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# Topologically Simple, Totally Disconnected, Locally Compact Infinite Matrix Groups

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**Abstract.** We construct uncountably many non-locally isomorphic examples of topologically simple nondiscrete totally disconnected locally compact groups. The new examples differ from known examples of such groups in that they have trivial quasi-centre, but also have infinite abelian locally normal subgroups. The examples are constructed as almost upper-triangular matrices modulo scalar matrices over finite fields, where 'almost upper-triangular' is defined with respect to one of an uncountable family of preorders generalising the orders $(\mathbb{Z}, \leq)$ and $(\mathbb{N}, \leq)$.

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## 1. Introduction

The present article contributes to the theory of totally disconnected, locally compact (t.d.l.c.) groups by constructing examples of topologically simple t.d.l.c. groups. The main result is the following.

**Theorem 1.1.** *There are $2^{\aleph_0}$ local isomorphism types of second-countable t.d.l.c. groups $G$ with the following properties:*

(i) *$G$ is topologically simple, with a proper dense normal subgroup;*

(ii) $\mathrm{QZ}(G) = \{1\}$*;*

(iii) *$G$ has an infinite abelian subgroup with open normaliser.*

The construction is of the form $G = \mathrm{AU}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q)$, where $\Lambda$ ranges over an uncountable family of totally preordered sets, and $\mathrm{AU}_\Lambda$ and $\mathrm{Z}_\Lambda$ refer to respectively the 'almost $\Lambda$-upper-triangular' and scalar matrices over the given finite field. Such a group has a compact open subgroup $\mathrm{U}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q)$ formed as $\Lambda$-upper-triangular matrices modulo scalar matrices.

**Remark 1.2.** Groups of infinite-dimensional almost upper-triangular matrices have been described previously. In the case when the preorder is $\mathbb{N}$ with its usual order, they were introduced and their representation theory investigated by A. M. Vershik and A. Zelevinsky in the early 1980s and by S. V. Kerov and A. M. Vershik in the 1990s, see [11]. The current state of the representation theory is given in [6], which also motivates study of the group and gives a short history of its representation

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theory with many more references. (The group called $\mathrm{AU}_{\mathbb{N}}(\mathbb{F}_q)$ in the present paper is called $\mathbb{GLB}$ in [6].) Local compactness of $\mathbb{GLB}$ and Haar measure are important for the representation theory.

Parabolic subgroups and the commutator subgroup of $\mathbb{GLB}$, and of similarly defined groups over more general rings, are described in [7, 8, 10]. It is noted in those papers that $\mathbb{GLB}$ is topologically simple modulo its centre. The novelty of the present paper is that we define uncountably many non-isomorphic infinite-matrix groups over each finite field and study their local structure.

We are grateful to W. Hołubowski for drawing our attention to the previous work on these groups after the first version of our paper was posted on the arXiv. ■

To explain the significance of Theorem 1.1 for the theory of t.d.l.c. groups, we recall some of the broader context.

A key aspect of the theory of locally compact groups is the relationship between the local structure of a group $G$, that is, the properties shared by all the neighbourhoods of the identity in $G$, and the global properties of $G$. For connected locally compact groups, the connection is strong and well-understood: every such group is a pro-Lie group, and the structure of connected pro-Lie groups is controlled to a great extent by the associated pro-Lie algebra, which precisely captures the local structure (see [9]). In the complementary case of t.d.l.c. groups, the key result on local structure is Van Dantzig's theorem: there is a base of neighbourhoods of the identity consisting of compact open subgroups, and so the local structure consists of the properties of a compact open subgroup that are invariant on passing to a closed subgroup of finite index. However, compared to (pro-)Lie groups, the general connection between local and global structure is less rigid and much less well-understood. A basic question to ask, by analogy with the importance of simple groups in Lie group theory and elsewhere, is which local structures can occur in (topologically) simple t.d.l.c. groups.

The class of t.d.l.c. groups includes the discrete groups, where there is nothing to say about local structure in the topological sense. However, if one moves sufficiently far away from the discrete case, a significant connection between local and global structure emerges; here we recall the framework introduced in [2] and [1]. We say two t.d.l.c. groups $G$ and $H$ are *locally isomorphic* if there is an open subgroup of $G$ that is isomorphic as a topological group to some open subgroup of $H$. This defines an equivalence relation on t.d.l.c. groups, and we can refer to the equivalence class of $G$ up to local isomorphism as its *local isomorphism type*. Suppose that $G$ is a t.d.l.c. group that is nondiscrete, but the quasi-centre $\mathrm{QZ}(G)$, that is, the set of elements with open centraliser, is discrete.

In this situation, the quotient $G/\mathrm{QZ}(G)$ is locally isomorphic to $G$ and has trivial quasi-centre, so one effectively reduces to the case where the quasi-centre is trivial. Given a group $G$ with trivial quasi-centre, the group of germs $\mathscr{L}(G)$ consists of all isomorphisms between open subgroups of $G$, modulo equality on an open set. Moreover $\mathscr{L}(G)$ is equipped with a t.d.l.c. group topology, which is the unique group topology such that the map $\mathrm{ad} : G \to \mathscr{L}(G)$ is an open embedding, where $\mathrm{ad}(g)$ is the isomorphism from $G$ to itself given by conjugation by $g$. In this context, the following dichotomy emerges: either there are no (topologically) simple groups locally isomorphic to $G$, or else the group $R := \mathrm{Res}(\mathscr{L}(G))$, defined as the intersection of all open normal subgroups of $\mathscr{L}(G)$, is open and (topologically) simple. In the latter case, $R$ is the largest (topologically) simple group locally isomorphic to $G$: given any topologically simple group $H$ locally isomorphic to $G$, there is an open embedding $\theta : H \to R$, and every open embedding of $R$ into a topologically simple group is surjective.

Examples are known of nondiscrete topologically simple t.d.l.c. groups with dense quasi-centre, see for example [12]. In the case of topologically simple t.d.l.c. groups with trivial quasi-centre, the known examples also have the local property that no nontrivial abelian subgroup has open normaliser. In fact, for nondiscrete compactly generated topologically simple t.d.l.c. groups $G$ it is known that $G$ has trivial quasi-centre ([1, Theorem 4.8]) and no nontrivial abelian subgroup of $G$ has open normaliser ([4, Theorem A]).

Returning to the groups $\mathrm{AU}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q)$ mentioned above, the general theory implies that there is a largest topologically simple group

$$\mathrm{RU}_\Lambda(\mathbb{F}_q) := \mathrm{Res}(\mathscr{L}(\mathrm{AU}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q)))$$

of the same local isomorphism type; the existence of nontrivial abelian subgroups with open normaliser ensures that $\mathrm{RU}_\Lambda(\mathbb{F}_q)$ cannot be compactly generated. The following natural question remains open.

**Question 1.3.** For which preorders $\Lambda$ and finite fields $\mathbb{F}_q$ is $\mathrm{AU}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q) = \mathrm{RU}_\Lambda(\mathbb{F}_q)$? When they are not equal, what is $\mathrm{RU}_\Lambda(\mathbb{F}_q)$?

The article is structured as follows: We define the matrix groups and their indexing sets in Section 2. In Section 3 we develop the algebraic properties of these groups and identify subgroups important for later results. The topology on the groups is described in Section 4. In Section 5 we show that $\mathrm{AU}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q)$ is topologically simple. In Section 6 we investigate the local structure, concluding with a proof of Theorem 1.1.

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## 2. Definitions

The groups of interest are infinite matrix groups in which the matrices are indexed by preorders of a certain form, which we will now define.

**Definition 2.1.** We recall that a *preorder* $\lesssim$ on a set $\Lambda$ is a reflexive transitive binary relation, where we write $i \lesssim j$ to mean $(i,j) \in \lesssim$; we say it is *total* if for all $i,j \in \Lambda$, at least one of $i \lesssim j$ and $j \lesssim i$ holds. The *complete* preorder is $\Lambda \times \Lambda$.

Fix a totally preordered set $(\Lambda, \lesssim)$. We abbreviate '$i \lesssim j$ and $j \lesssim i$' to $i \sim j$ and '$i \lesssim j$ but $j \not\lesssim i$' to $i \lnsim j$; the interval notation $[i,j]$ refers to $\{k \in \Lambda \mid i \lesssim k \lesssim j\}$.

The subset $\Lambda'$ of $\Lambda$ is *convex* if, whenever $i,j \in \Lambda'$, then $[i,j] \subseteq \Lambda'$. Equivalently, for all $i \in \Lambda \setminus \Lambda'$, either $i \lnsim j$ for all $j \in \Lambda'$, or else $i \gnsim j$ for all $j \in \Lambda'$. The *convex hull* of a subset $\Lambda'$ of $\Lambda$ is the smallest convex set that contains it. The totally preordered set $(\Lambda, \lesssim)$ is $\mathbb{Z}$*-like* if every finite subset of $\Lambda$ is contained in a finite convex subset of $\Lambda$.

Write $\mathcal{C}_\Lambda$ for the set of finite convex subsets of $\Lambda$, partially ordered by inclusion. ■

In the cases when $\Lambda$ is $\mathbb{N}$, $-\mathbb{N}$ or $\mathbb{Z}$ with their usual ordering, a finite convex subset is just an interval $[m,n]$ with $m<n$. For this article we take the convention $0 \notin \mathbb{N}$.

Next, we give a construction of $\mathbb{Z}$-like preorders that will be used to construct examples of groups.

**Proposition 2.2.** *Suppose that $(\Lambda, \lesssim)$ is $\mathbb{Z}$-like and that $\mathcal{P} = \{\mathfrak{p}_\alpha\}$ is a partition of $\Lambda$ into finite convex subsets. Denote the equivalence relation corresponding to $\mathcal{P}$ by $\sim_\mathcal{P}$. Then $(\Lambda, \lesssim^\mathcal{P})$ is $\mathbb{Z}$-like, where $\lesssim^\mathcal{P} = \lesssim \cup \mathcal{P}$.*

**Proof.** Let $i,j,k \in \Lambda$, where $i \in \mathfrak{p} \in \mathcal{P}$. If $i \sim_\mathcal{P} j$ and $j \lesssim k$, then either $k \lesssim i$, in which case $i \sim_\mathcal{P} k$ since $\mathfrak{p}$ is convex, or else $i \lesssim k$. A similar argument holds if $i \lesssim j$ and $j \sim_\mathcal{P} k$. Therefore $\lesssim^\mathcal{P} = \lesssim \cup \mathcal{P}$ is a transitive relation; since $\lesssim^\mathcal{P}$ contains the total preorder $\lesssim$, we conclude that $\lesssim^\mathcal{P}$ is a total preorder.

Let $\mathfrak{q}$ be a finite subset of $\Lambda$ and let $\mathfrak{q}'$ be the $\lesssim$-convex hull of $\mathfrak{q}$. Let $\mathfrak{p}_1$, ..., $\mathfrak{p}_n$ be the parts of $\mathcal{P}$ that intersect $\mathfrak{q}'$. Then $\mathfrak{q}'' = \bigcup_{l=1}^n \mathfrak{p}_l$ is a finite subset of $\Lambda$. To complete the proof, it suffices to show that $\mathfrak{q}''$ is $\lesssim^\mathcal{P}$-convex. Consider $i,j,k \in \Lambda$ such that

$$i \lesssim^\mathcal{P} j \lesssim^\mathcal{P} k$$

and suppose that $i,k \in \mathfrak{q}''$; we must show that $j \in \mathfrak{q}''$. We see that $i \sim_\mathcal{P} i'$ and $k \sim_\mathcal{P} k'$ for $i',k' \in \mathfrak{q}'$, and then by transitivity

$$i' \lesssim^\mathcal{P} j \lesssim^\mathcal{P} k'.$$

If $j$ is $\mathcal{P}$-equivalent to either $i'$ or $k'$, then clearly $j \in \mathfrak{q}''$. Otherwise, we have

$$i' \lesssim j \lesssim k,$$

and hence $j \in \mathfrak{q}'$, since $\mathfrak{q}'$ is $\lesssim$-convex. Thus $j \in \mathfrak{q}''$ as required. ■

We will use the following terms and notation when applying Proposition 2.2.

**Definition 2.3.** Given a finite convex subset $\Lambda' \subset \Lambda$, the partition

$$\{\Lambda'\} \sqcup \{[k,k] \subset \Lambda \mid k \in \Lambda \setminus \Lambda'\}$$

will be denoted by $\mathcal{P}\Lambda'$. We write $\Lambda + \Lambda'$ for the preordered set $(\Lambda, \lesssim^{\mathcal{P}\Lambda'})$ as defined in Proposition 2.2. ■

Note that, when $\Lambda'$ is a finite convex subset of $\Lambda$, the preorder $\Lambda + \Lambda'$ agrees with that of $\Lambda$ except that all elements of $\Lambda'$ are equivalent.

Next, we define matrix operations and associated notation. We will later focus on matrices over finite fields, but for the moment the definitions apply to any commutative ring $R$.

**Definition 2.4.** Fix a set $\Lambda$ and a commutative unital ring $R$; we write $R^*$ for the group of units of $R$. A $(\Lambda \times \Lambda)$-*matrix* over $R$ is a tuple $(a_{ij})_{i,j\in\Lambda}$ such that $a_{ij} \in R$ for all $i,j \in \Lambda$. Note that we can add any pair of $(\Lambda \times \Lambda)$-matrices entry by entry. We define a partial operation of multiplication of $(\Lambda \times \Lambda)$-matrices: the product of $(a_{ij})$ and $(b_{ij})$ is given by $(c_{ij})$ where

$$c_{ij} = \sum_{k\in\Lambda} a_{ik} b_{kj}, \tag{1}$$

subject to the requirement that the product is only defined if, for all $i, j \in \Lambda$, the sum defining $c_{ij}$ has only finitely many nonzero terms. It then follows that matrix multiplication is associative, for the same reason as for finite-dimensional matrices. Write $\mathrm{M}_\Lambda(R)$ for the set of $(\Lambda\times\Lambda)$-matrices equipped with the operation of addition (under which $\mathrm{M}_\Lambda(R)$ is an abelian group) and the partial operation of multiplication. When $\Lambda$ is finite $\mathrm{M}_\Lambda(R)$ is a ring; we write $\mathrm{GL}_\Lambda(R)$ for the group of units of this ring.

Suppose for the rest of this definition that $\Lambda$ is a $\mathbb{Z}$-like preordered set. The $(\Lambda\times\Lambda)$-matrix $(a_{ij})$ is *($\Lambda$)-nonsingular* if there is a finite convex $\Lambda' \subset \Lambda$ such that, for every finite convex subset $\Lambda'' \supset \Lambda'$, the $(\Lambda'' \times \Lambda'')$-submatrix $(a_{ij})_{i,j\in\Lambda''}$ is invertible over $R$. The matrix is $\Lambda$-*diagonal* if $a_{ij} = 0$ whenever $i \not\sim j$, and *scalar* if $a_{ij} = 0$ whenever $i \neq j$ and $a_{ii}$ is constant as $i$ ranges over $\Lambda$. We remark that any matrix can be multiplied on either side by a $\Lambda$-diagonal matrix, and moreover that every matrix commutes with the scalar matrices. In particular, the matrix $I = (\delta_{ij})$ is an identity element for multiplication on $\mathrm{M}_\Lambda(R)$. Also observe that, for each $k \in \Lambda$, the restriction of the $\Lambda$-diagonal matrices to the $[k,k]\times[k,k]$ 'block' $\{(i,j) \mid i,j \sim k\}$ produces a ring isomorphic to $\mathrm{M}_{[k,k]}(R)$. ∎

The following observation, which we will use without further comment, will be useful when working with nonsingular matrices.

**Lemma 2.5.** *Let $\Lambda$ be a $\mathbb{Z}$-like preordered set and let $\Lambda_0 \in \mathcal{C}_\Lambda$. Then a $(\Lambda\times\Lambda)$-matrix $(a_{ij})$ is $\Lambda$-nonsingular if and only if it is $(\Lambda+\Lambda_0)$-nonsingular.*

**Proof.** In the definition of $\Lambda$-nonsingularity, there is no loss of generality in taking the convex subset $\Lambda'$ to contain $\Lambda_0$. The convex subsets of $\Lambda$ and $\Lambda+\Lambda_0$ containing $\Lambda'$ are then the same, so the nonsingularity property is the same with respect to $\Lambda$ as with respect to $\Lambda+\Lambda_0$. ∎

We now define and name some subsets of $\mathrm{M}_\Lambda(R)$ that will turn out to be groups under the matrix multiplication.

**Definition 2.6.** The group of nonsingular $\Lambda$-diagonal matrices is denoted by $\Delta_\Lambda(R)$ and its subgroup of nonsingular scalar matrices by $\mathrm{Z}_\Lambda(R)$.

Given a $\mathbb{Z}$-like preordered set $\Lambda$, a $(\Lambda\times\Lambda)$-matrix $(a_{ij})$ is: *($\Lambda$-)upper-triangular* if $a_{ij} = 0$ whenever $i \gnsim j$; *strictly ($\Lambda$-)upper-triangular* if $a_{ij} = 0$ whenever $i \gtrsim j$; and *almost ($\Lambda$-)upper-triangular* if $a_{ij} = 0$ for all but finitely many pairs $i, j$ such that $i \gnsim j$. Write $\mathrm{U}_\Lambda(R)$ for the set of nonsingular upper-triangular $(\Lambda\times\Lambda)$-matrices; $\mathrm{U}^*_\Lambda(R)$ for the set of matrices $(\delta_{ij} + a_{ij})$ with $(a_{ij})$ strictly upper-triangular; and $\mathrm{AU}_\Lambda(R)$ for the set of nonsingular almost upper-triangular $(\Lambda\times\Lambda)$-matrices.

**Remark 2.7.** Let $(\Lambda, \lesssim)$ be a $\mathbb{Z}$-like preordered set and suppose that the $(\Lambda\times\Lambda)$-matrix $(a_{ij})$ is nonsingular and almost upper-triangular. Then there is $\Lambda' \in \mathcal{C}_\Lambda$ such that $a_{ij} = 0$ unless $i, j \in \Lambda'$ or $i \lesssim j$ and, recalling the notation of Definition 2.3, $(a_{ij})$ belongs to $\mathrm{U}_{\Lambda+\Lambda'}(R)$. Conversely, if $\Lambda' \in \mathcal{C}_\Lambda$, then $\mathrm{U}_{\Lambda+\Lambda'}(R) \leq \mathrm{AU}_\Lambda(R)$. We also note that $\mathrm{U}_{\Lambda+\Lambda'}(R) \leq \mathrm{U}_{\Lambda+\Lambda''}(R)$ given $\Lambda', \Lambda'' \in \mathcal{C}_\Lambda$ such that $\Lambda' \subseteq \Lambda''$, and that $\mathcal{C}_\Lambda$ forms a directed set under inclusion. It follows, therefore, that

$$\mathrm{AU}_\Lambda(R) = \varinjlim_{\Lambda'\in\mathcal{C}_\Lambda} \mathrm{U}_{\Lambda+\Lambda'}(R). \tag{2}$$

In particular, we deduce from (2) that for any $\Lambda' \in \mathcal{C}_\Lambda$, we have $\mathrm{AU}_\Lambda(R) = \mathrm{AU}_{\Lambda+\Lambda'}(R)$.

## 3. Intermediate results

To begin our study of these matrix groups, we show that they are closed under multiplication, and that products of such matrices can easily be understood in terms of products of finite submatrices.

**Lemma 3.1.** *Let $R$ be a commutative unital ring, let $\Lambda$ be a $\mathbb{Z}$-like preordered set and suppose that $(a_{ij})$ and $(b_{ij})$ are $\Lambda$-upper-triangular.*

(i) *The product $(c_{ij}) = (a_{ij})(b_{ij})$ is well-defined.*

(ii) *The matrix $(c_{ij})$ is $\Lambda$-upper-triangular.*

(iii) *For every convex subset, $\Lambda'$, of $\Lambda$, we have*

$$(c_{ij})_{i,j\in\Lambda'} = (a_{ij})_{i,j\in\Lambda'}(b_{ij})_{i,j\in\Lambda'}.$$

**Proof.** Suppose $i, j, k \in \Lambda$ are such that $a_{ik}b_{kj} \neq 0$. Then $a_{ik}, b_{kj} \neq 0$ and so $i \lesssim k$ and $k \lesssim j$. Hence $i \lesssim j$ and $k \in [i, j]$. Since $\Lambda$ is $\mathbb{Z}$-like, $[i, j]$ is finite, and we see that the sum

$$c_{ij} := \sum_{k\in\Lambda} a_{ik}b_{kj} \tag{3}$$

is well-defined for all $i, j \in \Lambda$ and is zero unless $i \lesssim j$. This proves (i) and (ii).

What is more, let $\Lambda'$ be a convex subset of $\Lambda$ and $i, j \in \Lambda'$. Then all nonzero terms of the sum on the right hand side of (3) arise from $k \in [i, j] \subseteq \Lambda'$, proving (iii). ∎

Invertible elements of $\mathrm{M}_\Lambda(R)$ have a simple characterization. As a result, the nonsingular upper-triangular matrices form a group with a natural semidirect product decomposition.

**Lemma 3.2.** *Let $R$ be a commutative unital ring, let $(\Lambda, \lesssim)$ be a $\mathbb{Z}$-like preordered set, and let $(a_{ij})$ be an upper-triangular $(\Lambda \times \Lambda)$-matrix over $R$. Then the following are equivalent:* (a) *The matrix $(a_{ij})$ is invertible in $\mathrm{M}_\Lambda(R)$;*

(b) *The matrix $(a_{ij})$ is nonsingular;*

(c) *For all $k \in \Lambda$, the finite matrix $(a_{ij})_{i,j\in[k,k]}$ is invertible.*

*Moreover, if $(a_{ij})$ is invertible, then*

$$(a_{ij})^{-1} = (r_{ij})(d_{ij})$$

*where $(r_{ij}) \in \mathrm{U}^*_\Lambda(R)$ and $(d_{ij}) \in \Delta_\Lambda(R)$. If $(a_{ij}) \in \mathrm{U}^*_\Lambda(R)$ then $(a_{ij})^{-1} \in \mathrm{U}^*_\Lambda(R)$, and if $(a_{ij})$ is a nonsingular diagonal matrix then $(a_{ij})^{-1} \in \Delta_\Lambda(R)$.*

**Proof.** Let us note first that lemma is clear when $(a_{ij})$ is a $\Lambda$-diagonal matrix. Indeed, when $(a_{ij})$ is $\Lambda$-diagonal its inverse matrix (if it exists) is the $\Lambda$-diagonal matrix $(d_{ij})$ such that $(d_{ij})_{i,j\in[k,k]} := ((a_{ij})_{i,j\in[k,k]})^{-1}$ for each $k$.

Given Lemma 3.1(iii), it is clear that (a) implies (b) and that (b) implies (c).

We next show that (c) implies (a), that is, under the assumption that $(a_{ij})_{i,j\in[k,k]}$ is invertible for all $k \in \Lambda$, that $(a_{ij})$ is invertible; in the process we will obtain the desired decomposition of $(a_{ij})$. We may suppose that $(a_{ij})$ is not diagonal. Let $(d_{ij})$ be the $\Lambda$-diagonal matrix such that $(d_{ij})_{i,j\in[k,k]} := ((a_{ij})_{i,j\in[k,k]})^{-1}$; note that if $(a_{ij}) \in \mathrm{U}^*_\Lambda(R)$ then $(d_{ij}) = (\delta_{ij})$. Then $(d_{ij})$ is invertible, so $(d_{ij}) \in \Delta_\Lambda(R)$, and the product $(d_{ij})(a_{ij}) =: (c_{ij})$ is defined, by Lemma 3.1. A calculation shows that $(c_{ij}) = (\delta_{ij} - s_{ij})$ where $(s_{ij})$ is strictly upper-triangular. For any given $i, j \in \Lambda$, the $(i, j)$-entry of $(s_{ij})^n$ is zero for all but finitely many $n \in \mathbb{N}$.

Specifically, the $(i,j)$-entry is given by the sum

$$\sum_{k_1,\ldots,k_n\in\Lambda'} s_{ik_1}s_{k_1k_2}\ldots s_{k_{n-1}k_n}s_{k_nj};$$

for a nonzero term we must have $i \lnsim k_1 \lnsim \cdots \lnsim k_n \lnsim j$, which in particular implies that $n < |[i,j]|$. Thus the infinite sum

$$(r_{ij}) := (\delta_{ij}) + \sum_{n\geq 1}(s_{ij})^n \tag{4}$$

is a well-defined matrix. It is then clear that $(r_{ij}) \in \mathrm{U}^*_\Lambda(R)$; moreover, $(r_{ij})(c_{ij}) = (\delta_{ij}) = (c_{ij})(r_{ij})$, that is, $(r_{ij}) = (\delta_{ij} - s_{ij})^{-1}$. Hence the product $(r_{ij})(d_{ij})$ is also a well-defined upper-triangular matrix, and we have

$$(r_{ij})(d_{ij})(a_{ij}) = (a_{ij})(r_{ij})(d_{ij}) = (\delta_{ij}).$$

Thus $(r_{ij})(d_{ij})$ is the inverse of $(a_{ij})$. ■

**Lemma 3.3.** *Let $R$ be a commutative unital ring and let $\Lambda$ be a $\mathbb{Z}$-like preordered set. Then $\mathrm{U}_\Lambda(R)$ is a group and $\mathrm{U}^*_\Lambda(R)$ and $\Delta_\Lambda(R)$ are subgroups of $\mathrm{U}_\Lambda(R)$, with $\mathrm{U}^*_\Lambda(R)$ normal, and $\mathrm{U}_\Lambda(R)$ decomposes as*

$$\mathrm{U}_\Lambda(R) = \mathrm{U}^*_\Lambda(R) \rtimes \Delta_\Lambda(R).$$

*Furthermore,* $\Delta_\Lambda(R) \cong \prod_{[k,k],\, k\in\Lambda} \mathrm{GL}_{n(k)}(R)$ *with* $n(k) = |[k,k]|$.

**Proof.** We see from Lemma 3.1 that $\mathrm{U}_\Lambda(R)$ is a multiplicative semigroup; by Lemma 3.2, every element of $\mathrm{U}_\Lambda(R)$ has an inverse in $\mathrm{U}_\Lambda(R)$, so $\mathrm{U}_\Lambda(R)$ is a group. It is easy to see that the sets $\mathrm{U}^*_\Lambda(R)$ and $\Delta_\Lambda(R)$ are closed under multiplication, and then Lemma 3.2 implies that $\mathrm{U}^*_\Lambda(R)$ and $\Delta_\Lambda(R)$ are subgroups of $\mathrm{U}_\Lambda(R)$. Given that every element of $\mathrm{U}_\Lambda(R)$ is an inverse of an element in $\mathrm{U}_\Lambda(R)$, the form of the inverse obtained in Lemma 3.2 shows that $\mathrm{U}_\Lambda(R) = \mathrm{U}^*_\Lambda(R)\Delta_\Lambda(R)$.

That $\mathrm{U}^*_\Lambda(R) \cap \Delta_\Lambda(R)$ is trivial is clear, and so to prove the semidirect product decomposition of $\mathrm{U}_\Lambda(R)$ it remains only to show that $\mathrm{U}^*_\Lambda(R)$ is a normal subgroup. For this, consider $(a_{ij}) \in \mathrm{U}_\Lambda(R)$ and $(\delta_{ij} - s_{ij}) \in \mathrm{U}^*_\Lambda(R)$. Lemma 3.1(iii) implies that, for every $k \in \Lambda$,

$$((a_{ij})(\delta_{ij} - s_{ij}))_{i,j\in[k,k]} = (a_{ij})_{i,j\in[k,k]},$$

and it follows, again by Lemma 3.1(iii), that

$$((a_{ij})(\delta_{ij} - s_{ij})(a_{ij})^{-1})_{i,j\in[k,k]} = (\delta_{ij})_{i,j\in[k,k]}.$$

Hence $\mathrm{U}^*_\Lambda(R)$ is normal.

For the direct product decomposition of $\Delta_\Lambda(R)$, observe that the minimal convex set $[k,k]$ is finite for every $k \in \Lambda$ and that, if an invertible matrix $(a_{ij})_{i,j\in[k,k]}$ is chosen for every such $[k,k]$, then the $(\Lambda\times\Lambda)$-matrix $(b_{ij})$ with

$$b_{ij} = \begin{cases} a_{ij}, & \text{if } i,j \in [k,k] \text{ for some } k \in \Lambda \\ 0, & \text{if } i \lnsim j \text{ or } j \lnsim i \end{cases}$$

belongs to $\Delta_\Lambda(R)$. Lemma 3.2 shows that every element of $\Delta_\Lambda(R)$ has this form and the claimed isomorphism follows. ■

The following observations, derived from Lemma 3.1(iii) and Lemma 3.3, will be useful later.

**Lemma 3.4.** *Let $\Lambda$ be a $\mathbb{Z}$-like preordered set and let $\Lambda'$ be a finite convex subset of $\Lambda$.*

(i) *Define* $\theta_{\Lambda'} : \mathrm{U}_\Lambda(R) \to \mathrm{GL}_{\Lambda'}(R);\ (a_{ij})_{i,j\in\Lambda} \mapsto (a_{ij})_{i,j\in\Lambda'}$.
*Then $\theta_{\Lambda'}$ is a group homomorphism with image $\mathrm{U}_{\Lambda'}(R)$. If $\Lambda'$ carries the complete preorder in $\Lambda$, in other words $\Lambda = \Lambda + \Lambda'$, then $\mathrm{U}_{\Lambda'}(R) = \mathrm{GL}_{\Lambda'}(R)$.*

(ii) *There is an injective group homomorphism*

$$\theta : \mathrm{U}_\Lambda(R) \to \prod_{\Lambda' \in \mathcal{C}_\Lambda} \mathrm{GL}_{\Lambda'}(R);\ (a_{ij})_{i,j\in\Lambda} \mapsto ((a_{ij})_{i,j\in\Lambda'})_{\Lambda'\in\mathcal{C}_\Lambda}.$$

**Proof.** (i) Lemma 3.3 ensures that $\mathrm{U}_\Lambda(R)$ is a subgroup of $\mathrm{M}_\Lambda(R)$ and that $\mathrm{U}_{\Lambda'}(R)$ is a subgroup of $\mathrm{GL}_{\Lambda'}(R)$. We then see that $\theta_{\Lambda'}$ is a group homomorphism by Lemma 3.1(iii). Given $(a_{ij})_{i,j\in\Lambda'} \in \mathrm{U}_{\Lambda'}(R)$, letting $(m_{ij})_{i,j\in\Lambda}$ be the matrix in $\mathrm{M}_\Lambda(R)$ given by

$$m_{ij} = \begin{cases} a_{ij}, & \text{if } i,j \in \Lambda' \\ \delta_{ij}, & \text{otherwise,} \end{cases}$$

then $(m_{ij}) \in \mathrm{U}_\Lambda(R)$ and $\theta_{\Lambda'}((m_{ij})_{i,j\in\Lambda}) = (a_{ij})_{i,j\in\Lambda'}$. Thus the image of $\theta_{\Lambda'}$ contains $\mathrm{U}_{\Lambda'}(R)$. On the other hand, given $(b_{ij}) \in \mathrm{U}_\Lambda(R)$, then $\theta_{\Lambda'}((b_{ij}))$ is still upper-triangular and nonsingular, hence it must be an element of $\mathrm{U}_{\Lambda'}(R)$. Hence $\theta_{\Lambda'}(\mathrm{U}_\Lambda(R)) = \mathrm{U}_{\Lambda'}(R)$.

If $\Lambda = \Lambda + \Lambda'$, then every matrix indexed by $\Lambda'$ is upper-triangular, so we have $\mathrm{U}_{\Lambda'}(R) = \mathrm{GL}_{\Lambda'}(R)$.

(ii) We obtain a group homomorphism $\theta$ from (i) by combining the homomorphisms $\theta_{\Lambda'}$ as $\Lambda'$ ranges over $\mathcal{C}_\Lambda$; it remains to prove that the kernel is trivial. For any nonidentity matrix $(a_{ij}) \in \mathrm{U}_\Lambda(R)$, there is some $i', j' \in \Lambda$ such that $a_{i'j'} \neq \delta_{i'j'}$; in particular, $(a_{ij})_{i,j\in[i',j']}$ is not an identity matrix. Since $\Lambda$ is $\mathbb{Z}$-like, we then have $[i,j] \in \mathcal{C}_\Lambda$, and hence $(a_{ij}) \notin \ker\theta$. ∎

To finish this section, we remark that the centre of $\mathrm{AU}_\Lambda(R)$ is now easily obtained from the centre of the general linear group.

**Lemma 3.5.** *Let $R$ be a commutative unital ring and let $\Lambda$ be a $\mathbb{Z}$-like preordered set. Then* $\mathrm{Z}(\mathrm{AU}_\Lambda(R)) = \mathrm{Z}_\Lambda(R)$.

**Proof.** It is clear that $\mathrm{Z}_\Lambda(R)$ is a central subgroup of $\mathrm{AU}_\Lambda(R)$. Conversely, recall (Remark 2.7) that $\mathrm{AU}_\Lambda(R)$ is the union of the subgroups $\mathrm{U}_{\Lambda+\Lambda'}(R)$ as $\Lambda'$ ranges over the finite convex subsets of $\Lambda$. Given a central element $(a_{ij})$ of $\mathrm{AU}_\Lambda(R)$, we see from Lemma 3.3 that $(a_{ij})_{i,j\in\Lambda'}$ is central in $\mathrm{GL}_{\Lambda'}(R)$ for all finite convex subsets $\Lambda'$ of $\Lambda$ such that $(a_{ij}) \in \mathrm{U}_{\Lambda+\Lambda'}(R)$. Hence $(a_{ij})_{i,j\in\Lambda'}$ is a scalar diagonal matrix. Given the freedom of choice of $\Lambda'$, we conclude that $(a_{ij})_{i,j\in\Lambda}$ is a scalar diagonal matrix, that is, $(a_{ij}) \in \mathrm{Z}_\Lambda(R)$. ∎

## 4. Topology

Suppose that $R$ is a topological ring. Equip $\mathrm{M}_\Lambda(R)$ with the product topology and $\mathrm{U}_\Lambda(R)$ with the subspace topology. Direct products of topological groups, such as in Lemma 3.4(ii), are also equipped with the product topology.

**Lemma 4.1.** *Let $R$ be a commutative unital topological ring and let $\Lambda$ be a $\mathbb{Z}$-like preordered set. Then $\mathrm{U}_\Lambda(R)$ is a topological group. Moreover, the map $\theta$ defined in Lemma 3.4(ii) is a closed topological embedding.*

**Proof.** To prove that $\mathrm{U}_\Lambda(R)$ is a topological group it suffices to show that the map

$$\pi(a,b) = ab^{-1} : \mathrm{U}_\Lambda(R) \times \mathrm{U}_\Lambda(R) \to \mathrm{U}_\Lambda(R)$$

is continuous. By definition of the product topology, $\pi$ is continuous if $\theta_{ij} \circ \pi$ is continuous for all $(i,j) \in \Lambda^2$, where $\theta_{ij}(a) = a_{ij}$ for $a = (a_{ij})$. For this, it suffices to show that $\theta_{\Lambda'} \circ \pi$ is continuous for every finite convex subset $\Lambda'$.

Lemma 3.4(i) implies that $\theta_{\Lambda'} \circ \pi = \pi \circ (\theta_{\Lambda'} \times \theta_{\Lambda'})$, and $\pi \circ (\theta_{\Lambda'} \times \theta_{\Lambda'})$ is continuous if the restriction of $\pi$ to $\theta_{\Lambda'}(\mathrm{U}_\Lambda(R))$, which is equal to $\mathrm{U}_{\Lambda'}(R)$, is continuous. Hence continuity of $\theta_{\Lambda'} \circ \pi$ amounts to $\mathrm{U}_{\Lambda'}(R)$ being a topological group for every finite convex set $\Lambda'$. Since $\mathrm{GL}_{\Lambda'}(R)$ is a topological group when equipped with the subspace topology for the product topology on $\mathrm{M}_{\Lambda'}(R)$, and since $\mathrm{U}_{\Lambda'}(R)$ is defined by a set of equations and hence is a closed subgroup of $\mathrm{GL}_{\Lambda'}(R)$, it follows that $\mathrm{U}_{\Lambda'}(R)$ is indeed a topological group.

The map $\theta$ is continuous because each homomorphism $\theta_{\Lambda'} : (a_{ij}) \mapsto (a_{ij})_{i,j\in\Lambda'}$ is continuous, and is a homeomorphism onto its range because each of the coordinate maps $\theta_{ij}$ (which determine the product topology on $\mathrm{M}_\Lambda(R)$) factors through $\theta_{\Lambda'}$ if $\Lambda'$ contains $i$ and $j$. The image is closed because it can be specified by a set of equations on the entries. ∎

If $R$ is finite and discrete, then $\mathrm{GL}_{\Lambda'}(R)$ is a finite discrete group for each finite convex $\Lambda'$ and we have the following immediate consequence.

**Corollary 4.2.** *Let $R$ be a finite commutative unital ring equipped with the discrete topology, and let $\Lambda$ be a $\mathbb{Z}$-like preordered set. Then $\mathrm{U}_\Lambda(R)$ is a profinite group. If $\Lambda$ is countable then $\mathrm{U}_\Lambda(R)$ is countably based.*

In the case that $R$ is discrete, the description of $\mathrm{AU}_\Lambda(R)$ as the direct limit of the groups $\mathrm{U}_{\Lambda+\Lambda'}(R)$ given in Equation (2) may be used to extend the topology on $\mathrm{U}_\Lambda(R)$ to $\mathrm{AU}_\Lambda(R)$. The topologies for $\mathrm{U}_{\Lambda+\Lambda'}(R)$ as $\Lambda'$ ranges over $\mathcal{C}_\Lambda$ are consistent with one another in the following sense. Given $\Lambda_1, \Lambda_2 \in \mathcal{C}_\Lambda$, then also $\Lambda_1 \cap \Lambda_2 \in \mathcal{C}_\Lambda$. The corresponding intersection

$$\mathrm{U}_{\Lambda+(\Lambda_1\cap\Lambda_2)}(R) = \mathrm{U}_{\Lambda+\Lambda_1}(R) \cap \mathrm{U}_{\Lambda+\Lambda_2}(R)$$

of upper-triangular groups, being determined by a condition on finitely many entries in $\mathrm{U}_{\Lambda+\Lambda_i}(R)$, is open in both $\mathrm{U}_{\Lambda+\Lambda_1}(R)$ and $\mathrm{U}_{\Lambda+\Lambda_2}(R)$ and carries the subspace topology in both. It follows that there is a unique group topology for $\mathrm{AU}_\Lambda(R)$ such that the embedding of $\mathrm{U}_{\Lambda+\Lambda'}(R)$ into $\mathrm{AU}_\Lambda(R)$ is continuous and open for all $\Lambda' \in \mathcal{C}_\Lambda$. If $R$ is finite, we see that the topology of $\mathrm{AU}_\Lambda(R)$ is locally profinite, that is, $\mathrm{AU}_\Lambda(R)$ is a t.d.l.c. group.

**Remark 4.3.** For each $\Lambda' \in \mathcal{C}_\Lambda$, the subgroup

$$\mathrm{U}_\Lambda^{\Lambda'}(R) := \{(a_{ij}) \in \mathrm{U}_\Lambda(R) \mid a_{ij} = \delta_{ij} \text{ if } i,j \in \Lambda'\}$$

is open in $\mathrm{AU}_\Lambda(R)$; in fact, these subgroups form a base of identity neighbourhoods in $\mathrm{AU}_\Lambda(R)$ because every finite subset of $\Lambda$ is contained in an element of $\mathcal{C}_\Lambda$.

## 5. Normal subgroups

We now consider the closed normal subgroups of $\mathrm{AU}_\Lambda(R)$. Just as for finite-dimensional matrix groups, there is a natural family of 'principal congruence subgroups' arising from the ring structure of $R$. Specifically, if $I$ is a proper ideal of $R$, then the map $(a_{ij}) \mapsto (a_{ij} + I)$ induces a group homomorphism from $\mathrm{AU}_\Lambda(R)$ to $\mathrm{AU}_\Lambda(R/I)$. Provided that $I$ is nonzero, the kernel of this map is a proper nontrivial closed normal subgroup. The question of how other closed normal subgroups of $\mathrm{AU}_\Lambda(R)$ relate to the principal congruence subgroups appears to be difficult.

To avoid this complication, from now until the end of the article we will specialise to the case where $R$ is a finite field. Fix a prime power $q$ and write $\mathbb{F}_q$ for the field of prime power order $q$, equipped with the discrete topology.

We recall a well-known simplicity result, which may be found in, for example, [5, §§103–105].

**Lemma 5.1.** *Let $F$ be a field and let $n \in \mathbb{N}$ such that $n \geq 2$; in the case $n = 2$, assume $|F| > 3$. Then every proper normal subgroup of $\mathrm{SL}_n(F)$ is central; every noncentral normal subgroup of $\mathrm{GL}_n(F)$ contains $\mathrm{SL}_n(F)$; and the quotient $\mathrm{PSL}_n(F)$ of $\mathrm{SL}_n(F)$ by its subgroup of scalar matrices is nonabelian and simple. If $\mathrm{PSL}_n(F) \cong \mathrm{PSL}_{n'}(F)$ for $n' \in \mathbb{N}$, then $n' = n$.*

The analogous result for $\mathrm{AU}_\Lambda(\mathbb{F}_q)$ can now be deduced.

**Theorem 5.2.** *Let $\Lambda$ be an infinite $\mathbb{Z}$-like preordered set. Then $\mathrm{Z}_\Lambda(\mathbb{F}_q)$ is the largest proper closed normal subgroup of $\mathrm{AU}_\Lambda(\mathbb{F}_q)$. In particular, $\mathrm{AU}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q)$ is topologically simple.*

**Proof.** It is clear that $\mathrm{Z}_\Lambda(\mathbb{F}_q)$ is a proper closed normal subgroup of $\mathrm{AU}_\Lambda(\mathbb{F}_q)$. To show that it is the unique largest one, it suffices to consider a closed normal subgroup $N$ of $\mathrm{AU}_\Lambda(\mathbb{F}_q)$ that is not contained in $\mathrm{Z}_\Lambda(\mathbb{F}_q)$ and show that $N = \mathrm{AU}_\Lambda(\mathbb{F}_q)$.

Given such $N$ and $(a_{ij}) \in N$ which is not a scalar matrix, there is a finite convex $\Lambda' \subset \Lambda$ with $|\Lambda'| \geq 3$ and such that $(a_{ij}) \in \mathrm{U}_{\Lambda+\Lambda'}(\mathbb{F}_q)$ and $(a_{ij})_{i,j\in\Lambda'}$ is not scalar. Suppose that $\Lambda'$ is any such finite convex subset of $\Lambda$. Then, applying the homomorphism $\theta_{\Lambda+\Lambda',\Lambda'}$ given in Lemma 3.4(i), we see that

$$N_{\Lambda'} := \theta_{\Lambda'}(\mathrm{U}_{\Lambda+\Lambda'}(\mathbb{F}_q) \cap N)$$

is noncentral and normal in $\mathrm{GL}_{\Lambda'}(\mathbb{F}_q)$. Hence $\mathrm{SL}_{\Lambda'}(\mathbb{F}_q) \leq N_{\Lambda'}$, by Lemma 5.1. Since $\mathbb{F}_q$ is a field, each matrix $(f_{ij})$ in $\mathrm{GL}_{\Lambda'}(\mathbb{F}_q)$ is a submatrix of a matrix $(g_{ij})$ in $\mathrm{U}_{\Lambda+\Lambda'}(\mathbb{F}_q)$ with $g_{ij} = f_{ij}$ for $i,j \in \Lambda'$ and $g_{ij} = 0$ when $i \neq j$ and $i$ or $j$ is not in $\Lambda'$. Moreover, $(g_{ij})$ may be chosen with $\theta_{\Lambda''}(g_{ij}) \in \mathrm{SL}_{\Lambda''}(\mathbb{F}_q)$ for some finite convex $\Lambda''$ strictly containing $\Lambda'$. Since $N_{\Lambda''} \geq \mathrm{SL}_{\Lambda''}(\mathbb{F}_q)$ by the previous argument and $(f_{ij}) = \theta_{\Lambda'}(g_{ij})$, it follows that $N_{\Lambda'} = \mathrm{GL}_{\Lambda'}(\mathbb{F}_q)$.

Consider now an arbitrary element $(c_{ij})$ of $\mathrm{AU}_\Lambda(\mathbb{F}_q)$ and suppose that $\Lambda' \in \mathcal{C}_\Lambda$ is sufficiently large that $(c_{ij}) \in \mathrm{U}_{\Lambda+\Lambda'}(\mathbb{F}_q)$. Then $(c_{ij})_{i,j\in\Lambda'}$ is an element of $\mathrm{GL}_{\Lambda'}(\mathbb{F}_q)$ and there is $(b_{ij}) \in N$ such that $(c_{ij})_{i,j\in\Lambda'} = \theta_{\Lambda'}(b_{ij})$. Since this holds for all sufficiently large $\Lambda'$, $(c_{ij})$ is approximated by elements of $N$ in the topology of entrywise convergence and, since $N$ is closed, it follows that $(c_{ij}) \in N$. This completes the proof that $N = \mathrm{AU}_\Lambda(\mathbb{F}_q)$.

In particular, any nontrivial closed normal subgroup of $\mathrm{AU}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q)$ has preimage equal to $\mathrm{AU}_\Lambda(\mathbb{F}_q)$ and $\mathrm{AU}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q)$ is topologically simple. ■

We conclude this section by showing that $\mathrm{AU}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q)$ is not simple. To this end consider the vector space

$$\mathrm{L}_\Lambda(\mathbb{F}_q) := \left\{(x_j) \in \mathbb{F}_q^\Lambda \mid \text{ there is } k \in \Lambda \text{ with } x_j = 0 \text{ for all } j \gtrsim k\right\}$$

and given an almost $\Lambda$-upper-triangular matrix $(a_{ij})$, write $\lambda((a_{ij}))$ for the linear operator on $\mathrm{L}_\Lambda(\mathbb{F}_q)$ obtained by multiplying on the left by $(a_{ij})$. Given $(x_l) \in \mathrm{L}_\Lambda(\mathbb{F}_q)$ and $k \in \Lambda$ such that $x_j = 0$ for all $j \gtrsim k$, we see that $\sum_{j\in\Lambda} a_{ij}x_j$ is a finite sum for all $i \in \Lambda$ and also $\sum_{j\in\Lambda} a_{ij}x_j = 0$ for all but finitely many $i \gtrsim k$, so $\lambda((a_{ij}))$ is well-defined. Now define

$$\mathcal{AU}_\Lambda(\mathbb{F}_q) = \{a \in \mathrm{AU}_\Lambda(\mathbb{F}_q) \mid \lambda(a - (\delta_{ij})) \text{ has finite rank}\}. \tag{5}$$

Then $\mathcal{AU}_\Lambda(\mathbb{F}_q)$ is closed under multiplication because the sum and product of finite rank operators have finite rank, and is closed under the inverse because every element of $\mathcal{AU}_\Lambda(\mathbb{F}_q)$ is equal to the identity on a finite-codimensional subspace of $\mathrm{L}_\Lambda(\mathbb{F}_q)$ and hence so is its inverse. Therefore $\mathcal{AU}_\Lambda(\mathbb{F}_q)$ is a subgroup of $\mathrm{AU}_\Lambda(\mathbb{F}_q)$. That it is a normal subgroup follows because the rank of an operator does not change under conjugation.

The subgroup $\mathcal{AU}_\Lambda(\mathbb{F}_q) \cap \Delta_\Lambda(\mathbb{F}_q)$ of $\Lambda$-diagonal matrices in $\mathcal{AU}_\Lambda(\mathbb{F}_q)$ has infinite index in $\mathrm{AU}_\Lambda(\mathbb{F}_q) \cap \Delta_\Lambda(\mathbb{F}_q)$ and so $\mathcal{AU}_\Lambda(\mathbb{F}_q)$ and $\mathcal{AU}_\Lambda(\mathbb{F}_q)\mathrm{Z}_\Lambda(\mathbb{F}_q)$ are proper subgroups of $\mathrm{AU}_\Lambda(\mathbb{F}_q)$. Since $\mathcal{AU}_\Lambda(\mathbb{F}_q)$ is not contained in $\mathrm{Z}_\Lambda(\mathbb{F}_q)$, Theorem 5.2 implies the following.

**Proposition 5.3.** *The group $\mathcal{AU}_\Lambda(\mathbb{F}_q)$ defined in* (5) *is a proper dense normal subgroup of* $\mathrm{AU}_\Lambda(\mathbb{F}_q)$.

## 6. Local structure of infinite matrix groups

'Local structure' of a t.d.l.c. group $G$ refers to properties of compact open subgroups of $G$ which are preserved under commensurability. In this last section of the article, we investigate how well the local structure distinguishes t.d.l.c. infinite matrix groups from one another and from some other known examples of topologically simple t.d.l.c. groups.

### 6.1. An uncountable number of non-isomorphic groups $\mathrm{AU}_\Lambda(\mathbb{F}_q)$

For each partition $\mathcal{Q}$ of $\mathbb{Z}$ (or $\mathbb{N}$ or $-\mathbb{N}$) into finite intervals define a $\mathbb{Z}$-like preorder

$$m \lesssim_{\mathcal{Q}} n \text{ if } m < n \text{ or } m, n \in \mathfrak{p} \text{ for some } \mathfrak{p} \in \mathcal{Q}$$

and denote $(\mathbb{Z}, \lesssim_{\mathcal{Q}})$ by $[\mathcal{Q}]$.

Furthermore, let $\sharp(\mathcal{Q}) = \{|\mathfrak{p}| \in \mathbb{N} \mid \mathfrak{p} \in \mathcal{Q}\}$.

A *local isomorphism* of topological groups $G$ and $H$ is an open embedding $\phi: U \to H$, where $U$ is an open neighbourhood of the identity in $G$ and $\phi$ is compatible with the group operations (as far as they are defined on $U$). We say two topological groups are *locally isomorphic* if there is a local isomorphism between them.

**Proposition 6.1.** *Let $\mathcal{Q}$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be partitions of $I$ into finite intervals where $I \in \{\mathbb{Z}, \mathbb{N}, -\mathbb{N}\}$.*

(i) *There is a continuous surjective homomorphism $\mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q) \to \mathrm{PGL}_n(\mathbb{F}_q)$ with $n > 1$ (or $n > 2$ if $q \leq 3$) if and only if $n \in \sharp(\mathcal{Q})$.*

(ii) *If $\mathrm{AU}_{[\mathcal{Q}_1]}(\mathbb{F}_q)$ is locally isomorphic to $\mathrm{AU}_{[\mathcal{Q}_2]}(\mathbb{F}_q)$ or $\mathrm{AU}_{[\mathcal{Q}_1]}(\mathbb{F}_q)/\mathbb{Z}_{[\mathcal{Q}_1]}(\mathbb{F}_q)$ is locally isomorphic to $\mathrm{AU}_{[\mathcal{Q}_2]}(\mathbb{F}_q)/\mathbb{Z}_{[\mathcal{Q}_1]}(\mathbb{F}_q)$, then $\sharp(\mathcal{Q}_1)\Delta\sharp(\mathcal{Q}_2)$ is finite.*

**Proof.** (i) It is shown in Lemma 3.3 that $\mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q) = \mathrm{U}^*_{[\mathcal{Q}]}(\mathbb{F}_q) \rtimes \Delta_{[\mathcal{Q}]}(\mathbb{F}_q)$ and that $\Delta_{[\mathcal{Q}]}(\mathbb{F}_q) \cong \prod_{\mathfrak{p}\in\mathcal{Q}} \mathrm{GL}_{|\mathfrak{p}|}(\mathbb{F}_q)$. In consequence there is a surjective homomorphism $\mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q) \to \mathrm{GL}_n(\mathbb{F}_q)$ for any $n = |\mathfrak{p}|$ with $\mathfrak{p} \in \mathcal{Q}$; since this homomorphism is realised by taking finitely many matrix coefficients, it is continuous. In turn, $\mathrm{PGL}_{|\mathfrak{p}|}(\mathbb{F}_q)$ is the quotient of $\mathrm{GL}_{|\mathfrak{p}|}(\mathbb{F}_q)$ by its centre, so in fact we obtain a continuous surjective homomorphism $\mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q) \to \mathrm{PGL}_n(\mathbb{F}_q)$.

For the converse, consider a continuous surjective homomorphism $\phi$ from $\mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q)$ to $\mathrm{PGL}_n(\mathbb{F}_q)$. Then $\phi(\mathrm{U}^*_{[\mathcal{Q}]}(\mathbb{F}_q))$ is a normal subgroup of $\mathrm{PGL}_n(\mathbb{F}_q)$, because $\mathrm{U}^*_{[\mathcal{Q}]}(\mathbb{F}_q)$ is normal in $\mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q)$, and then since $\mathrm{PGL}_n(\mathbb{F}_q)$ is simple, the only possibilities are that $\phi(\mathrm{U}^*_{[\mathcal{Q}]}(\mathbb{F}_q))$ is trivial or equal to $\mathrm{PGL}_n(\mathbb{F}_q)$. Since the commutator subgroups $\mathrm{U}^*_{[\mathcal{Q}]}(\mathbb{F}_q)^{(n)}$ in the descending series for $\mathrm{U}^*_{[\mathcal{Q}]}(\mathbb{F}_q)$ converge to the trivial subgroup, whereas $[\mathrm{PGL}_n(\mathbb{F}_q), \mathrm{PGL}_n(\mathbb{F}_q)] = \mathrm{PGL}_n(\mathbb{F}_q)$, it must be that $\phi(\mathrm{U}^*_{[\mathcal{Q}]}(\mathbb{F}_q))$ is trivial. Since $\mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q) = \mathrm{U}^*_{[\mathcal{Q}]}(\mathbb{F}_q) \rtimes \Delta_{[\mathcal{Q}]}(\mathbb{F}_q)$, it follows that $\phi$ restricts to a surjective homomorphism from $\Delta_{[\mathcal{Q}]}(\mathbb{F}_q)$.

Using the isomorphism $\Delta_{[\mathcal{Q}]}(\mathbb{F}_q) \cong \prod_{\mathfrak{p}\in\mathcal{Q}} \mathrm{GL}_{|\mathfrak{p}|}(\mathbb{F}_q)$, we can consider $g \in \Delta_{[\mathcal{Q}]}(\mathbb{F}_q)$ as a direct product $\Delta_{[\mathcal{Q}]}(\mathbb{F}_q) = \prod_{\mathfrak{p}\in\mathcal{Q}} K_{\mathfrak{p}}$ of subgroups $K_{\mathfrak{p}} \cong \mathrm{GL}_{|\mathfrak{p}|}(\mathbb{F}_q)$; note that $\Delta_{[\mathcal{Q}]}(\mathbb{F}_q)$ carries the product topology with respect to this decomposition. Then for each $\mathfrak{p} \in \mathcal{Q}$, $K_{\mathfrak{p}}$ is normal in $\Delta_{[\mathcal{Q}]}(\mathbb{F}_q)$, so $\phi(K_{\mathfrak{p}})$ is normal in $\mathrm{PGL}_n(\mathbb{F}_q)$. If $|\mathfrak{p}| \neq n$, we see by Lemma 5.1 that $\phi(K_{\mathfrak{p}})$ is trivial. On the other hand, $\phi(\Delta_{[\mathcal{Q}]}(\mathbb{F}_q))$ is nontrivial, so by continuity $\phi(\bigoplus_{\mathfrak{p}\in\mathcal{Q}} K_{\mathfrak{p}})$ is nontrivial. Hence there is some $\mathfrak{p} \in \mathcal{Q}$ with $|\mathfrak{p}| = n$ such that $\phi(K_{\mathfrak{p}})$ is nontrivial. In particular, $n \in \sharp(\mathcal{Q})$.

(ii) Let $\psi: U/Z_1 \to \mathrm{AU}_{[\mathcal{Q}_2]}(\mathbb{F}_q)/Z_2$ be a local isomorphism from $\mathrm{AU}_{[\mathcal{Q}_1]}(\mathbb{F}_q)/Z_1$ to $\mathrm{AU}_{[\mathcal{Q}_2]}(\mathbb{F}_q)/Z_2$, where either $Z_1 = Z_2 = \{1\}$ or $Z_i = \mathrm{Z}_{[\mathcal{Q}_i]}(\mathbb{F}_q)$. Since $\mathrm{U}_{[\mathcal{Q}_1]}(\mathbb{F}_q)/Z_1$ is an open profinite subgroup of $\mathrm{AU}_{[\mathcal{Q}_1]}(\mathbb{F}_q)/Z_1$, we can take $U/Z_1$ to be an open normal subgroup of $\mathrm{U}_{[\mathcal{Q}_1]}(\mathbb{F}_q)/Z_1$; since $\mathrm{U}_{[\mathcal{Q}_2]}(\mathbb{F}_q)/Z_2$ is open in $\mathrm{AU}_{[\mathcal{Q}_2]}(\mathbb{F}_q)/Z_2$, by restricting to a smaller domain we may assume that $\psi(U) \leq \mathrm{U}_{[\mathcal{Q}_2]}(\mathbb{F}_q)/Z_2$. Hence, by Remark 4.3, there is a finite interval $[m,n] \subset I$ such that

$$\mathrm{U}^{[m,n]}_{[\mathcal{Q}_2]}(\mathbb{F}_q)Z_2/Z_2 \leq \psi(U/Z_1) \leq \mathrm{U}_{[\mathcal{Q}_2]}(\mathbb{F}_q)/Z_2.$$

Each $\mathfrak{p} \in \mathcal{Q}_2$ yields a homomorphism $\phi_{\mathfrak{p}}: \mathrm{U}_{[\mathcal{Q}_2]}(\mathbb{F}_q)/Z_2 \to \mathrm{PGL}_{|\mathfrak{p}|}(\mathbb{F}_q)$ whose restriction to $\mathrm{U}^{[m,n]}_{[\mathcal{Q}_2]}(\mathbb{F}_q)Z_2/Z_2$, and hence to $\psi(U/Z_1)$, is surjective provided that $\mathfrak{p}$ is disjoint from $[m,n]$.

Supposing that $\mathfrak{p}$ is disjoint from $[m,n]$ and $|\mathfrak{p}|>2$, we can then extend $\phi_{\mathfrak{p}}\circ\psi$ to a surjective homomorphism

$$\widetilde{\phi_{\mathfrak{p}}} : U\mathrm{U}^*_{[\mathcal{Q}_1]}(\mathbb{F}_q)Z_1/Z_1 \to \mathrm{PGL}_{|\mathfrak{p}|}(\mathbb{F}_q),$$

where $\widetilde{\phi_{\mathfrak{p}}}(\mathrm{U}^*_{[\mathcal{Q}_1]}(\mathbb{F}_q)Z_1/Z_1)$ is trivial; we then see that $U\mathrm{U}^*_{[\mathcal{Q}_1]}(\mathbb{F}_q)\cap\Delta_{[\mathcal{Q}_1]}(\mathbb{F}_q)$ is an open normal subgroup of $\Delta_{[\mathcal{Q}_1]}(\mathbb{F}_q)$ that surjects onto $\mathrm{PGL}_{|\mathfrak{p}|}(\mathbb{F}_q)$, and hence $|\mathfrak{p}|\in\sharp(\mathcal{Q}_1)$. Thus $\sharp(\mathcal{Q}_1)\setminus\sharp(\mathcal{Q}_2)$ is finite; similarly, $\sharp(\mathcal{Q}_2)\setminus\sharp(\mathcal{Q}_1)$ is finite, so $\sharp(\mathcal{Q}_1)\Delta\sharp(\mathcal{Q}_2)$ is finite. ■

Let $\mathcal{P}(\mathbb{N})$ be the set of subsets of $\mathbb{N}$ and let $\sim$ be the equivalence relation of finite symmetric difference. Then $|\mathcal{P}(\mathbb{N})|=2^{\aleph_0}$ but each equivalence class is countable, so $|\mathcal{P}(\mathbb{N})/\sim|=2^{\aleph_0}$. Moreover, we see that for all $X\in\mathcal{P}(\mathbb{N})$, there is a partition $\mathcal{Q}$ of $\mathbb{Z}$ (or $\mathbb{N}$ or $-\mathbb{N}$) into finite intervals such that $\sharp(\mathcal{Q})=X$. We also note that there are only $2^{\aleph_0}$ isomorphism classes of first-countable profinite groups, and hence at most $2^{\aleph_0}$ local isomorphism classes of first-countable t.d.l.c. groups. Corollary 4.2 and Proposition 6.1 therefore imply

**Corollary 6.2.** *For each prime power $q$, there are $2^{\aleph_0}$ local isomorphism classes of topologically simple t.d.l.c. groups of the form $\mathrm{AU}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q)$ with $(\Lambda,\lesssim)$ a countable $\mathbb{Z}$-like preorder.*

The construction just given for $(\mathbb{Z},\lesssim_{\mathcal{Q}})$ also produces uncountably many local isomorphism classes of topologically simple groups $\mathrm{AU}_\Lambda(\mathbb{F}_q)$ where $\Lambda$ is of the form $(\mathbb{N},\lesssim_{\mathcal{Q}})$ or $(-\mathbb{N},\lesssim_{\mathcal{Q}})$. Considering isomorphism classes rather than local isomorphism classes, it is likely that $\sharp(\mathcal{Q})$ is not a sufficiently fine invariant to distinguish all non-isomorphisms between pairs of such groups, because there is no obvious isomorphism between $\mathrm{AU}_{[\mathcal{Q}_1]}(\mathbb{F}_q)$ and $\mathrm{AU}_{[\mathcal{Q}_2]}(\mathbb{F}_q)$ if all intervals appearing in $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have the same lengths but the lengths appear in a different order. For example, let

$$\mathcal{Q}_1=\{[20n,20n+9]\mid n\in\mathbb{Z}\}\cup\{\{m\}\mid m\in[20n+10,20n+19],\ n\in\mathbb{Z}\}\ \text{and}$$
$$\mathcal{Q}_2=\{[100n,100n+9]\mid n\in\mathbb{Z}\}\cup\{\{m\}\mid m\in[100n+10,100n+99],\ n\in\mathbb{Z}\};$$

it is not clear to us whether $\mathrm{AU}_{[\mathcal{Q}_1]}(\mathbb{F}_q)$ and $\mathrm{AU}_{[\mathcal{Q}_2]}(\mathbb{F}_q)$ are isomorphic.

### 6.2. Locally normal subgroups

Suppose that $G$ is a t.d.l.c. group and that $U$ is a compact open subgroup of $G$. A subgroup $H\leq U$ is *locally normal* if the normaliser of $H$ is open. This concept is defined in [3, 4] where the lattice of commensurability classes of locally normal subgroups is studied. We recall further concepts from [3, 4] before investigating locally normal subgroups in $\mathrm{AU}_\Lambda(\mathbb{F}_q)$.

**Definition 6.3.** Let $G$ be a t.d.l.c. group. The *quasi-centre* $\mathrm{QZ}(G)$ is the set of elements whose centraliser is open. We say $G$ is [A]*-semisimple* if $\mathrm{QZ}(G)=\{1\}$ and the only abelian locally normal subgroup is the trivial group.

Theorem 5.3 in [4] shows every compactly generated topologically simple t.d.l.c. group is [A]-semisimple. On the other hand, if we remove the condition of compact generation, there are examples of non-discrete topologically simple t.d.l.c. groups with open abelian subgroups, as in [12]; any such group is equal to its quasi-centre. The goal of this section is to show that the topologically simple groups $\mathrm{AU}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q)$ fall in between these two cases: they have trivial quasi-centre, but nontrivial abelian locally normal subgroups.

**Proposition 6.4.** *Let $(\Lambda,\lesssim)$ be an infinite $\mathbb{Z}$-like preorder. Then* $\mathrm{QZ}(\mathrm{AU}_\Lambda(\mathbb{F}_q)) = \mathrm{Z}_\Lambda(\mathbb{F}_q)$.

**Proof.** Since $\Lambda$ is infinite and every finite subset is contained in a finite convex subset, $\Lambda$ cannot have both maximal and minimal elements. Let us suppose $\Lambda$ has no maximal elements; the argument when $\Lambda$ has no minimal elements is similar. Consider $(x_{ij}) \in \mathrm{QZ}(\mathrm{AU}_\Lambda(\mathbb{F}_q))$. Recalling Remark 4.3, we see that $(x_{ij})$ centralises the open neighbourhood $\mathrm{U}_\Lambda^{\Lambda'}(\mathbb{F}_q)$ of the identity for some finite convex set $\Lambda' \subset \Lambda$.

Let $r,s \in \Lambda$ such that $r \lnsim s$ and $s \notin \Lambda'$; define a matrix $(e^{rs}_{ij})_{i,j\in\Lambda}$, where $e^{rs}_{ij} = 1$ if $(i,j) = (r,s)$ and $e^{rs}_{ij} = \delta_{ij}$ otherwise. We see that $(e^{rs}_{ij}) \in \mathrm{U}_\Lambda^{\Lambda'}(\mathbb{F}_q)$, so $(e^{rs}_{ij})$ commutes with $(x_{ij})$. The effect on $(x_{ij})$ of multiplying by $(e^{rs}_{ij})$ on the left is to add row $s$ to row $r$, whereas the effect of multiplying by $(e^{rs}_{ij})$ on the right is to add column $r$ to column $s$. Thus to have $(e^{rs}_{ij})(x_{ij}) = (x_{ij})(e^{rs}_{ij})$, we must have $x_{sj} = 0$ for all $j \neq s$; $x_{ir} = 0$ for all $i \neq r$; and $x_{rr} = x_{ss}$.

Given $r \in \Lambda$, by choosing $s \in \Lambda\setminus\Lambda'$ such that $r \lnsim s$, we conclude that $x_{ir} = 0$ for all $i \in \Lambda\setminus\{r\}$. Given $r_1, r_2 \in \Lambda$, we can choose $s \in \Lambda\setminus\Lambda'$ such that $r_1, r_2 \lnsim s$, and by conjugating $(x_{ij})$ by $(e^{r_n s}_{ij})$ for $n = 1,2$, we see that $x_{r_1r_1} = x_{ss} = x_{r_2r_2}$. Thus $(x_{ij})$ is a scalar matrix, so $(x_{ij}) \in \mathrm{Z}_\Lambda(\mathbb{F}_q)$, showing that $\mathrm{QZ}(\mathrm{AU}_\Lambda(\mathbb{F}_q)) = \mathrm{Z}_\Lambda(\mathbb{F}_q)$. The reverse inclusion is clear. ∎

**Corollary 6.5.** *Let $(\Lambda,\lesssim)$ be an infinite $\mathbb{Z}$-like preorder. Then*

$$\mathrm{QZ}(\mathrm{AU}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q)) = \{1\}.$$

**Proof.** Suppose that $x\mathrm{Z}_\Lambda(\mathbb{F}_q) \in \mathrm{AU}_\Lambda(\mathbb{F}_q)/\mathrm{Z}_\Lambda(\mathbb{F}_q)$ centralises the open subgroup $U$ and let $V$ be an open subgroup of $\mathrm{AU}_\Lambda(\mathbb{F}_q)$ such that $V/\mathrm{Z}_\Lambda(\mathbb{F}_q) \leq U$. Then the map $v \mapsto [x,v]$ is a continuous homomorphism $V \to \mathrm{Z}_\Lambda(\mathbb{F}_q)$ which has an open kernel because $\mathrm{Z}_\Lambda(\mathbb{F}_q)$ is discrete. Hence $x \in \mathrm{Z}_\Lambda(\mathbb{F}_q)$, by Proposition 6.4. ∎

On the other hand, we find that many of the groups constructed in the previous section have nontrivial abelian locally normal subgroups. The following special case will be sufficient for our purposes.

**Lemma 6.6.** *Let $\mathcal{Q}$ be a partition of $\mathbb{N}$ into finite intervals, such that $\{1\} \in \mathcal{Q}$. Then $\mathrm{AU}_{[\mathcal{Q}]}(\mathbb{F}_q)$ and $\mathrm{AU}_{[\mathcal{Q}]}(\mathbb{F}_q)/\mathrm{Z}_{[\mathcal{Q}]}(\mathbb{F}_q)$ each have an infinite abelian subgroup with open normaliser.*

**Proof.** We have a right action of $\mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q)$ on the space $\mathbb{F}_q^{\mathbb{N}}$ of row vectors over $\mathbb{F}_q$ indexed by $\mathbb{N}$, given by matrix multiplication on the right. Let $V$ be the subspace of vectors in $\mathbb{F}_q^{\mathbb{N}}$ whose first entry is zero. Then since $1 \lnsim_{[\mathcal{Q}]} i$ for all $i \in \mathbb{N}\setminus\{1\}$, we see that $\mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q)$ stabilises $V$, that is, $va \in V$ for all $v \in V$ and $a \in \mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q)$. It follows that

$$H := \{h \in \mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q) \mid \forall v \in V : vh = v\}$$

is a normal subgroup of $\mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q)$.

Let $H^* = H \cap \mathrm{U}^*_{[\mathcal{Q}]}(\mathbb{F}_q)$; by Lemma 3.3, $\mathrm{U}^*_{[\mathcal{Q}]}(\mathbb{F}_q)$ is also a normal subgroup of $\mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q)$, so $H^*$ is normal in $\mathrm{U}_{[\mathcal{Q}]}(\mathbb{F}_q)$. Routine calculations show that

$$H^* = \{(a_{ij})_{i,j\in\mathbb{N}} \mid a_{ii} = 1 \text{ and } a_{ij} = 0 \text{ if } i > j \text{ or } j > i > 1\},$$

that is, $H^*$ consists of those matrices that differ from the identity matrix only in the first row, and that also have $a_{11} = 1$; from this description it is easily seen that $H^*$ is infinite and abelian. At the same time, we see that $H^* \cap \mathrm{Z}_{[\mathcal{Q}]}(\mathbb{F}_q) = \{1\}$. Thus $H^*$ is an infinite abelian locally normal subgroup of $\mathrm{AU}_{[\mathcal{Q}]}(\mathbb{F}_q)$, while $H^*\mathrm{Z}_{[\mathcal{Q}]}(\mathbb{F}_q)/\mathrm{Z}_{[\mathcal{Q}]}(\mathbb{F}_q)$ is an infinite abelian locally normal subgroup of $\mathrm{AU}_{[\mathcal{Q}]}(\mathbb{F}_q)/\mathrm{Z}_{[\mathcal{Q}]}(\mathbb{F}_q)$. ∎

We can now prove the main theorem from the introduction.

**Proof of Theorem 1.1.** Since there are only $2^{\aleph_0}$ isomorphism types of first-countable profinite groups, there are at most $2^{\aleph_0}$ local isomorphism types of second-countable t.d.l.c. groups $G$ with the specified properties. It remains to exhibit $2^{\aleph_0}$ such local isomorphism types. We do so as follows. Fix a prime power $q$; let $\mathcal{P}(\mathbb{N})/\sim$ be the set of subsets of $\mathbb{N}$ modulo the equivalence relation of finite symmetric difference, and let $P \subseteq \mathcal{P}(\mathbb{N})$ be a set of representatives for the equivalence classes, one for each class, such that $1 \in X$ for all $X \in P$. Note that $|P| = 2^{\aleph_0}$. Now for each $X \in P$, choose a partition $\mathcal{Q}_X$ of $\mathbb{N}$ into finite intervals, such that $\{1\} \in \mathcal{Q}_X$ and $\sharp(\mathcal{Q}_X) = X$; and then let $G_X = \mathrm{AU}_{[\mathcal{Q}_X]}(\mathbb{F}_q)/\mathrm{Z}_{[\mathcal{Q}_X]}(\mathbb{F}_q)$. Our family of groups is then $\{G_X \mid X \in P\}$.

By Proposition 6.1(i), given $X, Y \in P$ distinct, then $G_X$ is not locally isomorphic to $G_Y$, so we have $2^{\aleph_0}$ distinct local isomorphism classes of groups. Let $G = G_X$ for $X \in P$. We see from Remark 2.7 and Corollary 4.2 that $\mathrm{AU}_{[\mathcal{Q}_X]}(\mathbb{F}_q)$ can be expressed as the limit of a countable directed system of profinite open subgroups, each of which is first-countable; thus $\mathrm{AU}_{[\mathcal{Q}_X]}(\mathbb{F}_q)$ is a second-countable t.d.l.c. group, so the quotient $G$ is also a second-countable t.d.l.c. group. By Theorem 5.2, $G$ is topologically simple, but by Proposition 5.3, $G$ has a proper dense normal subgroup. The quasi-centre of $G$ is trivial by Corollary 6.5. By Lemma 6.6, $G$ has an infinite abelian subgroup with open normaliser. All the required properties for the family $\{G_X \mid X \in P\}$ have now been proved. ∎

## References


[1] Y. Barnea, M. Ershov, T. Weigel: *Abstract commensurators of profinite groups*, Trans. Amer. Math. Soc. 363(10) (2011) 5381–5417.

[2] P.-E. Caprace, T. De Medts: *Simple locally compact groups acting on trees and their germs of automorphisms*, Transformation Groups 16 (2011) 375–411.

[3] P.-E. Caprace, C. D. Reid, G. A. Willis: *Locally normal subgroups of totally disconnected groups. I: General theory*, Forum Math. Sigma 5(11) (2017), e11, 76 pp.

[4] P.-E. Caprace, C. D. Reid, G. A. Willis, *Locally normal subgroups of totally disconnected groups. II: Compactly generated simple groups*, Forum Math. Sigma 5 (2017), e12, 89 pp.

[5] L. E. Dickson: *Linear Groups with an Exposition of the Galois Field Theory*, Teubner, Leipzig (1901).

[6] V. Gorin, S. V. Kerov, A. M. Vershik: *Finite traces and representations of the group of infinite matrices over a finite field*, Advances Math. 254(6) (2014) 331–395.

[7] C. K. Gupta, W. Hołubowski: *Commutator subgroup of Vershik-Kerov group*, Linear Algebra Appl. 436 (2012) 4279–4284.

[8] C. K. Gupta, W. Hołubowski: *Commutator subgroup of Vershik-Kerov group II*, Linear Algebra Appl. 471 (2015) 85–95.

[9] K. H. Hofmann, S. A. Morris: *The Lie Theory of Connected Pro-Lie Groups*, EMS Tracts in Mathematics 2, European Mathematical Society, Zurich (2007).

[10] W. Hołubowski: *Parabolic subgroups of Vershik-Kelov's group*, Proc. Amer. Math. Soc. 130 (2002) 2579–2582.

[11] S. V. Kerov, A. M. Vershik: *On an infinite-dimensional group over a finite field*, Functional Analysis Appl. 32 (1998) 147–152.

[12] G. A. Willis: *Compact open subgroups in simple totally disconnected groups*, J. Algebra 312 (2007) 405–417.

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