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A Different Perspective on H-like Lie Algebras

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Abstract. We characterize H-like Lie algebras in terms of subspaces of cones over conjugacy classes in $\mathfrak{so}(\mathbb{R}^q)$, translating the classification problem for H-like Lie algebras to an equivalent problem in linear algebra. We study properties of H-like Lie algebras, present new methods for constructing them, including tensor products and central sums, and we classify H-like Lie algebras whose associated J_Z-maps have real rank two for all nonzero Z.

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1. Introduction

A *nilmanifold* is a simply connected nilpotent Lie group endowed with a left-invariant Riemannian metric. Kaplan defined the class of nilmanifolds of *Heisenberg type* (also called *H-type*) [27]. Nilmanifolds in this class have many extraordinary symmetry properties ([3, 10]). Most notably, they admit solvable extensions, *Damek-Ricci spaces*, which are harmonic spaces but not necessarily symmetric spaces ([3, 8]).

It is natural to consider defining conditions for nilmanifolds weaker than those in the definition of H-type, with the hope that some nice properties of H-type nilmanifolds may be preserved in the larger class. Many generalizations of the H-type property have been studied. The *nonsingular* nilpotent Lie algebras were first studied by Métivier and later studied by others from both geometric and analytic perspectives under various names ([9, 28, 30, 32, 33]). For *pseudo H-type* or *generalized H-type* nilmanifolds, the inner product is assumed to be nondegenerate instead of positive definite, and even possibly sub-semi-Riemannian ([5, 18]). Lauret generalized the notion of H-type to *modified H-type* in [29], and this was further generalized in [26].

In analyzing properties of the length spectrum of two-step nilmanifolds, Gornet and Mast defined the notion of an *H-like* nilmanifold ([21]) as a generalization of an H-type nilmanifold. Some of the remarkable geometric properties possessed by H-type nilmanifolds have natural analogs held by H-like nilmanifolds ([7, 21]). In this work we study the metric Lie algebras associated to H-like nilmanifolds and examine their properties. (We will give a precise definition of H-like in Definition 1.1.)

Quite a few low-dimensional examples of H-like nilmanifolds, including continuous families of two-step nilmanifolds with center of dimension greater than one, have been found ([7, 21, 36]). There is a one-to-one correspondence between nilmanifolds

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and metric nilpotent Lie algebras. Two nilmanifolds are isometric if and only if the corresponding metric nilpotent Lie algebras are metrically isomorphic ([39]). In [7], a general method for constructing H-like Lie algebras using representations of $\mathfrak{su}(2)$ was given, and in [6] H-like Lie algebras defined by undirected, uncolored graphs were classified.

Fix a two-step metric nilpotent Lie algebra $(\mathfrak{n}, Q)$. Let $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$ and let $\mathfrak{v} = \mathfrak{z}^\perp$. (Note that some authors use $\mathfrak{z}$ to mean the center of $\mathfrak{n}$ instead of the commutator.) If $\dim \mathfrak{z} = p$ and $\dim \mathfrak{v} = q$, we say that $\mathfrak{n}$ is of type (p, q). Define $J : \mathfrak{z} \to \operatorname{End}(\mathfrak{v})$ by

$$\langle J(Z)X, Y\rangle = \langle Z, [X, Y]\rangle \qquad \text{for } X, Y \in \mathfrak{v}, Z \in \mathfrak{z}. \tag{1}$$

The map J is linear and $J_Z : \mathfrak{v} \to \mathfrak{v}$ is skew-symmetric for all Z. Furthermore, $J(\mathfrak{z})$ is a p-dimensional subspace of $\mathfrak{so}(\mathfrak{v})$ (see Lemma 3.8).

Conversely, given finite-dimensional inner product spaces $\mathbb{R}^p$ and $\mathbb{R}^q$ and a nonzero linear map $J : \mathbb{R}^p \to \mathfrak{so}(\mathbb{R}^q)$, there is a two-step metric nilpotent Lie algebra $(\mathfrak{n}, Q)$ defined by J. The underlying vector space is $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z} = \mathbb{R}^q \oplus \mathbb{R}^p$ and the inner product Q on $\mathfrak{n}$ is the orthogonal sum of the inner products on $\mathfrak{v}$ and $\mathfrak{z}$. The Lie bracket is defined by (1) and the assumptions that $[\mathfrak{v}, \mathfrak{v}]$ is contained in $\mathfrak{z}$ and $\mathfrak{z}$ is central. Two such nilpotent Lie algebras defined by $J : \mathbb{R}^p \to \mathfrak{so}(\mathbb{R}^q)$ and $J' : \mathbb{R}^p \to \mathfrak{so}(\mathbb{R}^q)$, define metrically isomorphic metric Lie algebras if and only if there are $A \in O(q)$ and $B \in O(p)$ so that $J'(BZ) = A \circ J(Z) \circ A^T$ for all $Z \in \mathfrak{z}$ (see [20]).

Because the maps $J(Z)$ are skew-symmetric, any nonzero eigenvalues of $J(Z)$ are purely imaginary, and the eigenvalues of $J(Z)$ completely determine $J(Z)$ up to conjugacy by an element of $GL(\mathfrak{v})$. We use a multiset S to record the eigenvalues of $J(Z)$. (A *multiset* is a set S endowed with a multiplicity function from S to $\mathbb{N}$.) Conjugacy classes of nonzero skew-symmetric maps in $\operatorname{End}(\mathfrak{v})$ are indexed by multisets S of eigenvalues that are purely imaginary with the multiplicities of bi and $-bi$ the same for all positive real numbers b. We will call such a multiset *admissible*.

Gornet and Mast originally defined H-like Lie algebras in terms of totally geodesic submanifolds. There are several equivalent characterizations of H-like Lie algebras (see [21] and [7]); we take one of these as part (3) in the following.

Definition 1.1. Let $(\mathfrak{n}, Q)$ be a two-step nilpotent Lie algebra endowed with inner product Q. Let S be an admissible multiset.

1. The metric Lie algebra $(\mathfrak{n}, Q)$ is *H-type* if for all $Z \in \mathfrak{z}$, $J_Z^2 = -\|Z\|^2 \operatorname{Id}_{\mathfrak{v}}$, where $\operatorname{Id}_{\mathfrak{v}}$ is the identity map for $\mathfrak{v}$.
2. The metric Lie algebra $(\mathfrak{n}, Q)$ has *constant J-spectrum* S if for all unit $Z \in \mathfrak{z}$, the map J_Z has constant spectrum S.
3. The metric Lie algebra $(\mathfrak{n}, Q)$ is *H-like* if $(\mathfrak{n}, Q)$ has constant J-spectrum and $\mathfrak{n}$ has no nontrivial abelian factors.

When $(\mathfrak{n}, Q)$ has constant J-spectrum, the rank of J_Z is constant for nonzero Z; this number is called the *J-rank* of $(\mathfrak{n}, Q)$. ∎

Examples are given in Section 2. It follows from the definitions that

$$\text{H-type} \Rightarrow \text{H-like} \Rightarrow \text{constant } J\text{-spectrum}.$$

In Gornet and Mast's definition of H-like they take $\mathfrak{z}$ to be the center of $\mathfrak{n}$ rather than the commutator, so that any abelian factor of $\mathfrak{n}$ lies in $\mathfrak{z}$. We take $\mathfrak{z}$ to be the commutator so that any abelian factor is contained in $\mathfrak{v}$.

A p-dimensional subspace W of $\mathfrak{so}(\mathbb{R}^q)$ defines a two-step metric nilpotent Lie algebra of type (p, q).

Definition 1.2. Let W be a p-dimensional subspace of $\mathfrak{so}(\mathbb{R}^q)$. Define the metric nilpotent Lie algebra $(\mathfrak{n}, Q)$ by taking as the underlying vector space $\mathfrak{n} = W \oplus \mathbb{R}^q$, and endowing this vector space with the inner product $\langle , \rangle^*$ so that W and $\mathbb{R}^q$ are orthogonal, the restriction of $\langle , \rangle^*$ to W is given by the restriction of the Frobenius inner product to W, and $\langle , \rangle^*$ is the standard inner product on $\mathbb{R}^q$; i.e., the one with respect to which the standard basis is orthonormal. The Lie bracket for $\mathfrak{n}$ is defined by assuming that W is central and letting $J : W \to \mathfrak{so}(\mathbb{R}^q)$ be the inclusion map: for $Z \in W$ and $X, Y \in \mathbb{R}^q$, $\langle [X, Y], Z \rangle^* = \langle Z(X), Y \rangle_{\mathbb{R}^q}$. Such a metric nilpotent Lie algebra is called the *standard two-step metric nilpotent Lie algebra defined by W.* ■

Eberlein showed that every two-step metric nilpotent Lie algebra is metrically isomorphic to a standard two-step metric nilpotent Lie algebra of type (p, q) (Proposition 3.1.2, [13]). The standard metric nilpotent Lie algebras defined by two p-dimensional subspaces W_1 and W_2 of $\mathfrak{so}(\mathbb{R}^q)$ are isomorphic if and only if there exists $g \in GL_q(\mathbb{R})$ so that $W_2 = gW_1g^T$ (Proposition 3.1.3, [13]).

Eberlein used this point of view to study a variety of problems involving two-step nilpotent Lie algebras, including moduli spaces, optimal metrics, metrics with geodesic flow invariant Ricci tensors, and Riemannian submersions and lattices ([13], [11], [12]). Gordon and Kerr used the correspondence between standard metric two-step nilpotent Lie algebras and subspaces of $\mathfrak{so}(\mathbb{R}^q)$ to characterize Carnot Einstein solvmanifolds in terms of subspaces of $\mathfrak{so}(\mathbb{R}^q)$ that they called "orthogonal uniform subspaces" ([19]).

Let $\mathfrak{v}$ be a q-dimensional vector space, where $q \geq 2$, let S be an admissible multiset of size q, and let $\mathcal{C}_S$ denote the conjugacy class of matrices in $\mathfrak{so}(\mathfrak{v})$ with eigenvalues given by the multiset S. Let $\mathbb{R}\mathcal{C}_S = \{rA : A \in \mathcal{C}_S, r \in \mathbb{R}\}$ be the cone over the conjugacy class $\mathcal{C}_S$. It is a subset of $\mathfrak{so}(\mathfrak{v})$. The main innovation of our work is to take the perspective of Eberlein, using standard nilpotent Lie algebras, and to re-interpret the definition of H-like using cones over conjugacy class in $\mathfrak{so}(\mathfrak{v})$. This makes some of the proofs we present here much more simple than they would be otherwise and provides a unifying perspective for known examples.

Our main observation is that for any admissible multiset S, there is a correspondence between H-like metric nilpotent Lie algebras of type (p, q) whose spectrum is a multiple of S and standard two-step metric nilpotent Lie algebras defined by p-dimensional subspaces of $\mathbb{R}\mathcal{C}_S$ in $\mathfrak{so}(\mathfrak{v})$.

Theorem 1.3. *Let S be an admissible multiset with size $q \geq 2$. Let $\mathbb{R}\mathcal{C}_S$ be the cone over the conjugacy class of elements of $\mathfrak{so}(\mathbb{R}^q)$ with spectrum S.*

1. *If W is a p-dimensional subspace of $\mathbb{R}\mathcal{C}_S$, then the standard two-step metric nilpotent Lie algebra of type (p, q) defined by W has constant J-spectrum S.*
2. *If $(\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, Q)$ is a two-step metric nilpotent Lie algebra of type (p, q) which has constant J-spectrum S, then $J(\mathfrak{z})$ is a p-dimensional subspace of $\mathbb{R}\mathcal{C}_S$, and $(\mathfrak{n}, Q)$ is homothetic to a standard two-step metric nilpotent Lie algebra.*

Therefore the problem of classifying H-like Lie algebras is equivalent to the problem of classifying linear subspaces of $\mathfrak{so}(\mathbb{R}^q)$ that are contained in subsets of the form $\mathbb{R}\mathcal{C}_S$. This kind of problem – finding sets of matrices so that nonlinear properties such as rank or spectrum are preserved under linear combinations – has been studied since at least as early as Hurwitz and Radon ([23, 35]). For example, see [2, 14, 15, 17, 24, 38] for subspaces of matrices having fixed or bounded rank, see [4, 16, 31] for subspaces of skew-symmetric matrices, and see [37] for subspaces of cones over conjugacy classes. Almost all of these results are over algebraically closed fields.

Vector spaces of matrices of maximal rank two were first classified in [1]. Early classification results are reviewed and obtained as special cases of a more general theorem in [14] (see Theorem 1.1, which assumes a nondegeneracy condition to exclude matrices with common rows or columns of zeros and that the underlying field is algebraically closed).

Theorem 1.4. [1] *A vector space of matrices of rank ≤ 2 is equivalent to one of the following:*

1. *a subspace of the space of matrices of the form* $\left\{ \begin{bmatrix} 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & * \\ * & * & \cdots & * & * \end{bmatrix} \right\}$,
2. $\mathfrak{so}(\mathbb{R}^3)$.

Even a general classification of skew-symmetric matrices of rank 4 analogous to the classification in Theorem 1.4 remains unknown.

In Definition 3.2, we define a class of metric Lie algebras by taking the tensor product of a J-map for a metric nilpotent Lie algebra with another linear map, and we show in Proposition 3.4 that if the original metric nilpotent Lie algebra is H-like and the linear map is symmetric, then the resulting Lie algebra is H-like. We show in Proposition 3.6 that if $(\mathfrak{n}, Q)$ is H-like, then the J map is unitary. In Propositions 4.1, 4.4 and 4.6 we give new methods for constructing H-like Lie algebras using subspaces of cones over conjugacy classes; these correspond to families of block diagonal matrices, Riemannian submersions with fibers in the center and central sums.

We classify H-like Lie algebras such that $\operatorname{rank}(J_Z) \leq 2$ for all $Z \in \mathfrak{n}$:

Theorem 1.5. *Suppose that $(\mathfrak{n}, Q)$ is an H-like metric nilpotent Lie algebra and* $\operatorname{rank}(J_Z) \leq 2$ *for all $Z \in \mathfrak{n}$. Then $(\mathfrak{n}, Q)$ is homothetically isomorphic to one of the following:*

1. $(\mathfrak{n}, Q) = (\mathfrak{f}_{3,2}, Q)$ *as in Example* 2.5*,*
2. *an almost abelian metric Lie algebra as in Example* 2.6*.*

In the first theorem of [6], the authors classify two-step nilpotent Lie algebras defined by graphs which admit H-like inner products and obtain the H-like Lie algebras listed in Theorem 1.5. All Lie algebras defined by a graph have $\operatorname{rank}(J_Z) \leq 2$ for all $Z \in \mathfrak{n}$, so the classification in [6] follows from Theorem 1.5. In higher dimensions, there are continuous families of nonhomothetic metric Lie algebras with $\operatorname{rank}(J_Z) \leq 2$ for all $Z \in \mathfrak{n}$, while there are only countably many Lie algebras defined by graphs. Hence, the classification in Theorem 1.5 is more general than that in [6].

Theorem 1.5 can be proved using Theorem 1.3 and Theorem 1.4. Instead we provide a new, self-contained proof which uses the language and properties of J-maps and uses the metric throughout the argument.

The paper is organized as follows. In Section 2 we review properties of multisets and present some examples of H-like Lie algebras. We discuss properties of H-like Lie algebras in Section 3. In Section 4, we prove Theorem 1.3 and use it to prove Propositions 4.1, 4.4 and 4.6. In Section 5, we prove Theorem 1.5.

2. Background and examples

We will denote the spectra of matrices as multisets with elements from $\mathbb{C}$. Recall that a multiset from $\mathbb{C}$ is a subset of $\mathbb{C}$ with multiplicities. For example, the set $\{i, -i, 0\}$ endowed with multiplicity function $m : S \to \mathbb{N}$ given by $m(i) = m(-i) = 2$, $m(0) = 1$, and $m(z) = 0$ for all other $z \in \mathbb{C}$ is a multiset. We say that the sum of the multiplicities of a multiset S is its *size* and we denote the size of S by $|S|$. The sum $S_1 \uplus S_2$ of multisets S_1 and S_2 is the multiset determined by summing the multiplicity functions for S_1 and S_2. For a multiset S and nonzero scalar k we define the multiset kS so that the multiplicity for kz in kS is always equal to the multiplicity of z in S.

Let S be an admissible multiset of size q. The Frobenius norm on $\mathfrak{so}(\mathbb{R}^q)$ is given by $\|A\|^2 = \operatorname{trace}(AA^T)$. All elements A of $\mathfrak{so}(\mathbb{R}^q)$ with spectrum S have the same Frobenius norm, so we define the norm of the multiset S by

$$N(S) = \|A\| = \sqrt{\sum_{a \in S} |a|^2 \cdot m(a)}, \tag{2}$$

where $m(a)$ is the multiplicity of a. For example, the multiset $S = \{i, -i, 3i, -3i\}$ with $m(i) = m(-i) = 2$ and $m(3i) = m(-3i) = 1$ has $N(S) = \sqrt{22}$.

Example 2.1. Suppose that $(\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, Q)$ is H-type. If $\|Z\| = 1$, then J_Z is nondegenerate with eigenvalues in the multiset S defined over $\{i, -i\}$, where i and $-i$ have multiplicity $\frac{1}{2} \dim \mathfrak{v}$. Then $N(S) = \sqrt{\dim \mathfrak{v}}$. ■

Deformations of inner products on H-type metric Lie algebras may give metrics on the same underlying Lie algebra that are H-like but not H-type. This occurs in the next example.

Example 2.2. Let $\mathfrak{v} \cong \mathbb{R}^4$ and $\mathfrak{z} \cong \mathbb{R}$ with Z a basis vector for $\mathfrak{z}$ and $\{X_1, Y_1, X_2, Y_2\}$ a basis for $\mathfrak{v}$. Let a and b be nonzero real numbers. Define $J : \mathfrak{z} \to \operatorname{End}(\mathfrak{v})$ by

$$J(Z) = \begin{bmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & b & 0 \end{bmatrix},$$

and let $(\mathfrak{n}, Q)$ be the resulting metric nilpotent Lie algebra with orthonormal basis $\{Z, X_1, Y_1, X_2, Y_2\}$. Clearly $(\mathfrak{n}, Q)$ has constant J-spectrum equal to the multiset $\{ai, -ai, bi, -bi\}$. When $|a| = |b| = 1$, we get the H-type inner product on the five-dimensional Heisenberg Lie algebra and when $|a| \neq |b|$ and both are nonzero, we get an H-like inner product that is not H-type. If we allow exactly one of the parameters to be zero, then $\mathfrak{n}$ is isomorphic to $\mathfrak{h}_3 \oplus \mathbb{R}^2$, and $(\mathfrak{n}, Q)$ has constant J-spectrum but is not H-like. ■

Gornet and Mast gave the following families of nonisometric H-like Lie algebras of type $(2,4)$ ([21]).

Example 2.3. Let $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$, and assume that $(c,d) \in \{\pm(-b,a), \pm(-a,b)\}$. Define for orthonormal Z_1, Z_2 in $\mathbb{R}^2$,

$$J(Z_1) = \begin{bmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{bmatrix}, \quad \text{and} \quad J(Z_2) = \begin{bmatrix} 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \\ -c & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \end{bmatrix}.$$

The spectrum of $J(Z)$ for unit Z in $\mathbb{R}^2$ is $\{ai, -ai, bi, -bi\}$. When $a = \pm b$, the resulting metric Lie algebra is H-type. When a and b are both nonzero and $|a| \neq |b|$, it is H-like but not H-type. ■

Remark 2.4. The previous two examples show that H-like metric nilpotent Lie algebras are not necessarily soliton: Soliton inner products on a fixed nilpotent Lie algebra are unique up to scaling, but there may be two nonhomothetic inner products on a fixed Lie algebra which are both H-like.

In the next example we consider the free two-step nilpotent Lie algebra on three generators.

Example 2.5. Let $\mathfrak{f}_{3,2}$ be the free two-step nilpotent Lie algebra on three generators. It has basis $\{E_1, E_2, E_3\} \cup \{F_1, F_2, F_3\}$, and the Lie bracket is determined by the relations

$$[E_1, E_2] = F_1, \quad [E_2, E_3] = F_2, \quad \text{and} \quad [E_1, E_3] = F_3.$$

Let Q_0 be the inner product that makes the basis orthonormal. Then $\mathfrak{v} = \mathrm{span}\{E_i\}_{i=1}^3$ and $\mathfrak{z} = \mathrm{span}\{F_i\}_{i=1}^3$. With respect to the basis $\mathcal{B} = \{E_1, E_2, E_3\}$ of $\mathfrak{v}$, the endomorphism $J_{a_1F_1+a_2F_2+a_3F_3}$ is given by

$$[J_{a_1F_1+a_2F_2+a_3F_3}]_{\mathcal{B}} = \begin{bmatrix} 0 & -a_1 & -a_3 \\ a_1 & 0 & -a_2 \\ a_3 & a_2 & 0 \end{bmatrix}.$$

The square of this mapping has eigenvalues $-(a_1^2 + a_2^2 + a_3^2)$ and 0, with multiplicities 2 and 1 respectively. Hence $(\mathfrak{f}_{3,2}, Q_0)$ has J-rank two and is H-like. Because the J_Z maps are always singular, it is not H-type. ■

The following family of metric Lie algebras is known to be H-like [6] (Example 6).

Example 2.6. Let $(\mathfrak{n}^m, Q_0)$ be the metric nilpotent Lie algebra with orthonormal basis $\{E_0, E_1, E_2, \ldots, E_m\} \cup \{F_1, \ldots, F_m\}$ with Lie bracket determined by relations

$$[E_0, E_k] = F_k \quad \text{for } k = 1, \ldots, m.$$

Note that $\mathrm{span}(\{E_1, E_2, \ldots, E_m\} \cup \{F_1, \ldots, F_m\})$ is a codimension one abelian ideal and that all bracket relations are determined by ad_{E_0}.

Such algebras are called *almost abelian.* If $Z = a_1F_1 + \cdots + a_kF_k$, the map J_Z^2 has eigenvalues $-(a_1^2 + \cdots + a_m^2)$ and 0, with multiplicities 2 and $m-1 = \dim(\mathfrak{v}) - 2$ respectively. Therefore, if $\|Z\| = 1$, the spectrum of J_Z is $\{i, -i, 0\}$. Hence $(\mathfrak{n}^m, Q_0)$ has J-rank 2 and is H-like. ■

Example 2.7. Let $(\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, Q)$ be a two-step metric nilpotent Lie algebra so that the isometry group of Q acts transitively on the unit sphere in $\mathfrak{z}$.
For any $Z \in \mathfrak{z}$, $J_{\phi(Z)} = \phi \circ J_Z \circ \phi^{-1}$, so J_Z and $J_{\phi(Z)}$ have the same spectrum. Hence $(\mathfrak{n}, Q)$ is H-like. ■

3. Properties of H-like Lie algebras

The next proposition describes how the property of having constant J-spectrum behaves under direct sums.

Proposition 3.1. *Let $(\mathfrak{n}_1, Q_1)$ and $(\mathfrak{n}_2, Q_2)$ be metric nilpotent Lie algebras which are abelian or two-step. Define the metric Lie algebra $(\mathfrak{n}_1 \oplus \mathfrak{n}_2, Q)$ with Q so that $Q|_{\mathfrak{n}_1} = Q_1, Q|_{\mathfrak{n}_2} = Q_2$ and $\mathfrak{n}_1 \perp \mathfrak{n}_2$. Assume $\mathfrak{n}_1 \oplus \mathfrak{n}_2$ is non-abelian. The following are equivalent:*

- *The direct sum $(\mathfrak{n}_1 \oplus \mathfrak{n}_2, Q)$ has constant J-spectrum.*
- *One of $(\mathfrak{n}_1, Q_1)$ or $(\mathfrak{n}_2, Q_2)$ is abelian, and the other has constant J-spectrum.*

Proof. Write $\mathfrak{n}_1 = \mathfrak{v}_1 \oplus \mathfrak{z}_1$ and $\mathfrak{n}_2 = \mathfrak{v}_2 \oplus \mathfrak{z}_2$.

Suppose that $(\mathfrak{n}_1 \oplus \mathfrak{n}_2, Q)$ has constant J-spectrum. If neither $\mathfrak{n}_1$ nor $\mathfrak{n}_2$ is abelian, then $\mathfrak{z}_1$ and $\mathfrak{z}_2$ are nontrivial, so there exist nonzero $Z_1 \in \mathfrak{z}_1 = [\mathfrak{v}_1, \mathfrak{v}_1]$ and nonzero $Z_2 \in \mathfrak{z}_2 = [\mathfrak{v}_2, \mathfrak{v}_2]$. By the constant spectrum hypothesis, the ranks of $J(Z_1 + Z_2)$, $J(Z_1)$, and $J(Z_2)$ are the same. But because $\mathfrak{v}_1 \cap \mathfrak{v}_2 = \{0\}$ in $\mathfrak{n}_1 \oplus \mathfrak{n}_2$, the rank of $J(Z_1 + Z_2)$ is the sum of the ranks of $J(Z_1)$ and $J(Z_2)$. Therefore one of them has rank zero, a contradiction to Z_1 and Z_2 being in the commutator. If $\mathfrak{n}_2$ is abelian, then clearly $\mathfrak{n}_1$ must have constant spectrum.

For the converse, suppose that $(\mathfrak{n}_1, Q_1)$ is two-step with constant spectrum S and $\mathfrak{n}_2$ is abelian. Then $(\mathfrak{n}_1, Q_1)$ has nontrivial commutator and $J^{\mathfrak{n}} = J^{\mathfrak{n}_1} \oplus 0^{\mathfrak{n}_2}$ where $0^{\mathfrak{n}_2}$ is the zero map on $\mathfrak{v}_2$. The spectrum of $J^{\mathfrak{n}}$ is $S \uplus T$, where T is $\{0\}$ with multiplicity $\dim(\mathfrak{v}_2)$. ■

A simple way to build new H-like Lie algebras from old is by taking the tensor product of a symmetric map with the J-map for an H-like Lie algebra. Before proving this we need to define the metric nilpotent Lie algebras corresponding to J-maps that are tensor products.

Definition 3.2. Suppose that $(\mathfrak{n}, Q)$ is a two-step metric nilpotent Lie algebra with $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. Let $J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ be the J-map for $(\mathfrak{n}, Q)$. Let $S : \mathbb{R}^m \to \mathbb{R}^m$, where $m > 1$, be a linear map which is symmetric with respect to the standard inner product $\langle \cdot, \cdot \rangle_{\text{standard}}$ on $\mathbb{R}^m$. Define the inner product Q_S on $\mathfrak{z} \oplus (\mathfrak{v} \otimes \mathbb{R}^m)$ so that $Q_S|_{\mathfrak{z}} = Q|_{\mathfrak{z}}$, $\mathfrak{z} \perp (\mathfrak{v} \otimes \mathbb{R}^m)$, and the restriction of Q_S to $\mathfrak{v} \otimes \mathbb{R}^m$ is determined by

$$\langle X_1 \otimes Y_1, X_2 \otimes Y_2 \rangle = \langle X_1, X_2 \rangle_Q \cdot \langle Y_1, Y_2 \rangle_{\text{standard}},$$

where $X_1, X_2 \in \mathfrak{v}$ and $Y_1, Y_2 \in \mathbb{R}^m$.

Define the map $J^S : \mathfrak{z} \to \mathrm{End}(\mathfrak{v} \otimes \mathbb{R}^m)$ by $J^S(Z) = J_Z \otimes S$. Let $\mathfrak{n} \otimes S = \mathfrak{z} \oplus (\mathfrak{v} \otimes \mathbb{R}^m)$. Make $\mathfrak{n} \otimes S$ into a Lie algebra by defining the bracket using the J map through equation (1). ■

Example 3.3. Let $(\mathfrak{h}_3, Q)$ be the Heisenberg metric Lie algebra with orthonormal basis $\{X, Y, Z\}$ where $[X, Y] = Z$. Take orthonormal basis $\{E_1, E_2\}$ for $\mathbb{R}^2$ and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the map with $S(E_1) = aE_1$ and $S(E_2) = bE_2$, where a and b are nonzero. Then the map $J^S : \mathfrak{z} \to \mathrm{End}(\mathrm{span}\{X, Y\} \otimes \mathbb{R}^2)$ as in Definition 3.2 sends Z to $J_Z \otimes S$. The resulting Lie algebra $\mathfrak{n} \otimes S$ as in Definition 3.2 is isometrically isomorphic to the one in Example 2.2. ■

Proposition 3.4. *Suppose that $(\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, Q)$ is a two-step metric nilpotent Lie algebra. Let $S : \mathbb{R}^m \to \mathbb{R}^m$ be a nonsingular linear map which is symmetric with respect to the standard inner product on $\mathbb{R}^m$, where $m > 1$.*
Then, as a multiset, the spectrum of $J^S(Z)$ is the multiset consisting of all products $\lambda\mu$, where $\lambda \in \mathrm{Spec}(J_Z)$ and $\mu \in \mathrm{Spec}(S)$, and the multiplicity of $\gamma \in \mathrm{Spec}(J^S(Z))$ is the sum of products of multiplicities, $m(\lambda_i)m(\mu_i)$ where $\lambda_i\mu_i = \gamma$. Hence, if $(\mathfrak{n}, Q)$ is H-like, then $\mathfrak{n} \otimes S$ is H-like.

Proof. First we show that $[\mathfrak{n} \otimes S, \mathfrak{n} \otimes S] = \mathfrak{z}$. From the definition of $\mathfrak{n} \otimes S$, the commutator for $\mathfrak{n} \otimes S$ is contained in the domain of J^S, that is $\mathfrak{z}$. Fix unit $Z \in \mathfrak{z} \subseteq \mathfrak{n}$. Then there exist orthogonal X and Y in $\mathfrak{v}$ so that $[X, Y] = Z$ in $\mathfrak{n}$. Choose unit U to be an eigenvector of S with (nonzero) eigenvalue λ. Then

$$\begin{aligned}\langle Z, [X \otimes U, Y \otimes U]\rangle &= \langle J^S_Z(X \otimes U), Y \otimes U\rangle = \langle J_Z X \otimes S(U), Y \otimes U\rangle \\ &= \langle J_Z X \otimes \lambda U, Y \otimes U\rangle = \langle Z, Z\rangle \cdot \langle \lambda U, U\rangle = \lambda \|Z\|^2 \neq 0.\end{aligned}$$

Hence $Z \in [\mathfrak{n} \otimes S, \mathfrak{n} \otimes S]$. Therefore $\mathfrak{z} \subseteq [\mathfrak{n} \otimes S, \mathfrak{n} \otimes S]$.
Because the eigenvalues of the tensor product of maps is the set of products of eigenvalues of each, it is immediate that if $(\mathfrak{n}, Q)$ has constant J-spectrum, then $\mathfrak{n} \otimes S$ has constant J-spectrum. ■

Example 3.5. The generalized Heisenberg groups defined by Goze and Haraguchi in [22] arise from tensor products. Their Lie algebras, which we will call *generalized Heisenberg Lie algebras*, are of the form $\mathfrak{m}^r \otimes I_p$, where $\mathfrak{m}^r$ is the Lie algebra of dimension $2r + 1$ from Example 2.6 and I_p is the $p \times p$ identity matrix. Taking $r = 1$ yields the $(2p + 1)$-dimensional Heisenberg algebra, and taking $p = 1$ yields $\mathfrak{m}^r$. Because the Lie algebras $\mathfrak{m}^r$ support H-like inner products, Proposition 3.4 implies that the generalized Heisenberg Lie algebras of Goze and Haraguchi admit H-like inner products. ■

We show that if $(\mathfrak{n}, Q)$ is an H-like metric Lie algebra, then the map J for $(\mathfrak{n}, Q)$ is unitary with respect to a rescaled Frobenius norm on the image space.

Proposition 3.6. *Suppose that $(\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, Q)$ is an H-like Lie algebra with spectrum S. Endow $\mathfrak{z}$ with the inner product that is the restriction of the inner product Q, and give $\mathrm{End}(\mathfrak{v})$ the rescaled Frobenius inner product*

$$\langle A, B\rangle_{\mathrm{End}(\mathfrak{v})} = \tfrac{1}{N(S)^2} \,\mathrm{trace}\, AB^T,$$

where $N(S)$ is as in equation (2), and the trace and transpose are taken with respect to the restriction of the inner product Q to $\mathfrak{v}$. Then, for all $Y, Z \in \mathfrak{z}$,

$$\langle J_Y, J_Z\rangle_{\mathrm{End}(\mathfrak{v})} = \langle Y, Z\rangle_Q. \qquad (3)$$

Proof. Let $Z \in \mathfrak{z}$ be a unit vector. Because $(\mathfrak{n}, Q)$ is H-like, trace $J_Z^2 = -N(S)^2$. This and skew-symmetry of J_Z give

$$\langle J_Z, J_Z \rangle_{\mathrm{End}(\mathfrak{v})} = \tfrac{1}{N(S)^2} \operatorname{trace} J_Z J_Z^T = -\tfrac{1}{N(S)^2} \operatorname{trace} J_Z^2 = 1.$$

Therefore J maps the unit sphere in $\mathfrak{z}$ into the unit sphere in $\mathrm{End}(\mathfrak{v})$ so it is unitary. ■

As a consequence, with respect to the unscaled Frobenius norm $\|\cdot\|_{\mathrm{Frobenius}}$, if $(\mathfrak{n}, Q)$ is H-like, and $\|Z\| = \|W\|$, then $\|J_Z\|_{\mathrm{Frobenius}} = \|J_W\|_{\mathrm{Frobenius}}$.

The next proposition describes how the spectrum of a vector in a subspace of the cone over a conjugacy class depends on its norm.

Lemma 3.7. *Let S be an admissible multiset of size q. Let $\mathbb{R}\mathcal{C}_S$ be the cone over the conjugacy class for S in $\mathfrak{so}(\mathbb{R}^q)$. Then any $A \in \mathbb{R}\mathcal{C}_S$ has spectrum $\frac{\|A\|_{Frobenius}}{N(S)} S$, where $\|A\|^2_{Frobenius} = \operatorname{trace}(AA^T)$.*

Proof. Clearly the statement holds when $A = 0$. Let $A \in \mathbb{R}\mathcal{C}_S$ be nonzero. Because A is in $\mathbb{R}\mathcal{C}_S$, its spectrum is a multiple of S, and we can rescale A by λ so that λA has spectrum S. As A is skew-symmetric, we may assume that $\lambda > 0$.

Choose $B \in \mathbb{R}\mathcal{C}_S$ with spectrum S. Then $\|B\| = N(S)$, where $N(S)$ is as in equation (2). Because they have the same spectrum, λA and B are conjugate. Conjugation preserves the Frobenius norm, so $\lambda\|A\| = \|B\|$. But $\lambda \operatorname{Spec}(A) = \operatorname{Spec}(B) = S$. Hence

$$\operatorname{Spec}(A) = \frac{1}{\lambda} S = \frac{\|A\|}{\|B\|} S = \frac{\|A\|}{N(S)} S. \qquad ■$$

We will use the following simple yet crucial lemma in the proof of the main theorem. It lies behind the correspondence between nilpotent Lie algebras of type (p, q) and p-dimensional subspaces of $\mathfrak{so}(\mathbb{R}^q)$ in [11].

Lemma 3.8. *Suppose that $(\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, Q)$ is a two-step metric nilpotent Lie algebra of type (p, q). Then $J(\mathfrak{z})$ is a p-dimensional subspace of $\mathrm{End}(\mathfrak{v})$.*

Proof. Let $\{Z_1, \ldots, Z_p\}$ be a basis of $\mathfrak{z}$. We want to show that $J(Z_1), \ldots, J(Z_p)$ are independent. Suppose that there are real numbers $a_1, ..., a_p$ so that $\sum_{i=1}^p a_i J(Z_i) = 0$. But then we have $J(\sum_{i=1}^p a_i Z_i) = 0$. Hence $J(Z) \equiv 0$ for $Z = \sum_{i=1}^p a_i Z_i$. But for $Z \in [\mathfrak{n}, \mathfrak{n}]$, $J_Z \equiv 0$ if and only if $Z = 0$. Because $\{Z_1, \ldots, Z_p\}$ is independent, $a_i = 0$ for all i. ■

Remark 3.9. It can be shown that if $(\mathfrak{n}, Q)$ is H-like, then the restriction of the Ricci endomorphism to $\mathfrak{z}$ is a constant times the identity (see [34, Lemma 1]).

4. The Main Theorem and some of its consequences

To prove Theorem 1.3, we show that the bijection between two-step metric nilpotent Lie algebras $(\mathfrak{n}, Q)$ of type (p, q) and p-dimensional subspaces of $\mathfrak{so}(\mathbb{R}^q)$ endowed with the natural inner product is a bijection between algebras with spectrum a multiple of S and p-dimensional subspaces of the cone $\mathbb{R}\mathcal{C}_S$ in $\mathfrak{so}(\mathbb{R}^q)$.

Proof of Theorem 1.3. Fix (p,q). Let S be an admissible multiset of size $q \geq 2$. Let W be a p-dimensional subspace of $\mathbb{R}\mathcal{C}_S \subseteq \mathfrak{so}(\mathbb{R}^q)$ and let $(\mathfrak{n}, Q)$ be the standard two-step metric nilpotent Lie algebra defined by W as in Definition 1.2. Choose $Z \in \mathfrak{z}$ with $\|Z\| = 1$. Then $J(Z) \in W \subseteq \mathbb{R}\mathcal{C}_S$, so its spectrum is a scalar multiple of S.
Suppose that $(\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, Q)$ is a two-step metric nilpotent Lie algebra of type (p,q) with constant J-spectrum S. By Lemma 3.8, $J(\mathfrak{z})$ is a p-dimensional subspace of $\mathfrak{so}(\mathfrak{v})$. By definition of constant J-spectrum, $J(Z)$ has spectrum S for all unit Z. Therefore the image of the unit sphere in $\mathfrak{z}$ is contained in $\mathcal{C}_S$. Because rescaling an element of $\mathfrak{so}(\mathfrak{v})$ by $\lambda \in \mathbb{R}$ rescales the spectrum by λ, all elements in $J(\mathfrak{z})$ are in the cone $\mathbb{R}\mathcal{C}_S$. ∎

The next proposition describes how to explicitly construct H-like Lie algebras using bases with a particular form. Recall that to form the sum $S = S_1 \uplus S_2 \uplus \cdots \uplus S_k$ of multisets $S_1, \ldots, S_k$, we add the multiplicity functions of $S_1, \ldots, S_k$.

Proposition 4.1. *For $i = 1, \ldots, k$, let $\mathfrak{v}_i$ be a vector space of dimension at least two, and let S_i be an admissible multiset with size equal to $\dim(\mathfrak{v}_i)$. Suppose that W is a p-dimensional subspace of $\mathfrak{so}(\oplus_{i=1}^k \mathfrak{v}_i)$ having a basis $\mathcal{B} = \{B_1, \ldots, B_p\}$ consisting of vectors $B_j = \oplus_{i=1}^k A_i^j$, $j = 1, \ldots, p$, where for each i,*

1. *$A_i^j \in \mathfrak{so}(\mathfrak{v}_i)$ for each j,*
2. *for all j the spectrum of A_i^j is S_i,*
3. *$\operatorname{span}\{A_i^j\}_{j=1}^p \subseteq \mathbb{R}\mathcal{C}_{S_i}$, and*
4. *$\{A_i^j : j = 1, \ldots, p\}$ is orthogonal with respect to the Frobenius inner product.*

Then W is a subspace of $\mathbb{R}\mathcal{C}_S$, where $S = S_1 \uplus S_2 \uplus \cdots \uplus S_k$ and $\mathbb{R}\mathcal{C}_S$ is the cone over the conjugacy class $\mathcal{C}_S$ in $\mathfrak{so}(\oplus_{i=1}^k \mathfrak{v}_i)$.

Proof. Let $\mathcal{B} = \{B_1, \ldots, B_p\}$ be a basis for W as in the statement of the proposition. Property (4) forces $\mathcal{B}$ to be orthogonal, and by Property (3), all of the vectors in $\mathcal{B}$ have spectrum $S = S_1 \uplus S_2 \uplus \cdots \uplus S_k$ and Frobenius norm $N(S)$, where N is the function associated to the multiset S as in equation (2). Let C in W be nonzero. We may write C as $C = \sum_{j=1}^p c_j (\oplus_{i=1}^k A_i^j) = \oplus_{i=1}^k \left(\sum_{j=1}^p c_j A_i^j \right)$. Then the norm of C is given by

$$\|C\|^2 = \sum_{i=1}^k \sum_{j=1}^p \|c_j A_i^j\|^2 = \sum_{j=1}^p c_j^2 \sum_{i=1}^k N(S_i)^2 = N(S)^2 \sum_{j=1}^p c_j^2,$$

which shows that $\sum_{j=1}^p c_j^2 = \|C\|^2 / N(S)^2$. The restriction of C to $\mathfrak{v}_i$ for fixed i is $C_i = \sum_{j=1}^p c_j A_i^j$. By the same reasoning as above, $\sqrt{\sum c_j^2} = \|C_i\| / N(S_i)$ for all i. Now we apply Lemma 3.7 to C_i in $\mathbb{R}\mathcal{C}_{S_i}$ and see that its spectrum is λS_i, where

$$\lambda = \frac{\|C_i\|}{N(S_i)} = \sqrt{\sum c_j^2}$$

is independent of i. Summing over i, we deduce that the spectrum of C is

$$\lambda S_1 \uplus \cdots \uplus \lambda S_k = \lambda(S_1 \uplus S_2 \uplus \cdots \uplus S_k) = \lambda S.$$

Thus, $W \subseteq \mathbb{R}\mathcal{C}_S$. ∎

Example 4.2. Take $\mathfrak{v}_1 = \mathbb{R}^2$ and $\mathfrak{v}_2 = \mathbb{R}^2$. Let a and b be nonzero. Take

$$A_1^1 = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \in \mathfrak{so}(\mathfrak{v}_1), \ A_1^2 = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \in \mathfrak{so}(\mathfrak{v}_2), \text{ and } B_1 = A_1^1 \oplus A_1^2 \subseteq \mathfrak{so}(\mathfrak{v}_1 \oplus \mathfrak{v}_2).$$

Let W be the subspace of $\mathfrak{so}(\mathfrak{v}_1 \oplus \mathfrak{v}_2)$ spanned by $A_1^1 \oplus A_1^2$. The resulting metric Lie algebra is isometrically isomorphic to $\mathfrak{h}_5$ endowed with an H-like inner product as in Ex. 2.2. Here $S_1 = \{ai, -ai\}, S_2 = \{bi, -bi\}$, and $S_1 \uplus S_2 = \{ai, -ai, bi, -bi\}$. ∎

Example 4.3. Let $(\mathfrak{f}_{3,2}, Q_0)$ and $J_{F_1}, J_{F_2}, J_{F_3} \in \mathfrak{so}(\mathbb{R}^3)$ be as in Example 2.5. By Proposition 4.1, any basis selected from the linearly dependent set

$$\{J_{F_k} \oplus J_{F_l} \ : \ k, l = 1, 2, 3\} \subseteq \mathfrak{so}(\mathbb{R}^3) \oplus \mathfrak{so}(\mathbb{R}^3) \subseteq \mathfrak{so}(\mathbb{R}^6)$$

defines a subspace of the cone over the conjugacy class for $\{i, i, -i, -i, 0, 0\}$. An example of this type appears in (4.7) of [16]. ∎

The next proposition shows that Riemannian submersions in a certain class map Lie algebras with constant spectrum onto Lie algebras with constant spectrum. In fact, by [12], the Riemannian submersion of the corresponding nilmanifolds has simply connected, flat, totally geodesic fibers.

Proposition 4.4. *Suppose that* $(\mathfrak{n}_1 = \mathfrak{v}_1 \oplus \mathfrak{z}_1, Q_1)$ *and* $(\mathfrak{n}_2 = \mathfrak{v}_2 \oplus \mathfrak{z}_2, Q_2)$ *are metric Lie algebras, and* $\phi : \mathfrak{n}_1 \to \mathfrak{n}_2$ *is a surjective homomorphism with* $\ker \phi \subseteq \mathfrak{z}_1$ *such that* $\langle X, Y \rangle = \langle \phi(X), \phi(Y) \rangle$ *for all* $X, Y \in (\ker \phi)^\perp$. *If* $(\mathfrak{n}_1, Q_1)$ *has constant spectrum* S, *then* $(\mathfrak{n}_2, Q_2)$ *has constant spectrum* S.

Proof. Let Z_2 be an arbitrary unit vector in $\mathfrak{z}_2$. Since the restriction ϕ_0 of ϕ to $(\ker \phi)^\perp$ is a bijection, it is invertible. Hence Z_2 has a unique pre-image $Z_1 = \phi_0^{-1}(Z_2)$ in $(\ker \phi)^\perp$ and this pre-image has length one. Because $(\mathfrak{n}_1, Q_1)$ has spectrum S, J_{Z_1} has spectrum S.
For any $Z \in (\ker \phi)^\perp$, $J_{\phi_0(Z)} = \phi_0 \circ J_Z \circ \phi_0^{-1}$. Hence $J_{Z_2} = \phi_0 \circ J_{Z_1} \circ \phi_0^{-1}$. Since J_{Z_2} is conjugate to J_{Z_1}, it also has spectrum S.
Thus, all vectors in the unit sphere in $\mathfrak{z}_2$ have spectrum S and $J(\mathfrak{z}_2) \subseteq \mathbb{R}\mathcal{C}_S$. ∎

The central sum construction glues two groups together along a subgroup; when we glue two nilpotent Lie algebras together along their centers we call this a *central sum.* This was called concatenation by Jablonski in [25].

Definition 4.5. Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be Lie algebras with centers $Z(\mathfrak{n}_1)$ and $Z(\mathfrak{n}_2)$ respectively. Let $\phi : Z(\mathfrak{n}_1) \to Z(\mathfrak{n}_2)$ be a bijective linear map.

(1) Define the Lie algebra $\mathfrak{n}_1 +_\phi \mathfrak{n}_2$, called *central sum of* $\mathfrak{n}_1$ *and* $\mathfrak{n}_2$ *defined by* ϕ, as $(\mathfrak{n}_1 \oplus \mathfrak{n}_2)/\mathfrak{i}$, where $\mathfrak{i}$ is the ideal $\{(W, -\phi(W)) : W \in Z(\mathfrak{n}_1)\}$.

(2) Suppose that Q_1 and Q_2 are inner products on $\mathfrak{n}_1$ and $\mathfrak{n}_2$ respectively, and that ϕ is an isometry with respect to Q_1 and Q_2. Let $\pi : \mathfrak{n}_1 \oplus \mathfrak{n}_2 \to \mathfrak{n}_1 +_\phi \mathfrak{n}_2$ be the natural projection map. Define the inner product Q on $\mathfrak{n}_1 +_\phi \mathfrak{n}_2$ so that the projections $\pi|_{\mathfrak{n}_1}$ and $\pi|_{\mathfrak{n}_2}$ are isometries. Then the metric nilpotent Lie algebra $(\mathfrak{n}_1 +_\phi \mathfrak{n}_2, Q)$ defined by J is called the *(metric) central sum of* $(\mathfrak{n}_1, Q_1)$ *and* $(\mathfrak{n}_2, Q_2)$ *defined by* ϕ. ∎

It is not hard to check that the inner product Q in Definition 4.5(2) is well-defined. This construction may be iterated. For example the $(2k+1)$-dimensional Heisenberg algebra may be viewed as a $(k-1)$-fold central sum of 3-dimensional Heisenberg algebras. In Section 6 of [7], the authors present families of H-like Lie algebras which are central sums. We show that in general, central sums of Lie algebras with constant spectrum again have constant spectrum.

An elementary way to define a subspace of the cone over a conjugacy class $\mathbb{R}C_S$ is by taking the direct sum of subspaces of $\mathbb{R}C_{S_1}$ and $\mathbb{R}C_{S_2}$, where $S = S_1 \uplus S_2$. Taking direct sums of subspaces translates to a natural construction of a metric Lie algebra called the metric central sum. Before we state the proposition we make some simple observations about the structure of metric central sums. Write $\mathfrak{n} = \mathfrak{n}_1 +_\phi \mathfrak{n}_2$. Because π is a surjective homomorphism,

$$[\mathfrak{n}, \mathfrak{n}] = \pi([\mathfrak{n}_1 \oplus \mathfrak{n}_2, \mathfrak{n}_1 \oplus \mathfrak{n}_2]) = \pi(\mathfrak{z}_1 \oplus \mathfrak{z}_2) = \pi(\mathfrak{z}_1).$$

The restriction of the canonical projection map $\pi : \mathfrak{n}_1 \oplus \mathfrak{n}_2 \to \mathfrak{n}$ to $\mathfrak{v}_1 \oplus \mathfrak{v}_2$ is an isometry, and the restriction of π to $\mathfrak{z}_1 \oplus \{0\}$ or $\{0\} \oplus \mathfrak{z}_2$ is an isometry to $\pi(\mathfrak{z}_1 \oplus \mathfrak{z}_2)$. We thus can identify $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$ with $\mathfrak{z}_1$ or $\mathfrak{z}_2$. We let $\mathfrak{v} = \mathfrak{z}^\perp$, so $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. After using the three isomorphisms to make appropriate identifications we can view the J map for $\mathfrak{n}$ as a map from $\mathfrak{z}_1$ to $\mathfrak{so}(\mathfrak{v}_1 \oplus \mathfrak{v}_2)$.

Proposition 4.6. *Suppose that* $(\mathfrak{n}_1 = \mathfrak{v}_1 \oplus \mathfrak{z}_1, Q_1)$ *and* $(\mathfrak{n}_2 = \mathfrak{v}_2 \oplus \mathfrak{z}_2, Q_2)$ *are metric nilpotent Lie algebras with centers* $Z(\mathfrak{n}_1)$ *and* $Z(\mathfrak{n}_2)$ *respectively.*
Let $\phi : Z(\mathfrak{n}_1) \to Z(\mathfrak{n}_2)$ *be an isometry.*

1. *Denote the* J *map for the metric central sum* $\mathfrak{n} = \mathfrak{n}_1 +_\phi \mathfrak{n}_2$ *by*

$$J^{\mathfrak{n}} : \mathfrak{z} \cong \mathfrak{z}_1 \to \mathfrak{so}(\mathfrak{v}) \cong \mathfrak{so}(\mathfrak{v}_1 \oplus \mathfrak{v}_2)$$

For $Z = \pi(Z_1) \in \mathfrak{z}$,

$$J^{\mathfrak{n}}(Z) = J^{\mathfrak{n}_1}(Z_1) \oplus J^{\mathfrak{n}_2}(\phi(Z_1)) \subseteq \mathfrak{so}(\mathfrak{v}_1) \oplus \mathfrak{so}(\mathfrak{v}_2) \subseteq \mathfrak{so}(\mathfrak{v}_1 \oplus \mathfrak{v}_2). \tag{4}$$

In particular, if $(\mathfrak{n}_1, Q_1)$ *and* $(\mathfrak{n}_2, Q_2)$ *have constant spectra* S_1 *and* S_2 *respectively, then* $(\mathfrak{n}_1 +_\phi \mathfrak{n}_2, Q)$ *has constant spectrum* $S_1 \uplus S_2$.

2. *The subspace of* $\mathfrak{so}(\mathfrak{v})$ *corresponding to* $(\mathfrak{n}_1 +_\phi \mathfrak{n}_2, Q)$ *is* $J^{\mathfrak{n}_1}(\mathfrak{z}_1) \oplus J^{\mathfrak{n}_2}(\mathfrak{z}_2) \subseteq \mathfrak{so}(\mathfrak{v}_1) \oplus \mathfrak{so}(\mathfrak{v}_2) \subseteq \mathfrak{so}(\mathfrak{v})$.

Proof. Suppose $\mathfrak{n}_1$ is type (p_1, q_1) and $\mathfrak{n}_2$ is type (p_2, q_2). Because $(\mathfrak{n}_1, Q_1)$ and $(\mathfrak{n}_2, Q_2)$ are H-like, $Z(\mathfrak{n}_1) = \mathfrak{z}_1$ and $Z(\mathfrak{n}_2) = \mathfrak{z}_2$. Because ϕ is an isometry, $p_1 = p_2$. Let S_1 and S_2 be the spectra for $(\mathfrak{n}_1, Q_1)$ and $(\mathfrak{n}_2, Q_2)$ respectively. From Theorem 1.3, $J(\mathfrak{z}_1)$ is a p_1-dimensional subspace of $\mathbb{R}\mathcal{C}_{S_1}$ contained in the cone over the conjugacy class $\mathcal{C}_{S_1}$, and $J(\mathfrak{z}_2)$ is a p_2-dimensional subspace of $\mathbb{R}\mathcal{C}_{S_2}$ contained in the cone over the conjugacy class $\mathcal{C}_{S_2}$.

In the following we write the map $J^{\mathfrak{n}}$ for $(\mathfrak{n}_1 +_\phi \mathfrak{n}_2, Q)$ in terms of $J^{\mathfrak{n}_1} : \mathfrak{z}_1 \to \mathfrak{so}(\mathfrak{v}_1)$ and $J^{\mathfrak{n}_2} : \mathfrak{z}_2 \to \mathfrak{so}(\mathfrak{v}_2)$. Let $X_1, Y_1 \in \mathfrak{v}_1, X_2, Y_2 \in \mathfrak{v}_2$, and $Z_1 \in \mathfrak{z}_1$. Denote their images under π with bars. Then

$$\begin{aligned}\langle J^{\mathfrak{n}}_{\overline{Z_1}} \overline{(X_1 + X_2)}, \overline{(Y_1 + Y_2)} \rangle &= \langle \overline{Z_1}, \overline{[X_1, Y_1]} \rangle + \langle \overline{\phi(Z_1)}, \overline{[X_2, Y_2]} \rangle \\ = \langle Z_1, [X_1, Y_1] \rangle + \langle \phi(Z_1), [X_2, Y_2] \rangle &= \langle J^{\mathfrak{n}_1}_{Z_1} X_1, Y_1 \rangle + \langle J^{\mathfrak{n}_2}_{\phi(Z_1)} X_2, Y_2 \rangle.\end{aligned}$$

Thus, equation (4) holds.

Let $Z = \pi(Z_1)$ be a unit vector in $\mathfrak{z}$. Because π maps $\mathfrak{z}_1$ onto $\mathfrak{z}$ isometrically, Z_1 is a unit vector in $\mathfrak{z}_1$. Hence $J(Z_1)$ has spectrum S_1. Because ϕ is an isometry, $\phi(Z_1)$ has norm one. Therefore, $J(\phi(Z_1))$ has spectrum S_2. It follows that $J^{\mathfrak{n}}(Z)$ as in equation (4) has spectrum $S_1 \uplus S_2$. By Theorem 1.3, $(\mathfrak{n}_1 +_\phi \mathfrak{n}_2, Q)$ is H-like with constant spectrum $S_1 \uplus S_2$. Statement (2) follows from equation (4). ■

Proposition 3.6 could have been used to complete the first part of the proof, along with the fact that $\{J(Z_i)\}_{i=1}^{p_1}$ and $\{J(\phi(Z_i))\}_{i=1}^{p_1}$ are orthogonal bases for $\mathfrak{so}(\mathfrak{v}_1)$ and $\mathfrak{so}(\mathfrak{v}_2)$ respectively.

The tensor product construction of $\mathfrak{n} \otimes S$ in Definition 3.2 is a central sum, with each summand corresponding to an element of a linearly independent set of eigenvectors for the symmetric map S.

5. Classification of H-like Lie algebras with J-rank two

Throughout this section we let $E(Z)$ denote the eigenspace for the nonzero eigenvalue of J_Z^2, where J is associated to a two-step metric nilpotent Lie algebra of J-rank two. Note that if $(\mathfrak{n}, Q)$ is a two-step metric nilpotent Lie algebra, $J_Z \not\equiv 0$ for all nonzero $Z \in \mathfrak{z}$. Note that when $(\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, Q)$ has no abelian factors (as in the H-like case), $\oplus_{i=1}^p E(Z_i) = \mathfrak{v}$, where $\{Z_i\}$ is a basis for $\mathfrak{z}$.

Lemma 5.1. *Let $(\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, Q)$ be a metric nilpotent Lie algebra. If $(\mathfrak{n}, Q)$ is H-like and has J-rank two, then for any independent vectors Z_1 and Z_2 in $\mathfrak{z}$,*

1. $E(Z_1) \neq E(Z_2)$
2. $E(Z_1) \cap E(Z_2)$ *is one-dimensional, and*
3. *if Z_1 and Z_2 are orthogonal, and nonzero X is in $E(Z_1) \cap E(Z_2)$, the vectors $J_{Z_1}X$ and $J_{Z_2}X$ are nonzero and orthogonal.*

Part (3) of the lemma is a special case of Theorem 3.8b of [7], which in turn relies on Lemma 3.3 of [21]. We give an alternate proof.

Proof. Assume that $(\mathfrak{n}, Q)$ is H-like and Z_1 and Z_2 are independent vectors in $\mathfrak{z}$. Then $Z_1, Z_2 \neq 0$. If $E(Z_1) = E(Z_2)$, then because $(\mathfrak{n}, Q)$ is H-like, $\lambda_1 J_{Z_1} = \lambda_2 J_{Z_2}$ for nonzero $\lambda_1, \lambda_2 \in \mathbb{R}$. It follows that $Z = \lambda_1 Z_1 - \lambda_2 Z_2$ has $J_Z \equiv 0$. Hence $Z = 0$, contradicting the independence of Z_1, Z_2.

Now assume $(\mathfrak{n}, Q)$ has J-rank two. We claim $E(Z_1) \cap E(Z_2) \neq \{0\}$. Otherwise $E(Z_1) + E(Z_2)$ is four-dimensional and $J_{Z_1+Z_2}$ will have rank four, a contradiction. Also, $E(Z_1) \cap E(Z_2)$ is not two-dimensional, because that would lead to the contradiction $E(Z_1) = E(Z_2)$.

Suppose that Z_1 and Z_2 above are orthogonal and $X \in E(Z_1) \cap E(Z_2)$ is nonzero. Then certainly $J_{Z_1}X$ and $J_{Z_2}X$ are nonzero. Since J_{Z_1} has rank two, there exists an orthogonal basis $\{X_1, X_2, \ldots, X_p\}$ for $\mathfrak{v}$ so that $X = X_1$, and with respect to this basis, $J_{Z_1} = X_1 \wedge X_2$ in $\mathfrak{so}(\mathfrak{v})$. Proposition 3.6 forces J_{Z_1} and J_{Z_2} to be orthogonal in $\mathfrak{so}(\mathfrak{v})$. Hence,

$$J_{Z_2} = \sum_{i<j,\ (i,j)\neq(1,2)} a_{ij} X_i \wedge X_j.$$

Evaluating J_{Z_2} at X gives $J_{Z_2}X = J_{Z_2}X_1 = \sum_{j=3}^q a_{1j}X_j$. Therefore $J_{Z_1}X$ and $J_{Z_2}X$ are orthogonal. ■

We have seen in Examples 2.5 and 2.6 that the free two-step nilpotent Lie algebra on three generators and certain two-step nilpotent Lie algebras with codimension one abelian ideals admit H-like inner products. In fact, these are all the H-like Lie algebras with J-rank two.

Theorem 5.2. *Suppose that $(\mathfrak{n}, Q)$ is H-like and has J-rank two.*

- *If $\cap_{Z\neq 0}E(Z)$ is nontrivial, then it is one-dimensional, and $(\mathfrak{n}, Q)$ is homothetic to $(\mathfrak{n}^p, Q_0)$, the metric Lie algebra with p-dimensional center as in Example* 2.6.
- *If $\cap_{Z\neq 0}E(Z) = \{0\}$, then $(\mathfrak{n}, Q)$ is homothetic to $(\mathfrak{f}_{3,2}, Q_0)$, the free two-step nilpotent Lie algebra endowed with the inner product Q_0 as in Example* 2.5.

Proof. We are classifying up to homothety, so we assume the nonzero elements of the spectrum are i and $-i$. We break our argument into cases depending on the value of $p = \dim \mathfrak{z}$.

First suppose that $p = 1$ and Z is a unit vector in $\mathfrak{z}$. If J_Z has any zero eigenvalues, then $\mathfrak{n}$ has an abelian factor, a contradiction. Thus, for unit Z, J_Z must be conjugate to a multiple of $X_1 \wedge X_2$. Hence $\mathfrak{n}$ is isomorphic to $\mathfrak{h}_3$. But $\mathfrak{h}_3$ is isomorphic to $\mathfrak{n}^1$, and after rescaling the metric, $(\mathfrak{n}, Q)$ is isomorphic to $(\mathfrak{n}^p, Q_0)$ with $p = 1$.

Now suppose that $p \geq 2$ and that $\cap_{Z\neq 0}E(Z)$ is one-dimensional. Let $\{Z_1, \ldots, Z_p\}$ be an orthonormal basis for $\mathfrak{z}$. Let X_1 be a unit vector spanning $\cap_{Z\neq 0}E(Z)$. Because of our assumption that nonzero eigenvalues are $\pm i$, $\{X_1, J_{Z_j}X_1\}$ is an orthonormal basis for $E(Z_j)$ for all j. By Lemma 5.1 part (3), the set $\{J_{Z_1}X_1, \ldots, J_{Z_p}X_1\}$ is independent and orthogonal. Hence $\{X_1, J_{Z_1}X_1, \ldots, J_{Z_p}X_1\}$ is an orthonormal basis for $\mathfrak{v} = \oplus_{i=1}^{p} E(Z_i)$. It is easy to check that $(\mathfrak{n}, Q)$ is isometrically isomorphic to $(\mathfrak{n}^p, Q_0)$.

Next suppose that $p \geq 2$ and $\dim \cap_{Z\neq 0}E(Z) \neq 1$. Part (1) of Lemma 5.1 implies $\cap_{Z\neq 0}E(Z) \subseteq E(Z_1)\cap E(Z_2)$ is at most one-dimensional and hence $\cap_{Z\neq 0}E(Z) = \{0\}$. If $p = 2$ then $E(Z_1) \cap E(Z_2) = \{0\}$ would contradict $(\mathfrak{n}, Q)$ having J-rank two. Hence we assume from now on that $p \geq 3$.

Let $p = 3$ and $\{Z_1, Z_2, Z_3\}$ be an orthonormal basis for $\mathfrak{z}$. Again by Lemma 5.1 we know that $E(Z_1) \cap E(Z_2), E(Z_2) \cap E(Z_3)$ and $E(Z_3) \cap E(Z_1)$ are one-dimensional. Choose unit vectors X_1, X_2, X_3 with

$$X_1 \in E(Z_1) \cap E(Z_2),\ X_2 \in E(Z_2) \cap E(Z_3), \text{ and } X_3 \in E(Z_3) \cap E(Z_1).$$

The vectors X_1, X_2, and X_3 are linearly independent because J_{Z_1}, J_{Z_2}, and J_{Z_3} are independent. Thus we have

$$[X_1, X_2] = \pm Z_2,\ [X_2, X_3] = \pm Z_3,\ [X_3, X_1] = \pm Z_1,$$

which describes $\mathfrak{f}_{3,2}$.

Now suppose $p > 3$. Let $\{Z_1, Z_2, Z_3, \ldots, Z_p\}$ be an orthonormal basis for $\mathfrak{z}$ and choose independent unit vectors X_1, X_2, X_3 with

$$X_1 \in E(Z_1) \cap E(Z_2),\ X_2 \in E(Z_2) \cap E(Z_3), \text{ and } X_3 \in E(Z_3) \cap E(Z_1)$$

as when $p = 3$. Consider $E(Z_4)$. By Lemma 5.1, $E(Z_4)$ intersects $E(Z_1)$ in a one-dimensional subspace spanned by some unit $Y_1 \in \mathfrak{v}$. Since the map $\mathbb{R}Z \mapsto E(Z)$ from the set of lines in $\text{span}\{Z_1, Z_2, Z_3\}$ to the set of two-planes in $\text{span}\{X_1, X_2, X_3\}$ is surjective and $Y_1 \in \text{span}\{X_1, X_3\}$, there exists unit Z in $\text{span}\{Z_1, Z_2, Z_3\}$ so that $E(Z) \perp Y_1$. Note that Z and Z_4 are independent.

Now consider the subspace $F = E(Z_4) \cap E(Z)$. By Lemma 5.1, it is one-dimensional. Because $J_{Z_1}, J_{Z_2}, J_{Z_3}$ and J_{Z_4} are independent, the subspace $E(Z_4) \cap \mathrm{span}\{X_1, X_2, X_3\}$ is one-dimensional. Hence it is spanned by the common element Y_1. But then

$$F = E(Z_4) \cap E(Z) \subseteq E(Z_4) \cap \mathrm{span}\{X_1, X_2, X_3\} = \mathrm{span}\{Y_1\}.$$

Since F is one-dimensional, it must be spanned by Y_1. But Y_1 is orthogonal to $E(Z)$, a contradiction. ∎

References

[1] M. D. Atkinson: *Primitive spaces of matrices of bounded rank II*, J. Austral. Math. Soc. Ser. A 34(3) (1983) 306–315.

[2] L. B. Beasley: *Spaces of matrices of equal rank*, Lin. Algebra Appl. 38 (1981) 227–237.

[3] J. Berndt, F. Tricerri, L. Vanhecke: *Generalized Heisenberg Groups and Damek-Ricci Harmonic Spaces*, Lecture Notes in Mathematics 1598, Springer, Berlin (1995).

[4] A. Boralevi, E. Mezzetti: *Planes of matrices of constant rank and globally generated vector bundles*, Ann. Inst. Fourier (Grenoble) 65(5) (2015) 2069–2089.

[5] P. Ciatti: *Scalar products on Clifford modules and pseudo-H-type Lie algebras*, Ann. Mat. Pura Appl. 178 (2000) 1–31.

[6] R. DeCoste, L. DeMeyer, M. Mainkar: *Graphs and metric 2-step nilpotent Lie algebras*, Adv. Geom. 18 (2018) 265–284.

[7] R. C. DeCoste, L. DeMeyer, M. B. Mast: *Characterizations of Heisenberg-like Lie algebras*, J. Lie Theory 21(3) (2011) 711–727.

[8] E. Damek, F. Ricci: *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. 27(1) (1992) 139–142.

[9] P. Eberlein: *Geometry of 2-step nilpotent groups with a left invariant metric*, Ann. Sci. École Norm. Sup. (4) 27(5) (1994) 611–660.

[10] P. Eberlein: *Geometry of 2-step nilpotent groups with a left invariant metric II*, Trans. Amer. Math. Soc. 343(2) (1994) 805–828.

[11] P. Eberlein: *The moduli space of 2-step nilpotent Lie algebras of type (p, q)*, in: *Explorations in Complex and Riemannian Geometry*, J. Bland, K.-T. Kim, and S. G. Krantz (eds.), Contemporary Mathematics 332, American Mathematical Society, Providence (2003) 37–72.

[12] P. Eberlein: *Riemannian submersions and lattices in 2-step nilpotent Lie groups*, Comm. Anal. Geom. 11(3) (2003) 441–488.

[13] P. Eberlein: *Geometry of 2-step nilpotent Lie groups*, in: *Modern Dynamical Systems and Applications*, M. Brin, B. Hasselblatt, and Y. Pesin (eds.), Cambridge University Press, Cambridge (2004) 67–101.

[14] D. Eisenbud, J. Harris: *Vector spaces of matrices of low rank*, Adv. in Math. 70(2) (1988) 135–155.

[15] P. Ellia, P. Menegatti: *Spaces of matrices of constant rank and uniform vector bundles*, Linear Algebra Appl. 507 (2016) 474–485.

[16] M. L. Fania, E. Mezzetti: *Vector spaces of skew-symmetric matrices of constant rank*, Linear Algebra Appl. 434(12) (2011) 2383–2403.

[17] H. Flanders: *On spaces of linear transformations with bounded rank*, J. London Math. Soc. 37 (1962) 10–16.

[18] M. Godoy Molina, A. Korolko, I. Markina: *Sub-semi-Riemannian geometry of general H-type groups*, Bull. Sci. Math. 137(6) (2013) 805–833.

[19] C. S. Gordon, M. M. Kerr: *New homogeneous Einstein metrics of negative Ricci curvature*, Ann. Global Anal. Geom. 19(1) (2001) 75–101.

[20] C. S. Gordon, E. N. Wilson: *Continuous families of isospectral Riemannian metrics which are not locally isometric*, J. Differential Geom. 47(3) (1997) 504–529.

[21] R. Gornet, M. B. Mast: *The length spectrum of Riemannian two-step nilmanifolds*, Ann. Sci. École Norm. Sup. (4) 33(2) (2000) 181–209.

[22] M. Goze, Y. Haraguchi: *Sur les r-systèmes de contact*, C. R. Acad. Sci. Paris Sér. I Math. 294(2) (1982) 95–97.

[23] A. Hurwitz: *Über die Komposition der quadratischen Formen*, Math. Ann. 88(1-2) (1922) 1–25.

[24] B. Ilic, J. M. Landsberg: *On symmetric degeneracy loci, spaces of symmetric matrices of constant rank and dual varieties*, Math. Ann. 314(1) (1999) 159–174.

[25] M. Jablonski: *Moduli of Einstein and non-Einstein nilradicals*, Geom. Dedicata 152(1) (2011) 63–84.

[26] C. Jang, T. Lee, K. Park: *Conjugate loci of 2-step nilpotent Lie groups satisfying $J_z^2 = \langle Sz, z\rangle A$*, J. Korean Math. Soc. 45(6) (2008) 1705–1723.

[27] A. Kaplan: *Fundamental solutions for a class of hypoelliptic PDE generated by compositions of quadratic forms*, Trans. Amer. Math. Soc. 258(1) (1980) 147–153.

[28] A. Kaplan, A. Tiraboschi: *Automorphisms of non-singular nilpotent Lie algebras*, J. Lie Theory 23(4) (2013) 1085–1100.

[29] J. Lauret: *Modified H-type groups and symmetric-like Riemannian spaces*, Differential Geom. Appl. 10(2) (1999) 121–143.

[30] J. Lauret, D. Oscari: *On non-singular 2-step nilpotent Lie algebras*, Math. Res. Lett. 21(3) (2014) 553–583.

[31] L. Manivel, E. Mezzetti: *On linear spaces of skew-symmetric matrices of constant rank*, Manuscripta Math. 117(3) (2005) 319–331.

[32] G. Métivier: *Hypoellipticité analytique sur des groupes nilpotents de rang* 2, Duke Math. J. 47(1) (1980) 195–221.

[33] D. Müller, A. Seeger: *Singular spherical maximal operators on a class of two step nilpotent Lie groups*, Israel J. Math. 141 (2004) 315–340.

[34] T. L. Payne: *The existence of soliton metrics for nilpotent Lie groups*, Geom. Dedicata 145 (2010) 71–88.

[35] J. Radon: *Lineare Scharen orthogonaler Matrizen*, Abh. Math. Sem. Univ. Hamburg 1(1) (1922) 1–14.

[36] M. Schroeder: *Metric Nilpotent Lie Algebras Defined by Graphs*, Master's thesis, Idaho State University, Pocatello (2015).

[37] M. Skrzyński: *On the linear capacity of algebraic cones*, Math. Bohem. 127(3) (2002) 453–462.

[38] R. Westwick: *Spaces of matrices of fixed rank*, Linear and Multilinear Algebra 20(2) (1987) 171–174.

[39] E. N. Wilson: *Isometry groups on homogeneous nilmanifolds*, Geom. Dedicata 12(3) (1982) 337–346.

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