof of Theorem 1.2: Let G be a connected complex simple Lie group with $\text{Lie}(G) = \mathfrak{g}$. Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$, θ be the corresponding Cartan involution of $\mathfrak{g}^{\mathbb{R}}$, and U be the connected Lie subgroup of G with Lie algebra $\mathfrak{u}$. Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$. Choose a system of positive roots Δ^+ in the set of all non-zero roots $\Delta = \Delta(\mathfrak{g},\mathfrak{h})$. Let Φ be the set of all simple roots in Δ^+. We have seen that up to unitary equivalence, the irreducible unitary representations of G with non-zero $(\mathfrak{g}\times\mathfrak{g},U)$-cohomology are in one-one correspondence with $p(\Phi)$, the power set of Φ. The one corresponding to $\Phi'\subset\Phi$ is $A_{\Phi'}$ which is the irreducible unitary representation of G associated with the θ-stable parabolic subalgebra $\mathfrak{q}_{\Phi'}\times\mathfrak{q}_{\Phi'}$ of $\mathfrak{g}$, where

$$\mathfrak{q}_{\Phi'}=\mathfrak{l}_{\Phi'}\oplus\mathfrak{u}_{\Phi'},\quad \mathfrak{l}_{\Phi'}=\mathfrak{h}\oplus\sum_{\substack{n_\psi(\alpha)=0\\ \forall\psi\in\Phi'}}\mathfrak{g}_\alpha,\quad \mathfrak{u}_{\Phi'}=\sum_{\substack{n_\psi(\alpha)>0\\ \text{for some }\psi\in\Phi'}}\mathfrak{g}_\alpha,\quad \alpha=\sum_{\psi\in\Phi}n_\psi(\alpha)\psi\in\Delta.$$

The Poincaré polynomial of $H^*(\mathfrak{g}\times\mathfrak{g},U;A_{\Phi',U})$ is given by

$$P(\Phi',t)=t^{\dim(\mathfrak{u}_{\Phi'})}(1+t)^{|\Phi'|}P(\mathfrak{l}_1,t)P(\mathfrak{l}_2,t)\cdots P(\mathfrak{l}_k,t),$$

where $\mathfrak{l}_1,\mathfrak{l}_2,\dots\mathfrak{l}_k$ are the simple factors of the semisimple part $[\mathfrak{l}_{\Phi'},\mathfrak{l}_{\Phi'}]$ of $\mathfrak{l}_{\Phi'}$. We begin with the following results regarding $P(\Phi',t)$.

Lemma 5.1. *If $\bar\nu$ is an automorphism of the Dynkin diagram of $\mathfrak{g}$, then we have $P(\Phi',t)=P(\bar\nu(\Phi'),t)$ for any $\Phi'\subset\Phi$.*

Proof. Note that

$$\{\alpha\in\Delta: n_\psi(\alpha)>0 \text{ for some } \psi\in\bar\nu(\Phi')\}=\bar\nu(\{\alpha\in\Delta: n_\psi(\alpha)>0 \text{ for some } \psi\in\Phi'\}),$$

and so $\dim(\mathfrak{u}_{\Phi'})=\dim(\mathfrak{u}_{\bar\nu(\Phi')})$. Also the Dynkin diagram of $[\mathfrak{l}_{\Phi'},\mathfrak{l}_{\Phi'}]$ is the subdiagram of the Dynkin diagram of $\mathfrak{g}$ consisting of the vertices in $\Phi\setminus\Phi'$. Now $\Phi\setminus\bar\nu(\Phi')=\bar\nu(\Phi\setminus\Phi')$ and $\bar\nu$ maps the subdiagram consisting of the vertices in $\Phi\setminus\Phi'$ onto the subdiagram consisting of the vertices in $\bar\nu(\Phi\setminus\Phi')$. So $[\mathfrak{l}_{\Phi'},\mathfrak{l}_{\Phi'}]\cong[\mathfrak{l}_{\bar\nu(\Phi')},\mathfrak{l}_{\bar\nu(\Phi')}]$. Hence the proof is complete. ■

Lemma 5.2. *If Φ', $\Phi''\subset\Phi$, then the degree of $P(\Phi',t)$ - the degree of $P(\Phi'',t)=$ dim$(\mathfrak{u}_{\Phi''})-$ dim$(\mathfrak{u}_{\Phi'})$.*

Proof. Recall that $P(\Phi',t)=t^{\dim(\mathfrak{u}_{\Phi'})}P(Y_{\Phi'},t)$, where $Y_{\Phi'}$ is the connected Lie subgroup of U with Lie algebra $\mathfrak{l}_{\Phi'}\cap\mathfrak{u}$, which is a compact real form of $\mathfrak{l}_{\Phi'}$. Hence the degree of $P(Y_{\Phi'},t)$ is the dimension of $Y_{\Phi'}$, which is equal to $\dim(\mathfrak{l}_{\Phi'})=\dim(\mathfrak{g})-2\dim(\mathfrak{u}_{\Phi'})$. Thus the degree of $P(\Phi',t)=\dim(\mathfrak{g})-\dim(\mathfrak{u}_{\Phi'})$. Similarly the degree of $P(\Phi'',t)=\dim(\mathfrak{g})-\dim(\mathfrak{u}_{\Phi''})$. So the degree of $P(\Phi',t)$ - the degree of $P(\Phi'',t)=\dim(\mathfrak{g})-\dim(\mathfrak{u}_{\Phi'})-\dim(\mathfrak{g})+\dim(\mathfrak{u}_{\Phi''})=\dim(\mathfrak{u}_{\Phi''})-\dim(\mathfrak{u}_{\Phi'})$. ■

Remark 5.3. If Φ', $\Phi''\subset\Phi$ with $\Phi'\subset\Phi''$, then $\dim(\mathfrak{u}_{\Phi'})\le\dim(\mathfrak{u}_{\Phi''})$. Thus $\dim(\mathfrak{u}_{\Phi'})\le\dim(\mathfrak{u}_{\Phi''})<$ the degree of $P(\Phi'',t)\le$ the degree of $P(\Phi',t)$. In particular, $\dim(\mathfrak{u}_{\{\alpha\}})\le\dim(\mathfrak{u}_{\Phi})<$ the degree of $P(\Phi,t)\le$ the degree of $P(\{\alpha\},t)$, for any $\alpha\in\Phi$. ■

Lemma 5.4. *Assume $f(t)=(1+t)(1+t^3)(1+t^5)\cdots(1+t^{2l+1})$ $(l\in\mathbb{N})$. Then the coefficients of t^2 and $t^{(l+1)^2-2}$ in $f(t)$ are zero and the coefficients of t^n $(0\le n\le(l+1)^2,\ n\ne2,\ (l+1)^2-2)$ in $f(t)$ are non-zero.*

Proof. We shall prove this by induction on l. For $l=1$, the result is obviously true. Assume that $l>1$ and the result is true for $l-1$. So the coefficients of t^2, t^{l^2-2} in $(1+t)(1+t^3)(1+t^5)\cdots(1+t^{2l-1})$ are zero and the coefficients of t^n $(0\le n\le l^2,\ n\ne2,\ l^2-2)$ are non-zero. Now the degree of $f(t)=(l+1)^2$, and so the coefficient of t^n is non-zero iff the coefficient of $t^{(l+1)^2-n}$ is non-zero.

Since $(l+1)^2-(l^2-2)=2l+3=1+3+(2l-1)$, $(l+1)^2-(l^2+2l)=1$, $(l+1)^2-(l^2+2m-1)=2l-2m+2=1+(2(l-m)+1)$, and finally also $(l+1)^2-(l^2+2m)=2(l-m)+1$ for all $1\le m\le l-1$, we have the coefficients of $t^{(l+1)^2-(l^2-2)}$, $t^{(l+1)^2-(l^2+2l)}$, $t^{(l+1)^2-(l^2+2m-1)}$, $t^{(l+1)^2-(l^2+2m)}$ in $f(t)$ are non-zero for all $1\le m\le l-1$, and thus the coefficients of $t^{(l^2-2)}$, $t^{(l^2+2l)}$, $t^{(l^2+2m-1)}$, $t^{(l^2+2m)}$ in $f(t)$ are non-zero for all $1\le m\le l-1$. Clearly the coefficient of t^2 and so the coefficient of $t^{(l+1)^2-2}$ are zero. Hence the result. ■

Now we shall return to the proof of Theorem 1.2. Note that $X:=G/U$ is a Riemannian globally symmetric space of type IV, and

$$H^k(\Gamma\backslash X;\mathbb{C})\cong\bigoplus_{\Phi'\subset\Phi} m_{\Phi'}H^k(\mathfrak{g}\times\mathfrak{g},U;A_{\Phi',U})$$

for all k and for any uniform lattice Γ of G, where $m_{\Phi'}$ is the multiplicity of $A_{\Phi'}$ in $L^2(\Gamma\backslash G)$. For the empty subset ϕ, $m_\phi=1$. Now by Theorem 1.1, for each $i=1,2$, or 3, there exists $\Gamma\in\mathcal{L}_i(G)$ such that $H^k(\Gamma\backslash X;\mathbb{C})$ contains a non-zero cohomology class which has no $H^k(\mathfrak{g}\times\mathfrak{g},U;A_{\phi,U})$-component, for some k (which depends only on $\mathfrak{g}$ if $\Gamma\in\mathcal{L}_1(G)$ or $\mathcal{L}_2(G)$, and it depends on $\mathfrak{g}$ and Γ if $\Gamma\in\mathcal{L}_3(G)$) given as $\dim(X(\bar\sigma))$ and $\dim(X(\bar\sigma\bar\theta))$ in Table 2. Now we shall determine possible $\Phi'\subset\Phi$ such that the non-zero cohomology class in $H^k(\Gamma\backslash X;\mathbb{C})$ has a $H^k(\mathfrak{g}\times\mathfrak{g},U;A_{\Phi',U})$-component, via case by case consideration. Note that

$$\dim(X(\bar\sigma))+\dim(X(\bar\sigma\bar\theta))=\dim_{\mathbb{C}}(\mathfrak{g})=m \text{ (say)}.$$

Let $q=\min\{\dim(X(\bar\sigma)),\ \dim(X(\bar\sigma\bar\theta))\}$. Then for $\Phi'\subset\Phi$, the degree of

$$P(\Phi',t)-(m-q)=\dim(\mathfrak{u}_{\Phi'})+\dim(\mathfrak{l}_{\Phi'})-m+q=q-\dim(\mathfrak{u}_{\Phi'}).$$

Thus $\dim(\mathfrak{u}_{\Phi'})\le q\le$ the degree of $P(\Phi',t)$ iff $\dim(\mathfrak{u}_{\Phi'})\le m-q\le$ the degree of $P(\Phi',t)$, and in this case $q-\dim(\mathfrak{u}_{\Phi'})=$ the degree of $P(\Phi',t)-(m-q)$. So the coefficient of t^q in $P(\Phi',t)$ is non-zero iff the coefficient of t^{m-q} in $P(\Phi',t)$ is non-zero. Hence it is sufficient to determine possible $\Phi'\subset\Phi$ such that the non-zero cohomology class in $H^k(\Gamma\backslash X;\mathbb{C})$ (where $k=\min\{\dim(X(\bar\sigma)),\ \dim(X(\bar\sigma\bar\theta))\}$) has a $H^k(\mathfrak{g}\times\mathfrak{g},U;A_{\Phi',U})$-component.

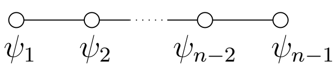
1. $\mathfrak{g}=\mathfrak{a}_{n-1}$, $n>1$. Then

$k=2p(n-p),\ p^2+(n-p)^2-1,\ \frac{(n-2)(n+1)}{2},\ \frac{n(n+1)}{2}\ (1\le p\le\frac{n}{2})$, if $n\in4\mathbb{Z}$;

$k=2p(n-p),\ p^2+(n-p)^2-1,\ \frac{(n-2)(n+1)}{2},\ \frac{n(n+1)}{2}\ (1\le p<\frac{n}{2})$, if $n\in2+4\mathbb{Z},\ n>2$;

$k=2p(n-p),\ p^2+(n-p)^2-1,\ \frac{n(n-1)}{2},\ \frac{(n-1)(n+2)}{2},\ (1\le p\le\frac{n-1}{2})$, if $n\in1+2\mathbb{Z}$.

$\mathfrak{a}_{n-1}$: $(n>1)$



Here $\Phi=\{\psi_1,\psi_2,\dots,\psi_{n-1}\}$, and $\dim(\mathfrak{u}_{\{\psi_j\}})=j(n-j)$ for all $1\le j\le n-1$.

$P(\{\psi_1\},t)=t^{n-1}(1+t)(1+t^3)(1+t^5)\cdots(1+t^{2n-3})=P(\{\psi_{n-1}\},t)$, and

$P(\{\psi_j\},t)=t^{j(n-j)}(1+t)(1+t^3)(1+t^5)\cdots(1+t^{2j-1})(1+t^3)(1+t^5)\cdots(1+t^{2n-2j-1})$

for all $2\le p\le n-2$. Also $P(\Phi,t)=t^{\frac{n(n-1)}{2}}(1+t)^{n-1}$. Now

$\dim(\mathfrak{u}_{\{\psi_j\}})-\dim(\mathfrak{u}_{\{\psi_{j-1}\}})=j(n-j)-(j-1)(n-j+1)=-j+n-j+1=n-2j+1.$

Thus $\dim(\mathfrak{u}_{\{\psi_j\}}) - \dim(\mathfrak{u}_{\{\psi_{j-1}\}}) > 0$ for all $1 < j < \frac{n+1}{2}$. That is,

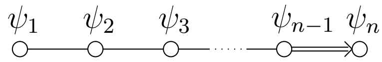
$$\dim(\mathfrak{u}_{\{\psi_1\}}) < \dim(\mathfrak{u}_{\{\psi_2\}}) < \cdots < \dim(\mathfrak{u}_{\{\psi_c\}}),$$

where $c = \frac{n-1}{2}$, if n is odd, and $c = \frac{n}{2}$, if n is even. Clearly

$$\dim(\mathfrak{u}_{\{\psi_{n-j}\}}) = \dim(\mathfrak{u}_{\{\psi_j\}}) \text{ for all } 1 \le j \le c.$$

First consider $k = 2(n-1)$. Note that $2(n-1) < 3(n-3) = \dim(\mathfrak{u}_{\{\psi_3\}})$ iff $n > 7$. So if $n > 7$, $2(n-1) < \dim(\mathfrak{u}_{\Phi'})$ for any $\Phi' \subset \Phi$ with $\psi_j \in \Phi'$ for some $3 \le j \le n-3$, since $\dim(\mathfrak{u}_{\Phi'}) \ge \dim(\mathfrak{u}_{\{\alpha\}})$ for any $\alpha \in \Phi'$. So the coefficient of $t^{2(n-1)}$ in $P(\Phi', t)$ is zero for any $\Phi' \subset \Phi$ with $\psi_j \in \Phi'$ for some $3 \le j \le n-3$, if $n > 7$. Also the coefficient of $t^{2(n-1)}$ in $P(\{\psi_1\}, t)$ is non-zero iff $n \ne 3$ (by Lemma 5.4). Since $2(n-1) = 2(n-2) + 2$, the coefficient of $t^{2(n-1)}$ in $P(\{\psi_2\}, t)$ is always zero (by Lemma 5.4). Now $\dim(\mathfrak{u}_{\{\psi_1,\psi_2\}}) = 2n-3 = \dim(\mathfrak{u}_{\{\psi_1,\psi_{n-1}\}})$, $\dim(\mathfrak{u}_{\{\psi_1,\psi_{n-2}\}}) = 3n-7$, $\dim(\mathfrak{u}_{\{\psi_2,\psi_{n-2}\}}) = 4n-12$. So the coefficients of $t^{2(n-1)}$ in $P(\{\psi_1,\psi_2\}, t)$, $P(\{\psi_1,\psi_{n-1}\}, t)$, $P(\{\psi_{n-1},\psi_{n-2}\}, t)$ are non-zero. Thus we do not get any significant result. The other values of k also do not give any significant result.

2. $\mathfrak{g} = \mathfrak{b}_n$ $(n \ge 2)$.

$\mathfrak{b}_n$:


$(n \ge 2)$

Here $\Phi = \{\psi_1, \psi_2, \ldots, \psi_n\}$, and $\dim(\mathfrak{u}_{\{\psi_j\}}) = 2j(n-j) + \frac{j(j+1)}{2}$ for all $1 \le j \le n$.

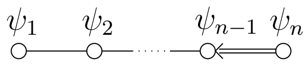
$$P(\{\psi_j\}, t) = t^{2j(n-j)+\frac{j(j+1)}{2}}(1+t)(1+t^3)(1+t^5)\cdots(1+t^{2j-1})(1+t^3)(1+t^7)\cdots \cdots(1+t^{4n-4j-1}) \text{ for all } 1 \le j \le n-1, \text{ and}$$

$$P(\{\psi_n\}, t) = t^{\frac{n(n+1)}{2}}(1+t)(1+t^3)(1+t^5)\cdots(1+t^{2n-1}).$$

Also $P(\Phi, t) = t^{n^2}(1+t)^n$.

In this case, we do not have Theorem 1.1. See Table 2.

3. $\mathfrak{g} = \mathfrak{c}_n$, $n \ge 3$. Then $k = 4p(n-p)$, $p(2p+1) + (n-p)(2n-2p+1)$, n^2, $n(n+1)$ $(1 \le p \le n-1)$, if $n \in 4\mathbb{Z}$ or $n \in 3+4\mathbb{Z}$; $k = 4p(n-p)$, $p(2p+1)+(n-p)(2n-2p+1)$, $(1 \le p \le n-1,\ p \ne \frac{n}{2})$, if $n \in 1+4\mathbb{Z}$ or $n \in 2+4\mathbb{Z}$.

$\mathfrak{c}_n$:


$(n \ge 3)$

Here $\Phi = \{\psi_1, \psi_2, \ldots, \psi_{n-1}, \psi_n\}$, and $\dim(\mathfrak{u}_{\{\psi_j\}}) = 2j(n-j) + \frac{j(j+1)}{2}$.

$$P(\{\psi_j\}, t) = t^{2j(n-j)+\frac{j(j+1)}{2}}(1+t)(1+t^3)(1+t^5)\cdots(1+t^{2j-1})(1+t^3)(1+t^7)\cdots \cdots(1+t^{4n-4j-1}) \text{ for all } 1 \le j \le n-1, \text{ and}$$

$$P(\{\psi_n\}, t) = t^{\frac{n(n+1)}{2}}(1+t)(1+t^3)(1+t^5)\cdots(1+t^{2n-1}).$$

Also $P(\Phi, t) = t^{n^2}(1+t)^n$. Now

$$\dim(\mathfrak{u}_{\{\psi_j\}}) - \dim(\mathfrak{u}_{\{\psi_{j-1}\}}) = 2j(n-j) + \frac{j(j+1)}{2} - 2(j-1)(n-j+1) - \frac{(j-1)j}{2}$$
$$= 2(n-j+1) - 2j + j = 2n + 2 - 3j.$$

So $\dim(\mathfrak{u}_{\{\psi_j\}}) - \dim(\mathfrak{u}_{\{\psi_{j-1}\}}) > 0$ for all $1 < j < \frac{2n+2}{3}$. Thus

$$\dim(\mathfrak{u}_{\{\psi_1\}}) < \dim(\mathfrak{u}_{\{\psi_2\}}) < \cdots < \dim(\mathfrak{u}_{\{\psi_c\}}) \geq \dim(\mathfrak{u}_{\{\psi_{c+1}\}}) > \dim(\mathfrak{u}_{\{\psi_{c+2}\}}) > \cdots \\ \cdots > \dim(\mathfrak{u}_{\{\psi_n\}}),$$

where $c = \frac{2n}{3}$, if $n \in 3\mathbb{Z}$, $c = \frac{2n+1}{3}$, if $n \in 1 + 3\mathbb{Z}$, and $c = \frac{2n-1}{3}$, if $n \in 2 + 3\mathbb{Z}$.

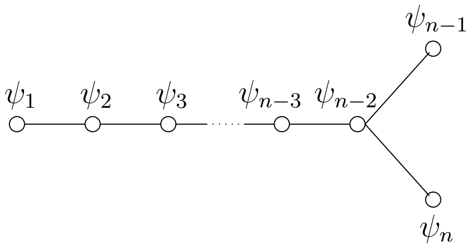
First consider $k = 4(n-1)$.

Note that $4(n-1) < \dim(\mathfrak{u}_{\{\psi_3\}})$, and $4(n-1) < \frac{n(n+1)}{2} = \dim(\mathfrak{u}_{\{\psi_n\}})$ iff $n \geq 6$. So if $n \geq 6$, $4(n-1) < \dim(\mathfrak{u}_{\Phi'})$ for any $\Phi' \subset \Phi$ with $\psi_j \in \Phi'$ for some $3 \leq j \leq n$, since $\dim(\mathfrak{u}_{\Phi'}) \geq \dim(\mathfrak{u}_{\{\alpha\}})$ for any $\alpha \in \Phi'$. So the coefficient of $t^{4(n-1)}$ in $P(\Phi', t)$ is zero for any $\Phi' \subset \Phi$, $\Phi' \neq \phi$, $\{\psi_1\}$, $\{\psi_2\}$, $\{\psi_1, \psi_2\}$, if $n \geq 6$.

Since $4(n-1) = \dim(\mathfrak{u}_{\{\psi_2\}}) + 1 = \dim(\mathfrak{u}_{\{\psi_1,\, \psi_2\}})$, the coefficients of $t^{4(n-1)}$ in $P(\{\psi_2\}, t)$ and $P(\{\psi_1,\, \psi_2\}, t)$ are non-zero. Now $4(n-1) - \dim(\mathfrak{u}_{\{\psi_1\}}) = 2n-3$, and $P(\{\psi_1\}, t) = t^{2n-1}(1+t)(1+t^3)(1+t^7)\cdots(1+t^{4(n-1)-1})$. Note that $2n-3 = \frac{4(n-1)}{2} - 1$ if n is odd, and $2n-3 = 3 + 7 + (\frac{4(n-6)}{2} - 1)$ if n is even. So if n is odd, then the coefficient of $t^{4(n-1)}$ in $P(\{\psi_1\}, t)$ is non-zero. And if n is even with $n \geq 12$, then the coefficient of $t^{4(n-1)}$ in $P(\{\psi_1\}, t)$ is non-zero. Also for $n = 6,\ 8$, or 10, the coefficient of $t^{4(n-1)}$ in $P(\{\psi_1\}, t)$ is zero. Thus if $n \geq 6$, the non-zero cohomology class in $H^{4(n-1)}(\Gamma\backslash X; \mathbb{C})$ has a $H^{4(n-1)}(\mathfrak{g}\times\mathfrak{g}, U; A_{\Phi', U})$-component, where $\Phi' = \{\psi_1\}$, or $\{\psi_2\}$, or $\{\psi_1,\ \psi_2\}$. If $n = 6, 8$, or 10, we can discard $\{\psi_1\}$ among these. This implies in particular that if $\mathrm{Lie}(G) = \mathfrak{c}_n (n \geq 6)$, then for each $i = 1, 2$, or 3, there is a uniform lattice $\Gamma \in \mathcal{L}_i(G)$, such that $L^2(\Gamma\backslash G)$ has an irreducible $A_{\Phi'}$-component for at least one Φ' given above. The other values of k do not give any significant result.

4. $\mathfrak{g} = \delta_n$, $n \geq 4$. Then $k = p(2n-p)$, $\frac{p(p-1)+(2n-p)(2n-p-1)}{2}$, $n(n-1)$, n^2 $(1 \leq p \leq n-1)$, if $n \notin 2 + 4\mathbb{Z}$; $k = p(2n-p)$, $\frac{p(p-1)+(2n-p)(2n-p-1)}{2}$ $(1 \leq p \leq n-1)$, if $n \in 2 + 4\mathbb{Z}$.

δ_n :
$(n \geq 4)$



Here $\Phi = \{\psi_1, \psi_2, \ldots, \psi_{n-2}, \psi_{n-1}, \psi_n\}$; $\dim(\mathfrak{u}_{\{\psi_j\}}) = 2j(n-j) + \frac{j(j-1)}{2}$ for all $1 \leq j \leq n, j \neq n-1$, and $\dim(\mathfrak{u}_{\{\psi_{n-1}\}}) = \dim(\mathfrak{u}_{\{\psi_n\}})$.

$$P(\{\psi_j\}, t) = t^{2j(n-j)+\frac{j(j-1)}{2}}(1+t)(1+t^3)(1+t^5)\cdots(1+t^{2j-1})(1+t^3)(1+t^7)\cdots \\ \cdots(1+t^{4n-4j-5})(1+t^{2n-2j-1}) \ \text{ for all } 1 \leq j \leq n-2, \text{ and}$$

$$P(\{\psi_n\}, t) = t^{\frac{n(n-1)}{2}}(1+t)(1+t^3)(1+t^5)\cdots(1+t^{2n-1}) = P(\{\psi_{n-1}\}, t).$$

Also $P(\Phi, t) = t^{n(n-1)}(1+t)^n$. Now

$$\begin{aligned} &\dim(\mathfrak{u}_{\{\psi_j\}}) - \dim(\mathfrak{u}_{\{\psi_{j-1}\}}) \\ &\qquad = 2j(n-j) + \frac{j(j-1)}{2} - 2(j-1)(n-j+1) - \frac{(j-1)(j-2)}{2} \\ &\qquad = 2(n-j+1) - 2j + j - 1 = 2n + 1 - 3j \ \text{ for all } 1 < j \leq n-2. \end{aligned}$$

Thus $\dim(\mathfrak{u}_{\{\psi_1\}}) < \dim(\mathfrak{u}_{\{\psi_2\}}) < \cdots < \dim(\mathfrak{u}_{\{\psi_c\}}) \geq \dim(\mathfrak{u}_{\{\psi_{c+1}\}})$
$> \dim(\mathfrak{u}_{\{\psi_{c+2}\}}) > \cdots > \dim(\mathfrak{u}_{\{\psi_{n-2}\}}) > \dim(\mathfrak{u}_{\{\psi_{n-1}\}}) = \dim(\mathfrak{u}_{\{\psi_n\}})$,

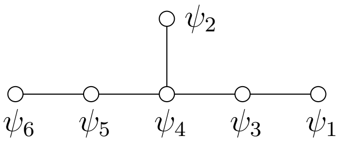
where $c = \frac{2n}{3}$, if $n \in 3\mathbb{Z}$, $c = \frac{2n-2}{3}$, if $n \in 1 + 3\mathbb{Z}$, and $c = \frac{2n-1}{3}$, if $n \in 2 + 3\mathbb{Z}$.
First consider $k = (2n-1)$. Note that

$$2n - 1 < 2j(n-j) < 2j(n-j) + \tfrac{j(j-1)}{2} = \dim(\mathfrak{u}_{\{\psi_j\}})$$

for all $2 \leq j \leq n-2$, and $2n - 1 < \frac{n(n-1)}{2} = \dim(\mathfrak{u}_{\{\psi_{n-1}\}}) = \dim(\mathfrak{u}_{\{\psi_n\}})$ iff $n > 4$. So if $n > 4$, $2n - 1 < \dim(\mathfrak{u}_{\Phi'})$ for any $\Phi' \subset \Phi$ with $|\Phi'| \geq 2$, as $\psi_j \in \Phi'$ for some $2 \leq j \leq n$, and $\dim(\mathfrak{u}_{\Phi'}) \geq \dim(\mathfrak{u}_{\{\alpha\}})$ for any $\alpha \in \Phi'$. Therefore the coefficient of t^{2n-1} in $P(\Phi', t)$ is zero for any $\Phi' \subset \Phi$, $\Phi' \neq \phi$, $\{\psi_1\}$, if $n > 4$. Since we have $2n - 1 = \dim(\mathfrak{u}_{\{\psi_1\}}) + 1$, the coefficient of t^{2n-1} in $P(\{\psi_1\}, t)$ is non-zero. Thus the non-zero cohomology class in $H^{2n-1}(\Gamma\backslash X; \mathbb{C})$ has only $H^{2n-1}(\mathfrak{g} \times \mathfrak{g}, U; A_{\{\psi_1\},U})$-component. This implies in particular that if $\mathrm{Lie}(G) = \delta_n (n > 4)$, then for each $i = 1, 2,$ or 3, there is a uniform lattice $\Gamma \in \mathcal{L}_i(G)$, such that $L^2(\Gamma\backslash G)$ has an irreducible $A_{\{\psi_1\}}$-component. If $n = 4$, then corresponding to the value $k = 2n - 1 = 7$, we can say that at least one $A_{\{\psi_j\}}$ $(j = 1, 3, \text{or } 4)$ will occur in $L^2(\Gamma\backslash G)$. The other values of k do not give any significant result.

5. $\mathfrak{g} = \mathfrak{e}_6$. Then $k = 26,\ 52,\ 32,\ 46,\ 36,\ 42,\ 38,\ 40$.

$\mathfrak{e}_6$:

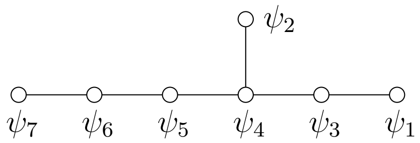


Here $\Phi = \{\psi_1, \psi_2, \ldots, \psi_6\}$, and
$P(\{\psi_1\}, t) = t^{16}(1+t)(1+t^3)(1+t^7)(1+t^{11})(1+t^{15})(1+t^9) = P(\{\psi_6\}, t)$,
$P(\{\psi_2\}, t) = t^{21}(1+t)(1+t^3)(1+t^5)(1+t^7)(1+t^9)(1+t^{11})$,
$P(\{\psi_3\}, t) = t^{25}(1+t)(1+t^3)^2(1+t^5)(1+t^7)(1+t^9) = P(\{\psi_5\}, t)$, and
$P(\{\psi_4\}, t) = t^{29}(1+t)(1+t^3)^3(1+t^5)^2$.
Also $P(\Phi, t) = t^{36}(1+t)^6$.

Clearly $\dim(\mathfrak{u}_{\{\psi_j\}}) < 26,\ 52,\ 32,\ 46,\ 36,\ 42,\ 38,\ 40 <$ the degree of $P(\{\psi_j\}, t)$ for all j except $j = 4$, and $\dim(\mathfrak{u}_{\{\psi_4\}}) < 32,\ 46,\ 36,\ 42,\ 38,\ 40 <$ the degree of $P(\{\psi_4\}, t)$. Not only these, but also the coefficients of t^k in $P(\{\psi_j\}, t)$ are non-zero for all $1 \leq j \leq 6$, $k = 32,\ 46,\ 36,\ 42,\ 38,\ 40$; and the coefficients of t^k in $P(\{\psi_j\}, t)$ are non-zero for all $1 \leq j \leq 6$, $j \neq 4$, $k = 26,\ 52$. Thus we do not get any significant result.

6. $\mathfrak{g} = \mathfrak{e}_7$. Then $k = 64,\ 69$.

$\mathfrak{e}_7$:



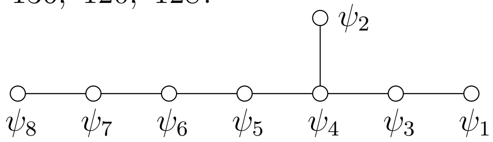
Here $\Phi = \{\psi_1, \psi_2, \ldots, \psi_7\}$, and
$P(\{\psi_1\}, t) = t^{33}(1+t)(1+t^3)(1+t^7)(1+t^{11})(1+t^{15})(1+t^{19})(1+t^{11})$,
$P(\{\psi_2\}, t) = t^{42}(1+t)(1+t^3)(1+t^5)(1+t^7)(1+t^9)(1+t^{11})(1+t^{13})$,
$P(\{\psi_3\}, t) = t^{47}(1+t)(1+t^3)^2(1+t^5)(1+t^7)(1+t^9)(1+t^{11})$,

$P(\{\psi_4\},t) = t^{53}(1+t)(1+t^3)^3(1+t^5)^2(1+t^7)$,
$P(\{\psi_5\},t) = t^{50}(1+t)(1+t^3)^2(1+t^5)^2(1+t^7)(1+t^9)$,
$P(\{\psi_6\},t) = t^{42}(1+t)(1+t^3)^2(1+t^7)(1+t^{11})(1+t^{15})(1+t^9)$, and
$P(\{\psi_7\},t) = t^{27}(1+t)(1+t^3)(1+t^9)(1+t^{11})(1+t^{15})(1+t^{17})(1+t^{23})$.
Also $P(\Phi,t) = t^{63}(1+t)^7$.

Clearly $\dim(\mathfrak{u}_{\{\psi_j\}}) < \dim(\mathfrak{u}_\Phi) < 64,\ 69 <$ the degree of $P(\Phi,t) <$ the degree of $P(\{\psi_j\},t)$ for all j. Not only these, but also the coefficients of t^k in $P(\{\psi_j\},t)$ are non-zero for all $1 \le j \le 7,\ k = 64,\ 69$. Thus we do not get any significant result.

7. $\mathfrak{g} = \mathfrak{e}_8$. Then $k = 112,\ 136,\ 120,\ 128$.

$\mathfrak{e}_8$:

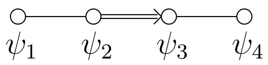


Here $\Phi = \{\psi_1, \psi_2, \ldots, \psi_8\}$, and
$P(\{\psi_1\},t) = t^{78}(1+t)(1+t^3)(1+t^7)(1+t^{11})(1+t^{15})(1+t^{19})(1+t^{23})(1+t^{13})$,
$P(\{\psi_2\},t) = t^{92}(1+t)(1+t^3)(1+t^5)(1+t^7)(1+t^9)(1+t^{11})(1+t^{13})(1+t^{15})$,
$P(\{\psi_3\},t) = t^{98}(1+t)(1+t^3)^2(1+t^5)(1+t^7)(1+t^9)(1+t^{11})(1+t^{13})$,
$P(\{\psi_4\},t) = t^{106}(1+t)(1+t^3)^3(1+t^5)^2(1+t^7)(1+t^9)$,
$P(\{\psi_5\},t) = t^{104}(1+t)(1+t^3)^2(1+t^5)^2(1+t^7)^2(1+t^9)$,
$P(\{\psi_6\},t) = t^{97}(1+t)(1+t^3)^2(1+t^5)(1+t^7)(1+t^{11})(1+t^{15})(1+t^9)$,
$P(\{\psi_7\},t) = t^{83}(1+t)(1+t^3)^2(1+t^9)(1+t^{11})(1+t^{15})(1+t^{17})(1+t^{23})$, and
$P(\{\psi_8\},t) = t^{57}(1+t)(1+t^3)(1+t^{11})(1+t^{15})(1+t^{19})(1+t^{23})(1+t^{27})(1+t^{35})$.
Also $P(\Phi,t) = t^{120}(1+t)^8$.

Clearly $\dim(\mathfrak{u}_{\{\psi_j\}}) < 112,\ 136,\ 120,\ 128 <$ the degree of $P(\{\psi_j\},t)$ for all j. Not only these, but also the coefficients of t^k in $P(\{\psi_j\},t)$ are non-zero for all $1 \le j \le 8$, $k = 112,\ 136,\ 120,\ 128$. Thus we do not get any significant result.

8. $\mathfrak{g} = \mathfrak{f}_4$. Then $k = 16,\ 36,\ 24,\ 28$.

$\mathfrak{f}_4$:

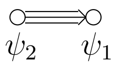


Here $\Phi = \{\psi_1, \psi_2, \psi_3, \psi_4\}$, and
$P(\{\psi_1\},t) = t^{15}(1+t)(1+t^3)(1+t^7)(1+t^{11})$,
$P(\{\psi_2\},t) = t^{20}(1+t)(1+t^3)^2(1+t^5)$,
$P(\{\psi_3\},t) = t^{20}(1+t)(1+t^3)^2(1+t^5)$, and
$P(\{\psi_4\},t) = t^{15}(1+t)(1+t^3)(1+t^7)(1+t^{11})$.
Also $P(\Phi,t) = t^{24}(1+t)^4$.

Clearly $\dim(\mathfrak{u}_{\{\psi_j\}}) < \dim(\mathfrak{u}_\Phi) = 24 < 28 =$ the degree of $P(\Phi,t) <$ the degree of $P(\{\psi_j\},t)$ for all j. But the coefficients of t^k in $P(\{\psi_j\},t)$ are non-zero only for $j = 2,\ 3$; $k = 24,\ 28$. Yet these values do not give any significant result. Now consider the values $k = 16,\ 36$. Clearly the coefficient of t^k ($k = 16,\ 36$) in $P(\Phi',t)$ is zero for any $\Phi' \subset \Phi$ such that Φ' contains ψ_2 or ψ_3. Also $16 < 20 = \dim(\mathfrak{u}_{\{\psi_1,\ \psi_4\}}) < 32 =$ the degree of $P(\{\psi_1, \psi_4\},t) < 36$, and the coefficients of t^k ($k = 16,\ 36$) in $P(\{\psi_1\},t)$, $P(\{\psi_4\},t)$ are non-zero. Thus the non-zero cohomology class in $H^k(\Gamma\backslash X;\mathbb{C})$ $(k = 16,\ 36)$ has a $H^k(\mathfrak{g}\times\mathfrak{g}, U; A_{\Phi',U})$-component, where $\Phi' = \{\psi_1\}$, or $\{\psi_4\}$.

This implies in particular that if $\mathrm{Lie}(G) = \mathfrak{f}_4$, then for each $i = 1, 2$, or 3, there is a uniform lattice $\Gamma \in \mathcal{L}_i(G)$, such that $L^2(\Gamma\backslash G)$ has an irreducible $A_{\Phi'}$-component for at least one Φ' given above.

9. $\mathfrak{g} = \mathfrak{g}_2$. Then $k = 6,\ 8$. $\mathfrak{g}_2$:



Here $\Phi = \{\psi_1, \psi_2\}$, $P(\{\psi_1\}, t) = t^5(1+t)(1+t^3)$, and $P(\{\psi_2\}, t) = t^5(1+t)(1+t^3)$. Also $P(\Phi, t) = t^6(1+t)^2$.

Clearly the coefficients of t^k in $P(\{\psi_1\}, t)$, $P(\{\psi_2\}, t)$, $P(\{\psi_1,\ \psi_2\}, t)$ are non-zero, $k = 6,\ 8$. Thus we do not get any significant result.

Thus the proof of Theorem 1.2 is complete. ■

Remark 5.5. (i) If $\mathfrak{g} = \mathfrak{a}_2$, then $k = 3,\ 4,\ 5$. Also the coefficients of t^4 in $P(\{\psi_1\}, t)$, $P(\{\psi_2\}, t)$ are zero, and the coefficient of t^4 in $P(\{\psi_1,\ \psi_2\}, t)$ is non-zero. This shows that if $\mathrm{Lie}(G) = \mathfrak{a}_2$, then for each $i = 1, 2$, or 3, there is a uniform lattice $\Gamma \in \mathcal{L}_i(G)$, such that $L^2(\Gamma\backslash G)$ has an irreducible $A_{\{\psi_1,\psi_2\}}$-component. That is, $A_{\{\psi_1,\psi_2\}}$ is an automorphic representation of G. See [32, Cor. 7.7] for $n = 3$.

(ii) Let G be a connected non-compact semisimple Lie group with finite centre, K be a maximal compact subgroup of G with θ, the corresponding Cartan involution, and $X = G/K$. Let σ be an involutive automorphism of G such that $\sigma\theta = \theta\sigma$, $G(\sigma) = \{g \in G : \sigma(g) = g\}$, $K(\sigma) = K \cap G(\sigma)$, and $X(\sigma) = G(\sigma)/K(\sigma)$. Let $\mathfrak{g} = \mathrm{Lie}(G)$, and $\mathfrak{g}(\sigma) = \mathrm{Lie}(G(\sigma))$. Let Γ be a torsion-free σ-stable uniform lattice in G such that $\Gamma(\sigma)\backslash X(\sigma)$ is embedded inside $\Gamma\backslash X$, where $\Gamma(\sigma) = \Gamma \cap G(\sigma)$. Let $C(\sigma, \Gamma)$ be the image of $\Gamma(\sigma)\backslash X(\sigma)$ in $\Gamma\backslash X$, and $\mathcal{P}(C(\sigma, \Gamma))$ be the Poincaré dual of the fundamental class $[C(\sigma, \Gamma)]$. Let $A_\mathfrak{q}$ be the irreducible unitary representation with trivial infinitesimal character associated with the θ-parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. Then we have

> *If G is simple, $A_\mathfrak{q}$ is not the trivial representation of G, and $A_\mathfrak{q}$ is discretely decomposable as a $(\mathfrak{g}(\sigma), K(\sigma))$-module, then $\mathcal{P}(C(\sigma, \Gamma))$ does not have a $A_\mathfrak{q}$-component* [25, Cor. 4.2].

This is a modification of [19, Th. 4.3], and is corollary of a more general result in [25, Th. 4.1], [18, Th. 1.2]. Now there is a classification of all pairs $(\mathfrak{g}, \mathfrak{g}(\sigma))$ and the modules $A_\mathfrak{q}$, such that $A_\mathfrak{q}$ is discretely decomposable as a $(\mathfrak{g}(\sigma), K(\sigma))$-module. See [20], the work of which is based on [16], [17]. According to this classification, if $\mathfrak{g} = \mathfrak{a}_{2n},\ \mathfrak{b}_n,\ \mathfrak{c}_n,\ \mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_8,\ \mathfrak{f}_4,\ \mathfrak{g}_2$, then there is no involutive automorphism σ and θ-stable parabolic subalgebra $\mathfrak{q}(\neq \mathfrak{g})$ such that $A_\mathfrak{q}$ is discretely decomposable as a $(\mathfrak{g}(\sigma), K(\sigma))$-module ([20, Th. 4.12]). If $\mathfrak{g} = \mathfrak{a}_{2n-1}$, and $\mathfrak{g}(\sigma) = \mathfrak{sp}(n, \mathbb{C})$, or $\mathfrak{su}^*(2n)$, then $A_\mathfrak{q}$ is discretely decomposable as a $(\mathfrak{g}(\sigma), K(\sigma))$-module iff $\mathfrak{q} = \mathfrak{q}_{\{\psi_1\}}$, or $\mathfrak{q}_{\{\psi_{2n-1}\}}$, or $\mathfrak{q} = \mathfrak{q}_\phi$. For any other $\mathfrak{g}(\sigma)$, no non-trivial representation $A_\mathfrak{q}$ is discretely decomposable as a $(\mathfrak{g}(\sigma), K(\sigma))$-module. If $\mathfrak{g} = \delta_n$, and $\mathfrak{g}(\sigma) = \mathfrak{so}(2n-1, \mathbb{C})$, or $\mathfrak{so}(2n-1, 1)$, then $A_\mathfrak{q}$ is discretely decomposable as a $(\mathfrak{g}(\sigma), K(\sigma))$-module iff $\mathfrak{q} = \mathfrak{q}_{\{\psi_n\}}$, or $\mathfrak{q} = \mathfrak{q}_\phi$. For any other $\mathfrak{g}(\sigma)$, no non-trivial representation $A_\mathfrak{q}$ is discretely decomposable as a $(\mathfrak{g}(\sigma), K(\sigma))$-module. See [20, Table C.4, C.5]. We know from the proof of Theorem 1.2 that if $\mathfrak{g} = \delta_4$, at least one $A_{\{\psi_j\}}$ $(j = 1, 3, \text{or } 4)$

is an automorphic representation. Now if we apply the above results for $\mathfrak{g} = \delta_4$, we see that either $A_{\{\psi_1\}}$, or $A_{\{\psi_3\}}$ is an automorphic representation. ∎

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