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Singularities of Intertwining Operators and Decompositions of Principal Series Representations

Taeuk Nam, Avner Segal, Lior Silberman

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Abstract. We show that, under certain assumptions, a parabolic induction $\mathrm{Ind}_B^G \lambda$ from the Borel subgroup B of a (real or p-adic) reductive group G decomposes into a direct sum of the form:

$$\mathrm{Ind}_B^G \lambda = \left(\mathrm{Ind}_P^G \mathrm{St}_M \otimes \chi_0\right) \oplus \left(\mathrm{Ind}_P^G \mathbf{1}_M \otimes \chi_0\right),$$

where P is a parabolic subgroup of G with Levi subgroup M of semi-simple rank 1, $\mathbf{1}_M$ is the trivial representation of M, St_M is the Steinberg representation of M and χ_0 is a certain character of M. We construct examples of this phenomenon for all simply-connected simple groups of rank at least 2.

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1. Introduction

Fixing our notation, let F be a local field, $\mathbb{G}$ a reductive F-group. We write $G = \mathbb{G}(F)$ in the analytic topology, and more generally use roman letters to denote the set of F-points of the corresponding algebraic subgroup. Accordingly let $P_0 \subset G$ be a minimal parabolic subgroup (formally $P_0 = \mathbb{P}_0(F)$ where $\mathbb{P}_0 \subset \mathbb{G}$ is a minimal parabolic subgroup defined over F, and similarly for other subgroups), and let $T \subset P_0$ be a Levi subgroup. The principal series of representations of G consists of the admissible representations $\mathrm{Ind}_{P_0}^G \lambda$ (normalized induction) as λ varies over the characters $\mathrm{Hom}_{\mathrm{cts}}(T, \mathbb{C}^\times)$.

Understanding the structure of these representations is a basic problem in the representation theory of G. Common questions about the structure include:

- Is $\mathrm{Ind}_{P_0}^G \lambda$ reducible?
- What is the length of its composition series?
- What are the composition factors? At least the irreducible subrepresentations and quotients?
- What is the composition series?

We specialize to the case of a quasi-split Chevalley group $\mathbb{G}$ defined over F, in which $P_0 = B$ is a Borel subgroup and T is a maximal torus of B of maximal split F-rank.

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We may as well assume $\text{rank}(G) > 1$. Let $\mathfrak{a}_{\mathbb{C}}^* = X^*(T) \otimes \mathbb{C}^\times = \text{Hom}_{\text{ur}}(T, \mathbb{C}^\times)$ be the space of unramified quasicharacters of T.

Fixing $\lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$, we study the induced representation $\text{Ind}_B^G \lambda$. We prove (Theorem 3.4) that, under certain assumptions on λ_0, the representation $\text{Ind}_B^G \lambda$ decomposes as the direct sum

$$\text{Ind}_B^G \lambda_0 = \left(\text{Ind}_P^G \text{St}_M \otimes \chi_0\right) \oplus \left(\text{Ind}_P^G \mathbf{1}_M \otimes \chi_0\right), \tag{1}$$

where:

- P is a parabolic subgroup of G with Levi subgroup M of semi-simple rank 1.
- $\mathbf{1}_M$ (resp. St_M) is the trivial (resp. Steinberg) representation of M.
- χ_0 is a character of M associated to the induction in stages from B to M.

Theorem 3.4 identifies the two invariant subspaces isomorphic to $\text{Ind}_P^G \text{St}_M \otimes \chi_0$ and $\text{Ind}_P^G \mathbf{1}_M \otimes \chi_0$ as eigenspaces of a certain intertwining operator. Furthermore, this shows that, under certain conditions, each of the two admits a unique irreducible subrepresentation. The proof is a generalization to [21, Lemma 3.1].

This decomposition is rather surprising since, for generic χ_0, and the associated $\lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$, only one of the following exact sequences hold

$$\text{Ind}_P^G \text{St}_M \otimes \chi_0 \hookrightarrow \text{Ind}_B^G \lambda_0 \twoheadrightarrow \text{Ind}_P^G \mathbf{1}_M \otimes \chi_0$$
$$\text{Ind}_P^G \mathbf{1}_M \otimes \chi_0 \hookrightarrow \text{Ind}_B^G \lambda_0 \twoheadrightarrow \text{Ind}_P^G \text{St}_M \otimes \chi_0.$$

The reason that these sequences split as in Equation (1) is that λ_0 lies in the intersection between two singularities of a certain standard intertwining operator $N(w, \lambda)$. Namely, $N(w, \lambda)$ admits a simple "pole" along a hyperplane H_1 and a simple "zero" along a hyperplane H_2 such that $\lambda_0 \in H_1 \cap H_2$. In such a case $N(w, \lambda_0)$ is not well defined. However, we show the existence of a line $\mathcal{S}$ along which $N(w, \lambda)$ is well-defined and continuous at λ_0. The limit of $N(w, \lambda)$ at λ_0 along $\mathcal{S}$ is an intertwining operator E of $\text{Ind}_B^G \lambda_0$. Furthermore, $\text{Ind}_P^G \text{St}_M \otimes \chi_0 \oplus \text{Ind}_P^G \mathbf{1}_M \otimes \chi_0$ is a decomposition of $\text{Ind}_B^G \lambda_0$ into eigenspaces of E.

In Section 4, we find an abundant amount of points where the assumptions of Theorem 3.4 are satisfied. We find distinct such λ_0 for every G and every Levi subgroup M as above. In fact, when $rank(G) > 2$, we show the existence of infinitely many such λ_0 (see Theroem 4.1). In particular, one has

Corollary 4.4 *For any simple group G and any simple root α, let $w_\alpha \in W$ be the corresponding simple reflection in the Weyl group and let ω_α be the associated fundamental weight. Let $\lambda_0 = -w_\alpha \cdot \omega_\alpha$. Then*

$$\text{Ind}_B^G \lambda_0 = \left(\text{Ind}_P^G \text{St}_M \otimes \chi_0\right) \oplus \left(\text{Ind}_P^G \mathbf{1}_M \otimes \chi_0\right). \tag{2}$$

We note here that Equation (1) implies that

$$\text{Ind}_B^G(-\lambda_0) = \left(\text{Ind}_P^G \text{St}_M \otimes (-\chi_0)\right) \oplus \left(\text{Ind}_P^G \mathbf{1}_M \otimes (-\chi_0)\right), \tag{3}$$

where we use additive notation for $\mathfrak{a}_{\mathbb{C}}^*$. This, again, is a decomposition into eigenspaces of the limit of $N(w^{-1}, \lambda)$ at $-\lambda_0$. In particular, each of $\text{Ind}_P^G \text{St}_M \otimes (-\chi_0)$ and $\text{Ind}_P^G \mathbf{1}_M \otimes (-\chi_0)$ admits a unique irreducible quotient and it is easy to find the Langlands operator (in the sense of [2, Cor. 4.6] or [11, Cor. 3.2]) for each.

One possible application for the results of this paper is to the computation of the residual spectrum of adelic groups. Namely, the irreducible subrepresentations of $\mathrm{Ind}_B^G \lambda$ can appear as local constituents of residual representations of $\mathbb{G}(\mathbb{A})$. In particular, the eigenvalue of the intertwining operator E on $\mathrm{Ind}_P^G \mathrm{St}_M \otimes \chi_0$ which appears in the proof of Theorem 3.4 dictates which irreducible subrepresentation of $\mathrm{Ind}_{\mathbb{B}(\mathbb{A})}^{\mathbb{G}(\mathbb{A})}(\lambda)$ will appear in the residual spectrum.

Such considerations have appeared in the computation of the residual spectrum of Sp_4 (see [8]), G_2 (see [10] and [21]) and quasi-split forms of $Spin_8$ (see [14] and [16, 17]). It is interesting to note that when $\mathbb{G} = Sp_4$ the unramified local constituents appear only in the non-square-integrable automorphic spectrum as can be seen by comparing [9, Theorem 5.4] with [8, Theorem 3.6(1)].

This paper is organized as follows:

- In Section 2 we discuss the assumptions we make on the group G and recall the definition and basic properties of the normalized intertwining operators used in this paper.
- In Section 3 we prove the main result of this paper (Theorem 3.4 and Corollary 3.6).
- In Section 4 we study a family of examples of points λ_0 for which Theorem 3.4 holds. In particular, for any simple group G and any simple root with respect to T we construct a different point λ_0 which satisfy the assumptions of Theorem 3.4.
- In Section 5 we discuss a generalization of Theorem 3.4 and Theorem 4.1 for decompositions with respect to larger Levi subgroups M.
- In Appendix A we prove a few simple results which did not fit into the body of the paper.

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2. Notation and preliminaries

2.1. Algebraic groups. Let F be a local field of characteristic 0. Let $\mathbb{G}$ be a semi-simple group over F.

It is known (see the next section) that the following assumption guarantees certain analytic properties of normalized intertwining operators. Accordingly, while our results likely hold in greater generality we suppose that:

- If F is Archimedean, assume that $\mathbb{G}$ is a connected, quasi-split, semi-simple, linear Lie group.
- If F is non-Archimedean, assume that $\mathbb{G}$ is a semi-simple Chevalley group in the sense of [18, pg. 21].

The papers [6, Theorem 5.3] (Archimedean case) and [20, Theorem 6.1, p. 953] (p-adic case) determine the analytic behavior of normalized intertwining operators under the hypotheses above. We believe that these necessary properties hold in greater generality; in any case the assumption on G could be replaced with hypotheses on the analytic behavior of the intertwining operators.

Fix a Borel subgroup and a maximal F-split torus $\mathbb{G} \supset \mathbb{B} \supset \mathbb{T}$. Also, let $\mathbb{N} \subset \mathbb{B}$ be the unipotent radical and let $G = \mathbb{G}(F)$, $B = \mathbb{B}(F)$, $T = \mathbb{T}(F)$, $N = \mathbb{N}(F)$.

Let $\Phi = \Phi(\mathbb{G} : \mathbb{T})$ be the set of roots of $\mathbb{G}$ with respect to $\mathbb{T}$, Φ^+ the roots occurring in $\mathbb{N}$, that is the positive roots with respect to the choice of $\mathbb{B}$. We also denote by $\Phi^\vee$ the set of coroots of $\mathbb{G}$ with respect to $\mathbb{T}$ and recall the map

$$\Phi \to \Phi^\vee, \quad \alpha \to \alpha^\vee.$$

Let $\Delta \subset \Phi^+$ be the corresponding set of simple roots. We denote the relative semisimple rank of $\mathbb{G}$ by $n = |\Delta|$.

Recall that $N_G(T)$ surjects onto the Weyl group $W = W(\mathbb{G} : \mathbb{T}) = N_{\mathbb{G}}(\mathbb{T})/C_{\mathbb{G}}(\mathbb{T})$, which is generated by the involutions $\{w_\alpha\}_{\alpha \in \Delta}$.

Let $X^*(T) = \mathrm{Hom}_F(\mathbb{T}, \mathbb{G}_m) \cong \mathbb{Z}^n$ denote the group of F-rational characters of $\mathbb{T}$. Let $\mathfrak{a}_{\mathbb{R}}^* = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}_{\mathrm{ur}}(T, \mathbb{R}^\times)$ be the space of unramified real characters of the topological group T and let $\mathfrak{a}_{\mathbb{C}}^* = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Hom}_{\mathrm{ur}}(T, \mathbb{C}^\times)$ be the space of unramified complex characters of T. Let $X_*(T) = \mathrm{Hom}_F(\mathbb{G}_m, \mathbb{T}) \cong \mathbb{Z}^n$ denote the set of F-rational cocharacters of $\mathbb{T}$.

Then we recall the pairing $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$, which extends to a pairing $\langle \cdot, \cdot \rangle : \mathfrak{a}_{\mathbb{C}}^* \times X_*(T) \to \mathbb{C}$. We also recall that this pairing is W-invariant, that is

$$\langle w \cdot \gamma_1, w \cdot \gamma_2^\vee \rangle = \langle \gamma_1, \gamma_2^\vee \rangle \quad \forall w \in W,\ \gamma_1 \in \mathfrak{a}_{\mathbb{C}}^*,\ \gamma_2^\vee \in X_*(T). \tag{4}$$

The set of fundamental weights $\{\omega_\alpha \mid \alpha \in \Delta\} \subset X^*(T)$ given by $\langle \omega_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$, is basis for $\mathfrak{a}_{\mathbb{R}}^*$ and hence gives rise to an identification $\mathfrak{a}_{\mathbb{R}}^* \cong \mathbb{R}^n$ and $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}^n$ as vector spaces via the map

$$\lambda = (s_1, ..., s_n) \mapsto \sum_{i=1}^{n} s_i \cdot \omega_{\alpha_i}. \tag{5}$$

It is convenient to use the inner product on $V = \mathfrak{a}_{\mathbb{R}}^*$ underlying the pairing $V \times V^\vee \to \mathbb{R}$ given by $\langle \cdot, \cdot \rangle$. Indeed, $\mathfrak{a}_{\mathbb{R}}^*$ is equipped with an inner product $(\cdot, \cdot)$ such that

$$\langle \gamma_1, \gamma_2^\vee \rangle = 2\frac{(\gamma_1, \gamma_2)}{(\gamma_2, \gamma_2)} \quad \forall \gamma_1, \gamma_2 \in \Phi. \tag{6}$$

In particular, this inner product is W-invariant.

Finally we recall the correspondence

$$\begin{array}{ccc} \{\Theta \subseteq \Delta\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{Standard parabolic} \\ \text{subgroups of } G \end{array} \right\} \\ \Theta & \longrightarrow & P_\Theta = M_\Theta U_\Theta \\ \Theta_M & \longleftarrow & P = MU. \end{array}$$

We denote the Weyl group of $M = M_\Theta$ by W_M or W_Θ, namely,

$$W_M = W_\Theta = \langle w_\alpha \mid \alpha \in \Theta \rangle \tag{7}$$

For a Levi subgroup M of G, let

$$\mathfrak{a}^*_{M,\mathbb{C}} = X^*(M) \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Hom}_{\mathrm{ur}}\left(M, \mathbb{C}^\times\right). \tag{8}$$

Let $K \subset G$ be a maximal compact subgroup (specifically the group $\mathbb{G}(\mathcal{O}_F)$ when F is non-Archimedean and $\mathbb{G}$ is defined over $\mathcal{O}$) and recall the *Iwasawa decomposition* $G = PK$ for all parabolic subgroups P.

2.2. Representation theory and intertwining operators. For any reductive group M we write $\mathbf{1}_M$ for the trivial representation of M.

For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ we write $\mathrm{Ind}_B^G \lambda$ for the (normalized) induction of λ (thought of as a character of B) to G. Recall that for all $w \in W$ we have an intertwining operator

$$M(w,\lambda) : \mathrm{Ind}_B^G \lambda \to \mathrm{Ind}_B^G (w \cdot \lambda)$$

defined by analytic continuation of the following integral (which converges absolutely in the positive Weyl chamber)

$$M(w,\lambda) f_\lambda(g) = \int\limits_{N \cap wNw^{-1} \backslash N} f_\lambda\left(w^{-1} u g\right) du.$$

We collect here some necessary results regarding the intertwining operators; a more detailed discussion may be found in [16, sec. 3] or [17, sec. 3].

- (*Gindikin–Karpelevich formula*) Let $f^0_\lambda \in \mathrm{Ind}_B^G \lambda$ denote the spherical (K-invariant) vector, normalized so that $f^0_\lambda(1) = 1$. Then

$$M(w,\lambda) f^0_\lambda = \left(\prod_{\alpha>0,\ w\cdot\alpha<0} \frac{\zeta(\langle \lambda, \alpha^\vee \rangle)}{\zeta(\langle \lambda, \alpha^\vee \rangle + 1)} \right) f^0_{w\cdot\lambda}, \tag{9}$$

  where $\zeta(s)$ is the local ζ-function of F.

- The operators $$N(w,\lambda) = \left(\prod_{\alpha>0,\ w\cdot\alpha<0} \frac{\zeta(\langle \lambda, \alpha^\vee \rangle + 1)}{\zeta(\langle \lambda, \alpha^\vee \rangle)} \right) M(w,\lambda) \tag{10}$$

  (to be called *normalized intertwining operators*) satisfy the following cocycle condition:

$$\forall w, w' \in W : N(ww', \lambda) = N(w, w' \cdot \lambda) \circ N(w', \lambda). \tag{11}$$

  By construction, we clearly have:

$$N(w,\lambda) f^0_\lambda = f^0_{w\cdot\lambda}. \tag{12}$$

- *(Induction in stages)* Given a simple reflection w_α, $N(w_\alpha, \lambda)$ factors through induction in stages. Namely, given the embedding $\iota_\alpha : SL_2(F) \to G$ associated to the simple root α, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Ind}_B^G \lambda & \xrightarrow{N(w_\alpha,\lambda)} & \mathrm{Ind}_B^G (w_\alpha \cdot \lambda) \\ \downarrow{\scriptstyle \iota_\alpha^*} & & \downarrow{\scriptstyle \iota_\alpha^*} \\ \mathrm{Ind}_{\mathcal{B}}^{SL_2(F)} (\langle \lambda, \alpha^\vee \rangle) & \xrightarrow{N(w_\square, \langle \lambda, \alpha^\vee \rangle)} & \mathrm{Ind}_{\mathcal{B}}^{SL_2(F)} (\langle w_\square \cdot \lambda, \alpha^\vee \rangle), \end{array} \tag{13}$$

where $\mathcal{B}$ is the standard Borel subgroup $SL_2(F)$, $w_\square = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the non-trivial Weyl element of $SL_2(F)$ and the vertical maps in the diagram should be understood as the pull-back map.

- (*Representations of* $SL_2(F)$) We consider the representation $\pi_s = \mathrm{Ind}_{\mathcal{B}}^{SL_2(F)} |\omega|^s$, where ω is the unique fundamental weight on the torus of $SL_2(F)$. The representation π_s is irreducible for $s \neq \pm 1$. For $s = \pm 1$ we have the following exact sequences

$$\begin{aligned} 0 \longrightarrow \mathbf{1}_{SL_2(F)} \longrightarrow \mathrm{Ind}_{\mathcal{B}}^{SL_2(F)} |\omega|^{-1} \longrightarrow \mathrm{St}_{SL_2(F)} \longrightarrow 0 \\ 0 \longrightarrow \mathrm{St}_{SL_2(F)} \longrightarrow \mathrm{Ind}_{\mathcal{B}}^{SL_2(F)} |\omega|^{+1} \longrightarrow \mathbf{1}_{SL_2(F)} \longrightarrow 0, \end{aligned} \tag{14}$$

where $\mathrm{St}_{SL_2(F)}$ denotes the Steinberg representations of $SL_2(F)$. Note that these sequences do not split. Furthermore, writing the Laurent series of $N(w_\square, s)$ around $s = -1$ and $s = +1$ yields

$$N(w_\square, s) = \sum_{i=0}^{\infty} (s-1)^i \mathcal{A}_i \quad N(w_\square, s) = \sum_{i=-1}^{\infty} (s+1)^i \mathcal{C}_i, \tag{15}$$

where
$$\begin{aligned} Im(\mathcal{A}_0) = \mathbf{1}_{SL_2(F)}, \quad Ker(\mathcal{A}_0) = \mathrm{St}_{SL_2(F)} \\ Im(\mathcal{C}_{-1}) = \mathrm{St}_{SL_2(F)}, \quad Ker(\mathcal{C}_{-1}) = \mathbf{1}_{SL_2(F)}. \end{aligned} \tag{16}$$

2.3. The Langlands Subrepresentation Theorem. We recall here the Langlands subrepresentation theorem. See [2, Chapter IV, Sec. XI.2], [11] or [1] for more details. Note that most sources describe the quotient version of the Langlands classification theorem rather than the subrepresentation version we use here. By taking contragredients, the two versions are equivalent.

Let Q be a standard parabolic subgroup of G with Levi subgroup L. Let

$$\mathfrak{a}_L^+ = \left\{ \lambda \in \mathfrak{a}_{L,\mathbb{R}}^* \;\middle|\; \langle \lambda, \alpha^\vee \rangle < 0 \ \forall \alpha \in \Delta \setminus \Theta_L \right\}.$$

A representation σ of L is called *tempered* if σ is a direct summand in a parabolic induction from a square-integrable representation. A *standard module* is an induction $\mathrm{Ind}_Q^G(\sigma \otimes \lambda)$, where σ is a tempered representation of L and $\lambda \in \mathfrak{a}_L^+$.

Theorem 2.1 ([11] Lemma 2.4). *Let* $\mathrm{Ind}_Q^G(\sigma \otimes \lambda)$ *be a standard module. Then* $\mathrm{Ind}_Q^G(\sigma \otimes \lambda)$ *admits a unique irreducible subrepresentation* τ *and* τ *is the kernel of* $N(w_L, \lambda)$*, where* w_L *is the shortest representative in* W *of the class of the longest element in* $W_L \backslash W$.

The operator $N(w_L, \lambda)$ is called the *Langlands operator* for the standard module $\mathrm{Ind}_Q^G(\sigma \otimes \lambda)$. We note the following useful corollary of Theorem 2.1.

Corollary 2.2. *Let* $\lambda \in \mathfrak{a}_{T,\mathbb{R}}^*$ *be anti-dominant, in the sense that* $\langle \lambda, \alpha^\vee \rangle \leq 0$ *for all* $\alpha \in \Delta$. *Then* $\mathrm{Ind}_B^G \lambda$ *admits a unique irreducible subrepresentation.*

In order to prove Corollary 2.2, we need the following fact:

Lemma 2.3. *The representation* $\pi = Ind_B^G \mathbf{1}_T$ *is irreducible.*

Proof. Harish-Chandra's commuting algebra theorem states that the algebra $End_G(\pi)$ is generated by $N(w, \mathbf{1}_M, 0)$ where $w \in \mathrm{Stab}_W(\mathbf{1}_T) = W$. However, a simple calculation (see [4, 5, 19] for Archimedean F and [12, 13] for non-Archimedean F) shows that $N(w, \mathbf{1}_M, 0) = Id$ for any $w \in \mathrm{Stab}_W(\mathbf{1}_T)$ and hence $End_G(\pi) \cong \mathbb{C}$. On the other hand, π is unitary of finite length and hence isomorphic to a direct sum of irreducible representation $\oplus_{i=1}^{l} \sigma_i$. It follows that $\dim(End_G(\pi)) \geq l$. Hence $l = 1$ and π is irreducible. ■

Proof of Corollary 2.2. Let $\Theta_L = \{\alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle = 0\}$, $P = P_{\Theta_L}$ and let $L = M_{\Theta_L}$ be the (maximal) standard Levi subgroup such that the restriction of λ to L^{der} is trivial. By Lemma 2.3, $\mathrm{Ind}_{B\cap L}^{L} \lambda$ is an irreducible representation and can, in fact, be written as $\sigma \otimes \lambda'$, where σ is a tempered representation of L and $\lambda \in \mathfrak{a}_L^+$. Corollary 2.2 then follows from Theorem 2.1. ■

3. Decomposition with respect to Levi subgroups of semi-simple rank 1

In this section we prove our main result of this paper, Theorem 3.4. Before stating and proving it, we start by setting up some notations and listing the assumptions of this theorem. While this list of assumptions may seem incomprehensible at first glance, in Section 4 we prove the existence of points $\lambda_0 \in \mathfrak{a}_{T,\mathbb{C}}^*$ such that $\mathrm{Ind}_B^G \lambda_0$ decompose as in Theorem 3.4. In fact, we show that if $rank(G) > 2$, then there are infinitely many such points λ_0.

We fix a simple root $\alpha \in \Delta$ and make the following notations:

- Let $H_1 = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \langle \lambda, \alpha^\vee \rangle = 1\}$, this is a hyperplane in $\mathfrak{a}_{T,\mathbb{C}}^*$.
- Let $P = P_{\{\alpha\}}$ and $M = M_{\{\alpha\}}$.
- Let A_M denote the central torus of M and $M^{der} = [M, M]$ be the derived group of M. We have $A_M \subset T$ and hence $\mathfrak{a}_{M,\mathbb{C}}^* \hookrightarrow \mathfrak{a}_{T,\mathbb{C}}^*$. In fact, the image of this embedding can be identified as those elements $\lambda \in \mathfrak{a}_{T,\mathbb{C}}^*$ satisfying $\langle \lambda, \alpha^\vee \rangle = 0$. Any character of M is a trivial extension of a character of A_M. Namely, of the form $\chi \boxtimes \mathbf{1}_{M^{der}}$, where χ is a character of A_M, trivial on $A_M \cap M^{der}$. Under these notations, it holds that

$$\chi_0 = \left(\lambda_0 - \frac{\alpha}{2}\right)\Bigg|_{A_M} \boxtimes \mathbf{1}_{M^{der}}. \tag{17}$$

  Alternatively, χ_0 is a character of M such that

$$\chi_0\Bigg|_T = \lambda_0 - \frac{\alpha}{2}. \tag{18}$$

Assumptions (A1)–(A5):

(A1) Fix $\lambda_0 \in H_1$ such that $\mathrm{Stab}_W(w_\alpha \cdot \lambda_0) \neq \{1\}$. Note that $w_\alpha \notin \mathrm{Stab}_W(w_\alpha \cdot \lambda_0)$.

(A2) Fix $1 \neq w_0 \in \mathrm{Stab}_W(w_\alpha \cdot \lambda_0)$ and assume that $N(w_0, w_\alpha \cdot \lambda_0) = Id$.

We denote

$$H_{-1} = \{\lambda \in \mathfrak{a}_{\mathbb{R}}^* \mid \langle w_0 w_\alpha \cdot \lambda, \alpha^\vee \rangle = -1\} = \{\lambda \in \mathfrak{a}_{\mathbb{R}}^* \mid \langle w_\alpha w_0 w_\alpha \cdot \lambda, \alpha^\vee \rangle = 1\}. \quad (19)$$

Note that $\lambda_0 \in H_1 \cap H_{-1}$.

(A3) Assume that $H_1 \neq H_{-1}$. Equivalently, assume that w_0 does not commute with w_α (see Lemma A.1 and Lemma A.3 in Appendix A).

(A4) Assume that $\mathrm{Ind}_P^G \mathbf{1}_M \otimes \chi_0$ admits a unique irreducible subrepresentation.

(A5) Assume that $\mathrm{Ind}_P^G \mathrm{St}_M \otimes \chi_0$ admits a unique irreducible subrepresentation.

Lemma 3.1. *There exists (infinitely many) lines $\mathcal{S} \subset \mathfrak{a}_{\mathbb{R}}^*$ such that:*

- $\mathcal{S} \cap H_1 = \mathcal{S} \cap H_{-1} = \{\lambda_0\}$.
- *The angle between $\mathcal{S}$ and H_1 is not supplementary to the angle between $\mathcal{S}$ and H_{-1}.*

Proof. Let $\overline{\Lambda_1}$ and $\overline{\Lambda_{-1}}$ denote non-zero vectors perpendicular to H_1 and H_{-1} respectively. Since H_1 and H_{-1} are affine hyperplanes in $\mathfrak{a}_{\mathbb{C}}^*$ passing through λ_0 they are determined by $\overline{\Lambda_1}$ and $\overline{\Lambda_{-1}}$. In particular, by (A3), $\overline{\Lambda_1}$ is not parallel to $\overline{\Lambda_{-1}}$.

Consider the plane $\mathcal{P} \subset \mathfrak{a}_{\mathbb{C}}^*$ spanned by $\overline{\Lambda_1}$ and $\overline{\Lambda_{-1}}$ and let $\overline{v} \in \mathcal{P}$ be a non-zero vector which is not perpendicular to either $\overline{\Lambda_1}$ or $\overline{\Lambda_{-1}}$ and such that the angles $\angle\left(\overline{\Lambda_1}, \overline{v}\right)$ and $\angle\left(\overline{\Lambda_{-1}}, \overline{v}\right)$ are not supplementary. The line $\mathcal{S}$ passing through λ_0 and parallel to $\overline{v}$ satisfies the requirements.

A possible positioning of H_1, H_{-1}, $\mathcal{S}$, $\overline{\Lambda_1}$, $\overline{\Lambda_{-1}}$ and $\overline{v}$ (illustrating the situation in [17, Subsection 4.4]) can be found in Figure 1. ■

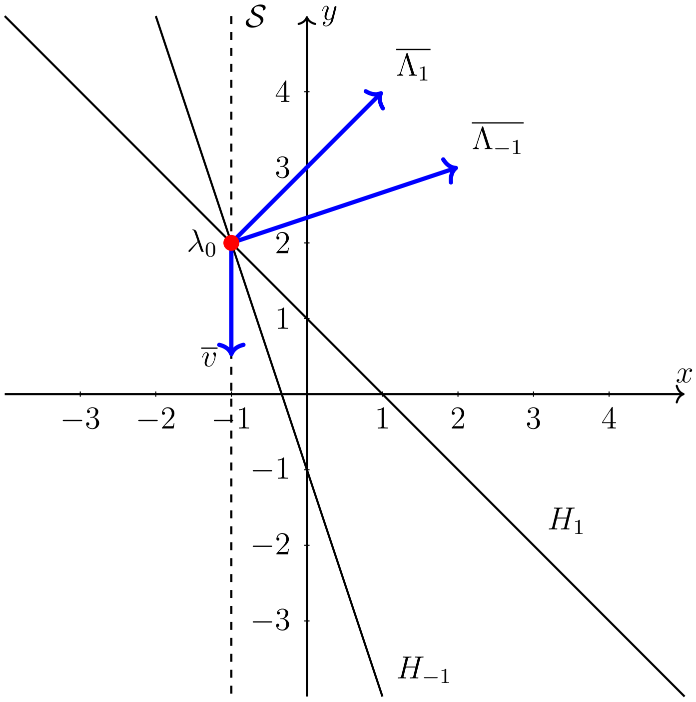


Figure 1: Illustration of a possible positioning of H_1, H_{-1}, $\mathcal{S}$, $\overline{\Lambda_1}$, $\overline{\Lambda_{-1}}$ and $\overline{v}$ in $\mathcal{P}$

Remark 3.2. We point out that $\mathcal{S}$ is not necessarily contained in the plane $\mathcal{P}$.

Remark 3.3. We note that, by Equation (6) and Equation (4), one may choose $\Lambda_1 = \frac{2\alpha}{(\alpha,\alpha)}$ and $\Lambda_{-1} = \frac{2\gamma}{(\gamma,\gamma)}$, with $\gamma = (w_0 w_\alpha)^{-1} \cdot \alpha$. We also note, that by the W-invariance of the inner product, it follows that $\|\Lambda_1\| = \|\Lambda_{-1}\|$, where $\|\cdot\|$ is the norm induced from $(\cdot,\cdot)$.

Theorem 3.4. *Assume that the data* $(\lambda_0, \alpha, w_0) \in \mathfrak{a}^*_{T,\mathbb{R}} \times \Delta \times \mathrm{Stab}_W(\lambda_0)$ *satisfy assumptions* (A1)–(A4). *Then*

$$\mathrm{Ind}_B^G \lambda_0 = \left(\mathrm{Ind}_P^G \mathrm{St}_M \otimes \chi_0\right) \oplus \left(\mathrm{Ind}_P^G \mathbf{1}_M \otimes \chi_0\right). \tag{20}$$

Furthermore, assuming (A5), *each of* $\mathrm{Ind}_P^G \mathrm{St}_M \otimes \chi_0$ *and* $\mathrm{Ind}_P^G \mathbf{1}_M \otimes \chi_0$ *admits a unique irreducible subrepresentation and the maximal semi-simple subrepresentation of* $\mathrm{Ind}_B^G \lambda_0$ *is of length* 2.

The proof of this theorem is a generalization of the argument in [21, Lemma 3.1].

Proof. In order to prove the theorem, we compute the limit of $N(w_\alpha w_0 w_\alpha, \lambda)$ at λ_0 *along* a fixed line $\mathcal{S}$, as in Lemma 3.1, and show that the direct summands in Equation (20) are both eigenspaces of that operator.

We start by fixing a coordinate system. Let $\ell_1, ..., \ell_n : \mathfrak{a}^*_{T,\mathbb{R}} \to \mathbb{C}$ denote a set of affine functions such that:

1. $\ell_i(\lambda_0) = 0$ for all $1 \leq i \leq n$. In particular, $\ell_i(\lambda - \lambda_0)$ is a linear functional on $\mathfrak{a}^*_{T,\mathbb{R}}$.
2. $\{\nabla \ell_i \mid 1 \leq i \leq n\}$ forms an orthogonal system in $\mathfrak{a}^*_{T,\mathbb{R}}$.
3. $\ell_1(\lambda) = \langle \lambda, \alpha^\vee \rangle - 1$.
4. $\ell_2(\lambda) = \langle w_0 w_\alpha \cdot \lambda, \alpha^\vee \rangle + 1$.

This can be done due to Assumption (A3). Note that any meromorphic function φ in the neighborhood of λ_0 has a Laurent expansion of the form

$$\varphi(\lambda) = \sum_{\vec{k} \in \mathbb{Z}^n} \left(\prod_{i=1}^n \ell_i(\lambda)^{k_i} \right) \varphi_{\vec{k}}$$

with $\varphi_{\vec{k}}$ having the same range as φ (in what follows, we consider operator-valued meromorphic functions).

We start by writing the Laurent expansions of some normalized standard intertwining operators in the neighborhood of λ_0:

$$N(w_\alpha, \lambda) = \sum_{i=0}^{\infty} \left(\langle \lambda, \alpha^\vee \rangle - 1\right)^i A_i = \sum_{i=0}^{\infty} \ell_1(\lambda)_i A_i,$$

$$N(w_0, w_\alpha \cdot \lambda) = \sum_{\vec{k} \in \mathbb{N}^n} \left(\prod_{i=1}^n \ell_i(\lambda)^{k_i} \right) B_{\vec{k}},$$

$$N(w_\alpha, (w_0 w_\alpha) \cdot \lambda) = \sum_{i=-1}^{\infty} \left(\left\langle \lambda, (w_0 w_\alpha)^{-1} \cdot \alpha^\vee \right\rangle + 1 \right)^i C_i = \sum_{i=-1}^{\infty} \ell_2(\lambda)^i C_i.$$

Here
$$A_i \in \mathrm{Hom}_{\mathbb{C}}\left(\mathrm{Ind}_B^G \lambda_0, \mathrm{Ind}_B^G (w_\alpha \cdot \lambda_0)\right),$$
$$B_{\vec{k}} \in \mathrm{End}_{\mathbb{C}}\left(\mathrm{Ind}_B^G (w_\alpha \cdot \lambda_0)\right),$$
$$C_i \in \mathrm{Hom}_{\mathbb{C}}\left(\mathrm{Ind}_B^G w_\alpha \cdot \lambda_0, \mathrm{Ind}_B^G (\lambda_0)\right).$$

Note that A_0, $B_{\vec{0}}$ and C_{-1} are G-equivariant but the rest of the operators A_i, $B_{\vec{k}}$ and C_i need not be G-equivariant. We further note that, by Assumption (A2) $B_{\vec{0}} = Id$. On the other hand, by Equation (13) and Equation (16):

$$Im\,(A_0) = \mathrm{Ind}_P^G (\mathbf{1}_M \otimes \chi_0), \quad Ker\,(A_0) = \mathrm{Ind}_P^G (\mathrm{St}_M \otimes \chi_0)$$
$$Im\,(C_{-1}) = \mathrm{Ind}_P^G (\mathrm{St}_M \otimes \chi_0), \quad Ker\,(C_{-1}) = \mathrm{Ind}_P^G (\mathbf{1}_M \otimes \chi_0).$$

It follows, from Equation (11), that

$$N (w_\alpha, (w_\alpha w_0 w_\alpha) \cdot \lambda) \circ N (w_\alpha, (w_\alpha w_0) \cdot \lambda) = Id$$
$$N (w_\alpha, (w_\alpha w_0) \cdot \lambda) \circ N (w_\alpha, (w_\alpha w_0 w_\alpha) \cdot \lambda) = Id$$

for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. By evaluating the leading terms of both the left-hand side and right-hand side of these equations, we conclude that

$$C_{-1}A_0 = 0 = A_0 C_{-1} \quad \text{and} \quad C_0 A_0 - C_{-1}A_1 = Id = A_0 C_0 - A_1 C_{-1}. \tag{21}$$

Note that

$$\begin{aligned} N (w_\alpha w_0 w_\alpha, \lambda) &= N (w_\alpha, w_0 w_\alpha \cdot \lambda) \circ N (w_0, w_\alpha \cdot \lambda) \circ N (w_\alpha, \lambda) \\ &= \left[\sum_{i=-1}^{\infty} \ell_2 (\lambda)^i C_i\right] \circ \left[\sum_{\vec{k}\in\mathbb{N}^n} \left(\prod_{i=1}^{n} \ell_i (\lambda)^{k_i}\right) B_{\vec{k}}\right] \circ \left[\sum_{i=0}^{\infty} \ell_1 (\lambda)_i A_i\right] \\ &= \frac{1}{\ell_2(\lambda)} C_{-1}A_0 + \frac{\ell_1(\lambda)}{\ell_2(\lambda)} C_{-1}A_1 + C_{-1}\left(\sum_{i=1}^{n} \frac{\ell_i(\lambda)}{\ell_2(\lambda)} B_{\hat{e}_i}\right) A_0 + C_0 A_0 \\ &\qquad + \sum_{\vec{k}\in\mathbb{N}^n,\ \left|\vec{k}\right|\geq 2} \left(\frac{\prod_{i=1}^{n} \ell_i(\lambda)^{k_i}}{l_2(\lambda)} N_{\vec{k}}\right) \end{aligned}$$

is a Laurent series for $N (w_\alpha w_0 w_\alpha, \lambda)$ in a neighborhood of λ_0, where:

- $\hat{e}_i = (\delta_{i,j})_{j=1,\ldots,n}$ are the standard basis vectors in $\mathbb{R}^n$.
- For $\vec{k} \in \mathbb{Z}^n$, we write $\left|\vec{k}\right| = \sum_{i=1}^{n} |k_i|$.
- $N_{\vec{k}}$ is the corresponding $\vec{k}$-coefficient in the Laurent series of $N (w_\alpha w_0 w_\alpha, \lambda)$; when $\left|\vec{k}\right| \geq 2$, these coefficients will not play a role in the following computations.

Restricting $N (w_\alpha w_0 w_\alpha, \lambda)$ (in the λ variable) to $\mathcal{S}$ yields:

$$N(w_\alpha w_0 w_\alpha, \lambda)\Big|_{\mathcal{S}} = \frac{\ell_1(\lambda)}{\ell_2(\lambda)} C_{-1}A_1 + C_{-1}\left(\sum_{i=1}^{n} \frac{\ell_i(\lambda)}{\ell_2(\lambda)} B_{\hat{e}_i}\right) A_0 + C_0 A_0 + \sum_{\substack{\vec{k}\in\mathbb{N}^n \\ |\vec{k}|\geq 2}} \left(\frac{\prod_{i=1}^{n} \ell_i(\lambda)^{k_i}}{l_2(\lambda)} N_{\vec{k}}\right) \quad (22)$$

For a vector $v \neq 0$, parallel to $\mathcal{S}$, we define

$$\kappa_i = \lim_{\lambda\to\lambda_0}\left[\frac{\ell_i(\lambda)}{\ell_2(\lambda)}\Big|_{\mathcal{S}}\right] = \frac{(\nabla\ell_i, v)}{(\nabla\ell_2, v)} = \frac{\langle\nabla\ell_i, v\rangle}{\langle\nabla\ell_2, v\rangle}. \quad (23)$$

The fact that these limits indeed exist, i.e. $(\nabla\ell_2, v) \neq 0$, is due to our assumptions on $\mathcal{S}$. Indeed, the fact that $\mathcal{S}\cap H_{-1} = \{\lambda_0\}$ implies that the angle between $\mathcal{S}$ and H_{-1} is non-zero, and hence so is the inner product $(\nabla\ell_2, v)$.
Note that κ_i is independent of the choice of v. Taking the limit of $N(w_\alpha w_0 w_\alpha, \lambda)$ at λ_0 along $\mathcal{S}$ yields

$$\begin{aligned}\mathscr{E} &= \lim_{\lambda\to\lambda_0}\left[N(w_\alpha w_0 w_\alpha, \lambda)\Big|_{\mathcal{S}}\right] \\ &= \kappa_1 C_{-1}A_1 + C_{-1}BA_0 + C_0A_0 = -\kappa_1 Id + (\kappa_1+1)C_0A_0 + C_{-1}BA_0, \end{aligned} \quad (24)$$

where $B = \sum_{i=1}^{n}\kappa_i B_{\hat{e}_i}$. We note that $\mathscr{E}\in \mathrm{End}_G\left(\mathrm{Ind}_B^G\lambda_0\right)$. Define

$$\mathscr{P} = \frac{1}{1+\kappa_1}(Id - \mathscr{E}) \in \mathrm{End}_G\left(\mathrm{Ind}_B^G\lambda_0\right). \quad (25)$$

This is well defined, i.e. $\kappa_1\neq -1$. Indeed, by our assumptions on $\mathcal{S}$, the fact that $\|\nabla\ell_1\| = \|\nabla\ell_2\| \neq 0$ (see Remark 3.3) and the angle between $\mathcal{S}$ and H_1 is not supplementary to the angle between $\mathcal{S}$ and H_{-1} implies that $(\nabla\ell_i, v) \neq -(\nabla\ell_2, v)$ and hence $\kappa_1\neq -1$. Applying Equation (21) yields

$$\begin{aligned}\mathscr{E}^2 &= \kappa_1^2 Id - 2\kappa_1(\kappa_1+1)C_0A_0 - 2\kappa_1C_{-1}BA_0 + (\kappa_1+1)^2 C_0A_0C_0A_0 \\ &\quad + (\kappa_1+1)C_0A_0C_{-1}BA_0 + (\kappa_1+1)C_{-1}BA_0C_0A_0 + C_{-1}BA_0C_{-1}BA_0 \\ &= \kappa_1^2 Id - 2\kappa_1(\kappa_1+1)C_0A_0 - 2\kappa_1C_{-1}BA_0 \\ &\quad + (\kappa_1+1)^2 C_0A_0(Id + C_{-1}A_1) + (\kappa_1+1)C_{-1}BA_0(Id + C_{-1}A_1) \\ &= \kappa_1^2 Id + \left(1-\kappa_1^2\right)C_0A_0 + (1-\kappa_1)C_{-1}BA_0 = \kappa_1 Id + (1-\kappa_1)\mathscr{E}\end{aligned}$$

and hence

$$\begin{aligned}\mathscr{P}^2 &= \frac{1}{(1+\kappa_1)^2}(Id-\mathscr{E})^2 = \frac{1}{(1+\kappa_1)^2}(Id - 2\mathscr{E} + \kappa_1 Id + (1-\kappa_1)\mathscr{E}) \\ &= \frac{1}{1+\kappa_1}(Id-\mathscr{E}) = \mathscr{P}.\end{aligned}$$

It follows that $\mathscr{P}$ is a projection. Since $\mathscr{P}$ is a G-equivariant projection, we have

$$\mathrm{Ind}_B^G\lambda_0 = \mathrm{Im}\,\mathscr{P}\oplus\mathrm{Ker}\,\mathscr{P}. \quad (26)$$

It remains to prove that $KerA_0 = Im\mathscr{P}$ and $ImA_0 \cong Ker\mathscr{P}$. Note that, a priori, ImA_0 is realized as a subrepresentation of $\mathrm{Ind}_B^G(w_\alpha \cdot \lambda_0)$ and not of $\mathrm{Ind}_B^G(\lambda_0)$. The isomorphism allows us to identify $Ker\mathscr{P}$ with ImA_0. Since

$$\mathscr{P} = Id - \left(C_0 - \frac{1}{\kappa_1+1}C_{-1}B\right)A_0$$

it follows that $KerA_0 \subseteq Im\mathscr{P}$. Assume that $KerA_0 \subsetneq Im\mathscr{P}$. In other words, $A_0 \circ \mathscr{P} \neq 0$. Note that $Id - \mathscr{P}$ is a projection on $Ker\mathscr{P}$. It holds that

$$A_0 = A_0 \circ \mathscr{P} + A_0 \circ (Id - \mathscr{P}). \tag{27}$$

We note that, since $\mathscr{E}v^0 = v^0$, it holds that $v^0 \in Ker\mathscr{P}$ and hence $A_0 \circ (Id - \mathscr{P}) \neq 0$. It follows that ImA_0 has at least two irreducible subrepresentations in contradiction with the fact that, by Assumption (A4), it has a unique irreducible subrepresentation. We conclude that $Im\mathscr{P} = KerA_0$. It follows from Equation (26) that the following sequences are exact

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Im\mathscr{P} & \longrightarrow & \mathrm{Ind}_B^G\lambda_0 & \longrightarrow & Ker\mathscr{P} & \longrightarrow & 0 \\
 & & \| & & \| & & & & \\
0 & \longrightarrow & KerA_0 & \longrightarrow & \mathrm{Ind}_B^G\lambda_0 & \longrightarrow & ImA_0 & \longrightarrow & 0.
\end{array}$$

Hence, $Ker\mathscr{P} \cong ImA_0$. ∎

Remark 3.5. It follows from the proof that $\mathrm{Ind}_P^G(\mathbf{1}_M \otimes \chi_0)$ is the eigenspace of $\mathscr{E}$ of eigenvalue 1 and $\mathrm{Ind}_P^G(\mathrm{St}_M \otimes \chi_0)$ is eigenspace of eigenvalue $-\kappa_1 \neq 1$. We note here that the decomposition in Equation (20) and the projection $\mathscr{P}$ in Equation (25) are independent of $\mathcal{S}$ and only the eigenvalues of $\mathscr{E}$ depend on $\mathcal{S}$. ∎

Using induction in stages, Equation (13), Assumption (A4) may be replaced with the following weaker assumption:

(A4') Let L be a standard Levi containing w_α and w_0 (and hence $M \subset L$) and assume that $\mathrm{Ind}_{P\cap L}^L \mathbf{1}_M \otimes \chi_0$ admits a unique irreducible subrepresentation.

Corollary 3.6. *Under the assumptions* (A1)–(A3) *and* (A4') *Equation* (20) *holds.*

Proof. Indeed, the conditions of Theorem 3.4 applies to $\mathrm{Ind}_{B\cap L}^L \lambda_0$ and hence

$$\mathrm{Ind}_{B\cap L}^L \lambda_0 = \mathrm{Ind}_{P\cap L}\left[\mathrm{Ind}_{P\cap L}^L(\mathrm{St}_M \otimes \chi_0) \oplus \mathrm{Ind}_{P\cap L}^L(\mathbf{1}_M \otimes \chi_0)\right].$$

Applying induction by stages yields

$$\begin{aligned}
\mathrm{Ind}_B^G \lambda_0 &= \mathrm{Ind}_Q^G\left(\mathrm{Ind}_{B\cap L}^L \lambda_0\right) \\
&= \mathrm{Ind}_Q^G\left(\mathrm{Ind}_{P\cap L}^L \mathrm{St}_M \otimes \chi_0\right) \oplus \mathrm{Ind}_Q^G\left(\mathrm{Ind}_{P\cap L}^L \mathbf{1}_M \otimes \chi_0\right) \\
&= \mathrm{Ind}_P^G(\mathrm{St}_M \otimes \chi_0) \oplus \mathrm{Ind}_P^G(\mathbf{1}_M \otimes \chi_0),
\end{aligned}$$

where Q is the standard parabolic subgroup whose Levi subgroup is L. ∎

We also record the following useful by-product of the proof of Theorem 3.4.

Corollary 3.7. *Under the assumptions of Theorem* 3.4*, the normalized intertwining operator* $N(w_\alpha w_0 w_\alpha, \lambda)\Big|_{\mathcal{S}}$ *is holomorphic at* λ_0.

Proof. Recall from Equation (22) that the restriction of $N(w_\alpha w_0 w_\alpha, \lambda)$ to $\mathcal{S}$ is given by the following series near λ_0,

$$N(w_\alpha w_0 w_\alpha, \lambda)\Big|_{\mathcal{S}} = \frac{\ell_1(\lambda)}{\ell_2(\lambda)} C_{-1} A_1 + C_{-1}\left(\sum_{i=1}^{n} \frac{\ell_i(\lambda)}{\ell_2(\lambda)} B_{\hat{e}_i}\right) A_0 + C_0 A_0 + \sum_{\substack{\vec{k}\in\mathbb{N}^n \\ |\vec{k}|\geq 2}} \left(\frac{\prod_{i=1}^{n} \ell_i(\lambda)^{k_i}}{l_2(\lambda)} N_{\vec{k}}\right).$$

Further note that, since $\lambda \in \mathcal{S}$, this is a function of one complex variable and since $\ell_2(\lambda)$ is non-zero on $\mathcal{S} \setminus \{\lambda_0\}$, this function is holomorphic on a punctured neighborhood of λ_0. On the other hand, $\ell_i(\lambda_0) = 0$ for any $1 \leq i \leq n$ and hence the limit of $N(w_\alpha w_0 w_\alpha, \lambda)\Big|_{\mathcal{S}}$ at λ_0 exists. Indeed, it was evaluated in Equation (24). By the Riemann Extension Theorem, the restriction $N(w_\alpha w_0 w_\alpha, \lambda)\Big|_{\mathcal{S}}$ is holomorphic at λ_0. ■

4. Existence of $\boldsymbol{\lambda_0}$

One question which arises from the discussion in Section 3 is whether there exist points λ_0 which satisfy the assumptions of Theorem 3.4. In this section, we show that for any simple group G (satisfying the assumptions in Section 2) and any simple root of G, one can choose λ_0 as in Theorem 3.4. We prove:

Theorem 4.1. *Fix* $\alpha \in \Delta$, $\lambda' \in \mathfrak{a}_{\mathbb{R}}^*$ *and* $S \subset \Delta \setminus \{\alpha\}$ *satisfying:*

1. *There exists* $\beta \in S$ *such that* $\langle \beta, \alpha^\vee \rangle \neq 0$ *(i.e.* α *and* β *are neighbors in the Dynkin diagram of* G*).*
2. $\langle \lambda', \alpha^\vee \rangle = -1$.
3. $\langle \lambda', \beta^\vee \rangle = 0 \quad \forall \beta \in S$.
4. $\langle \lambda', \beta^\vee \rangle < 0 \quad \forall \beta \notin S \cup \{\alpha\}$.

Then, for $\lambda_0 = w_\alpha \cdot \lambda'$ *and* $M = M_{\{\alpha\}}$*, it holds that*

$$\mathrm{Ind}_B^G \lambda_0 = \left(\mathrm{Ind}_P^G \mathrm{St}_M \otimes \chi_0\right) \oplus \left(\mathrm{Ind}_P^G \mathbf{1}_M \otimes \chi_0\right),$$

where χ_0 *is as in Equation* (17)*. Furthermore,* $\mathrm{Ind}_P^G \mathbf{1}_M \otimes \chi_0$ *admits a unique irreducible subrepresentation and if* St_M *is irreducible, then so does* $\mathrm{Ind}_P^G \mathrm{St}_M \otimes \chi_0$.

Remark 4.2. Note that the set of λ' satisfying the conditions in Theorem 4.1 is a non-empty set of dimension $n - |S| - 1$. ■

Remark 4.3. We note here that the Steinberg representation of $SL_2(\mathbb{R})$ has length 2. ■

By choosing $S = \Delta \setminus \{\alpha\}$ in Theorem 4.1 we have:

Corollary 4.4. *For any group G, as in* Section 2*, and any simple root $\alpha \in \Delta$, let $\lambda_0 = -w_\alpha \cdot \omega_\alpha$. Then*

$$\operatorname{Ind}_B^G \lambda_0 = \left(\operatorname{Ind}_P^G \operatorname{St}_M \otimes \chi_0\right) \oplus \left(\operatorname{Ind}_P^G \mathbf{1}_M \otimes \chi_0\right), \tag{28}$$

where χ_0 is chosen as in Equation (17)*.*

Remark 4.5. The decompositions appearing in [15, Sec. 8] [21, Lem. 3.1], [10, pg. 1260-1 CASE 1], [14, Lem. 5.12] and [17, Subsec. 4.4] are all special cases of Corollary 4.4.

Proof of Theorem 4.1 In order to prove Theorem 4.1, we construct a system of equalities and inequalities, System I, whose solutions are guaranteed to satisfy the assumptions of Theorem 3.4. We then show that this system is equivalent to the system, System IV, given by the assumptions of Theorem 4.1. We list the assumptions of Theorem 3.4 and reinterpret some of them as inequalities that will compose our system; other assumptions (i.e. (A1) and (A3) will be quoted verbatim in System I.

Assumptions:

(A1) Fix $\lambda_0 \in H_1$ such that $\operatorname{Stab}_W (w_\alpha \cdot \lambda_0) \neq \{1\}$.

(A2) Fix $1 \neq w_0 \in \operatorname{Stab}_W (w_\alpha \cdot \lambda_0)$ such that $N(w_0, w_\alpha \cdot \lambda_0) = Id$.

Assume that $\lambda' = w_\alpha \cdot \lambda_0$ lies in the anti-dominant chamber.
Let $S = \{\gamma \in \Delta \mid \langle \lambda', \gamma^\vee \rangle = 0\}$. It follows from (A1) that $S \neq \emptyset$ and that $w_0 \in W_S$. By induction in stages, it holds that

$$\operatorname{Ind}_B^G \lambda' = \operatorname{Ind}_{P_S}^G \left(\operatorname{Ind}_{B \cap M_S}^{M_S} \mathbf{1}\right) \otimes \lambda'. \tag{29}$$

We prove in Lemma 2.3 that $\operatorname{Ind}_{B \cap M_S}^{M_S} \mathbf{1}$ is irreducible. It is also spherical and hence, by Equation (13) and Equation (12), it follows that $N(w_0, w_\alpha \cdot \lambda_0) = Id$.

(A3) Assume that w_0 does not commute with w_α.

(A4) Assume that $\operatorname{Ind}_P^G \mathbf{1}_M \otimes \chi_0$ admits a unique irreducible subrepresentation.

Let $\lambda' = w_\alpha \cdot \lambda_0$ and S be as in Equation (29). Since $\operatorname{Ind}_P^G \mathbf{1}_M \otimes \chi_0$ embeds into $\operatorname{Ind}_B^G \lambda'$, Corollary 2.2 implies (A4).

(A5) Assume that $\operatorname{Ind}_P^G \operatorname{St}_M \otimes \chi_0$ admits a unique irreducible subrepresentation.

If $\langle \chi_0, \beta^\vee \rangle \leq 0$ for any $\beta \in \Phi^+ \setminus \{\alpha\}$ and St_M is irreducible, then (A5) follows from the Langlands' subrepresentation theorem since St_M is tempered (indeed, it is a discrete series representation) and $\operatorname{Ind}_P^G \operatorname{St}_M \otimes \chi_0$ is a standard module.

We summarize this discussion by the following system of equalities and inequalities:

System I: Pick $w \in W$ and $\lambda \in \mathfrak{a}_{\mathbb{R}}^*$ such that:

$$(1)\ [w, w_\alpha] \neq 1, \quad (2)\ w w_\alpha \cdot \lambda = w_\alpha \cdot \lambda, \quad (3)\ \langle \lambda, \alpha^\vee \rangle = 1,$$

$$(4)\ \left\langle \lambda - \frac{\alpha}{2}, \beta^\vee \right\rangle \leq 0 \quad \forall \beta \in \Phi^+ \setminus \{\alpha\}, \quad (5)\ \langle w_\alpha \cdot \lambda, \beta^\vee \rangle \leq 0 \quad \forall \beta \in \Delta.$$

We now argue that System I is equivalent to the system in the statement of Theorem 4.1, System IV. We do this in stages by showing the equivalence of System I, System II, System III and System IV.

Note that $\langle \lambda, \alpha^\vee \rangle = 1$ implies $w_\alpha \cdot \lambda = \lambda - \alpha$. We make a change of variables $\lambda' = w_\alpha \cdot \lambda$ and get an equivalent system:

System II: Pick $w \in W$ and $\lambda' \in \mathfrak{a}^*_{\mathbb{R}}$ such that:

$$(1)\ [w, w_\alpha] \neq 1, \quad (2)\ w\lambda' = \lambda', \quad (3)\ \langle \lambda', \alpha^\vee \rangle = -1,$$
$$(4)\ \langle \lambda', \beta^\vee \rangle \leq -\frac{1}{2} \langle \alpha, \beta^\vee \rangle \quad \forall \beta \in \Phi^+ \setminus \{\alpha\}, \quad (5)\ \langle \lambda', \beta^\vee \rangle \leq 0 \quad \forall \beta \in \Delta.$$

Since λ' is anti-dominant, $Stab_W(\lambda')$ is generated by simple reflections. In particular, $Stab_W(\lambda') = \langle w_\beta \mid \langle \lambda', \beta^\vee \rangle = 0,\ \beta \in \Delta \rangle$ not trivial if and only if λ' is on a wall of the chamber.

We now consider the following system:

System III: Pick a subset $S \subset \Delta \setminus \{\alpha\}$ and $\lambda' \in \mathfrak{a}^*_{\mathbb{R}}$ such that:

$$(1) \text{ There exists } \beta \in S \text{ such that } \langle \beta, \alpha^\vee \rangle \neq 0, \quad (2)\ \langle \lambda', \alpha^\vee \rangle = -1,$$
$$(3)\ \langle \lambda', \beta^\vee \rangle = 0 \quad \forall \beta \in S, \quad (4)\ \langle \lambda', \beta^\vee \rangle \leq 0 \quad \forall \beta \notin S \cup \{\alpha\},$$
$$(5)\ \langle \lambda', \beta^\vee \rangle \leq -\frac{1}{2} \langle \alpha, \beta^\vee \rangle \quad \forall \beta \in \Phi^+ \setminus \{\alpha\}.$$

The set of solutions of this system equals the set of solutions of System II as will be explained now.

- Let $w \in W$ and $\lambda' \in \mathfrak{a}^*_{\mathbb{R}}$ constitute a solution of **System II**. We automatically see that II.3 implies III.2, II.5 implies III.4 and II.4 implies III.5. Let
$$S = \{\beta \mid \langle \lambda', \beta^\vee \rangle = 0\}.$$
  This choice automatically guarantees System III.3. It remains to show that System III.1 holds.

  Assume that $\langle \beta, \alpha^\vee \rangle = 0$ for all $\beta \in S$. II.5 implies that λ' is anti-dominant and hence $Stab_W(\lambda') = \langle w_\beta \mid \beta \in S \rangle$. II.2 implies that $Stab_W(\lambda')$ is non-trivial. In fact, it follows that $S \neq \emptyset$. If $\langle \beta, \alpha^\vee \rangle = 0$ for all $\beta \in S$ it would imply that $[w, w_\alpha] = 1$ for all $w \in Stab_W(\lambda')$ contradicting II.1.
- Let $S \subset \Delta \setminus \{\alpha\}$ and $\lambda' \in \mathfrak{a}^*_{\mathbb{R}}$ constitute a solution of System III. We automatically see that III.2 implies II.3 and III.5 implies II.4. Also, III.2, III.3 and III.4 implies II.5 and, in particular, λ' lies in the anti-dominant chamber.

  Again, $Stab_W(\lambda') = \langle w_\beta \mid \beta \in S \rangle$ and III.1 implies that there exists $w \in Stab_W(\lambda')$ such that $[w, w_\alpha] \neq 1$ (say, $w = w_\beta$) so II.1 and II.2 hold. In particular, any solution of System II is attained this way.

It is shown in Lemma A.4 that, in fact, III.5 is redundant. Hence, System III is equivalent to the following system:

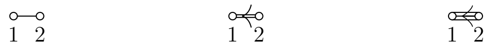
System IV: Pick a subset $S \subset \Delta \setminus \{\alpha\}$ and $\lambda' \in \mathfrak{a}^*_{\mathbb{R}}$ such that:

$$(1) \text{ There exists } \beta \in S \text{ such that } \langle \beta, \alpha^\vee \rangle \neq 0, \quad (2)\ \langle \lambda', \alpha^\vee \rangle = -1,$$
$$(3)\ \langle \lambda', \beta^\vee \rangle = 0\ \forall \beta \in S, \quad (4)\ \langle \lambda', \beta^\vee \rangle < 0 \quad \forall \beta \notin S \cup \{\alpha\}.$$

■

We now wish to consider a few particular examples of G, α and λ_0 given by Theorem 4.1. For simplicity, we assume that F is non-Archimedean.

Example 4.6. We consider simple, connected, simply-connected, split groups of rank 2. In this case, G is either of type A_2, $B_2 = C_2$ or G_2. Namely, its Dynkin diagram is one of the following:



Type A_2 Type $B_2 = C_2$ Type G_2

For each of these groups, and every $\alpha_i \in \Delta$, S may only be $S = \Delta \setminus \{\alpha_i\}$. The possible λ_0 given by Theorem 4.1 are listed in the following table.

	α_1	α_2
A_2	$\lambda_0 = (1, -1)$	$\lambda_0 = (-1, 1)$
$B_2 = C_2$	$\lambda_0 = (1, -1)$	$\lambda_0 = (-2, 1)$
G_2	$\lambda_0 = (1, -1)$	$\lambda_0 = (-3, 1)$

For each of these points, we get a decomposition of the form

$$\mathrm{Ind}_B^G \lambda_0 = \left(\mathrm{Ind}_{P_{\{\alpha_i\}}}^G \mathbf{1}_{M_{\{\alpha_i\}}} \otimes \chi_0\right) \oplus \left(\mathrm{Ind}_{P_{\{\alpha_i\}}}^G \mathrm{St}_{M_{\{\alpha_i\}}} \otimes \chi_0\right),$$

as in Corollary 4.4. However, some of these points could be associated to a degenerate principal series representation induced from the other maximal parabolic. Namely, there exist an s such that $I_{P_{\{\alpha_{3-i}\}}}(s) = \mathrm{Ind}_{P_{\{\alpha_{3-i}\}}}^G \delta_{P_{\{\alpha_{3-i}\}}}^s$ is a subrepresentation of $\mathrm{Ind}_B^G \lambda_0$. These degenerate principal series are given in the following table:

	α_1	α_2
A_2	$I_{P_{\{\alpha_2\}}}\left(\frac{1}{6}\right)$	$I_{P_{\{\alpha_1\}}}\left(\frac{1}{6}\right)$
$B_2 = C_2$	$I_{P_{\{\alpha_2\}}}(0)$	
G_2	$I_{P_{\{\alpha_2\}}}\left(-\frac{1}{10}\right)$	

Let $\pi_1 \oplus \pi_{-\kappa_1}$ be the maximal semi-simple subrepresentation of $\mathrm{Ind}_B^G \lambda_0$, where:

- κ_1 is given by Equation (23).
- $-\kappa_1$ is the eigenvalue of $\mathrm{Ind}_{P_{\{\alpha_i\}}}^G \mathrm{St}_{M_{\{\alpha_i\}}} \otimes \chi_0$ under the operator $\mathscr{E}$ defined in Equation (24) with respect to the line given by

$$\mathcal{S} = \begin{cases} \{(s, -1) \mid s \in \mathbb{C}\}, & i = 1 \\ \{(-1, s) \mid s \in \mathbb{C}\}, & i = 2 \end{cases}$$

  and the Weyl element $w_0 = w_{\alpha_{3-i}}$.
- π_1 is the unique irreducible subrepresentation of $\mathrm{Ind}_{P_{\{\alpha_i\}}}^G \mathbf{1}_{M_{\{\alpha_i\}}}$ and $\pi_{-\kappa_1}$ is the unique irreducible subrepresentation of $\mathrm{Ind}_{P_{\{\alpha_i\}}}^G \mathrm{St}_{M_{\{\alpha_i\}}} \otimes \chi_0$.

Obviously, π_1 is a subrepresentation of $I_{P_{\{\alpha_{3-i}\}}}(s)$. We wish to determine whether $\pi_{-\kappa_1}$ is also a subrepresentation of $I_{P_{\{\alpha_{3-i}\}}}(s)$ or not. We answer this question for the p-adic case (in the Archimedean case the results are similar, while the arguments are more involved).

- A_2 **case**: In this case, $I_{P_{\{\alpha_1\}}}\left(-\frac{1}{6}\right) = I_{P_{\{\alpha_2\}}}\left(\frac{1}{6}\right)$ and $I_{P_{\{\alpha_1\}}}\left(\frac{1}{6}\right) = I_{P_{\{\alpha_2\}}}\left(-\frac{1}{6}\right)$ are both irreducible. Hence, π_{-1} is not a subrepresentation of any of these degenerate principal series representations.

- $B_2 = C_2$ **case**: This case was studied in [8, pg. 9]. In this case, we have $\pi_1 \oplus \pi_{-1}$ as a subrepresentation of $I_{P_{\{\alpha_2\}}}(0)$. In order to see this, one can compare the multiplicity of the exponent λ_0 in the Jacquet functor (along N) of $\operatorname{Ind}_B^G \lambda_0$, π_1, π_{-1} and $I_{P_{\{\alpha_2\}}}(0)$ (2, 1, 1 and 2 respectively).

- G_2 **case**: This case was studied in [21, Lem. 3.1]. It is shown there that $\pi_1 \oplus \pi_{-2}$ is a subrepresentation of $I_{P_{\{\alpha_2\}}}\left(-\frac{1}{10}\right)$. This can be shown by comparing the multiplicity of the exponent λ_0 in the Jacquet functor (along N) of $\operatorname{Ind}_B^G \lambda_0$, π_1, π_{-2} and $I_{P_{\{\alpha_2\}}}\left(-\frac{1}{10}\right)$ (2, 1, 1 and 2 respectively).

In what follows, we use the following notations on the Dynkin diagrams:

- We use $\bullet$ to denote the simple root α.
- We use $\times$ to denote simple roots in S.
- We use $\circ$ to denote other simple roots.
- The k-vertex in a Dynkin diagram is associated to the simple root denoted α_k (using the conventions of [3]). We further denote by ω_k the k^{th} fundamental weight and by $w_k = w_{\alpha_k}$ the simple reflection associated to α_k.

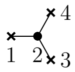
Example 4.7. Let $G = SL_4(F)$ with the standard choice of B, T and the enumeration of simple roots. The group G is of type A_3 and have the following Dynkin diagram:

○—○—○
1 2 3

For $\alpha = \alpha_1$, we have two possible choices for the set S: either $\{\alpha_2\}$ or $\{\alpha_2, \alpha_3\}$. For $\alpha = \alpha_2$, we have three possible choices for S: either $\{\alpha_1\}$, $\{\alpha_3\}$ or $\{\alpha_1, \alpha_3\}$. The analysis for $\alpha = \alpha_3$ is similar to the case of α_1. The possible values of λ_0 are given in the following table:

α		S	w	λ_0
α_1	●—×—○ 1 2 3	$S = \{\alpha_2\}$	w_2	$\lambda_0 = -w_1 \cdot \omega_1 - t\omega_3$ $= (1, -1, -t) \quad \forall t > 0$
α_1	●—×—× 1 2 3	$S = \{\alpha_2, \alpha_3\}$	w_2, w_{23}, w_{32}, w_{232}	$\lambda_0 = -w_1 \cdot \omega_1$ $= (1, -1, 0)$
α_2	×—●—○ 1 2 3	$S = \{\alpha_1\}$	w_1	$\lambda_0 = -w_2 \cdot \omega_2 - t\omega_3$ $= (-1, 1, -t-1) \quad \forall t > 0$
α_2	○—●—× 1 2 3	$S = \{\alpha_3\}$	w_3	$\lambda_0 = -t\omega_1 - w_2 \cdot \omega_2$ $= (-t-1, 1, -1) \quad \forall t > 0$
α_2	○—●—× 1 2 3	$S = \{\alpha_1, \alpha_3\}$	w_1, w_3, w_{13}	$\lambda_0 = -w_2 \cdot \omega_2 = (-1, 1, -1)$

Example 4.8. Another interesting example occurs in the case where G is a quasi-split group of type D_4. The Dynkin diagram of the absolute root system of G, together with our choice of α and S, is given by



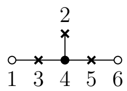
In this case, it follows, from Theorem 4.1, that

$$\operatorname{Ind}_B^G \lambda_0 = \left(\operatorname{Ind}_{P_{\{\alpha_2\}}}^G \mathbf{1}_{M_{\{\alpha_2\}}} \otimes \chi_0\right) \oplus \left(\operatorname{Ind}_{P_{\{\alpha_2\}}}^G \operatorname{St}_{M_{\{\alpha_2\}}} \otimes \chi_0\right),$$

where $\lambda_0 = -w_2 \cdot \omega_2 = (-1,1,-1,-1)$ and $\iota_{M_{\{\alpha_2\}}}(\chi_0) = \left(-\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2}\right)$. In particular, let $\pi_1 \oplus \pi_{-2}$ be the maximal semi-simple subrepresentation of $\operatorname{Ind}_B^G \lambda_0$. Note that the eigenvalue -2 is computed with respect to $w_0 = w_1 w_3 w_4$ and $\mathcal{S} = (-1,s,-1,-1)$.

As in the rank 2 case, $\operatorname{Ind}_B^G \lambda_0$ contains a degenerate principal series representation. Namely, $\operatorname{Ind}_{P_{\{\alpha_1,\alpha_3,\alpha_4\}}}^G \delta_{P_{\{\alpha_1,\alpha_3,\alpha_4\}}}^{-\frac{1}{10}}$ is a subrepresentation of $\operatorname{Ind}_B^G \lambda_0$. It is clear that π_1 is a subrepresentation of $\operatorname{Ind}_{P_{\{\alpha_1,\alpha_3,\alpha_4\}}}^G \delta_{P_{\{\alpha_1,\alpha_3,\alpha_4\}}}^{-\frac{1}{10}}$ and the question is whether π_{-2} is also a subrepresentation. This question was studied in detail in [17, Subsec. 4.4] and it is shown there that π_{-2} is a subrepresentation of $\operatorname{Ind}_{P_{\{\alpha_1,\alpha_3,\alpha_4\}}}^G \delta_{P_{\{\alpha_1,\alpha_3,\alpha_4\}}}^{-\frac{1}{10}}$ when the relative root system of G is of type G_2 and that π_1 is the unique irreducible subrepresentation of $\operatorname{Ind}_{P_{\{\alpha_1,\alpha_3,\alpha_4\}}}^G \delta_{P_{\{\alpha_1,\alpha_3,\alpha_4\}}}^{-\frac{1}{10}}$ when the relative root system of G is of type B_3 or D_4.

Example 4.9. As another example, let G be the split, simply-connected, simple group of type E_6. The Dynkin diagram of G, together with our choice of α and S, is given by



Let $\lambda' = (-1,0,0,-1,0,-1)$ and $\lambda_0 = w_\alpha \cdot \lambda' = (-1,-1,-1,1,-1,-1)$. By Theorem 4.1, it holds that

$$\operatorname{Ind}_B^G \lambda_0 = \left(\operatorname{Ind}_P^G \operatorname{St}_M \otimes \chi_0\right) \oplus \left(\operatorname{Ind}_P^G \mathbf{1}_M \otimes \chi_0\right)$$

and the maximal semi-simple subrepresentation of $\operatorname{Ind}_B^G \lambda_0$ can be written as $\pi_1 \oplus \pi_{-1}$. The degenerate principle series $\Pi = \operatorname{Ind}_{P_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6\}}}^G \delta_{P_{\{\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6\}}}^{-\frac{3}{14}}$ is a subrepresentation of $\operatorname{Ind}_B^G \lambda_0$ so the maximal semi-simple subrepresentation of Π is either π_1 or $\pi_1 \oplus \pi_{-1}$. It is shown in [7] that, in fact, π_1 is the unique irreducible subrepresentation of Π.

5. Decomposition with respect to Levi subgroups of higher semi-simple rank

In this section, we discuss a generalization of Theorem 3.4. This generalization allows to consider points λ_0 where one could apply Corollary 3.6 to triples (λ_0, α, w_0) with more than one simple root α. In such a case, one would be able to prove a finer

decomposition of $\mathrm{Ind}_B^G \lambda_0$ into a direct sum of generalized degenerate principal series. In Subsection 5.1 we provide a framework for such decompositions and in Subsection 5.2 we list various examples for such a decomposition.
In this section we fix a labeling $\Delta = \{\alpha_1, ..., \alpha_n\}$ of the Dynkin diagram according to the conventions of [3]. When there is no source of confusion, we identify the index k with α_k.

5.1. Commuting projections. Let $\Theta = \{\alpha_{m_1}, ..., \alpha_{m_k}\} \subset \Delta$, with $1 \le k \le n$, such that the Dynkin sub-diagram induced by Θ is totally disconnected. Also, let

$$\mathcal{P}(k) = \{X \subset \{1, ..., k\}\}.$$

We recall the parabolic subgroup, $P_\Theta = \bigcap_{i=1}^k P_{\Delta \setminus \{\alpha_{m_i}\}}$, associated to Θ.
For $X \in \mathcal{P}(k)$, let $\mathrm{St}_X = (\otimes_{i\in X} \mathbf{1}_{m_i}) \otimes (\otimes_{i\notin X} \mathrm{St}_{m_i})$, where $\mathbf{1}_j$ and St_j are the trivial and Steinberg representations of $M_j = M^{der}_{\{\alpha_{m_j}\}}$. This is a representation of M^{der}_Θ. We consider a character χ_0 of M as a character of A_M, then $\mathrm{St}_X \otimes \chi_0$ is a representation of M.

Corollary 5.1. *Assume that λ_0 satisfies assumptions* (A1), (A2), (A3) *and* (A4') *with respect to each triple $\left(\lambda_0, \alpha_{m_i}, w_0^{(i)}\right)$ for $\Theta = \{\alpha_{m_1}, ..., \alpha_{m_k}\} \subset \Delta$. For each $1 \le i \le k$, let $\mathscr{P}_i$ be the projection on $\mathrm{Ind}_B^G \lambda_0$ constructed in Equation* (25) *for $\left(\lambda_0, \alpha_{m_i}, w_0^{(i)}\right)$. Further assume that the projections $\mathscr{P}_i$ are mutually commuting.*

Then
$$\mathrm{Ind}_B^G \lambda_0 = \bigoplus_{X \in \mathcal{P}(k)} \mathrm{Ind}_{P_\Theta}^G (\mathrm{St}_X \otimes \chi_0), \tag{30}$$

where $\chi_0 \in X^*(M_\Theta)$ such that its normalized Jacquet functor $\mathcal{J}_T^{M_\Theta} \chi_0$ along N satisfies $\mathcal{J}_T^{M_\Theta} \chi_0 = \lambda_0$.

Proof. For $X \in \mathcal{P}$, let $\mathscr{P}_X = \prod_{i \in X} \mathscr{P}_i \prod_{i \notin X} (Id - \mathscr{P}_i)$.
One checks that $\{\mathscr{P}_X \mid X \in \mathcal{P}(k)\}$ is a set of mutually orthogonal (and hence commuting) projections on $\mathrm{Ind}_B^G \lambda_0$ such that

$$\sum_{X \in \mathcal{P}(k)} \mathscr{P}_X = Id. \tag{31}$$

It follows that
$$\mathrm{Ind}_B^G \lambda_0 = \bigoplus_{X \in \mathcal{P}(k)} \mathrm{Im}(\mathscr{P}_X). \tag{32}$$

On the other hand, for $X \in \mathcal{P}(k)$, we have

$$\begin{aligned} \mathrm{Im}(\mathscr{P}_X) &= \bigcap_{i\in X} \mathrm{Im}(\mathscr{P}_i) \cap \bigcap_{i \notin X} \mathrm{Im}(Id - \mathscr{P}_i) \\ &= \bigcap_{i \in X} \left(\mathrm{Ind}_{P_i}^G \mathbf{1}_i \otimes \chi_0\right) \cap \bigcap_{i\notin X} \left(\mathrm{Ind}_{P_i}^G \mathrm{St}_i \otimes \chi_0\right) = \mathrm{Ind}_P^G \mathrm{St}_X \otimes \chi_0. \end{aligned} \tag{33}$$

The proof is complete. ■

Remark 5.2. Note that if $w_{\alpha_1} w_0^{(1)} w_{\alpha_1}$, $w_2 w_0^{(2)} w_2$,..., $w_k w_0^{(k)} w_k$ are all commuting, then so are $\mathscr{P}_1$, $\mathscr{P}_2$,..., $\mathscr{P}_k$. ■

Remark 5.3. If the projections $\mathscr{P}_1, ..., \mathscr{P}_k$ were not commuting, one can show that the resulting enodmorphisms $\mathscr{P}_X$ would be unipotent and not projective. This shows that some of the (not necessarily irreducible) constituents $\operatorname{Ind}_P^G \operatorname{St}_X \otimes \chi_0$ of $\operatorname{Ind}_B^G \lambda_0$ are not direct summands of $\operatorname{Ind}_B^G \lambda_0$. ■

5.2. Examples. We now wish to use Corollary 3.6, Theorem 4.1, Corollary 5.1 and Remark 5.2 in order to find points λ_0 and triples $\left(\lambda_0, \alpha_{m_i}, w_0^{(i)}\right)$ which satisfy the assumptions of Corollary 5.1. This discussion is aimed at describing "generic examples" and is not intended to be exhaustive, only to set the ground for possible future application.

We interpret the conditions of Theorem 4.1 in view of Corollary 3.6.

System V: Pick a subset $S \subset \Delta \setminus \{\alpha\}$ and $\lambda' \in \mathfrak{a}_{\mathbb{R}}^*$ such that:

$$(1)\ \text{There exist } \beta \in S \text{ such that } \langle \beta, \alpha^\vee \rangle \neq 0, \quad (2)\ \langle \lambda', \alpha^\vee \rangle = -1,$$

$$(3)\ \langle \lambda', \beta^\vee \rangle = 0 \quad \forall \beta \in S, \quad (4)\ \langle \lambda', \beta^\vee \rangle \leq 0 \quad \forall \beta \notin S \cup \{\alpha\}.$$

For the sake of this computation, it is more convenient to consider triples $(\lambda_0, \alpha_{m_i}, S_i)$, where $S_i \subset \Delta$ satisfies System V, and let $w_0^{(i)} \in W_S$ be as in the proof of Theorem 4.1. A choice of $(\lambda_0, (\alpha_{m_1}, S_1), ..., (\alpha_{m_k}, S_k)) \in \mathfrak{a}_{\mathbb{C}}^* \times (\Delta, \mathcal{P}(\Delta))^k$ is called *nice data* if each triple $(\lambda_0, \alpha_{m_i}, S_i)$ satisfies the conditions of Theorem 4.1 and that $[w_i, w_j] = 1$ for any $w_i \in W_{S_i \cup \{\alpha_{m_i}\}}$ and $w_j \in W_{S_j \cup \{\alpha_{m_j}\}}$ (see (7) for the definition of W_{S_i}).

By abuse of notations, we call $((\alpha_{m_1}, S_1), ..., (\alpha_{m_k}, S_k)) \in (\Delta, \mathcal{P}(\Delta))^k$ *nice data* if there exists $\lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$ so that $(\lambda_0, (\alpha_{m_1}, S_1), ..., (\alpha_{m_k}, S_k))$ is a nice data.

From now on, we consider only groups of type A_n, with $n \geq 5$, or E_n for the following reasons:

- The underlying graph of type B_n, C_n, G_2 or F_4 is the same as that of A_n, hence it is enough to consider only types A_n, D_n and E_n.
- If G is of type D_n, let α_{n-1} and α_n denote the "horns" of the Dynkin diagram of type D_n. Generically $w_{n-1} w_0^{(n-1)} w_{n-1}$ and $w_n w_0^{(n)} w_n$ will not commute. Hence, a "generic" choice of vertices on the Dynkin diagram of type D_n can be done in the diagram of type A_{n-1}.

  It should be noted that, for particular choices of $w_0^{(n-1)}$ and $w_0^{(n)}$, these words might commute.
- For similar reasons, we consider only the cases where $\{\alpha_{m_i}\} \cup S_i$ are disjoint and have no neighboring vertices. In particular, we assume that $rank\,(G) \geq 5$.

Furthermore, we will assume that each of the $\{\alpha_{m_i}\} \cup S_i$ is connected. Under this assumption, we may replace the last item by the assumption that the sets $\{\alpha_{m_i}\} \cup S_i$ are disjoint and that $\left\{\alpha_{m_i}, \alpha_{m_j}\right\} \cup S_i \cup S_j$ is disconnected for all i and j.

We point out that each choice of sets S_i constitutes a function $S : \{1, ..., k\} \to \mathcal{P}(n)$, and the space of such functions admits a partial order induced from the containment order on $\mathcal{P}(n)$. We say that a nice data $(\lambda_0, (\alpha_{m_1}, S_1), ..., (\alpha_{m_k}, S_k))$ is *minimal* if there is no nice data $(\lambda_0, (\alpha_{m_1}, S_1'), ..., (\alpha_{m_k}, S_k'))$ such that $S' < S$ in the order on the space of functions $\{1, ..., k\} \to \mathcal{P}(n)$. One, similarly, defines *maximal* nice data.

We note that for any nice data $(\lambda_0, (\alpha_{m_1}, S_1), ..., (\alpha_{m_k}, S_k))$, there exist a minimal nice data $(\lambda_0, (\alpha_{m_1}, S'_1), ..., (\alpha_{m_k}, S'_k))$ and a maximal nice data $(\lambda_0, (\alpha_{m_1}, S''_1), ..., (\alpha_{m_k}, S''_k))$ such that $S' \leq S \leq S''$.

We also note that if $(\lambda_0, (\alpha_{m_1}, S_1), ..., (\alpha_{m_k}, S_k))$ is a minimal nice data, then S_i is a singleton for any $1 \leq i \leq k$.

Below, we describe an algorithm to construct minimal nice data $(\lambda_0, (\alpha_{m_1}, S_1), ..., (\alpha_{m_k}, S_k))$ and construct all nice data for the group $SL_6(F)$. Then, we turn to describe all maximal nice data for groups of type E_n.

Remark 5.4. We point out that in order to prove the decomposition in Equation (30), it is enough to study only minimal nice data. However, other nice data are useful to study the behavior of intertwining operators $N_w(\lambda_0)$ for various $w \in \mathrm{Stab}_W(\lambda_0)$. ■

We start by determining the simple roots $\alpha_{m_1}, ..., \alpha_{m_k}$ and singletons $S_1, ..., S_k$ which could participate in a nice data and then describe all values of $\lambda_0 \in \mathfrak{a}^*_{\mathbb{C}}$ such that $(\lambda_0, (\alpha_{m_1}, S_1), ..., (\alpha_{m_k}, S_k))$ comprises nice data.

A sequence $(\gamma_i, \gamma'_i)_{i=1}^k \in (\Delta \times \Delta)^k$ is said to be a *well-separated data* if it satisfies the following conditions:

- $\gamma_i \not\perp \gamma'_i$ for all $1 \leq i \leq k$.
- $\gamma_i \perp \gamma_j$ and $\gamma_i \perp \gamma'_j$ for all $1 \leq i \leq k$ and any $j \neq i$.

Here $\gamma \perp \delta$ if $(\gamma, \delta) = 0$. Choosing $\alpha_{m_i} = \gamma_i$ and $S_i = \{\gamma'_i\}$ (or vice versa) and choosing $\lambda_0 = (s_1, ..., s_n) \in \mathfrak{a}^*_{\mathbb{C}}$ as follows results in nice data $(\lambda_0, (\alpha_{m_1}, S_1), ..., (\alpha_{m_k}, S_k))$:

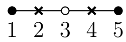
- For any $1 \leq i \leq k$, let $s_{m_i} = -1$.
- For any $t \in \bigcup_{i=1}^k S_i$, let $s_t = 0$.
- For any $t \notin \bigcup_{i=1}^k S_i$, choose $s_t \leq 0$.

Below, we describe all nice data for the group $SL_6(F)$ (of type A_5) and all maximal nice data for the groups of type E_n. For other groups, one can follow the algorithm described above. In order to mark our choice of α_i and S_i we use the following markings on the Dynkin diagram of G (similar to the notations used in Section 4):

- We use $\bullet$ to denote the simple roots $\alpha_{m_1}, ..., \alpha_{m_k}$ in Θ.
- We use $\times$ to denote simple roots which lie in one of the S_i. Note that since we assume that $\{\alpha_{m_i}, \alpha_{m_j}\} \cup S_i \cup S_j$ is disconnected, there is no source of confusion in using the same notation for the different sets S_i.
- We use $\circ$ to denote simple roots not in $\Theta \cup \bigcup_{i=1}^k S_i$.
- The k-vertex in a Dynkin diagram is associated to the simple root denoted α_k. We further denote by ω_k the k^{th} fundamental weight and by $w_k = w_{\alpha_k}$ the simple reflection associated to α_k.

Example 5.5. Let G be of type A_5 (i.e. $G = SL_6(F)$). There are four possible choices of vertices α_{m_i}. All of these are associated with the well-separated data $((\alpha_1, \alpha_2), (\alpha_4, \alpha_5))$.

(1) Choosing the 1^{st} and 5^{th} vertices in the Dynkin diagram:



Here, $\Theta = \{\alpha_1, \alpha_5\}$, $S_1 = \{\alpha_2\}$, $S_2 = \{\alpha_4\}$ and

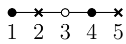
$$P_\Theta = \left\{ \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix} \in SL_6(F) \right\}$$

The possible associated points are $\lambda_0 = -(w_1 \cdot \omega_1 + w_5 \cdot \omega_5) - t\omega_3$, where $t \geq 0$. The two projections in Corollary 5.1 are the ones associated to $N(w_1w_2w_1)$ and $N(w_5w_4w_5)$. The decomposition which follows is

$$\begin{aligned} \operatorname{Ind}_B^G \lambda_0 &= \bigoplus_{X\subseteq\{1,5\}} \operatorname{Ind}_{P_\Theta}^G \operatorname{St}_X \otimes\chi_0 \\ &= \operatorname{Ind}_{P_\Theta}^G (\mathbf{1}_1 \otimes \mathbf{1}_5 \otimes\chi_0) \oplus \operatorname{Ind}_{P_\Theta}^G (\mathbf{1}_1 \otimes \operatorname{St}_5 \otimes\chi_0) \\ &\quad \oplus \operatorname{Ind}_{P_\Theta}^G (\operatorname{St}_1 \otimes \mathbf{1}_5 \otimes\chi_0) \oplus \operatorname{Ind}_{P_\Theta}^G (\operatorname{St}_1 \otimes \operatorname{St}_5 \otimes\chi_0), \end{aligned}$$

where $\mathcal{J}_T^{M_\Theta}\chi_0 = \lambda_0$.

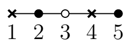
(2) Choosing the 1^{st} and 4^{th} vertices in the Dynkin diagram:



The associated points are $\lambda_0 = -(w_1 \cdot \omega_1 + w_4 \cdot \omega_4) - t\omega_3$, where $t \geq 0$. The two projections in Corollary 5.1 are the ones associated to $N(w_1w_2w_1)$ and $N(w_4w_5w_4)$. The decomposition which follows is

$$\operatorname{Ind}_B^G \lambda_0 = \bigoplus_{X\subseteq\{1,4\}} \operatorname{Ind}_{P_\Theta}^G \operatorname{St}_X \otimes\chi_0.$$

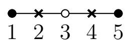
(3) Choosing the 2^{nd} and 5^{th} vertices in the Dynkin diagram:



The associted point is $\lambda_0 = -(w_2 \cdot \omega_2 + w_5 \cdot \omega_5) - t\omega_3$, where $t \geq 0$. The two projections in Corollary 5.1 are the ones associated to $N(w_2w_1w_2)$ and $N(w_5w_4w_5)$. The decomposition which follows is

$$\operatorname{Ind}_B^G \lambda_0 = \bigoplus_{X\subseteq\{2,5\}} \operatorname{Ind}_{P_\Theta}^G \operatorname{St}_X \otimes\chi_0.$$

(4) Choosing the 2^{nd} and 4^{th} vertices in the Dynkin diagram:

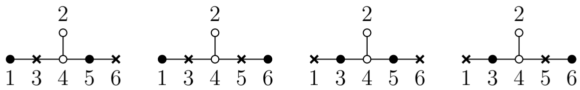


The associted point is $\lambda_0 = -(w_2 \cdot \omega_2 + w_4 \cdot \omega_4) - t\omega_3$, where $t \geq 0$. The two projections in Corollary 5.1 are the ones associated to $N(w_2w_1w_2)$ and $N(w_4w_5w_4)$. The decomposition which follows is

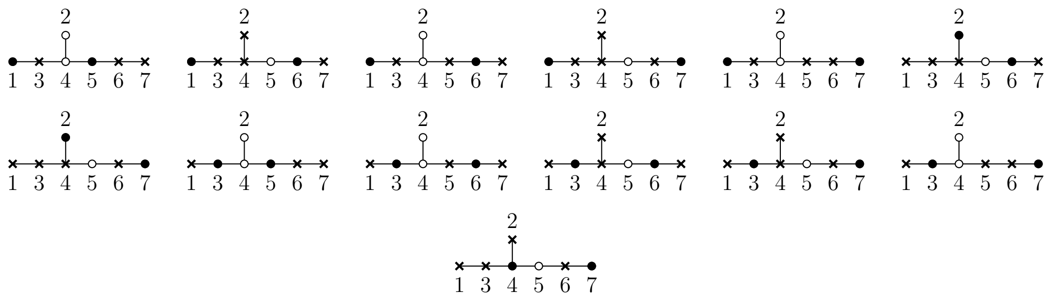
$$\operatorname{Ind}_B^G \lambda_0 = \bigoplus_{X\subseteq\{2,4\}} \operatorname{Ind}_{P_\Theta}^G \operatorname{St}_X \otimes\chi_0.$$

We now list all possible maximal nice data for groups of type E_n. As it happens, the conditions for a well-separated choice allow only choices with $k = 2$.

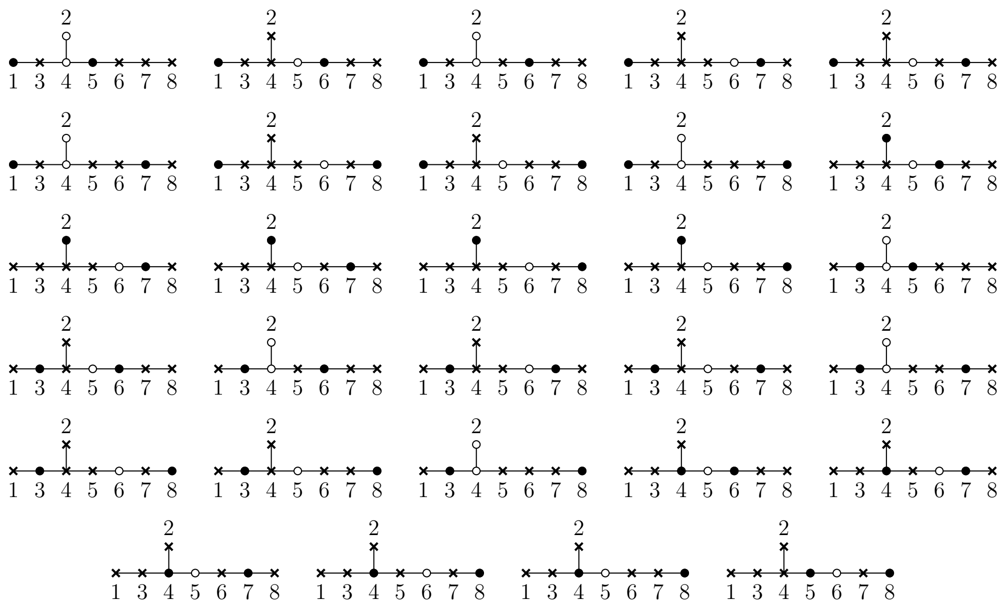
- If G is of type E_6, the possible choices are given by the following diagrams:



- If G is of type E_7, the possible choices are given by the following diagrams:



- If G is of type E_8, the possible choices are given by the following diagrams:



Appendix

A. Some Facts on Root Systems and Weyl Groups

In this section we record a few simple but useful facts about the action of the Weyl group on the root system for which we weren't able to locate a convenient reference. We retain the notations of Section 2.

Lemma A.1. *Let $w \in W$ and $\alpha \in \Delta$. If w and w_α commute, then*

$$w(\alpha) \in \{\alpha, -\alpha\} \quad \text{and} \quad w(\alpha^\vee) \in \{\alpha^\vee, -\alpha^\vee\}.$$

Proof. Indeed,

$$w_\alpha\left(w\left(\alpha\right)\right)=w\left(\alpha\right)-\left\langle w\left(\alpha\right),\alpha^\vee\right\rangle\alpha\ \text{ and }\ w\left(w_\alpha\left(\alpha\right)\right)=w\left(-\alpha\right)=-w\left(\alpha\right).$$

Since $w_\alpha w=ww_\alpha$ it follows that $w\left(\alpha\right)=\frac{1}{2}\left\langle w\left(\alpha\right),\alpha^\vee\right\rangle\alpha$ and hence, since the root system is reduced, it follows that $w\left(\alpha\right)\in\left\{\alpha,-\alpha\right\}$. Similarly $w\left(\alpha^\vee\right)\in\left\{\alpha^\vee,-\alpha^\vee\right\}$. ■

Lemma A.2. *Let $w\in W$ and $\alpha\in\Delta$. Assume that w and w_α commute. Then*

$$\left\langle ww_\alpha\lambda,\alpha^\vee\right\rangle=\pm\left\langle\lambda,\alpha^\vee\right\rangle\quad\forall\lambda\in\mathfrak{a}_{\mathbb{R}}^*.$$

Proof. We start by noting that w^{-1} also commutes with w_α. Hence

$$\left\langle ww_\alpha\lambda,\alpha^\vee\right\rangle=\left\langle w_\alpha\lambda,w^{-1}\alpha^\vee\right\rangle=\left\langle w_\alpha\lambda,\pm\alpha^\vee\right\rangle=\pm\left\langle\lambda,w_\alpha\alpha^\vee\right\rangle=-\mp\left\langle\lambda,\alpha^\vee\right\rangle.$$ ■

Lemma A.3. *Assume that*

$$\left\{\lambda\in\mathfrak{a}_{\mathbb{R}}^*\mid\left\langle w_\alpha ww_\alpha\lambda,\alpha^\vee\right\rangle=1\right\}=\left\{\lambda\in\mathfrak{a}_{\mathbb{R}}^*\mid\left\langle\lambda,\alpha^\vee\right\rangle=1\right\}.$$

Then w and w_α commute.

Proof. Fix $\lambda_0\in H_1=\left\{\lambda\in\mathfrak{a}_{\mathbb{R}}^*\mid\left\langle\lambda,\alpha^\vee\right\rangle=1\right\}$ and consider the vector space $V=H_1-\lambda_0=\alpha^\perp$. It follows that

$$\left\{\lambda\in\mathfrak{a}_{\mathbb{R}}^*\mid\left\langle w_\alpha ww_\alpha\lambda,\alpha^\vee\right\rangle=0\right\}=\left\{\lambda\in\mathfrak{a}_{\mathbb{R}}^*\mid\left\langle\lambda,\alpha^\vee\right\rangle=0\right\}.$$

Namely, $\alpha^\perp=\left(w_\alpha ww_\alpha\alpha\right)^\perp$. Since the root system is reduced, we conclude that $\alpha=\pm w_\alpha ww_\alpha\alpha$, or in other words $w\alpha=\pm\alpha$. It follows that $w\alpha^\vee=\pm\alpha^\vee$ (same sign). We show that $ww_\alpha=w_\alpha w$ by examining the action of both sides on $\mathfrak{a}_{\mathbb{R}}^*$. Indeed,

$$ww_\alpha\lambda=w\left(\lambda-\left\langle\lambda,\alpha^\vee\right\rangle\alpha\right)=w\lambda\mp\left\langle\lambda,\alpha^\vee\right\rangle\alpha=w\lambda-\left\langle w\lambda,\alpha^\vee\right\rangle\alpha=w_\alpha w\lambda.$$ ■

Lemma A.4. *Fix $\alpha\in\Delta$ and $\lambda\in\mathfrak{a}_{\mathbb{R}}^*$ such that $\left\langle\lambda',\beta^\vee\right\rangle\le0$ for all $\beta\in\Delta$ and, in particular, $\left\langle\lambda',\alpha^\vee\right\rangle=-1$. Then, for any $\beta\in\Phi^+\setminus\{\alpha\}$, it holds that*

$$\left\langle\lambda',\beta^\vee\right\rangle\le-\frac{1}{2}\left\langle\alpha,\beta^\vee\right\rangle. \tag{34}$$

Proof. We recall the inner product $(\cdot,\cdot)$ on $V=\mathfrak{a}_{\mathbb{R}}^*$, fixed in Equation (6). The inequality $\left\langle\lambda',\beta^\vee\right\rangle\le-\frac{1}{2}\left\langle\alpha,\beta^\vee\right\rangle$ is equivalent to

$$\left(\lambda,\beta\right)\le-\frac{1}{2}\left(\alpha,\beta\right). \tag{35}$$

We note that for $\beta,\gamma\in\Delta$ it holds that

$$\left(\beta,\omega_\gamma\right)=\begin{cases}\frac{(\beta,\beta)}{2}, & \beta=\gamma\\ 0, & \beta\ne\gamma\end{cases},$$

where ω_γ is the fundamental weight associated with γ.

Fix $\beta \in \Phi^+ \setminus \{\alpha\}$ and write

$$\beta = \sum_{\gamma\in\Delta} n_\gamma(\beta)\,\gamma, \quad \lambda' = \sum_{\gamma\in\Delta} m_\gamma(\lambda')\,\omega_\gamma, \quad S = \{\gamma \in \Delta \mid (\lambda',\gamma) = 0\}.$$

We note that, by the definition of Φ^+, $n_\gamma(\beta) \geq 0$ for all $\gamma \in \Delta$.
Since $(\alpha,\gamma) \leq 0$ for all $\gamma \in \Delta \setminus \{\alpha\}$, it holds that

$$\frac{(\alpha,\beta)}{(\alpha,\alpha)} = \sum_{\gamma\in\Delta} n_\gamma(\beta)\frac{(\alpha,\gamma)}{(\alpha,\alpha)} \leq n_\alpha(\beta).$$

Hence, it holds that $n_\alpha(\beta)\cdot(\alpha,\alpha) - (\alpha,\beta) \geq 0$.
On the other hand, by our assumption on λ', $m_\gamma(\lambda') = \frac{2(\lambda',\gamma)}{(\gamma,\gamma)} < 0$ we have for any $\gamma \in \Delta \setminus (S \cup \{\alpha\})$. In particular, it follows that

$$\sum_{\gamma\in\Delta\setminus(S\cup\{\alpha\})} n_\gamma(\beta)\frac{(\gamma,\gamma)}{2} m_\gamma(\lambda') \leq 0 \leq \frac{n_\alpha(\beta)\cdot(\alpha,\alpha) - (\alpha,\beta)}{2}. \tag{36}$$

It follows that

$$\begin{aligned}(\lambda',\beta) &= \sum_{\gamma\in\Delta} n_\gamma(\beta)(\lambda',\gamma) = \sum_{\gamma\in\Delta} n_\gamma(\beta)\frac{(\gamma,\gamma)}{2} m_\gamma(\lambda') \\ &= \sum_{\gamma\in\Delta\setminus(S\cup\{\alpha\})} n_\gamma(\beta)\frac{(\gamma,\gamma)}{2} m_\gamma(\lambda') - n_\alpha(\beta)\frac{(\alpha,\alpha)}{2} \leq -\frac{1}{2}(\alpha,\beta).\end{aligned}$$

Therefore Equation (35) holds, which completes the proof. ■

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Taeuk Nam, Department of Mathematics, University of British Columbia, Vancouver, BC, V6T 1Z2, Canada, taeuk.nam@alumni.ubc.ca.

Avner Segal, Department of Mathematics, Bar Ilan University, Ramat Gan 5290002, Israel, segalavner@gmail.com.

Lior Silberman, Department of Mathematics, University of British Columbia, Vancouver, BC, V6T 1Z2, Canada, lior@math.ubc.ca.

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