Journal of Lie Theory
Volume **30** (2020) 997–1026

Transfer of Characters in the Theta Correspondence with One Compact Member

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Communicated by M. Pevzner

Abstract. For an irreducible dual pair $(\mathrm{G}, \mathrm{G}') \subseteq \mathrm{Sp}(W)$ with one member compact and two representations $\Pi \leftrightarrow \Pi'$ appearing in the Howe duality, we give an expression of the character $\Theta_{\Pi'}$ of Π' via the character of Π. We compute the value of $\Theta_{\Pi'}$ on the maximal compact torus T' of G' for the dual pair $(\mathrm{G} = \mathrm{U}(n, \mathbb{C}), \mathrm{G}' = \mathrm{U}(p, q, \mathbb{C}))$, which are explicit in low dimensions. For $(\mathrm{G} = \mathrm{U}(1, \mathbb{C}), \mathrm{G}' = \mathrm{U}(1, 1, \mathbb{C}))$, we determine the value of the character on both Cartan subgroups of G'.

Mathematics Subject Classification: 22E45, 22E46, 22E30.

Key Words: Howe correspondence, characters, oscillator semigroup, reductive dual pairs.

1. Introduction

For a finite dimensional representation (Π, V) of a group G, the character of Π, denoted by Θ_Π, is defined by:

$$\Theta_\Pi : \mathrm{G} \ni g \to \mathrm{tr}(\Pi(g)) \in \mathbb{C}.$$

In general, to determine precisely the character of such representations is a hard problem, but in few cases, in particular for a compact connected group, the formula is explicit. Indeed, let G be a compact connected Lie group, T a Cartan subgroup of G, $\mathfrak{g}$ and $\mathfrak{t}$ the Lie algebras of G and T respectively, $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{t}_\mathbb{C}$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$, $\Phi(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ (resp. $\Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$) be the set of roots (resp. positive roots) of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{t}_\mathbb{C}$ and $\mathscr{W} = \mathscr{W}(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ be the corresponding Weyl group. According to H. Weyl, all the irreducible representations (Π, V) of G are finite dimensional and parametrised by a linear form λ on $\mathfrak{t}_\mathbb{C}$: this linear form is called the highest weight of Π. Moreover, the character of Π is given by the following formula:

$$\Theta_\Pi(\exp(x)) = \sum_{\omega \in \mathscr{W}} \mathrm{sgn}(\omega) \frac{e^{\omega(\lambda+\rho)(x)}}{\prod\limits_{\alpha \in \Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})} (e^{\frac{\alpha(x)}{2}} - e^{-\frac{\alpha(x)}{2}})} \qquad (x \in \mathfrak{t}^{\mathrm{reg}}), \tag{1}$$

where ρ is a linear form on $\mathfrak{t}_\mathbb{C}$ given by $\rho = \frac{1}{2} \sum\limits_{\alpha \in \Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})} \alpha$.

In the 50's, for a real reductive Lie group G, Harish-Chandra extended the concept of characters for a certain class of representation of G called quasi-simple (see [6,

ISSN 0949–5932 / \$2.50

Section 10]). More precisely, for such a representation $(\Pi, \mathscr{H})$ of G, he proved (see [7, Section 3]) that the map:

$$\Theta_{\Pi} : \mathscr{C}_c^{\infty}(\mathrm{G}) \ni \Psi \to \mathrm{tr}(\Pi(\Psi)) = \mathrm{tr} \int_{\mathrm{G}} \Psi(g)\Pi(g)dg \in \mathbb{C}$$

is well-defined and continuous. The map Θ_{Π} is called the global character of Π. Moreover, he proved (see [8, Theorem 2]) that this distribution is given by a locally integrable function on G (still denoted by Θ_{Π}) which is analytic on the set of regular points of G, i.e.

$$\Theta_{\Pi}(\Psi) = \int_{\mathrm{G}} \Theta_{\Pi}(g)\Psi(g)dg \qquad (\Psi \in \mathscr{C}_c^{\infty}(\mathrm{G})).$$

An explicit formula for the function Θ_{Π} on $\mathrm{G}^{\mathrm{reg}}$ (regular points of G) is hard to get. We recall briefly some well known facts on those characters. Let G be reductive group, K be a maximal compact subgroup of G such that $\mathrm{rk}(\mathrm{K}) = \mathrm{rk}(\mathrm{G})$ and T be a Cartan subgroup of K (which is also a Cartan subgroup for G by our assumptions on ranks). As before, we denote by $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{t}$ their Lie algebras and by $\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$ their complexifications.

1. If $(\Pi, \mathscr{H})$ is a discrete series representation of G of Harish-Chandra parameter $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$, the character Θ_{Π} of Π is given by (see [9]):

$$\Theta_{\Pi}(\exp(x)) = (-1)^{\frac{\dim(\mathrm{G}\,/\,\mathrm{K})}{2}} \sum_{\omega \in \mathscr{W}(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})} \varepsilon(\omega) \frac{e^{\omega(\lambda(x))}}{\prod\limits_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})} (e^{\frac{\alpha(x)}{2}} - e^{-\frac{\alpha(x)}{2}})} \quad (x \in \mathfrak{t}^{\mathrm{reg}}),$$

   where $\mathscr{W}(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is the compact Weyl group of K.

2. Assume that (G, K) is a symmetric pair of hermitian type. If $(\Pi, \mathscr{H}_{\Pi})$ is an irreducible unitary representation of G of highest weight $\lambda - \rho$, the character Θ_{Π} of Π is given by (see [4, Corollary 2.3]):

$$\prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \setminus \Phi^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})} (e^{\frac{\alpha(x)}{2}} - e^{-\frac{\alpha(x)}{2}}) \Theta_{\Pi}(\exp(x)) = \sum_{\omega \in \mathscr{W}_{\lambda}^{\mathfrak{k}}} (-1)^{l_{\lambda}(\omega)} \Theta(\mathrm{K}, \Lambda(\omega, \lambda))(\exp(x)),$$

   where $x \in \mathfrak{t}^{\mathrm{reg}}$ and $\mathscr{W}_{\lambda}^{\mathfrak{k}}$ is defined in [4, Def. 2.1], and $\Theta(\mathrm{K}, \Lambda(\omega, \lambda))(\exp(x))$ is the character of a K-representation of highest weight $\Lambda(\omega, \lambda)$, where $\Lambda(\omega, \lambda)$ is defined in [4, Corollary 2.3].

We also mention a conjecture of A. Kirillov (see [18]), which should hold for a really general Lie group, and a paper of H. Hecht ([10])

Let $(W, \langle \cdot, \cdot \rangle)$ be a real symplectic space, $\mathrm{Sp}(W)$ its corresponding group of isometries, $\widetilde{\mathrm{Sp}(W)}$ the metaplectic group and $(\omega, \mathscr{H})$ the corresponding metaplectic representation (see Section 2). For a subgroup $\mathrm{H} \in \mathrm{Sp}(\mathrm{W})$, we denote by $\widetilde{\mathrm{H}}$ its preimage in $\widetilde{\mathrm{Sp}(W)}$ and by $\mathscr{R}(\widetilde{\mathrm{H}}, \omega)$ the set of equivalence classes of irreducible admissible representations of $\widetilde{\mathrm{H}}$ which are infinitesimally equivalent to a quotient of ω^{∞}. For an irreducible reductive dual pair $(\mathrm{G}, \mathrm{G}')$ in $\mathrm{Sp}(W)$, R. Howe proved (see [16]) that there exists a bijection between $\mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ and $\mathscr{R}(\widetilde{\mathrm{G}'}, \omega)$.

In this paper, we assume that G is compact. In that case, it turns out that the situation is easier: the representations Π and Π' are just subrepresentations of ω^{∞}. Our goal here is to determine the character $\Theta_{\Pi'}$ using the character Θ_{Π} of Π. By projecting onto the Π-isotypic component in $\mathscr{H}^{\infty}$, we get in Proposition 2.3 that for all $\Psi \in \mathscr{C}_c^{\infty}(\widetilde{\mathrm{G}})$:

$$\Theta_{\Pi}(\Psi) = \mathrm{tr} \int_{\widetilde{\mathrm{G}'}} \int_{\widetilde{\mathrm{G}}} \overline{\Theta_{\Pi}(\widetilde{g})} \Psi(\widetilde{g'}) \omega^{\infty}(\widetilde{g}\widetilde{g'}) d\widetilde{g} d\widetilde{g'}.$$

So, formally, we have: $\Theta_{\Pi'}(\widetilde{g'}) = \int_{\widetilde{\mathrm{G}}} \overline{\Theta_{\Pi}(\widetilde{g})} \Theta(\widetilde{g}\widetilde{g'}) d\widetilde{g} \qquad (\widetilde{g'} \in \widetilde{\mathrm{G}}'^{\mathrm{reg}}),$

where the last equality is an equality as distributions on $\mathscr{C}_c^{\infty}(\widetilde{\mathrm{G}})$ and where Θ is the character of ω (see Section 2). To avoid the problem of non-continuity of Θ, we use the Oscillator semigroup introduced by Howe (see [15] or Section 3) and denoted by $\widetilde{\mathrm{Sp}(W_{\mathbb{C}})^{++}}$. The extension of Θ on $\widetilde{\mathrm{Sp}(W_{\mathbb{C}})^{++}}$ is holomorphic and $\widetilde{\mathrm{Sp}(W)} \cdot \widetilde{\mathrm{Sp}(W_{\mathbb{C}})^{++}} \subseteq \widetilde{\mathrm{Sp}(W_{\mathbb{C}})^{++}}$. In particular, we get for $\widetilde{g'} \in \widetilde{\mathrm{G}'}^{\mathrm{reg}}$ that:

$$\Theta_{\Pi'}(\widetilde{g'}) = \lim_{\substack{\widetilde{p} \to 1 \\ \widetilde{p} \in \widetilde{\mathrm{G}'^{++}}}} \int_{\widetilde{\mathrm{G}}} \overline{\Theta_{\Pi}(\widetilde{g})} \Theta(\widetilde{g}\widetilde{g'}\widetilde{p}) d\widetilde{g},$$

where $\mathrm{G}'^{++} = \mathrm{G}'_{\mathbb{C}} \cap \mathrm{Sp}(W_{\mathbb{C}})^{++}$. This is Theorem 4.3. The character of the representation Π can be obtained using Weyl's character formula (see equation (1)) together with a paper of M. Kashiwara and M. Vergne [17], where they give explicitly the weights of the representations appearing in the correspondence. Moreover, using [32] or [1], we get an explicit formula for the restriction of the character Θ on $\mathrm{T} \cdot \mathrm{T}'^{++}$, where T (resp. T') is a compact torus of G (resp. G') and $\mathrm{T}'^{++} = \mathrm{T}'_{\mathbb{C}} \cap \mathrm{Sp}(W_{\mathbb{C}})^{++}$ (see Corollary 5.3).

We focus our attention on the case $(\mathrm{G} = \mathrm{U}(n, \mathbb{C}), \mathrm{G}' = \mathrm{U}(p, q, \mathbb{C}))$. We highlight the result for $n = 1$ (Proposition 6.4):

$$\Theta_{\Pi'_k}(\widetilde{t'}) = \begin{cases} \prod_{i=1}^{p+q} t_i^{\frac{1}{2}} \sum_{h=1}^{p} \dfrac{t_h^{p-(k+1)}}{\prod_{h \neq j} (t_h - t_j)} & \text{if } k \leqslant p-1 \\ -\prod_{i=1}^{p+q} t_i^{\frac{1}{2}} \sum_{h=p+1}^{p+q} \dfrac{t_h^{p-(k+1)}}{\prod_{h \neq j} (t_h - t_j)} & \text{otherwise} \end{cases}$$

In Section 7, we work with the pair $(\mathrm{G} = \mathrm{U}(1, \mathbb{C}), \mathrm{G}' = \mathrm{U}(1, 1, \mathbb{C}))$ and determine the value of the character $\Theta_{\Pi'}$ on the non-compact torus of $\mathrm{U}(1, 1, \mathbb{C})$ (unique up to conjugation, see Section 7). A result of Jian-Shu Li [21], it follows that Π' is a discrete series representation. In particular, the value of $\Theta_{\Pi'}$ could also be obtained using [10, Theorem 2.17], but we do not use of this result in our computations of $\Theta_{\Pi'}$.

In Section 8, we recall a conjecture of T. Przebinda (see [27]) concerning the transfer of characters for a general dual pair $(\mathrm{G}, \mathrm{G}')$ and discuss in few words an ongoing project linked with recents works of T. Przebinda [28]. We refer the readers to [22] and [5] on transfer of other invariants of representations such as associated cycles and generalized Whittaker models.

Acknowledgements. A part of this paper was done during my thesis at the University of Lorraine under the supervision of Angela Pasquale (University of Lorraine) and Tomasz Przebinda (University of Oklahoma). I would like to thank them for the ideas and time they shared with me. I finished this paper during my stay at the National University of Singapore, as a Research Fellow under the supervision of Hung Yean Loke. I am supported by the grant R-146-000-261-114 (Reductive dual pair correspondences and supercuspidal representations). I would also like to thank the anonymous referee for remarks and suggestions improving the article.

2. Metaplectic representation and Howe's duality theorem

We recall the construction of the metaplectic representation using the so-called Stone-Von Neumann theorem. Brieffly, let $(W, \langle\cdot,\cdot\rangle)$ be a real symplectic space and $\mathrm{H(W)}$ the space $W \oplus \mathbb{R}$ with group multiplication:

$$(w_1, \lambda_1).(w_2, \lambda_2) = (w_1 + w_2, \lambda_1 + \lambda_2 + \tfrac{1}{2}\langle w_1, w_2\rangle), \qquad (w_1, w_2 \in W, \lambda_1, \lambda_2 \in \mathbb{R}).$$

Clearly, $\mathscr{Z}(\mathrm{H(W)}) = \{(0,\lambda), \lambda \in \mathbb{R}\} \approx \mathbb{R}$. According to the Stone-Von Neumann theorem, for every non trivial character Ψ of $\mathscr{Z}(\mathrm{H(W)})$, there exists, up to equivalence, a unique irreducible unitary representation of $\mathrm{H(W)}$ with central character Ψ. The group of isometries of $(W, \langle\cdot,\cdot\rangle)$, denoted by $\mathrm{Sp}(W)$, acts naturally on $\mathrm{H(W)}$ by

$$g.(w,\lambda) = (g(w), \lambda) \qquad (g \in \mathrm{Sp}(W), (w,\lambda) \in \mathrm{H(W)}).$$

By fixing an irreducible unitary representation $(\Pi_\lambda, \mathscr{H}_\lambda)$ of $\mathrm{H(W)}$ with central character $\Psi_\lambda, \lambda \in \mathbb{R}$, we get that the map:

$$\Pi_{\lambda,g}(h) = \Pi_\lambda(g^{-1}(h)) \qquad (g \in \mathrm{Sp}(W), h \in \mathrm{H(W)}),$$

is an irreducible unitary representation of $\mathrm{H(W)}$ with central character Ψ_λ, and then, by application of the Stone- Von Neumann theorem, there exists an operator $\omega_\lambda(g)$ such that:

$$\omega_\lambda(g)\Pi_\lambda(h)\omega_\lambda(g)^{-1} = \Pi_\lambda(g^{-1}(h)).$$

In particular, we get a projective representation of $\mathrm{Sp}(W)$. One can prove that we get a representation $(\omega, \mathscr{H})$ of a non-trivial double cover of $\mathrm{Sp}(W)$, that we will denote by $\widetilde{\mathrm{Sp}(W)}$ (see [35]).

In this section, we give an explicit realisation of the metaplectic representation (using a paper of A-M. Aubert and T. Przebinda [1]). In particular, we get a formula for the character of this representation (one can also check the paper of T. Thomas [32]).

Let χ be the character of $\mathbb{R}$ given by $\chi(r) = e^{2i\pi r}$. We denote by $\mathfrak{sp}(W)$ the Lie algebra of $\mathrm{Sp}(W)$, i.e.

$$\mathfrak{sp}(W) = \{X \in \mathrm{End}(W), \langle X(w), w'\rangle + \langle w, X(w')\rangle = 0, (\forall w, w' \in W)\}.$$

Let J be an element of $\mathfrak{sp}(W)$ satisfying $J^2 = -\mathrm{Id}$ and such that the symmetric bilinear form (w, w') defined by $(w,w') = \langle J(w), w'\rangle$ is positive definite. For all $g \in \mathrm{Sp}(W)$, we denote by J_g the automorphism of W given by $J_g = J^{-1}(g-1)$. One can check easily that the adjoint J_g^* of J_g with respect to the form $(\cdot,\cdot)$ is given by $J_g^* = Jg^{-1}(1-g)$ and that the restriction of J_g to $J_g(W)$ is well defined and invertible. The metaplectic group is defined as:

$$\widetilde{\mathrm{Sp}(W)} = \left\{\widetilde{g} = (g,\xi) \in \mathrm{Sp}(W) \times \mathbb{C}^*, \xi^2 = i^{\dim_{\mathbb{R}}(g-1)W} \det(\mathrm{J_g})^{-1}_{\mathrm{J_g(W)}}\right\}. \tag{2}$$

The covering map $\pi : \widetilde{\mathrm{Sp}(W)} \ni (g,\xi) \to g \in \mathrm{Sp}(W)$ is the first projection and the multiplication law is defined by:

$$(g_1,\xi_1).(g_2,\xi_2) = (g_1g_2, \xi_1\xi_2 C(g_1,g_2)),$$

where the cocycle $C' : \mathrm{Sp}(W)\times\mathrm{Sp}(W) \to \mathbb{C}$ is defined in [1, Proposition 4.13]. Using [28, Equation (3)], we get that the absolute value of C' satisfies, for every $g_1, g_2 \in \mathrm{Sp}(W)$, the following equations:

$$|C'(g_1,g_2)| = \sqrt{\left|\frac{\det(\mathrm{J_{g_1}})_{\mathrm{J_{g_1}(W)}}\det(\mathrm{J_{g_2}})_{\mathrm{J_{g_2}(W)}}}{\det(\mathrm{J_{g_1g_2}})_{\mathrm{J_{g_1g_2}(W)}}}\right|}, \qquad \frac{C'(g_1,g_2)}{|C'(g_1,g_2)|} = \chi\left(\frac{1}{8}\,\mathrm{sgn}(q_{g_1,g_2})\right),$$

where $c(g), g\in\mathrm{Sp}(W)$, is the Cayley transform of g defined on the space $(g-1)W$ by

$$c(g) : (g-1)W \ni (g-1)w \to (g+1)w + \mathrm{Ker}(g-1) \in W/\mathrm{Ker}(g-1),$$

(see [1, Section 2.3] for more details) and where $\mathrm{sgn}(q_{g_1,g_2})$ is the signature of the form q_{g_1,g_2} defined by:

$$q_{g_1,g_2}(u,v) = \tfrac{1}{2}\left(\langle c(g_1)u,v\rangle + \langle c(g_2)u,v\rangle\right) \qquad (u,v \in (g_1-1)W \cap (g_2-1)W). \quad (3)$$

To simplify the notations, for all $g \in \mathrm{Sp}(W)$, we denote by $\chi_{c(g)}$ the form on $(g-1)W$ given by $\chi_{c(g)}(u) = \chi\left(\frac{1}{8}\langle c(g)u,u\rangle\right)$.

We now construct the metaplectic representation. We denote by $\mathrm{S(W)}$ the Schwarz space corresponding to W and by $t : \mathrm{Sp}(W) \to \mathrm{S^*(W)}$, $\Theta : \widetilde{\mathrm{Sp}(W)} \to \mathbb{C}^*$ and $T : \widetilde{\mathrm{Sp}(W)} \to \mathrm{S^*(W)}$ defined by

$$t(g) = \chi_{c(g)}\mu_{(g-1)W} \qquad \Theta(\widetilde{g}) = \xi, \qquad T(\widetilde{g}) = \Theta(\widetilde{g})t(g), \qquad (\widetilde{g} = (g,\xi)),$$

where $\mu_{(g-1)W} \in \mathrm{S^*(W)}$ is the Lebesgue measure on the space $(g-1)W$ such that the volume with respect to $(\cdot,\cdot)$ of the corresponding unit cube is 1.

We now fix a complete polarisation $W = X \oplus Y$, i.e. a direct sum of two maximal isotropic subspaces of W. The Weyl transform $\mathscr{K} : \mathrm{S(W)} \to \mathrm{S(X\times X)}$ given by:

$$\mathscr{K}(\eta)(x,x') = \int_Y \eta(x-x'+y)\chi\left(\frac{1}{2}\langle y, x+x'\rangle\right)dy$$

is an isomorphism and the extension of $\mathscr{K}$ to the corresponding space of tempered distributions $\mathscr{K} : \mathrm{S^*(W)} \to \mathrm{S^*(X\times X)}$ is still an isomorphism. Similarly, the map $\mathrm{Op} : \mathrm{S(X\times X)} \to \mathrm{Hom(S(X), S^*(X))}$ given by:

$$\mathrm{Op}(K)v(x) = \int_X K(x,x')v(x')dx'$$

extends to isomorphism $\mathrm{Op} : \mathrm{S^*(X\times X)} \to \mathrm{Hom(S(X), S^*(X))}$. According to [1, Section 4.8], for every $\Psi \in \mathrm{L^2(W)}$, $\mathrm{Op}\circ\mathscr{K}(\Psi)$ is an Hilbert-Schmidt operator on $\mathrm{L^2(X)}$ and the map

$$\mathrm{Op}\circ\mathscr{K} : \mathrm{L^2(W)} \to \mathrm{HS(L^2(X))}$$

is an isometry. We denote by $\omega : \widetilde{\mathrm{Sp}(W)} \to \mathrm{U(L^2(X))}$, defined by $\omega = \mathrm{Op}\circ\mathscr{K}\circ T$, a unitary representation of $\widetilde{\mathrm{Sp}(W)}$, called *metaplectic representation*. Moreover, the function Θ defined previously is the character of $(\omega, \mathrm{L^2(X)})$ and the space of smooth vectors is $\mathrm{S(X)}$, the Schwartz space of X.

Remark 2.1. We denote by $\mathrm{Sp}(W)^c$ the subspace of $\mathrm{Sp}(W)$ defined by $\mathrm{Sp}(W)^c = \{g \in \mathrm{Sp}(W), \det(\mathrm{g}-1) \neq 0\}$. We denote by $\widetilde{\mathrm{Sp}(W)}^c$ the preimage of $\mathrm{Sp}(W)^c$ in $\widetilde{\mathrm{Sp}(W)}$. For every $\widetilde{g} = (g, \xi) \in \widetilde{\mathrm{Sp}(W)}^c$, we get:

$$\Theta(\widetilde{g}) = \left(i^{\dim_{\mathbb{R}} W} \det(\mathrm{J}(\mathrm{g}-1))^{-1}\right)^{\frac{1}{2}} = \det(\mathrm{i}(\mathrm{g}-1))^{-\frac{1}{2}}.$$

A dual pair in $\mathrm{Sp}(W)$ is a pair of subgroups $(\mathrm{G}, \mathrm{G}')$ of $\mathrm{Sp}(W)$ which are mutually centralizer in $\mathrm{Sp}(W)$, i.e. $C_{\mathrm{Sp}(W)}(\mathrm{G}) = \mathrm{G}'$ and $C_{\mathrm{Sp}(W)}(\mathrm{G}') = \mathrm{G}$. The dual pair is said to be reductive if the action of G and G′ on W is reductive. If we have a decomposition of W as an orthogonal sum $W = W_1 \oplus W_2$ where W_1 and W_2 are $\mathrm{G} \cdot \mathrm{G}'$-invariant, then $(\mathrm{G}_{|W_i}, \mathrm{G}'_{|W_i})$ is a dual pair in $\mathrm{Sp}(W_{|W_i}), i = 1, 2$. If we cannot find such a decomposition, the dual pair is said to be irreducible.
The irreducible reductive dual pairs in the symplectic group had been classified by R. Howe [13]. In this paper, we assume that the group G is compact. In this case, $(\mathrm{G}, \mathrm{G}')$ is one of the following dual pairs

1. $(\mathrm{U}(n, \mathbb{C}), \mathrm{U}(p, q, \mathbb{C})) \subseteq \mathrm{Sp}(2n(p+q), \mathbb{R})$,
2. $((n, \mathbb{R}), \mathrm{Sp}(2m, \mathbb{R})) \subseteq \mathrm{Sp}(2nm, \mathbb{R})$,
3. $(\mathrm{U}(n, \mathbb{H}), {}^*(m, \mathbb{H})) \subseteq \mathrm{Sp}(4nm, \mathbb{R})$.

For the computations in the Section 6, we will focus our attention on the first one.

Remark 2.2. (1) For a dual pair $(\mathrm{G}, \mathrm{G}')$ in $\mathrm{Sp}(W)$, we denote by $\widetilde{\mathrm{G}} = \pi^{-1}(\mathrm{G})$ and $\widetilde{\mathrm{G}}'$ the preimages of G and G′ in $\widetilde{\mathrm{Sp}(W)}$. In [13], R. Howe proved that $(\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}')$ is a dual pair in $\widetilde{\mathrm{Sp}(W)}$. With the precise definition we gave for $\widetilde{\mathrm{Sp}(W)}$ in equation (2), we can see that easily. Indeed, we need to prove that for all $g \in \mathrm{G}$ and $g' \in \mathrm{G}'$, we have $C(g, g') = C(g', g)$. Obviously, $|C(g, g')| = |C(g', g)|$, and because $q_{g,g'} = q_{g',g}$, the result follows.
(2) If the group G is compact, then $\widetilde{\mathrm{G}}$ is also compact.

From now on, we assume that $(\mathrm{G}, \mathrm{G}')$ is an irreducible reductive dual pair in $\mathrm{Sp}(W)$ with G compact. The Howe duality theorem can be stated in a easier way when we assume that one member is compact. As before, we consider a complete polarisation of W of the form $X \oplus Y$ and we realise the metaplectic representation ω on the space $\mathrm{L}^2(\mathrm{X})$. The space of smooth vector is the Schwartz space S(X) and under the action of $\widetilde{\mathrm{G}}$, we get the following decomposition:

$$\mathrm{S}(\mathrm{X}) = \bigoplus_{(\Pi, \mathrm{V}_\Pi) \in \widehat{\widetilde{\mathrm{G}}}_\omega} \mathrm{V}(\Pi),$$

where $\widehat{\widetilde{\mathrm{G}}}_\omega$ is the set of irreducible unitary representations of $\widetilde{\mathrm{G}}$ such that we have $\mathrm{Hom}_{\widetilde{\mathrm{G}}}(\Pi, \omega^\infty) \neq \{0\}$ and $V(\Pi)$ is the Π-isotypic component in S(X), i.e. the closure with respect to the topology on S(X) of the space $\{T(V_\Pi), T \in \mathrm{Hom}_{\widetilde{\mathrm{G}}}(\Pi, \omega^\infty)\}$.
Because $\widetilde{\mathrm{G}}'$ commute with $\widetilde{\mathrm{G}}$, the group $\widetilde{\mathrm{G}}'$ acts on $V(\Pi)$ for every $\Pi \in \widehat{\widetilde{\mathrm{G}}}_\omega$, and as a $\widetilde{\mathrm{G}} \times \widetilde{\mathrm{G}}'$-module, we get the following decomposition:

$$\mathrm{S}(\mathrm{X}) = \bigoplus_{(\Pi, \mathrm{V}_\Pi) \in \widehat{\widetilde{\mathrm{G}}}_\omega} \Pi \otimes \Pi',$$

where Π' is an irreducible unitary representation of $\widetilde{\mathrm{G}}'$. The map

$$\theta : \widehat{\widetilde{\mathrm{G}}}_\omega \ni \Pi \to \Pi' = \theta(\Pi) \to \widehat{\widetilde{\mathrm{G}}'}_\omega$$

is one-to-one and usually called Howe's correspondence.

Let $(\Pi, V_\Pi) \in \widehat{\widetilde{\mathrm{G}}}_\omega$ and $\Pi' = \theta(\Pi)$ the corresponding representation of $\widetilde{\mathrm{G}}'$. We denote by $\mathscr{P}_\Pi : \mathrm{S(X)} \to \mathrm{V}(\Pi)$ the projection onto the Π-isotypic component. According to [34, Section 1.4], the map $\mathscr{P}_\Pi$ is given by the formula:

$$\mathscr{P}_\Pi = d_\Pi \int_{\widetilde{\mathrm{G}}} \overline{\Theta_\Pi(\widetilde{g})} \omega^\infty(\widetilde{g}) d\widetilde{g} = \omega^\infty(d_\Pi \overline{\Theta_\Pi}),$$

where $d_\Pi = \dim_{\mathbb{C}}(V_\Pi)$ is the dimension of the representation Π. We get the following result for the global character of Π'.

Proposition 2.3. *For every compactly supported function $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}}')$, we get:*

$$\Theta_{\Pi'}(\Psi) = \mathrm{tr} \int_{\widetilde{\mathrm{G}}'} \left(\int_{\widetilde{\mathrm{G}}} \overline{\Theta_\Pi(\widetilde{g})} \omega^\infty(\widetilde{g}\widetilde{g}') d\widetilde{g} \right) \Psi(\widetilde{g}') d\widetilde{g}'.$$

Proof. For such a function Ψ, we have:

$$\mathrm{tr}(\mathscr{P}_\Pi \omega(\Psi)) = \mathrm{tr}(\mathrm{Id}_{V_\Pi} \otimes \Pi'(\Psi)) = d_\Pi \Theta_{\Pi'}(\Psi),$$

and then,

$$\begin{aligned}\Theta_{\Pi'}(\Psi) &= \frac{1}{d_\Pi} \mathrm{tr}(\mathscr{P}_\Pi \omega^\infty(\Psi)) = \mathrm{tr} \int_{\widetilde{\mathrm{G}}'} \Psi(\widetilde{g}') \mathscr{P}_\Pi \omega^\infty(\widetilde{g}') d\widetilde{g}' \\ &= \mathrm{tr} \int_{\widetilde{\mathrm{G}}'} \left(\int_{\widetilde{\mathrm{G}}} \overline{\Theta_\Pi(\widetilde{g})} \omega^\infty(\widetilde{g}\widetilde{g}') d\widetilde{g} \right) \Psi(\widetilde{g}') d\widetilde{g}'.\end{aligned}$$

■

Using that Θ is the character of ω, we get formally:

$$\Theta_{\Pi'}(\widetilde{g}) = \int_{\widetilde{\mathrm{G}}} \overline{\Theta_\Pi(\widetilde{g})} \Theta(\widetilde{g}\widetilde{g}') d\widetilde{g} \qquad (\widetilde{g}' \in \widetilde{\mathrm{G}}').$$

Because the character Θ is not continuous, the second member of the previous equation could not make sense. To avoid this problem, we use the Oscillator semigroup introduced by R. Howe (see [15]).

3. Howe's oscillator semigroup

Let $(W, \langle \cdot, \cdot \rangle)$ be a (finite dimensional) real symplectic vector space and $(W_{\mathbb{C}}, \langle \cdot, \cdot \rangle)$ its complexification. For every $w \in W_{\mathbb{C}}$, we consider the decomposition $w = a + ib$, $a, b \in W$ and we denote by $\overline{w} = a - ib$ the conjugate with respect to the decomposition $W_{\mathbb{C}} = W \oplus iW$. By extension, we get a symplectic form $\langle \cdot, \cdot \rangle$ on $W_{\mathbb{C}}$.

Lemma 3.1. *The form $\mathrm{H} : \mathrm{W}_{\mathbb{C}} \times \mathrm{W}_{\mathbb{C}} \to \mathbb{C}$ defined by*

$$\mathrm{H}(\mathrm{w}, \mathrm{w}') = \mathrm{i}\langle \mathrm{w}, \overline{\mathrm{w}}' \rangle \tag{4}$$

is hermitian.

Proof. Straightforward verification. ∎

We define now the subset $\mathrm{Sp}(W_{\mathbb{C}})^{++}$ of $\mathrm{Sp}(W_{\mathbb{C}})$ by:

$$\mathrm{Sp}(W_{\mathbb{C}})^{++} = \{g \in \mathrm{Sp}(W_{\mathbb{C}})\,;\, \mathrm{H}(\mathrm{w},\mathrm{w}) > \mathrm{H}(\mathrm{g}(\mathrm{w}),\mathrm{g}(\mathrm{w})), (\forall \mathrm{w} \in \mathrm{W}_{\mathbb{C}} \setminus \{0\})\}\,. \qquad (5)$$

Similarly, we denote by $\mathfrak{sp}(W_{\mathbb{C}})^{++}$ the subset of $\mathrm{End}(W_{\mathbb{C}})$ given by:

$$\mathfrak{sp}(W_{\mathbb{C}})^{++} = \{z = x + iy\,;\, x, y \in \mathfrak{sp}(W), \det(\mathrm{z}-1) \neq 0, \langle \mathrm{yw}, \mathrm{w}\rangle > 0, \mathrm{w} \in \mathrm{W} \setminus \{0\}\}\,.$$

Lemma 3.2. *Fix an element $z = x+iy$ with $x, y \in \mathfrak{sp}(W)$ such that* $\det(\mathrm{z}-1) \neq 0$*.*
Then,
$$\mathrm{H}(\mathrm{w},\mathrm{w}) > \mathrm{H}(\mathrm{c}(\mathrm{z})\mathrm{w}, \mathrm{c}(\mathrm{z})\mathrm{w}) \qquad (\forall \mathrm{w} \in \mathrm{W}_{\mathbb{C}} \setminus \{0\})$$
if and only if
$$\langle yw, w\rangle > 0 \qquad (\forall w \in W \setminus \{0\}).$$
Finally, we obtain $c(\mathfrak{sp}(W_{\mathbb{C}})^{++}) = \mathrm{Sp}(W_{\mathbb{C}})^{++}$.

Proof. Fix $z = x + iy$, with $x, y \in \mathfrak{sp}(W)$. We have

$$\mathrm{H}(\mathrm{c}(\mathrm{z})\mathrm{w}, \mathrm{c}(\mathrm{z})\mathrm{w}) = \mathrm{H}(\overline{\mathrm{c}(\mathrm{z})}^{-1}\mathrm{c}(\mathrm{z})\mathrm{w}, \mathrm{w})$$

and then $\mathrm{H}(\mathrm{w},\mathrm{w}) > \mathrm{H}(\mathrm{c}(\mathrm{z})\mathrm{w}, \mathrm{c}(\mathrm{z})\mathrm{w}) \Leftrightarrow \mathrm{H}((1 - \overline{\mathrm{c}(\mathrm{z})}^{-1}\mathrm{c}(\mathrm{z}))\mathrm{w}, \mathrm{w}) > 0$. Or,

$$\begin{aligned}
1-\overline{c(z)}^{-1}c(z)) &= 1 - \left((\bar z+1)(\bar z-1)^{-1}\right)^{-1}(z+1)(z-1)^{-1}\\
&= 1 - (\bar z-1)(\bar z+1)^{-1}(z+1)(z-1)^{-1}\\
&= 1 - (\bar z+1)^{-1}(\bar z-1)(z+1)(z-1)^{-1}\\
&= (\bar z+1)^{-1}(\bar z+1)(z-1)(z-1)^{-1} - (\bar z+1)^{-1}(\bar z-1)(z+1)(z-1)^{-1}\\
&= (\bar z+1)^{-1}\left((\bar z+1)(z-1) - (\bar z-1)(z+1)\right)(z-1)^{-1}
\end{aligned}$$

By definition of z we get $(\bar z+1)(z-1) - (\bar z-1)(z+1) = 4iy$. So,

$$1 - \overline{c(z)}^{-1}c(z) = 4i(\bar z+1)^{-1}y(z-1)^{-1}.$$

Then, for all $w \in W_{\mathbb{C}} \setminus \{0\}$, we get:

$$\begin{aligned}
&\mathrm{H}(\mathrm{w},\mathrm{w}) > \mathrm{H}(\mathrm{c}(\mathrm{z})\mathrm{w}, \mathrm{c}(\mathrm{z})\mathrm{w}) \Leftrightarrow\ \mathrm{H}((1 - \overline{\mathrm{c}(\mathrm{z})}^{-1}\mathrm{c}(\mathrm{z}))\mathrm{w}, \mathrm{w}) > 0\\
&\quad\Leftrightarrow\ 4i\mathrm{H}((\bar{\mathrm{z}}+1)^{-1}\mathrm{y}(\mathrm{z}-1)^{-1}\mathrm{w}, \mathrm{w}) > 0 \qquad \Leftrightarrow\ -4\langle(\bar{\mathrm{z}}+1)^{-1}\mathrm{y}(\mathrm{z}-1)^{-1}\mathrm{w}, \overline{\mathrm{w}}\rangle > 0\\
&\quad\Leftrightarrow\ -4\langle y(z-1)^{-1}w, (-\bar z+1)^{-1}\overline{w}\rangle > 0 \qquad \Leftrightarrow\ 4\langle yw', \overline{w}'\rangle > 0
\end{aligned}$$

with $w' = (z-1)^{-1}w$. As $\langle yw', \overline{w}'\rangle \in \mathbb{R}_+^*$, by writing w' as $w' = w_1' + iw_2'$, we get:

$$\langle yw', \overline{w}'\rangle = \langle y(w_1'), w_1'\rangle + \langle y(w_2'), w_2'\rangle. \qquad ∎$$

A proof of the previous lemma can also be found in Proposition 1.2 of [11].

Proposition 3.3. *The set $\mathrm{Sp}(W_{\mathbb{C}})^{++}$ is a subsemigroup of $\mathrm{Sp}(W_{\mathbb{C}})$, which does not contain the identity but stable under $g \to \bar g^{-1}$. Moreover, we have*

$$\mathrm{Sp}(W_{\mathbb{C}})^{++}.\,\mathrm{Sp}(W) = \mathrm{Sp}(W).\,\mathrm{Sp}(W_{\mathbb{C}})^{++} \subseteq \mathrm{Sp}(W_{\mathbb{C}})^{++} \qquad (6)$$

and the set $\mathrm{Sp}(W_{\mathbb{C}})^{++} \cup \mathrm{Sp}(W)$ is a subsemigroup of $\mathrm{Sp}(W_{\mathbb{C}})$. To conclude, the symplectic group $\mathrm{Sp}(W)$ is contained in the closure of $\mathrm{Sp}(W_{\mathbb{C}})^{++}$.

Proof. Fix g and g' in $\mathrm{Sp}(W_{\mathbb{C}})^{++}$. Obviously we have $gg' \in \mathrm{Sp}(W_{\mathbb{C}})$. For every $w \in W_{\mathbb{C}}$, we have $\mathrm{H(gg'w, gg'w) < H(g'w, g'w) < H(w, w)}$, which implies that $gg' \in \mathrm{Sp}(W_{\mathbb{C}})^{++}$. The subspace $\mathfrak{sp}(W_{\mathbb{C}})^{++}$ is stable under the map $z \to -\bar{z}$ and $\overline{c(z)}^{-1} = c(-\bar{z})$. Then, if $g \in \mathrm{Sp}(W_{\mathbb{C}})^{++}$, we get $\bar{g}^{-1} \in \mathrm{Sp}(W_{\mathbb{C}})^{++}$.
Now, fix $g \in \mathrm{Sp}(W_{\mathbb{C}})^{++}$ and $h \in \mathrm{Sp}(W)$. For all $w \in W_{\mathbb{C}}$, we have $\overline{h(w)} = h(\bar{w})$ and then:

$$H(gh(w), \overline{gh(w)}) < H(h(w), \overline{h(w)}) = i\langle h(w), \overline{h(w)}\rangle = i\langle h(w), h(\bar{w})\rangle = H(w, w).$$

In particular, $gh \in \mathrm{Sp}(W_{\mathbb{C}})^{++}$. Finally, for every element $g \in \mathrm{Sp}(W)$,

$$g = -c(0)g = \lim_{y \to 0,\ \langle y\cdot, \cdot\rangle > 0} -c(iy)g$$

which proves that every elements of $\mathrm{Sp}(W)$ is a limit of elements in the semigroup $\mathrm{Sp}(W_{\mathbb{C}})^{++}$. ■

Remark 3.4. Let $z = X + iY$ with $z \in \mathfrak{sp}(W_{\mathbb{C}})^{++}$. Then, for all $w \in W$,

$$\chi_z(w) = e^{\frac{i\pi}{2} w^t J(X+iY)w} = e^{\frac{i\pi}{2} w^t JXw} e^{-\frac{\pi}{2} w^t JYw}.$$

The matrix $Y \in \mathfrak{sp}(W)$, the form $\langle Y\cdot, \cdot\rangle$ is positive and JY is symmetric and positive definite. Then, there exists a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_{2n})$ and a matrix $O \in (2n, \mathbb{R})$ such that $JY = O^t DO$. So,

$$\begin{aligned}\int_W \chi_{iY}(w)dw &= \int_W e^{-\frac{\pi}{2} w^t JYw} dw = \int_W e^{-\frac{\pi}{2} w^t O^t DOw} dw = \int_W e^{-\frac{\pi}{2} Y^t DY} |\mathrm{det(O)}|\mathrm{dY} \\ &= \int_W e^{-\frac{\pi}{2} \sum\limits_{k=1}^{2n} d_k Y_k^2} |\mathrm{det(O)}|\mathrm{dY} = \prod_{\mathrm{k=1}}^{\mathrm{2n}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{\pi}{2}\mathrm{d_k Y_k^2}} \mathrm{dY_k} = \prod_{\mathrm{k=1}}^{\mathrm{2n}} \frac{1}{\sqrt{\mathrm{d_k}}} = \mathrm{det}^{-\frac{1}{2}}(\mathrm{D}).\end{aligned}$$

Using that $|\chi_{X+iY}(w)| = e^{-\frac{\pi}{2} w^t JYw}$, we get that the integral $\int_W \chi_{X+iY}(w)dw$ is absolutely convergent. More precisely, we get:

$$\int_W \chi_{X+iY}(w)dw = \mathrm{det}^{-\frac{1}{2}}\left(\frac{1}{2}(\mathrm{X + iY})\right).$$

From now on, we denote by $\Lambda(X + iY)$ the previous determinant, i.e.

$$\Lambda(X + iY) = \mathrm{det}^{-\frac{1}{2}}\left(\frac{1}{2}(\mathrm{X + iY})\right). \tag{7}$$

Even if the complex symplectic group $\mathrm{Sp}(W_{\mathbb{C}})$ is simply connected, the complex manifold $\mathrm{Sp}(W_{\mathbb{C}})^{++}$ is not simply connected. We define on $\mathrm{Sp}(W_{\mathbb{C}})^{++}$ a non-trivial cover, denoted by $\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++}$, by

$$\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++} = \left\{(g, \xi)\,;\, g \in \mathrm{Sp}(W_{\mathbb{C}})^{++}, \xi^2 = \mathrm{det(i(g-1))^{-1}}\right\},$$

and let $C : \mathrm{Sp}(W_{\mathbb{C}})^{++} \times \mathrm{Sp}(W_{\mathbb{C}})^{++} \to \mathbb{C}$ defined by:

$$C(g_1, g_2) = \mathrm{det}^{-\frac{1}{2}}\left(\frac{1}{2\mathrm{i}}(\mathrm{c(g_1) + c(g_2)})\right). \tag{8}$$

Remark 3.5. As explained in [1], the cocycle C' defined in Section 2 is continuous, and it's also obvious that the cocycle C defined in equation (8) is continuous. The cocycles C and C' match to form a continuous function on $\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++} \cup \widetilde{\mathrm{Sp}(W)}$. Indeed, because of the continuity of those cocycles, it is enough to prove that they match on a dense subset of $\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++} \cup \widetilde{\mathrm{Sp}(W)}$.
The subset Ω of $(\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++} \cup \widetilde{\mathrm{Sp}(W)}) \times (\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++} \cup \widetilde{\mathrm{Sp}(W)})$ consisting of elements (g_1, g_2) satisfying $\mathrm{det}(\mathrm{g}_1 - 1)\mathrm{det}(\mathrm{g}_2 - 1)\mathrm{det}(\mathrm{g}_1\mathrm{g}_2 - 1) \neq 0$ is dense. We get:

$$\begin{aligned} c(g_1) + c(g_2) &= (g_1 + 1)(g_1 - 1)^{-1} + (g_2 + 1)(g_2 - 1)^{-1} \\ &= (g_1 - 1)^{-1}\left((g_1 + 1)(g_2 - 1) + (g_1 - 1)(g_2 + 1)\right)(g_2 - 1)^{-1} \\ &= 2(g_1 - 1)^{-1}(g_1 g_2 - 1)(g_2 - 1)^{-1} \end{aligned}$$

and then the cocycle C extends to a continuous map on Ω. Moreover, by using the notations of [1, Proposition 4.13],

$$\begin{aligned} \gamma(q_{g_1,g_2}) &= \int_W \chi\left(\frac{1}{2}\langle c(g_1)w, w\rangle + \langle c(g_2)w, w\rangle\right) dw = \int_{\mathbb{R}^{2n}} e^{\frac{i\pi}{2} w^T \mathrm{B}_{g_1,g_2} w} dw \\ &= \int_{\mathbb{R}^{2n}} e^{-\frac{\pi}{2i} w^T \mathrm{B}_{g_1,g_2} w} dw = \mathrm{det}^{-\frac{1}{2}}\left(\frac{1}{2\mathrm{i}} \mathrm{B}_{\mathrm{g}_1,\mathrm{g}_2}\right) = \Lambda(\mathrm{c}(\mathrm{g}_1) + \mathrm{c}(\mathrm{g}_2)) \end{aligned}$$

where B_{g_1,g_2} is the matrix of form q_{g_1,g_2} defined in equation (3). It follows that the cocycles C and C' match on Ω and then, the result follows.

Theorem 3.6. *The function* $\Theta : \widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++} \ni (g, \xi) \to \xi \in \mathbb{C}$ *is holomorphic, and we have the following equality:*

$$\frac{\Theta(\widetilde{g_1}\widetilde{g_2})}{\Theta(\widetilde{g_1})\Theta(\widetilde{g_2})} = C(g_1, g_2) \qquad \left(\widetilde{g_1}, \widetilde{g_2} \in \widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++} \cup \widetilde{\mathrm{Sp}(W)}\right). \tag{9}$$

Moreover, for every function $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}(W)})$, *we get:*

$$\int_{\widetilde{\mathrm{Sp}(W)}} \Theta(\widetilde{g})\Psi(\widetilde{g}) d\mu_{\widetilde{\mathrm{Sp}(W)}}(\widetilde{g}) = \lim_{\substack{\widetilde{p} \to 1 \\ \widetilde{p} \in \widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++}}} \int_{\widetilde{\mathrm{Sp}(W)}} \Theta(\widetilde{p}\widetilde{g})\Psi(\widetilde{g}) d\mu_{\widetilde{\mathrm{Sp}(W)}}(\widetilde{g}). \tag{10}$$

Proof. The equation (9) has been proved in [1, Lemma 4.17 and Lemma 4.24] for $(\tilde{g}_1, \tilde{g}_2) \in \widetilde{\mathrm{Sp}(W)}$. The same arguments can be applied to prove this equality for $(\tilde{g}_1, \tilde{g}_2) \in \widetilde{\mathrm{Sp}(W)}^{++}$. The general case follows from Remark 3.5.
We now prove the second part of the theorem. We first assume that the support of Ψ is contained in the image of $\widetilde{c}(0)\widetilde{c}$. Then,

$$\begin{aligned} &\int_{\widetilde{\mathrm{Sp}(W)}} \Theta(\widetilde{p}\widetilde{g})\Psi(\widetilde{g}) d\widetilde{g} = \int_{\widetilde{\mathrm{Sp}(W)}} \Theta(\widetilde{p}\widetilde{c}(0)\widetilde{g})\Psi(\widetilde{c}(0)\widetilde{g}) d\widetilde{g} = \int_{\mathrm{supp}\,\Psi} \Theta(\widetilde{p}\widetilde{c}(0)\widetilde{g})\Psi(\widetilde{c}(0)\widetilde{g}) d\widetilde{g} \\ &= \int_{\mathfrak{sp}(W)} \Theta(\widetilde{p}\widetilde{c}(0)\widetilde{c}(x))\Psi(\widetilde{c}(0)\widetilde{c}(x)) j(x) dx = \int_{\mathfrak{sp}(W)} \Theta(\widetilde{c}(iy)\widetilde{c}(x))\Psi(\widetilde{c}(0)\widetilde{c}(x)) j(x) dx \end{aligned}$$

where $\widetilde{p}\widetilde{c}(0) = \widetilde{c}(iy)$ with $y \in \mathfrak{sp}(W)$ such that $\langle y\cdot, \cdot\rangle > 0$ and $j(x)$ is the function defined in [25, Lemma 3.11]. In particular, $y \to 0$ when $\widetilde{p} \to 1$.

But, according to equation (7), $\Theta(\widetilde{c}(iy)\widetilde{c}(x)) = \Theta(\widetilde{c}(iy))\Theta(\widetilde{c}(x))\Lambda(x+iy)$.

We denote by ψ the function of $\mathfrak{sp}(W)$ given by $\psi(x) = \Theta(\widetilde{c}(x))\Psi(\widetilde{c}(x))j(x)$ (we notice easily that $\psi \in \mathscr{C}_c^\infty(\mathfrak{sp}(W))$). We get:

$$\int_{\mathfrak{sp}(W)} \Lambda(x+iy)\psi(x)dx = \int_{\mathfrak{sp}(W)}\int_W \chi_{x+iy}(w)\psi(x)dwdx$$

$$= \int_W \int_{\mathfrak{sp}(W)} \chi_x(w)\chi_{iy}(w)\psi(x)dxdw = \int_W \chi_{iy}(w)\int_{\mathfrak{sp}(W)} \chi_x(w)\psi(x)dxdw$$

and then $\displaystyle\int_{\mathfrak{sp}(W)} \chi_x(w)\psi(x)dx = \int_{\mathfrak{sp}(W)} \psi(x)e^{2i\pi\tau(w)(x)}dx = \widehat{\psi}\left(\frac{1}{4}\tau(w)\right)$,

where $\tau : W \to \mathfrak{sp}(W)^*$ is the moment map and $\widehat{\psi}$ is the Fourier transform of ψ on $\mathfrak{sp}(W)$. Then,

$$\int_{\mathfrak{sp}(W)} \Lambda(x+iy)\psi(x)dx = \int_W \chi_{iy}(w)\widehat{\psi}\left(\frac{1}{4}\tau(w)\right)dw.$$

For all $w \in W\setminus\{0\}$, we have $\chi_{iy}(w) = e^{\frac{2i\pi}{4}\langle iy(w),w\rangle} = e^{-\frac{\pi}{2}\langle yw,w\rangle} < 1$, because $\langle yw,w\rangle > 0$ for every non zero $w \in W$. Finally, we get:

$$\lim_{y\to 0}\int_{\mathfrak{sp}(W)} \Lambda(x+iy)\psi(x)dx = \int_W \widehat{\psi}\left(\frac{1}{4}\tau(w)\right)dw.$$

Using that $\lim\limits_{y\to 0}\Theta(\widetilde{c}(iy)) = \Theta(\widetilde{c}(0))$, we get:

$$\lim_{y\to 0}\int_{\mathfrak{sp}(W)} \Theta(\widetilde{c}(iy))\Lambda(x+iy)\psi(x)dx = \int_W \widehat{\psi}\left(\frac{1}{4}\tau(w)\right)dw.$$

Finally, we have proved that the limit we considered in equation (10) exists. Now, we determine this limit. For every $x \in \mathfrak{sp}(W)$, we denote by B the matrix of the bilinear form $\langle x\cdot,\cdot\rangle$. We remark that the matrix of the form $\langle J\cdot,\cdot\rangle$ is the identity matrix. For all $t \geqslant 0$, we have:

$$\begin{aligned}\Lambda(x+itJ) &= \int_W \chi_{x+itJ}(w)dw = \int_W e^{\frac{i\pi}{2}\langle (x+itJ)w,w\rangle}dw = \int_W e^{\frac{i\pi}{2}\langle (x+itJ)w,w\rangle}dw \\ &= \int_W e^{\frac{i\pi}{2}w^t(B+itI)w}dw = \int_W e^{\frac{-\pi}{2}w^t(-iB+tI)w}dw = \det{}^{-\frac{1}{2}}\left(\frac{1}{2}(-\mathrm{iB}+\mathrm{tI})\right)\end{aligned}$$

We know that the eigenvalues of x are real numbers. So, for all $t > 0$,

$$|\det(-\mathrm{iB}+\mathrm{tI})| > |\det(\mathrm{iB})| \ \text{ i.e. } \ |\det(-\mathrm{iB}+\mathrm{tI})|^{-\frac{1}{2}} < |\det(\mathrm{iB})|^{-\frac{1}{2}}.$$

Then, $|\Lambda(x+itJ)| \leq |\Lambda(x)|$. Using that the function Λ is locally integrable, we get:

$$\lim_{t\to 0^+}\int_{\mathfrak{sp}(W)} \Lambda(x+itJ)\psi(x)dx = \int_{\mathfrak{sp}(W)} \Lambda(x)\psi(x)dx.$$

The choice of such function Ψ is done without loss of generality. Indeed, as explained in [1, Proof of Theorem 4.28], we can find elements $\tilde{h}_1,\ldots,\tilde{h}_n \in \widetilde{\mathrm{Sp}(W)}$ such that:

$$\widetilde{\mathrm{Sp}(W)} = \bigsqcup_{i=1}^n \tilde{h}_i\widetilde{\mathrm{Sp}(W)}^c.$$

Then, using a continuous partition of unity subordinate to the finite open cover $\{\widetilde{h_i\mathrm{Sp}(W)^c}\}_{1\leq i\leq n}$, one can easily check that we can reduce the problem to the case $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}(W)^c})$, and it's what we have done before. ■

Remark 3.7. We first extend the map T on the semigroup. For every element $g \in \mathrm{Sp}(W_{\mathbb{C}})^{++}$, we have $\det(\mathrm{g}-1)$. We define the map $T : \widetilde{\mathrm{Sp}(W_{\mathbb{C}})^{++}} \to \mathrm{S}^*(\mathrm{W})$ by

$$T(\widetilde{g} = (g,\xi)) = \Theta(\widetilde{g})\chi_{c(g)}\mu_W.$$

We denote by $\mathrm{Cont}(\mathrm{L}^2(\mathrm{X}))$ the semigroup of contractions on the Hilbert space $\mathrm{L}^2(\mathrm{X})$. One can prove that

$$\omega = \mathrm{Op}\circ\mathscr{K}\circ T : \widetilde{\mathrm{Sp}(W_{\mathbb{C}})^{++}} \to \mathrm{Cont}(\mathrm{L}^2(\mathrm{X}))$$

is a semigroup homomorphism. Moreover, for all $\widetilde{p} \in \widetilde{\mathrm{Sp}(W_{\mathbb{C}})^{++}}$, the operator $\omega(\widetilde{p})$ is of trace class and $\mathrm{tr}\,\omega(\widetilde{p}) = \Theta(\widetilde{p})$.

4. A general formula for $\Theta_{\Pi'}$

Let us start this section with comments concerning some particular integrals. As shown in [1, Section 4.8], for all $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}(W)})$,

$$\int_{\widetilde{\mathrm{Sp}(W)}} \Psi(\widetilde{g})T(\widetilde{g})d\widetilde{g} \tag{11}$$

is in $\mathrm{S}(\mathrm{W})$. For $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}(W)})$ with $\mathrm{supp}(\Psi) \subseteq \mathrm{Im}(\widetilde{\mathrm{c}})$ and $\phi \in \mathrm{S}(\mathrm{W})$, we have:

$$\left(\int\limits_{\widetilde{\mathrm{Sp}(W)}} \Psi(\widetilde{g})T(\widetilde{g})d\widetilde{g}\right)(\phi) = \int\limits_{W}\int\limits_{\mathfrak{sp}(W)} \Psi(\widetilde{c}(X))\Theta(\widetilde{c}(X))\phi(w)j_{\mathfrak{sp}}(X)\chi\left(\frac{1}{4}\langle Xw,w\rangle\right)dXdw$$

$$= \int\limits_{W}\left(\int\limits_{\mathfrak{sp}(W)} \varphi(X)\chi\left(\frac{1}{4}\langle Xw,w\rangle\right)dX\right)\phi(w)dw = \int\limits_{W} \widehat{\varphi}\circ\tau_{\mathfrak{sp}}(w)\phi(w)dw$$

where $\varphi(X) = \Psi(\widetilde{c}(X))\Theta(\widetilde{c}(X))j_{\mathfrak{sp}}(X) \in \mathscr{C}_c^\infty(\mathfrak{sp}(W))$. Then, $\widehat{\phi} \in \mathrm{S}(\mathfrak{sp}(\mathrm{W}))$ and $\lambda(w) = \widehat{\phi}\circ\tau_{\mathfrak{sp}}(w) \in \mathrm{S}(\mathrm{W})$.
Similarly, for all $\widetilde{p} \in \widetilde{\mathrm{Sp}(W_{\mathbb{C}})^{++}}$ and $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}(W)})$, one can prove that there exists $\lambda_{\widetilde{p}} \in \mathrm{S}(\mathrm{W})$ such that for every $\phi \in \mathrm{S}(\mathrm{W})$, we have:

$$\left(\int_{\widetilde{\mathrm{Sp}(W)}} \Psi(\widetilde{g})T(\widetilde{p}\widetilde{g})d\widetilde{g}\right)(\phi) = \int_W \lambda_{\widetilde{p}}(w)\phi(w)dw. \tag{12}$$

The link between the functions $\lambda_{\widetilde{p}}$ and λ is given by the following equality:

$$\lambda_{\widetilde{p}}(w) = T(\widetilde{p})\natural\lambda(w) \qquad (w \in W). \tag{13}$$

Lemma 4.1. *For all $\widetilde{g} \in \widetilde{\mathrm{Sp}(W)}^c$ and $\widetilde{h} \in \widetilde{\mathrm{Sp}(W_{\mathbb{C}})^{++}}$, we get:*

$$C(g,h)\chi_{c(gh)}(w) = \int_W \chi_{c(g)}(u)\chi_{c(h)}(w-u)\chi\left(\frac{1}{2}\langle u,w\rangle\right)du \qquad (\forall w \in W).$$

Proof. We have $T(\widetilde{gh}) = T(\widetilde{g})\natural T(\widetilde{h})$, i.e. $T(\widetilde{gh})\natural\phi = T(\widetilde{g})\natural T(\widetilde{h})\natural\phi$ for all $\phi \in \mathrm{S(W)}$. For all $w \in W$, we have:

$$\begin{aligned} T(\widetilde{gh})\natural\phi(w) &= \int_W \Theta(\widetilde{gh})\chi_{c(gh)}(u)\phi(w-u)\chi\left(\frac{1}{2}\langle u,w\rangle\right)du \\ &= \Theta(\widetilde{g})\Theta(\widetilde{h})C(g,h)\int_W \chi_{c(gh)}(u)\phi(w-u)\chi\left(\frac{1}{2}\langle u,w\rangle\right)du \\ &= -\Theta(\tilde{g})\Theta(\tilde{h})C(g,h)\int_W \chi_{c(gh)}(w-v)\chi\left(-\frac{1}{2}\langle v,w\rangle\right)dv, \end{aligned}$$

and

$$(T(\widetilde{g})\natural T(\widetilde{h}))\natural\phi(w) = T(\widetilde{g})\natural(T(\widetilde{h})\natural\phi)(w) = \int_W \Theta(\widetilde{g})\chi_{c(g)}(u)T(\widetilde{h})\natural\phi(w-u)\chi\left(\frac{1}{2}\langle u,w\rangle\right)du$$

$$= \int_W \Theta(\widetilde{g})\chi_{c(g)}(u)\left(\int_W \Theta(\widetilde{h})\chi_{c(h)}(v)\phi(w-u-v)\chi\left(\frac{1}{2}\langle v,w-u\rangle\right)dv\right)\chi\left(\frac{1}{2}<u,w>\right)du$$

$$= \int_W \Theta(\widetilde{g})\chi_{c(g)}(u)\left(\int_W \Theta(\widetilde{h})\chi_{c(h)}(w-u-z)\phi(z)\chi\left(-\frac{1}{2}\langle z,w-u\rangle\right)dz\right)\chi\left(\frac{1}{2}\langle u,w\rangle\right)du$$

$$= \Theta(\widetilde{g})\Theta(\widetilde{h})\int_W \phi(z)\left(\int_W \chi_{c(g)}(u)\chi_{c(h)}(w-u-z)\chi\left(-\frac{1}{2}\langle z,w-u\rangle\right)\chi\left(\frac{1}{2}\langle u,w\rangle\right)du\right)dz$$

$$= C(g,h)\chi_{c(gh)}(w-v)\chi\left(-\frac{1}{2}\langle v,w\rangle\right).$$

Then, for all $v, w \in W$, we get: $\quad C(g,h)\chi_{c(gh)}(w-v)\chi\left(-\frac{1}{2}\langle v,w\rangle\right)$

$$= \int_W \chi_{c(g)}(u)\chi_{c(h)}(w-u-v)\chi\left(-\frac{1}{2}\langle v,w-u\rangle\right)\chi\left(\frac{1}{2}\langle u,w\rangle\right)du.$$

We get the result by taking $v = 0$. ∎

Proposition 4.2. *For every $\widetilde{p} \in \widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++}$ and $\Psi, \Phi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}(W)})$, we have that*

$$\int_{\widetilde{\mathrm{Sp}(W)}} \Psi(\widetilde{g}) \int_{\widetilde{\mathrm{Sp}(W)}} \Phi(\widetilde{h})T(\widetilde{g}\widetilde{h}\widetilde{p})d\widetilde{h}d\widetilde{g}$$

is a Schwartz function $\phi_{\widetilde{p}}$ given by

$$\phi_{\widetilde{p}}(w) = \int_{\widetilde{\mathrm{Sp}(W)}} \Psi(\widetilde{g}) \int_{\widetilde{\mathrm{Sp}(W)}} \Phi(\widetilde{h})\Theta(\widetilde{g}\widetilde{h}\widetilde{p})\chi_{c(ghp)}(w)d\widetilde{h}d\widetilde{g}.$$

Proof. For all $\phi \in \mathrm{S(W)}$, we have:

$$\left(\int_{\widetilde{\mathrm{Sp}(W)}} \Psi(\widetilde{g}) \int_{\widetilde{\mathrm{Sp}(W)}} \Phi(\widetilde{h})T(\widetilde{g}\widetilde{h}\widetilde{p})d\widetilde{h}d\widetilde{g}\right)(\phi) = \int_{\widetilde{\mathrm{Sp}(W)}} \Psi(\widetilde{g}) \int_{\widetilde{\mathrm{Sp}(W)}} \Phi(\widetilde{h})T(\widetilde{g}\widetilde{h}\widetilde{p})\natural\phi(0)d\widetilde{h}d\widetilde{g}$$

$$= \int_{\widetilde{\mathrm{Sp}(W)}} \Psi(\widetilde{g})T(\widetilde{g})\natural\left(\int_{\widetilde{\mathrm{Sp}(W)}} \Phi(\widetilde{h})T(\widetilde{h}\widetilde{p})\natural\phi d\widetilde{h}\right)(0)d\widetilde{g}$$

$$= \int_{\widetilde{\mathrm{Sp}(W)}} \Psi(\widetilde{g})\int_W \Theta(\widetilde{g})\chi_{c(g)}(w)\left(\int_{\widetilde{\mathrm{Sp}(W)}} \Phi(\widetilde{h})T(\widetilde{h}\widetilde{p})\natural\phi d\widetilde{h}\right)(-w)dwd\widetilde{g}$$

$$= \int\limits_{\widetilde{\mathrm{Sp}(W)}} \Psi(\widetilde{g}) \int\limits_{W} \Theta(\widetilde{g})\chi_{c(g)}(w) \int\limits_{\widetilde{\mathrm{Sp}(W)}} \Phi(\widetilde{h}) \left(\int\limits_{W} \Theta(\widetilde{h}\widetilde{p})\chi_{c(hp)}(u)\phi(-w-u)\chi(-\tfrac{1}{2}\langle u, w\rangle)du \right) d\widetilde{h}dwd\widetilde{g}$$

$$= \int\limits_{W} \phi(v) \left(\int\limits_{\widetilde{\mathrm{Sp}(W)}} \int\limits_{\widetilde{\mathrm{Sp}(W)}} \int\limits_{W} \Psi(\widetilde{g})\Phi(\widetilde{h})\Theta(\widetilde{g})\Theta(\widetilde{h}\widetilde{p})\chi_{c(g)}(u-v)\chi_{c(hp)}(u)\chi\left(\tfrac{1}{2}\langle u, w\rangle\right) dud\widetilde{h}d\widetilde{g} \right) dv$$

$$= \int\limits_{W} \phi(v) \left(\int\limits_{\widetilde{\mathrm{Sp}(W)}} \int\limits_{\widetilde{\mathrm{Sp}(W)}} \Psi(\widetilde{g})\Phi(\widetilde{h})\Theta(\widetilde{g})\Theta(\widetilde{h}\widetilde{p})C(g, hp)\chi_{chp}(v)d\widetilde{h}d\widetilde{g} \right) dv$$

$$= \int\limits_{W} \phi(v) \left(\int\limits_{\widetilde{\mathrm{Sp}(W)}} \int\limits_{\widetilde{\mathrm{Sp}(W)}} \Psi(\widetilde{g})\Phi(\widetilde{h})\Theta(\widetilde{g}\widetilde{h}\widetilde{p})\chi_{c(ghp)}(v)d\widetilde{g}d\widetilde{h} \right) dv$$

(where the last equality is obtained using Lemma (4.1)). ■

Now, we are able to state and prove the following theorem.

Theorem 4.3. *For every function $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}}')$, we get:*

$$\Theta_{\Pi'}(\Psi) = \lim_{\substack{\widetilde{p}\to 1 \\ \widetilde{p}\in\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++}}} \int_{\widetilde{\mathrm{G}}'} \int_{\widetilde{\mathrm{G}}} \overline{\Theta_\Pi(\widetilde{g})}\Theta(\widetilde{g}\widetilde{g}'\widetilde{p})\Psi(\widetilde{g}')d\widetilde{g}d\widetilde{g}'.$$

Then, as a distributions on $\widetilde{\mathrm{G}}'$, we have:

$$\Theta_{\Pi'}(\widetilde{g}') = \lim_{\substack{\widetilde{p}\to 1 \\ \widetilde{p}\in\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++}}} \int_{\widetilde{\mathrm{G}}} \overline{\Theta_\Pi(\widetilde{g})}\Theta(\widetilde{g}\widetilde{g}'\widetilde{p})d\widetilde{g}. \tag{14}$$

Proof. According to Proposition 4.2 there exists a function $\lambda_{\widetilde{p}} \in \mathrm{S(W)}$ such that

$$\left(\int_{\widetilde{\mathrm{G}}'} \int_{\widetilde{\mathrm{G}}} \overline{\Theta_\Pi(\widetilde{g})}T(\widetilde{g}\widetilde{g}'\widetilde{p})\Psi(\widetilde{g}')d\widetilde{g}d\widetilde{g}' \right)(\phi) = \int_W \lambda_{\widetilde{p}}(w)\phi(w)dw \qquad (\phi \in \mathrm{S(W)}).$$

Similarly, there exists $\lambda \in \mathrm{S(W)}$ such that

$$\left(\int_{\widetilde{\mathrm{G}}'} \int_{\widetilde{\mathrm{G}}} \overline{\Theta_\Pi(\widetilde{g})}T(\widetilde{g}\widetilde{g}')\Psi(\widetilde{g}')d\widetilde{g}d\widetilde{g}' \right)(\phi) = \int_W \lambda(w)\phi(w)dw \qquad (\phi \in \mathrm{S(W)}).$$

Using equation (13) we have for all $w \in W$ that $\lambda_{\widetilde{p}}(w) = T(\widetilde{p})\natural\lambda(w)$. Moreover, $\delta_0 = T(\widetilde{1}) = \lim\limits_{\substack{\widetilde{p}\to 1 \\ \widetilde{p}\in\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++}}} T(\widetilde{p})$, and then, using [1, Section 4.5], we get

$$\lim_{\substack{\widetilde{p}\to 1 \\ \widetilde{p}\in\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++}}} \lambda_{\widetilde{p}}(0) = \lim_{\substack{\widetilde{p}\to 1 \\ \widetilde{p}\in\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++}}} T(\widetilde{p})\natural\lambda(0) = \delta_0\natural\lambda(0) = \lambda(0).$$

Then, using [14, Theorem 3.5.4], we get:

$$\begin{aligned}\Theta_{\Pi'}(\Psi) &= \operatorname{tr}\int_{\widetilde{\mathrm{G}}'}\int_{\widetilde{\mathrm{G}}}\overline{\Theta_\Pi(\widetilde{g})}\Psi(\widetilde{g}')\omega(\widetilde{g}\widetilde{g}')d\tilde{g}d\tilde{g}' \\ &= \operatorname{tr}\operatorname{Op}\circ\mathscr{K}\int_{\widetilde{\mathrm{G}}'}\int_{\widetilde{\mathrm{G}}}\overline{\Theta_\Pi(\widetilde{g})}\Psi(\widetilde{g}')T(\widetilde{g}\widetilde{g}')d\tilde{g}d\tilde{g}' \\ &= \left(\int_{\widetilde{\mathrm{G}}'}\int_{\widetilde{\mathrm{G}}}\overline{\Theta_\Pi(\widetilde{g})}\Psi(\widetilde{g}')T(\widetilde{g}\widetilde{g}')d\tilde{g}d\tilde{g}'\right)(0) = \lambda(0) \\ &= \lim_{\substack{\widetilde{p}\to 1 \\ \widetilde{p}\in\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++}}}\lambda_{\widetilde{p}}(0) = \lim_{\substack{\widetilde{p}\to 1 \\ \widetilde{p}\in\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++}}}\int_{\widetilde{\mathrm{G}}'}\int_{\widetilde{\mathrm{G}}}\overline{\Theta_\Pi(\widetilde{g})}\Theta(\widetilde{g}\widetilde{g}'\widetilde{p})\Psi(\widetilde{g}')d\tilde{g}d\tilde{g}'. \qquad \blacksquare\end{aligned}$$

From now on, we assume that G is connected. For every $\widetilde{p}\in\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++}$ and $\widetilde{g}'\in\widetilde{\mathrm{G}}'$, we define the function $\mathrm{F}_{\widetilde{p},\widetilde{g}'}:\widetilde{\mathrm{G}}\to\mathbb{C}$ by:

$$\mathrm{F}_{\widetilde{p},\widetilde{g}'}(\widetilde{g}) = \overline{\Theta_\Pi(\widetilde{g})}\Theta(\widetilde{g}\widetilde{g}'\widetilde{p}).$$

We easily prove that for every element $g\in\mathrm{G}$, we have $\mathrm{F}_{\widetilde{p},\widetilde{g}'}((g,\xi)) = \mathrm{F}_{\widetilde{p},\widetilde{g}'}((g,-\xi))$, and in particular, we get a function $\mathrm{H}_{\widetilde{\mathrm{p}},\widetilde{\mathrm{g}}'}$ on G by:

$$\mathrm{H}_{\widetilde{\mathrm{p}},\widetilde{\mathrm{g}}'}(\mathrm{pr}(\widetilde{\mathrm{g}})) = \mathrm{F}_{\widetilde{\mathrm{p}},\widetilde{\mathrm{g}}'}(\widetilde{\mathrm{g}}) \qquad (\widetilde{\mathrm{g}}\in\widetilde{\mathrm{G}}).$$

By a standard result of differential geometry (see [33, Lemma A.4.2.11]), we get

$$\int_{\widetilde{\mathrm{G}}}\mathrm{F}_{\widetilde{p},\widetilde{g}'}(\widetilde{g})d\tilde{g} = 2\int_{\mathrm{G}}\mathrm{H}_{\widetilde{\mathrm{p}},\widetilde{\mathrm{g}}'}(\mathrm{g})\mathrm{dg},$$

where dg is the normalized Haar measure on G.

From now on, we assume that G is connected. By Weyl's integration formula (see [19, Theorem 8.60]), we get:

$$\int_{\mathrm{G}}\mathrm{H}_{\widetilde{\mathrm{p}},\widetilde{\mathrm{g}}'}(\mathrm{g})\mathrm{dg} = \int_{\mathrm{T}}\left(\int_{\mathrm{G}/\mathrm{T}}\mathrm{H}_{\widetilde{\mathrm{p}},\widetilde{\mathrm{g}}'}(\mathrm{gtg}^{-1})\mathrm{dg}\,\mathrm{T}\right)|\mathrm{D}(\mathrm{t})|^2\mathrm{dt}$$

where D is the Weyl denominator. We define $\mathrm{G}'^{++} = \mathrm{G}'_{\mathbb{C}}\cap\mathrm{Sp}(W_{\mathbb{C}})^{++}$ and denote by $\widetilde{\mathrm{G}'}^{++}$ the preimage in $\widetilde{\mathrm{Sp}(W_{\mathbb{C}})}^{++}$. For every element $\widetilde{p}\in\widetilde{\mathrm{G}}'^{++}$, we prove easily that the function $\mathrm{H}_{\widetilde{\mathrm{p}},\widetilde{\mathrm{g}}'}$ in invariant by conjugation. In particular, we get:

$$\int_{\mathrm{T}}\left(\int_{\mathrm{G}/\mathrm{T}}\mathrm{H}_{\widetilde{\mathrm{p}},\widetilde{\mathrm{g}}'}(\mathrm{gtg}^{-1})\mathrm{dg}\,\mathrm{T}\right)|D(t)|^2dt = \int_{\mathrm{T}}\mathrm{H}_{\widetilde{\mathrm{p}},\widetilde{\mathrm{g}}'}(\mathrm{t})|\mathrm{D}(\mathrm{t})|^2\mathrm{dt}$$

Using Theorem 4.3, we get:

Proposition 4.4. *Assume that* G *is compact and connected. For every regular element* $\widetilde{g}'\in\widetilde{\mathrm{G}}'$*, the character* $\Theta_{\Pi'}$ *of* Π' *is given by the following formula:*

$$\Theta_{\Pi'}(\widetilde{g}') = \lim_{\substack{\widetilde{p}\to 1 \\ \widetilde{p}\in\widetilde{\mathrm{G}}'^{++}}}\int_{\mathrm{T}}\mathrm{H}_{\widetilde{\mathrm{p}},\widetilde{\mathrm{g}}'}(\mathrm{t})|\Delta(\mathrm{t})|^2\mathrm{dt}. \tag{15}$$

Using a result of Kashiwara and Vergne [17], we obtain the weights of the representations $\Pi\in\widehat{\widetilde{\mathrm{G}}}_\omega$. Then, using the Weyl character formula (equation (1)), we get a formula for the character Θ_Π. What we need now, is an explicit realisation of the character Θ of the metaplectic representation.

5. A restriction of Θ to a maximal compact subgroup

We recall here the main ideas of [29, Section 2]. Here we want an explicit formula for the character Θ of the metaplectic representation on a maximal compact subgroup of $\mathrm{Sp}(W)$. We know that for any positive complex structure J on W, the subgroup $\mathrm{Sp}(W)^J$ of symplectic matrices which commute with J is a maximal compact subgroup of $\mathrm{Sp}(W)$. More precisely, for every compact dual pair $(\mathrm{G}, \mathrm{G}')$ (with G compact), there exists a complex structure J of $\mathrm{Sp}(W)$ such that $\mathrm{G} \,.\, \mathrm{T}' \subseteq \mathrm{Sp}(W)^J$, where T' is the maximal compact Cartan subgroup of G' (we will construct this element J explicitly for the dual pair $(\mathrm{U}(n, \mathbb{C}), \mathrm{U}(p, q, \mathbb{C}))$ in Section 6).

We fix a positive complex structure J on W, and we denote by $W_{\mathbb{C}}$ the complexification of W. With respect to the endomorphism J, we get a decomposition of $W_{\mathbb{C}}$ of the form

$$W_{\mathbb{C}} = W_{\mathbb{C}}^+ \oplus W_{\mathbb{C}}^-$$

where $W_{\mathbb{C}}^+$ (resp. $W_{\mathbb{C}}^-$) is the i-eigenspace (resp. $-i$-eigenspace) for J. One can prove easily that the restriction of the form H defined in equation (4) to the space $W_{\mathbb{C}}^+$ is positive definite. We denote by $\mathrm{U} = \mathrm{U}(W_{\mathbb{C}}^+, \mathrm{H}_{|\mathrm{w}_{\mathbb{C}}})$ the subgroup of $\mathrm{GL}(W_{\mathbb{C}}^+)$ which preserve the form $\mathrm{H}_{|\mathrm{w}_{\mathbb{C}}}$.

We define a two fold cover of U, denoted by $\widetilde{\mathrm{U}}$, as

$$\widetilde{\mathrm{U}} = \left\{(u, \xi), \xi^2 = \det(\mathrm{u}), \mathrm{u} \in \mathrm{U}\right\} \subseteq \mathrm{GL}(W_{\mathbb{C}}^+) \times \mathbb{C}^*. \tag{16}$$

Then, $\widetilde{\mathrm{U}}$ is a group (endowed with the pointwise multiplication). More precisely, it's a connected two-fold covering of U.

Proposition 5.1. *The map* $\mathrm{Sp}(W)^J \ni g \to g_{|_{W_{\mathbb{C}}^+}} \in \mathrm{U}$ *is a group isomorphism and lifts to an isomorphism*

$$\widetilde{\mathrm{Sp}(W)}^J \ni (g, \xi) \to (u, \xi \det(\mathrm{g}-1)_{|_{(\mathrm{g}-1)\mathrm{W}_{\mathbb{C}}^+}}) \in \widetilde{\mathrm{U}}.$$

Then, the restriction of the metaplectic cover to $\mathrm{Sp}(W)^J$ *is isomorphic to the covering*

$$\widetilde{\mathrm{U}} \ni (u, \xi) \to u \in \mathrm{U}$$

Proof. The proof of this result can be found in [29, Proposition 1]. ■

According to equation (14), we need a formula for Θ not only on $\mathrm{Sp}(W)^J$, but on an analogue subset in the oscillator semigroup. Briefly, the map

$$(\mathrm{Sp}(W_{\mathbb{C}})^{++})^J \ni g \to g_{|_{W_{\mathbb{C}}^+}} \in \mathrm{GL}(W_{\mathbb{C}}^+)$$

is well define and bijective. We now define a subgroup $\mathrm{GL}(W_{\mathbb{C}}^+)^{++}$ of $\mathrm{GL}(W_{\mathbb{C}}^+)$ as

$$\mathrm{GL}(W_{\mathbb{C}}^+)^{++} = \left\{h \in \mathrm{GL}(W_{\mathbb{C}}^+), \mathrm{H}_{|\mathrm{w}_{\mathbb{C}}}(\mathrm{w}, \mathrm{w}) > \mathrm{H}_{|\mathrm{w}_{\mathbb{C}}}(\mathrm{hw}, \mathrm{hw}), 0 \neq \mathrm{w} \in \mathrm{W}_{\mathbb{C}}^+\right\}$$

As in equation (16), we define a non-trivial double cover of $\mathrm{GL}(W_{\mathbb{C}}^+)^{++}$ by

$$\widetilde{\mathrm{GL}(W_{\mathbb{C}}^+)}^{++} = \left\{(h, \xi) \in \mathrm{GL}(W_{\mathbb{C}}^+)^{++} \times \mathbb{C}^*, \xi^2 = \det(\mathrm{h})\right\}.$$

The group structure on $\widetilde{\mathrm{GL}(W_{\mathbb{C}}^+)}^{++}$ is given by the coordinate-wise multiplication. More particularly, we get the following proposition.

Proposition 5.2. *The set* $\widetilde{\mathrm{GL}(W_{\mathbb{C}}^{+})}^{++} \cup \widetilde{\mathrm{U}}$ *is a semigroup. Moreover, the map*

$$(\mathrm{Sp}(W_{\mathbb{C}})^{++})^{J} \cup \widetilde{\mathrm{Sp}(W)}^{J} \ni g \to g_{|_{W_{\mathbb{C}}^{+}}} \in \widetilde{\mathrm{GL}(W_{\mathbb{C}}^{+})}^{++} \cup \widetilde{\mathrm{U}}$$

is a semigroup isomorphism.

Corollary 5.3 gives us the character Θ on the subsemigroup $(\mathrm{Sp}(W_{\mathbb{C}})^{++})^{J} \cup \widetilde{\mathrm{Sp}(W)}^{J}$.

Corollary 5.3. *The restriction of* Θ *on the subsemigroup* $(\widetilde{\mathrm{Sp}(W_{\mathbb{C}}^{+})}^{++})^{J} \cup \widetilde{\mathrm{Sp}(W)}^{J}$ *is given by*

$$\Theta(\widetilde{k}) = \lim_{\substack{\widetilde{h} \to \widetilde{k} \\ \widetilde{h} \in \widetilde{\mathrm{GL}(W_{\mathbb{C}}^{+})}^{++}}} \Theta(\widetilde{h}) = \lim_{\substack{\widetilde{h} \to \widetilde{k} \\ \widetilde{h} \in \widetilde{\mathrm{GL}(W_{\mathbb{C}}^{+})}^{++}}} \frac{\xi}{\det(1 - \mathrm{h})} \qquad (\widetilde{h} = (h, \xi)). \tag{17}$$

6. The dual pair $(\mathrm{G} = \mathrm{U}(n, \mathbb{C}), \mathrm{G}' = \mathrm{U}(p, q, \mathbb{C}))$

Let (V, b) be a n-dimensional vector space over $\mathbb{C}$ endowed with a positive definite hermitian form b and $\mathscr{B}$ be a basis of V such that $\mathrm{Mat}(b, \mathscr{B}) = \mathrm{Id}_n$. We denote by $\mathrm{U}(V, b)$ the group of isometries of b, i.e.

$$\mathrm{U}(V, b) = \{g \in \mathrm{GL}(V), b(gu, gv) = b(u, v), (\forall u, v \in V)\}. \tag{18}$$

By writing the endomorphisms in the basis $\mathscr{B}$, we get that the right hand side of equation (18) can be written as:

$$\{g \in \mathrm{GL}(n, \mathbb{C}), g^{*}g = \mathrm{Id}_n\}, \tag{19}$$

where $g^{*} = g^{-1^{t}}$. We denote by $\mathrm{G} = \mathrm{U}(n, \mathbb{C})$ the group defined in equation (19), by $\mathfrak{g} = \mathfrak{u}(n, \mathbb{C})$ the Lie algebra of $\mathrm{U}(n, \mathbb{C})$. The maximal torus T of $\mathrm{U}(n, \mathbb{C})$ is given by $\mathrm{T} = \{\mathrm{diag}(t_1, \ldots, t_n), t_i \in \mathrm{S}^1\}$ and its Lie algebra $\mathfrak{t}$ is defined as:

$$\mathfrak{t} = \bigoplus_{k=1}^{n} i\mathbb{R}\, \mathrm{E}_{k,k}.$$

One knows (see [19, Chapter II]) that the roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ are given by

$$\Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm(e_i - e_j), 1 \leq i < j \leq n\},$$

where $e_k(\mathrm{diag}(h_1, \ldots, h_n)) = h_k$. Similarly, let (V', b') be a $p+q$-dimensional vector space over $\mathbb{C}$ endowed with a non-degenerate hermitian form b' of signature (p, q) and let $\mathscr{B}'$ be a basis of V' such that $\mathrm{Mat}(b', \mathscr{B}') = \mathrm{Id}_{p,q}$. We denote by $\mathrm{U}(V', b')$ the group of isometries of b', i.e.

$$\mathrm{U}(V', b') = \{g \in \mathrm{GL}(V'), b'(gu, gv) = b'(u, v), (\forall u, v \in V')\}, \tag{20}$$

and by $\mathrm{U}(p, q, \mathbb{C})$ the following group

$$\{g \in \mathrm{GL}(p, q, \mathbb{C}), g^{*}\, \mathrm{Id}_{p,q}\, g = \mathrm{Id}_{p,q}\}. \tag{21}$$

Let $\mathrm{K}' = \mathrm{U}(p, \mathbb{C}) \times \mathrm{U}(q, \mathbb{C})$ be the maximal compact subgroup of G'.

Using the paper of Kashiwara and Vergne [17] (we can also use the Appendix of [26]), the weights of the representations of $\Pi \in \widehat{\widetilde{\mathrm{U}(n, \mathbb{C})}}_{\omega}$ which appears in the correspondence are given by the following formula:

$$\lambda = \sum_{a=1}^{n} \frac{q-p}{2} e_a - \sum_{a=1}^{r} \nu_a e_{n+1-a} + \sum_{a=1}^{s} \mu_a e_a, \tag{22}$$

where $0 \leqslant r \leqslant p$, $0 \leqslant s \leqslant q$, $r+s \leqslant n$, and integers $\nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_s$ which satisfy $\nu_1 \geqslant \ldots \geqslant \nu_r > 0$ and $\mu_1 \geqslant \ldots \geqslant \mu_s > 0$. The weights λ can be written as

$$\lambda = \sum_{a=1}^{n} \left(\frac{q-p}{2} + \lambda_a \right) e_a \tag{23}$$

where $\lambda_i \in \mathbb{Z}$, $\lambda_1 \geq \ldots \geq \lambda_n$ with at most q of the integers λ_i are positives and p negatives. It is easily proved that, for $\mathrm{G} = \mathrm{U}(n, \mathbb{C})$, we have:

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})} \alpha = \sum_{a=1}^{n} \frac{n-2a+1}{2} e_a$$

Using Corollary 5.3, we give a formula for the character Θ on $\mathrm{T} \,.\, \mathrm{T}'^{++}$, where T and T' are diagonal Cartan subgroups of G and G' respectively and $\mathrm{T}'^{++} = \mathrm{T}'_{\mathbb{C}} \cap \mathrm{Sp}(W_{\mathbb{C}})^{++}$.

Proposition 6.1. (1) *The set* T'^{++} *is given by*

$$\mathrm{T}'^{++} = \{\mathrm{diag}(t_1, \ldots, t_{p+q})\,;\ |t_i| < 1 \text{ for } 1 \leq i \leq p, |t_i| > 1 \text{ for } p < i \leq p+q\}\,. \tag{24}$$

(2) *For all* $\widetilde{t} \in \widetilde{\mathrm{T}}$ *and* $\widetilde{t'} \in \mathrm{T}'^{++}$, *the character* Θ *is given by:*

$$\Theta(\widetilde{tt'}) = \frac{(-1)^{nq} \prod_{b=1}^{n} t_b^p \left(\prod_{a=1}^{p+q} t'_a \right)^{\frac{n}{2}} \left(\prod_{b=1}^{n} t_b^{\frac{q-p}{2}} \right)}{\prod_{a=1}^{p} \prod_{b=1}^{n} (t_b - t'_a) \prod_{a=p+1}^{p+q} \prod_{b=1}^{n} \left(t_b - \frac{1}{t'_a} \right)}. \tag{25}$$

Proof. (1): See Appendix A.

(2): We consider $W = \mathrm{M}((p+q) \times n, \mathbb{C})$ as a real vector space endowed with the following form

$$\langle w, w' \rangle = \mathrm{Im}(\bar{\mathrm{w}}'^{\mathrm{t}} \mathrm{I}_{\mathrm{p,q}} \mathrm{w}).$$

This form is symmetric and non-degenerate. Moreover, the map $J(w) = iI_{p,q}w$ is a posiitive definite complex structure on W. The maps

$$\mathrm{G} \times W \ni (g, X) \to gX \in W, \qquad \mathrm{G}' \times W \ni (g', X) = Xg'^{-1}$$

give embeddings of G and G' into $\mathrm{Sp}(W)$. For every matrix $\mathrm{E}_{a,b} \in W$, we have:

$$(tt')\, \mathrm{E}_{a,b} = t'_a t_b^{-1}\, \mathrm{E}_{a,b}\,.$$

By definition of J, we get: $J(\mathrm{E}_{a,b}) = \begin{cases} i\,\mathrm{E}_{a,b} & \text{if } 1 \leq a \leq p \\ -i\,\mathrm{E}_{a,b} & \text{if } a > p \end{cases}.$

Then, the eigenvalues of tt' are of the form

$$\left\{ t'_a t_b^{-1}\,;\, 1 \leq a \leq p, 1 \leq b \leq n \right\} \cup \left\{ \overline{t'_a t_b^{-1}}\,;\, p+1 \leq a \leq p+q, 1 \leq b \leq n \right\}.$$

Finally, using equation (17), we get:

$$\Theta(\widetilde{tt'}) = \frac{\left(\prod_{a=1}^{p}\prod_{b=1}^{n} t'_a t_b^{-1} \prod_{a=p+1}^{p+q}\prod_{b=1}^{n} \overline{t'_a t_b^{-1}}\right)^{\frac{1}{2}}}{\prod_{a=1}^{p}\prod_{b=1}^{n}(1-t'_a t_b^{-1}) \prod_{a=p+1}^{p+q}\prod_{b=1}^{n}(1-\overline{t'_a t_b^{-1}})} = \frac{\left(\prod_{a=1}^{p}(t'_a)^n \prod_{a=p+1}^{p+q}(\overline{t'_a})^n \prod_{b=1}^{n} t_b^{q-p}\right)^{\frac{1}{2}}}{\prod_{a=1}^{p}\prod_{b=1}^{n}(1-t'_a t_b^{-1}) \prod_{a=p+1}^{p+q}\prod_{b=1}^{n}(1-\overline{t'_a t_b^{-1}})}$$

In particular, for every element $\widetilde{t'}$ of $\widetilde{\mathrm{T}}'$, we get:

$$\Theta(\widetilde{t'}) = \frac{\left(\prod_{a=1}^{p}(t'_a)^n \prod_{a=p+1}^{p+q}(\overline{t'_a})^n\right)^{\frac{1}{2}}}{\prod_{a=1}^{p}\prod_{b=1}^{n}(1-t'_a) \prod_{a=p+1}^{p+q}\prod_{b=1}^{n}(1-\overline{t'_a})}$$

According to equation (9), we get $\Theta(\widetilde{\widetilde{t}\widetilde{t'}}) = \Theta(\widetilde{t})\Theta(\widetilde{t'})\Lambda(c(t)+c(t'))$.
The rest of the proof is a straightforward computation. ∎

Proposition 6.2. *For every regular element $\widetilde{t'}$ in $\widetilde{\mathrm{T}}'$, we get:*

$$\Theta_{\Pi'}(\widetilde{t'}) = \frac{(-1)^{nq+\frac{(n-1)n}{2}}\left(\prod_{a=1}^{p+q} t'_a\right)^{\frac{n}{2}}}{(2i\pi)^n n!}$$
$$\lim_{\substack{r\to 1\\ 0<r<1}} \sum_{w\in\mathscr{W}} \mathrm{sgn}(w) \int_{\mathrm{S}^1}\cdots\int_{\mathrm{S}^1} \frac{\prod_{b=1}^{n} t_b^{p-n-1} \prod_{a=1}^{n} t_{w(a)}^{a-\lambda_a} \prod_{1\le i<j\le n}(t_i-t_j)}{\prod_{a=1}^{p}\prod_{b=1}^{n}(t_b - rt'_a) \prod_{a=p+1}^{p+q}\prod_{b=1}^{n}\left(t_b - \frac{1}{r\overline{t'_a}}\right)} \prod_{k=1}^{n} dt_k$$
$$= K(t') \lim_{\substack{r\to 1\\ 0<r<1}} \sum_{w\in\mathscr{W}}\sum_{\beta\in\mathscr{S}_n} \mathrm{sgn}(w)\,\mathrm{sgn}(\beta) \prod_{b=1}^{n}\int_{\mathrm{S}^1} \frac{t_b^{p-n-2+w^{-1}(b)-\lambda_{w^{-1}(b)}+\beta^{-1}(b)}}{\prod_{a=1}^{p}(t_b - rt'_a)\prod_{a=p+1}^{p+q}\left(t_b - \frac{1}{r\overline{t'_a}}\right)} dt_b$$

where $K(t') = \dfrac{(-1)^{nq+\frac{(n-1)n}{2}}\left(\prod_{a=1}^{p+q} t'_a\right)^{\frac{n}{2}}}{(2i\pi)^n n!}$.

Proof. Using equation (15), we get

$$\Theta_{\Pi'}(\widetilde{g}') = \lim_{\substack{\widetilde{p}\to 1\\ \widetilde{p}\in\widetilde{\mathrm{G}}_{\mathbb{C}}'^{++}}} \int_{\mathrm{T}} \mathrm{H}_{\widetilde{\mathrm{p}},\widetilde{\mathrm{g}}'}(\mathrm{t})|\Delta(\mathrm{t})|^2 \mathrm{d}\mu_{\mathrm{T}}(\mathrm{t}).$$

The torus T of $\mathrm{U}(n,\mathbb{C})$ is isomorphic to $\mathrm{S}^{1\otimes n}$. Under this identification, we get

$$d\mu_{\mathrm{T}} = \bigotimes_{i=1}^{n} d\mu_{\mathrm{S}^1} \quad \text{and} \quad d\mu_{\mathrm{S}^1}(z) = \frac{dz}{2i\pi z}.$$

Moreover, $$D(\widetilde{t} = \widetilde{\exp(x)}) = \prod_{\alpha>0}(e^{\frac{\alpha}{2}(x)} - e^{-\frac{\alpha}{2}(x)}) = \prod_{1\le i<j\le n}(t_i^{\frac{1}{2}}t_j^{-\frac{1}{2}} - t_i^{-\frac{1}{2}}t_j^{\frac{1}{2}})$$
$$= \prod_{i=1}^{n} t_i^{-\frac{n-1}{2}} \prod_{1\le i<j\le n}(t_i - t_j), \tag{26}$$

and by using the Vandermonde's determinant formula, we get

$$\prod_{1\leq i<j\leq n}(t_j-t_i)=\sum_{\beta\in\mathscr{S}_n}\operatorname{sgn}(\beta)\prod_{i=1}^{n}t_{\beta_i}^{i-1}. \tag{27}$$

The rest of the proof is a straightforward computation using equation (1), Proposition 6.1 and equation (23). ■

We now give a technical lemma concerning the integrals which appears in the previous proposition (the proof is obvious using residue theorem).

Lemma 6.3. *Let $a_1,\dots,a_p$ be p-complex numbers such that $|a_i|<1$ for all $i\in[|1,p|]$. Similarly, we consider $a_{p+1},\dots,a_{p+q}\in\mathbb{C}$ such that $|a_i|>1$ for all $i\in[|p+1,p+q|]$. Moreover, we assume that $a_i\neq a_j, i\neq j$. Then, we get:*

$$\frac{1}{2i\pi}\int_{\mathrm{S}^1}\frac{t^k}{\prod\limits_{i=1}^{p+q}(t-a_i)}dt=\begin{cases}\sum\limits_{h=1}^{p}\dfrac{a_h^k}{\prod\limits_{j\neq h}(a_h-a_j)} & \text{if } k\geqslant 0\\ -\sum\limits_{h=p+1}^{p+q}\dfrac{a_h^k}{\prod\limits_{j\neq h}(a_h-a_j)} & \text{otherwise}\end{cases}$$

Let us now fix $n=1$. In this case, the weights λ of the representations Π are of the form $\lambda=\left(\frac{q-p}{2}+k\right)e_1, k\in\mathbb{Z}$. The reason why we voluntarily change the notations here is because the set of irreducible genuine representations of $\widetilde{\mathrm{U}(1,\mathbb{C})}$ is isomorphic to the unitary dual of $\mathrm{U}(1,\mathbb{C})$, which is isomorphic to $\mathbb{Z}$ via the isomorphism:

$$\mathbb{Z}\ni k\to(x\to e^{2i\pi kx})\in\widehat{\mathrm{U}(1,\mathbb{C})}.$$

We denote by Π_k the representation of $\widetilde{\mathrm{U}(1,\mathbb{C})}$ of highest weight $\left(\frac{q-p}{2}+k\right)e_1$ and by Π'_k the corresponding representation of $\widetilde{\mathrm{U}(p,q,\mathbb{C})}$. Using Proposition 6.2 and Lemma 6.3, we get the following proposition.

Proposition 6.4. *The character $\Theta_{\Pi'_k}$ of the representation Π'_k of $\mathrm{U}(p,q,\mathbb{C})$ is given, for every $\widetilde{t'}\in\widetilde{\mathrm{T}}'$, by:*

$$\Theta_{\Pi'_k}(\widetilde{t'})=\begin{cases}\prod\limits_{i=1}^{p+q}t_i^{\frac{1}{2}}\sum\limits_{h=1}^{p}\dfrac{t_h^{p-(k+1)}}{\prod\limits_{h\neq j}(t_h-t_j)} & \text{if } k\leqslant p-1\\ -\prod\limits_{i=1}^{p+q}t_i^{\frac{1}{2}}\sum\limits_{h=p+1}^{p+q}\dfrac{t_h^{p-(k+1)}}{\prod\limits_{h\neq j}(t_h-t_j)} & \text{otherwise}\end{cases} \tag{28}$$

The Weyl group $\mathscr{W}$ (resp. $\mathscr{W}(\mathfrak{k})$) of G' (resp. K') is isomorphic to $\mathscr{S}_{p+q}$ (resp. $\mathscr{S}_p\times\mathscr{S}_p$). For every element $h\in\{1,\dots,p+q\}$, we denote by $\mathscr{W}^h$ the stabilizer of h, i.e.

$$\mathscr{W}^h=\{\sigma\in\mathscr{W},\sigma(h)=h\}.$$

We define similarly $\mathscr{W}(\mathfrak{k})^h$.

Proposition 6.5. *We get, up to a constant, the following result:*

$$D(X)\Theta_{\Pi'}(\exp(X)) = \begin{cases} \sum\limits_{\mu\in A^{p+1}} \mathrm{sgn}(\mu) \sum\limits_{\omega\in\mathscr{S}_p\times\mathscr{S}_q} \mathrm{sgn}(\omega) e^{i\omega(\mu\lambda_1+\xi)(x)} & \textit{if } k > p-1 \\ \sum\limits_{\mu\in A^{1}} \mathrm{sgn}(\mu) \sum\limits_{\omega\in\mathscr{S}_p\times\mathscr{S}_q} \mathrm{sgn}(w) e^{i\omega(\mu\lambda_2+\xi)(x)} & \textit{otherwise} \end{cases}$$

where

- $\lambda_1 = \sum\limits_{i=1}^{p}(i-1)e_i + (p-(k+1))e_{p+1} + \sum\limits_{i=p+2}^{p+q}(i-2)e_i$,
- $\lambda_2 = (p-(k+1))e_1 + \sum\limits_{a=2}^{p+q}(a-2)e_a$,
- A^1 *(resp.* A^{p+1}*) is a system of representatives of* $\mathscr{W}^1/\mathscr{W}(\mathfrak{k})^1$ *(resp.* $\mathscr{W}^{p+1}/\mathscr{W}(\mathfrak{k})^{p+1}$*)*,
- $\xi = \sum\limits_{k=1}^{p+q} \frac{p+q-2}{2} e_k$.

Proof. We assume first that $k > p-1$. According to equation (26), we have:

$$\prod_{\alpha\in\Phi^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})} (e^{\frac{\alpha(x)}{2}} - e^{-\frac{\alpha(x)}{2}}) = \prod_{i=1}^{m} t_i^{-\frac{p+q-1}{2}} \prod_{1\leq i<j\leq p+q} (t_i - t_j).$$

For all $h \in \{1, \ldots, p+q\}$, we get:

$$\frac{\prod\limits_{1\leq i<j\leq p+q} (t_i - t_j)}{\prod\limits_{k\neq h} (t_h - t_k)} = (-1)^{h-1} \prod_{\substack{1\leq i<j\leq p+q \\ i,j\neq h}} (t_i - t_j)$$

and then

$$\prod_{\alpha\in\Phi^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})} (e^{\frac{\alpha(x)}{2}} - e^{-\frac{\alpha(x)}{2}}) \Theta_{\Pi'_k}(\widetilde{\exp(x)}) = \prod_{i=1}^{p+q} t_i^{-\frac{p+q-2}{2}} \sum_{h=p+1}^{p+q} (-1)^{h+1} t_h^{p-(k+1)} \prod_{\substack{1\leqslant i<j\leqslant p+q \\ i\neq h, j\neq h}} (t_i - t_j).$$

For all $h \in \{1, \ldots, p+q\}$, we denote by $\tilde{t_k}, 1 \leq k \leq p+q-1$ the following elements

$$\tilde{t_k} = \begin{cases} t_k & \text{if } k < h \\ t_{k+1} & \text{otherwise} \end{cases}.$$

Then, up to a ± 1, we get:

$$\begin{aligned} \prod_{\substack{1\leqslant i<j\leqslant p+q \\ i\neq h, j\neq h}} (t_i - t_j) &= \prod_{1\leqslant i<j\leqslant p+q-1} (\tilde{t_i} - \tilde{t_j}) = \sum_{\sigma\in\mathscr{S}_{p+q-1}} \mathrm{sgn}(\sigma) \prod_{a=1}^{p+q-1} \tilde{t}_{\sigma(a)}^{a-1} \\ &= \sum_{\sigma\in\mathscr{S}^h_{p+q}} \mathrm{sgn}(\sigma) \prod_{a=1}^{h-1} t_{\sigma(a)}^{a-1} \prod_{a=h+1}^{p+q} t_{\sigma(a)}^{a-2}. \end{aligned}$$

Finally, we prove that:

$$\begin{aligned} &\sum_{h=p+1}^{p+q} \sum_{\sigma\in\mathscr{S}^h_{p+q}} \mathrm{sgn}(\sigma)(-1)^{h+1} t_h^{p-(k+1)} \prod_{a=1}^{h-1} t_{\sigma(a)}^{a-1} \prod_{a=h+1}^{p+q} t_{\sigma(a)}^{a-2} \\ &\qquad = (-1)^p \sum_{\mu\in A^{p+1}} \mathrm{sgn}(\mu) \sum_{\omega\in\mathscr{S}_p\times\mathscr{S}_q} \mathrm{sgn}(\omega) e^{i\omega(\mu\lambda)(x)} \end{aligned}$$

where $\lambda_1 = \sum\limits_{i=1}^{p}(i-1)e_i + (p-(k+1))e_{p+1} + \sum\limits_{i=p+2}^{p+q}(i-2)e_i$.

The proof is similar if $k \leq p-1$. ■

We recall briefly some well-known facts from [30] (see also [2]) concerning the Fourier transform of co-adjoint orbits. To simplify the notations, we assume that G is a semi-simple connected Lie group such that $\mathrm{rk}(\mathrm{K}) = \mathrm{rk}(\mathrm{G})$, where K is a maximal compact subgroup of G. We denote by Ad^* the natural co-ajoint action of G on $\mathfrak{g}^*$. For every $\lambda \in \mathfrak{g}^*$, we denote by G_λ the G-orbit associated to λ. On the space G_λ, we have a natural measure $d\beta_\lambda$, usually called the Liouville measure on G_λ (see [2, Section 7.5]).

The Fourier transform of G_λ, denoted by $\mathrm{F}_{\mathrm{G}_\lambda}$, is the generalized function on $\mathfrak{g}$ defined by:

$$\mathrm{F}_{\mathrm{G}_\lambda}(X) = \int_{\mathrm{G}_\lambda} e^{if(X)} d\beta_\lambda(f) \qquad (X \in \mathfrak{g}).$$

From [30, page 217], if $\lambda \in \mathfrak{t}^{*\mathrm{reg}}$, we have, up to a constant, the following equality:

$$\left(\prod_{\alpha \in \Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})} \alpha(X)\right) \mathrm{F}_{\mathrm{G}_\lambda}(X) = \sum_{\omega \in \mathscr{W}(\mathfrak{k})} \varepsilon(\omega) e^{i\lambda(\omega(X))} \qquad (X \in \mathfrak{t}^{\mathrm{reg}}).$$

If the weight λ is not regular, see [2, Theorem 7.24]. To simplify the notations, we denote by $\pi(X)$ the quantity $\prod\limits_{\alpha \in \Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})} \alpha(X)$.

Corollary 6.6. *Using the notations of Proposition* 6.5*, we have, up to a constant:*

$$\pi(X)^{-1} D(X) \Theta_{\Pi'}(\exp(X)) = \begin{cases} \sum\limits_{\mu \in A^{p+1}} \mathrm{sgn}(\mu)\, \mathrm{F}_{\mathrm{G}_{\mu(\lambda_1+\xi)}}(X) & \text{if } p < k+1 \\ \sum\limits_{\mu \in A^1} \mathrm{sgn}(\mu)\, \mathrm{F}_{\mathrm{G}_{\mu(\lambda_2+\xi)}}(X) & \text{if } k < -q+1 \end{cases}$$

Remark 6.7. (1) For the dual pair $(\mathrm{G} = \mathrm{U}(1,\mathbb{C}), \mathrm{G}' = \mathrm{U}(1,\mathbb{C}))$, using equation (28), we get:

$$\Theta_{\Pi'_k}(\widetilde{t'}) = t'^{-k-\frac{1}{2}} = \Theta_{\Pi_k}(t'^{-1}).$$

In particular, to be precise, with our method, we don't get $\Theta_{\Pi'}(\widetilde{t'})$ but $\Theta_{\Pi'}(\widetilde{t'}^{-1})$ (or $\Theta_{\Pi'}(\widetilde{t'})$ because the representation Π' is unitary): in the embedding of $(\mathrm{G}, \mathrm{G}')$ in $\mathrm{Sp}(W)$, $g' \in \mathrm{G}'$ acts on $w \in W$ as $g'.w = wg'^{-1}$.

(2) The function $\mathfrak{t} \to X \to \pi(X)^{-1} D(X) \in \mathbb{C}$ is well-known in the literature, usually denoted by $p(x)$ (see [18] or [30]). More precisely, $p(X)$ can be defined as

$$p(X) = \det{}^{\frac{1}{2}}\left(\frac{\sinh(\mathrm{ad}(\mathrm{X}/2))}{\mathrm{ad}(\mathrm{X}/2)}\right).$$

7. The case of $(\mathbf{G} = \mathbf{U}(1,\mathbb{C}), \mathbf{G}' = \mathbf{U}(1,1,\mathbb{C}))$

As recalled in equation (21), the unitary group $\mathrm{U}(p,q,\mathbb{C})$ is defined by

$$\mathrm{U}(p,q,\mathbb{C}) = \{A \in \mathrm{GL}(p+q,\mathbb{C}),\ A^* \,\mathrm{Id}_{p,q}\, A = \mathrm{Id}_{p,q}\}.$$

We fix the convention that $p \leq q$. In this case, there is, up to conjugaison, $q+1$ Cartan subgroups in $\mathrm{U}(p,q,\mathbb{C})$ (see [12]). More precisely, the Cartan subgroups $\mathrm{H_k}, 0 \leq \mathrm{k} \leq \mathrm{q}$ are of the form $\mathrm{H_k} = \mathrm{H_k^-}\mathrm{H_k^+}$, where $\mathrm{H_k^-}$ and $\mathrm{H_k^+}$ are the subgroups of $\mathrm{U}(p,q,\mathbb{C})$ are defined by

$$\mathrm{H_k^+} = \left\{\mathrm{H_k^+}(\mathrm{t_1}, \dots, \mathrm{t_k}), \mathrm{t_i} \in \mathbb{R}, 1 \leq \mathrm{i} \leq \mathrm{k}\right\}$$

where $\mathrm{H_k^+}(\mathrm{t_1}, \dots, \mathrm{t_k})$ is given by

$$\mathrm{H_k^+}(\mathrm{t_1}, \dots, \mathrm{t_k}) = \begin{pmatrix} \mathrm{Id}_{p-k} & & & & & & & 0 \\ & \mathrm{ch}(t_k) & & & & & \mathrm{sh}(t_k) & \\ & & \mathrm{ch}(t_{k-1}) & & & \mathrm{sh}(t_{k-1}) & & \\ & & & \ddots & \iddots & & & \\ & & & \mathrm{ch}(t_1) & \mathrm{sh}(t_1) & & & \\ & & & \mathrm{sh}(t_1) & \mathrm{ch}(t_1) & & & \\ & & & \iddots & \ddots & & & \\ & & \mathrm{sh}(t_{k-1}) & & & \mathrm{ch}(t_{k-1}) & & \\ & \mathrm{sh}(t_k) & & & & & \mathrm{ch}(t_k) & \\ 0 & & & & & & & \mathrm{Id}_{q-k} \end{pmatrix}$$

and

$$\mathrm{H_k^-} = \{\mathrm{diag}(\mathrm{e}^{\mathrm{i}\phi_1}, \dots, \mathrm{e}^{\mathrm{i}\phi_{\mathrm{p-k}}}, \mathrm{e}^{\mathrm{i}\theta_{\mathrm{k}}}, \dots, \mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_1}, \dots, \mathrm{e}^{\mathrm{i}\theta_{\mathrm{k}}}, \mathrm{e}^{\mathrm{i}\tau_{\mathrm{q-k}}}, \dots, \mathrm{e}^{\mathrm{i}\tau_1}), \phi_{\mathrm{i}}, \theta_{\mathrm{i}}, \tau_{\mathrm{i}} \in \mathbb{R}\}.$$

We now work with the pair $(\mathrm{G}(V, \mathrm{b_V}), \mathrm{G}(\mathrm{V}', \mathrm{b_{V'}})) = (\mathrm{U}(1,\mathbb{C}), \mathrm{U}(1,1,\mathbb{C})) \subseteq \mathrm{Sp(W)}$, where $W = (\mathbb{C} \otimes \mathbb{C}^2)_{\mathbb{R}}$ and $\langle \cdot, \cdot \rangle = \mathrm{Im}(\mathrm{b_V} \otimes \mathrm{b_{V'}})$. We denote by $\mathrm{H_1}$ and $\mathrm{H_2}$ the two Cartan subgroups of G' (up to conjugation), with $\mathrm{H_1}$ compact.

Let $\mathscr{B}' = \{v_1, v_2\}$ be a basis of V' such that:

$$\mathrm{F} = \mathrm{Mat}(\mathscr{B}', \mathrm{b_{V'}}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, $\mathrm{U}(1,1,\mathbb{C}) = \{g \in \mathrm{GL}(2,\mathbb{C}), g^* \mathrm{F}\, g = \mathrm{F}\}$, and

$$\begin{aligned} \mathfrak{u}(1,1,\mathbb{C}) &= \{A \in \mathrm{M}(2,\mathbb{C}), A\,\mathrm{F} + \mathrm{F}\,A^* = 0\} \\ &= \mathbb{R}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \end{aligned}$$

In particular, we have $\mathfrak{u}(1,1,\mathbb{C})_{\mathbb{C}} = \mathbb{C}\,\mathrm{Id}_2 \oplus \mathfrak{sl}(2,\mathbb{C})$. We fix the $\mathfrak{sl}(2,\mathbb{C})$-triple (h,e,f) defined by $h = E_{1,1} - E_{2,2}$, $e = E_{1,2}$, $f = E_{2,1}$ and let

$$C = \exp\left(i\frac{\pi}{4}(e+f)\right).$$

More particularly, we have:

$$C = \exp\begin{pmatrix} 0 & \frac{i\pi}{4} \\ \frac{i\pi}{4} & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{ch}(\frac{i\pi}{4}) & \mathrm{sh}(\frac{i\pi}{4}) \\ \mathrm{sh}(\frac{i\pi}{4}) & \mathrm{ch}(\frac{i\pi}{4}) \end{pmatrix} = \begin{pmatrix} \cos(\frac{\pi}{4}) & i\sin(\frac{\pi}{4}) \\ i\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

so $C \notin \mathfrak{u}(1,1,\mathbb{C})$ (the choice of the matrix C is natural, see [31, Section 2]).

Lemma 7.1. *Let*

$$\mathfrak{h}_1 = \mathbb{R}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad \mathfrak{h}_2 = \mathbb{R}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Then, $\mathrm{H}_1 = \exp(\mathfrak{h}_1)$ *and* $\mathrm{H}_2 = \exp(\mathfrak{h}_2)$ *are the two non-conjugate Cartan subgroups of* $\mathrm{U}(1,1,\mathbb{C})$ *and* H_1 *is compact. Moreover, we have*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = C\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} C^{-1},$$

where the choice of the matrix C *is motivated by a work of Koranyi-Wolf* (*see* [31, Section 2]).

Remark 7.2. More particularly, the subgroups H_1 and H_2 are given by

$$\mathrm{H}_1 = \left\{ \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix}, \theta_1, \theta_2 \in \mathbb{R} \right\}$$

and
$$\begin{aligned} \mathrm{H}_2 &= \left\{ \begin{pmatrix} e^{i\theta_1}\operatorname{ch}(X) & i\operatorname{sh}(X) \\ -i\operatorname{sh}(X) & e^{i\theta_1}\operatorname{ch}(X) \end{pmatrix}, \theta_1, X \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_1} \end{pmatrix}, \theta_1 \in \mathbb{R} \right\} . \left\{ \begin{pmatrix} \operatorname{ch}(X) & i\operatorname{sh}(X) \\ -i\operatorname{sh}(X) & \operatorname{ch}(X) \end{pmatrix}, X \in \mathbb{R} \right\} = \mathrm{T}_2\,\mathrm{A}_2\,. \end{aligned}$$

The set A_2 is the split part of H_2 (see [34, Section 2.3.6]).

Proposition 7.3. *For all element* $\widetilde{g} \in \widetilde{\mathrm{G}}'^{++}$, *we get:*

$$\Theta(\widetilde{g}) = \det(\mathrm{g})^{\frac{1}{2}}\det(\mathrm{g}-1)^{-1}.$$

Proof. Let $g \in \mathrm{U}(1,1,\mathbb{C})$ $\left(\leftrightarrow \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{R})\right)$. We get:

$$\begin{aligned} \det_{\mathrm{W}_\mathbb{C}}(\mathrm{i}(\mathrm{g}-1)) &= \det_{\mathrm{W}_\mathbb{C}}(\mathrm{g}-1) = \det_{\mathbb{C}^2}(\mathrm{g}-1)\det_{\mathbb{C}^2}(\mathrm{g}^*-1) \\ &= \det(\mathrm{g}-1)\det(\mathrm{g}^{-1}-1) = \det(\mathrm{g}-1)\det(\mathrm{g}^{-1})\det(1-\mathrm{g}) \\ &= \det(\mathrm{g})^{-1}\det(\mathrm{g}-1)^2. \end{aligned}$$

■

We can now determine $\Theta_{\Pi'_k}$ on H_2 (more particularly on A_2). According to equation (15), we get for $X > 0$:

$$\begin{aligned} &\Theta_{\Pi'_k}\left(\begin{pmatrix} \operatorname{ch}(X) & i\operatorname{sh}(X) \\ -i\operatorname{sh}(X) & \operatorname{ch}(X) \end{pmatrix}\right) = \Theta_{\Pi'_k}\left(\exp\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}\right) \\ &= \int_{\mathrm{S}^1} z^{-k}\Theta\left(z\exp\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}\right) dz \\ &= \int_{\mathrm{S}^1} z^{-k}\Theta\left(z\exp C\begin{pmatrix} -X & 0 \\ 0 & X \end{pmatrix} C^{-1}\right) dz = \int_{\mathrm{S}^1} z^{-k}\Theta\left(\begin{pmatrix} ze^{-X} & 0 \\ 0 & ze^{X} \end{pmatrix}\right) dz \\ &= \int_{\mathrm{S}^1} z^{-k+1}\det\begin{pmatrix} ze^{-X}-1 & 0 \\ 0 & ze^{X}-1 \end{pmatrix}^{-1} \mathrm{dz} = \int_{\mathrm{S}^1} \frac{\mathrm{z}^{-\mathrm{k}+1}}{(\mathrm{ze}^{\mathrm{X}}-1)(\mathrm{ze}^{-\mathrm{X}}-1)}\,\mathrm{dz} \\ &= \int_{\mathrm{S}^1} \frac{z^{-k+1}}{(z-e^{-X})(z-e^{X})}\,dz = \begin{cases} \dfrac{e^{X(k-1)}}{e^{X}-e^{-X}} & \text{if } k \geq 1, \\[2ex] \dfrac{e^{-X(k-1)}}{e^{-X}-e^{X}} & \text{otherwise.} \end{cases} \end{aligned}$$

More generally, for all $\theta \in \mathbb{R}$ and $X \in \mathbb{R}^+$, we get:

$$\begin{aligned}
&\Theta_{\Pi'_k}\left(\begin{pmatrix} e^{i\theta}\,\mathrm{ch}(X) & ie^{i\theta}\,\mathrm{sh}(X) \\ -ie^{i\theta}\,\mathrm{sh}(X) & e^{i\theta}\,\mathrm{ch}(X)\end{pmatrix}\right) = \Theta_{\Pi'_k}\left(\exp\left(\begin{pmatrix} i\theta & 0 \\ 0 & i\theta\end{pmatrix} + \begin{pmatrix} 0 & iX \\ -iX & 0\end{pmatrix}\right)\right) \\
&= \int_{\mathrm{S}^1} z^{-k}\Theta\left(z\exp\left(\begin{pmatrix} i\theta & 0 \\ 0 & i\theta\end{pmatrix} + \begin{pmatrix} 0 & iX \\ -iX & 0\end{pmatrix}\right)\right)dz \\
&= \int_{\mathrm{S}^1} z^{-k}\Theta\left(z\exp\left(C\left(\begin{pmatrix} i\theta & 0 \\ 0 & i\theta\end{pmatrix} + \begin{pmatrix} -X & 0 \\ 0 & X\end{pmatrix}\right)C^{-1}\right)\right)dz \\
&= \int_{\mathrm{S}^1} z^{-k}\Theta\left(\begin{pmatrix} ze^{-X}e^{i\theta} & 0 \\ 0 & ze^{i\theta}e^{X}\end{pmatrix}\right)dz = \int_{\mathrm{S}^1} z^{-k+1}e^{i\theta}\det\begin{pmatrix} ze^{i\theta}e^{-X}-1 & 0 \\ 0 & ze^{i\theta}e^{X}-1\end{pmatrix}^{-1}\mathrm{dz} \\
&= \int_{\mathrm{S}^1} \frac{z^{-k+1}e^{i\theta}}{(ze^{i\theta}e^{X}-1)(ze^{i\theta}e^{-X}-1)}\,dz = \int_{\mathrm{S}^1} \frac{e^{-i\theta}z^{-k+1}}{(z-e^{-i\theta}e^{-X})(z-e^{-i\theta}e^{X})}\,dz \\
&= \begin{cases} \dfrac{e^{i\theta(k-1)}e^{X(k-1)}}{e^{X}-e^{-X}} & \text{if } k \geq 1, \\[2ex] \dfrac{e^{i\theta(k-1)}e^{-X(k-1)}}{e^{-X}-e^{X}} & \text{otherwise.} \end{cases}
\end{aligned}$$

We got similar results for $X < 0$.

8. A conjecture of Przebinda

In [27], Przebinda investigated the correspondence of characters for a general dual pair. We recall here the Howe's duality theorem in this context. Let $(W, \langle\cdot,\cdot\rangle)$ be a symplectic vector space over $\mathbb{R}$, $\widetilde{\mathrm{Sp}(W)}$ the corresponding metaplectic group, $(\omega, \mathscr{H})$ the metaplectic representation of $\widetilde{\mathrm{Sp}(W)}$, $(\mathrm{G}, \mathrm{G}')$ be a dual pair in $\mathrm{Sp}(W)$ and $(\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}')$ the corresponding dual pair in $\widetilde{\mathrm{Sp}(W)}$.

We denote by $\mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ the set of equivalence classes of irreducible admissible representations of $\widetilde{\mathrm{G}}$ which are infinitisemally equivalent to a quotient of ω^∞. In [16], Howe proved that there exists a bijection between $\mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ and $\mathscr{R}(\widetilde{\mathrm{G}}', \omega)$ whose graph is $\mathscr{R}(\widetilde{\mathrm{G}} \cdot \widetilde{\mathrm{G}}', \omega)$. We denote by θ the following one-to-one map:

$$\theta : \mathscr{R}(\widetilde{\mathrm{G}}, \omega) \ni \Pi \to \Pi' = \theta(\Pi) \in \mathscr{R}(\widetilde{\mathrm{G}}', \omega).$$

Let $(\mathrm{G}, \mathrm{G}')$ be an irreducible reductive dual pair in $\mathrm{Sp}(W)$. Without loss of generality, we assume that $\mathrm{rk}(\mathrm{G}) \leq \mathrm{rk}(\mathrm{G}')$. We denote by $\{\mathrm{H_1}, \dots, \mathrm{H_n}\}$ the set of conjugacy classes of Cartan subgroups of G. As explained in [34, Section 2.3.6], for every $1 \leq i \leq n$, there exists a decomposition of $\mathrm{H_i}$ of the form

$$\mathrm{H_i} = \mathrm{T_i}\,\mathrm{A_i},$$

where T_i is compact and A_i is the split part of $\mathrm{H_i}$ (by convention, $\mathrm{H_1}$ is the compact Cartan subgroup, i.e. $\mathrm{A_1} = \{\mathrm{Id}\}$).

For every $1 \leq i \leq n$, we denote by $\mathrm{A}'_i = C_{\mathrm{Sp}(W)}(\mathrm{A}_i)$ and $\mathrm{A}''_i = C_{\mathrm{Sp}(W)}(\mathrm{A}'_i)$ (in particular, $\mathrm{A}'_1 = \mathrm{Sp}(W)$ and $\mathrm{A}''_1 = Z(\mathrm{Sp}(W)) = \{\pm 1\}$). Then, $(\mathrm{A}'_i, \mathrm{A}''_i)$ is a reductive dual pair in $\mathrm{Sp}(W)$ (not irreducible in general).

We define a measure $\overline{dw_i}$ on the quotient $\mathrm{A}_i^{''} \backslash W$ given by:

$$\int_W \phi(w)dw = \int_{\mathrm{A}'' \backslash W} \int_{\mathrm{A}_i'} \phi(a'w)d\mu_{\mathrm{A}_i'}(a')\overline{dw}.$$

In [27, Section 2], T. Przebinda defines the following distribution on $\mathrm{A}_i^{'}$:

$$\mathrm{Chc}(\Psi) = \int_{\mathrm{A}_i^{''} \backslash W} \left(\int_{\mathrm{A}_i^{'}} \Psi(\widetilde{g})T(\widetilde{g})d\widetilde{g} \right)(w)\overline{dw} \qquad (\Psi \in \mathscr{C}_c^\infty(\mathrm{A}_i^{'})).$$

As mentioned in Section 4, the integral

$$\int_{\mathrm{A}_i^{'}} \Psi(\widetilde{g})T(\widetilde{g})d\widetilde{g} \in \mathrm{S(W)},$$

in particular, $\mathrm{Chc}(\Psi)$ is well defined. Moreover, for all $\widetilde{h} \in \widetilde{\mathrm{H}}_i^{\mathrm{reg}}$, the intersection of the wave front set $\mathrm{WF(Chc)}$ of the distribution Chc with the conormal bundle of the embedding

$$\widetilde{\mathrm{G}'} \ni \widetilde{g}' \to \widetilde{h}\widetilde{g}' \in \widetilde{A_i'}$$

is empty. In particular, there is a unique restriction of the distribution Chc to $\widetilde{\mathrm{G}'}$. We denote by $\mathrm{Chc}_{\widetilde{h}}$ this restriction. In [27, Conjecture 2.18], T. Przebinda conjectured the following result:

Conjecture 8.1. (T. Przebinda) We denote by G_1' the connected component at identity of G'. We assume that $(\Theta_{\Pi'})_{\widetilde{\mathrm{G}'} \backslash \widetilde{\mathrm{G}_1'}} = 0$. For every $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}_1'})$, the character $\Theta_{\Pi'}$ of Π' is given by:

$$\Theta_{\Pi'}(\Psi) = \mathrm{K}_\Pi \sum_{i=1}^{n} \frac{1}{|\mathscr{W}(\mathrm{H_i})|} \int_{\widetilde{\mathrm{H}}_i^{\mathrm{reg}}} \overline{\Theta_\Pi(\widetilde{h})} |D(\widetilde{h})|^2 \, \mathrm{Chc}_{\widetilde{h}}(\Psi)d\widetilde{h}. \qquad (29)$$

where K_Π is a complex number depending of Π (one can check [27, Definition 2.17]).

We now explain how we can get characters by double lifting starting with a compact dual pair. To simplify the notations, we will present that for the dual pair of unitary groups (this is an ongoing project [23]). Let $(\mathrm{G} = \mathrm{U}(1,\mathbb{C}), \mathrm{G}' = \mathrm{U}(1,1,\mathbb{C}))$ in $\mathrm{Sp}(W_\mathbb{R})$, where $W_\mathbb{R} = (\mathbb{C}^1 \otimes_\mathbb{C} \mathbb{C}^{1,1})_\mathbb{R}$ and $(\mathrm{G}_1, \mathrm{G}_1') = (\mathrm{U}(1,1,\mathbb{C}), \mathrm{U}(m, m+1))$ in $\mathrm{Sp}(W_\mathbb{R}^m)$, where $W_\mathbb{R}^1 = (\mathbb{C}^{1,1} \otimes_\mathbb{C} \mathbb{C}^{m,1+m})_\mathbb{R}$. We denote by $(\omega, \mathscr{H})$ the metaplectic representation of $\widetilde{\mathrm{Sp}(W_\mathbb{R})}$, by $(\omega_m, \mathscr{H}_m)$ the metaplectic representation of $\widetilde{\mathrm{Sp}(W_\mathbb{R}^m)}$ and by θ and θ_m the two bijections

$$\theta : \mathscr{R}(\widetilde{\mathrm{G}}, \omega) \to \mathscr{R}(\widetilde{\mathrm{G}'}, \omega) \qquad \theta_m : \mathscr{R}(\widetilde{\mathrm{G}_1}, \omega_m) \to \mathscr{R}(\widetilde{\mathrm{G}_1'}, \omega_m).$$

For two positive integers r, s, we denote by $\det^{\frac{\mathrm{r-s}}{2}} - \mathrm{cov}$ the double cover of $\mathrm{U}(r, s, \mathbb{C})$ given by:

$$\left\{(g, \xi) \in \mathrm{U}(r, s, \mathbb{C}) \times \mathbb{C}^*, \xi^2 = \det(\mathrm{g})^{\mathrm{r-s}}\right\}.$$

According to [24, Section 1.2], we have:

$$\widetilde{\mathrm{G}} \approx \det^{\frac{1}{2}} - \mathrm{cov}, \qquad \widetilde{\mathrm{G}'} \approx \det^0 - \mathrm{cov}, \qquad \widetilde{\mathrm{G}_1} \approx \det^0 - \mathrm{cov} \qquad \widetilde{\mathrm{G}_1'} \approx \det^{\frac{1}{2}} - \mathrm{cov}.$$

where $\det^0 - \mathrm{cov}$ is the trivial cover.

Let $\Pi \in \mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ such that $\theta(\Pi) \neq \{0\}$. According to a result of J-S Li [21], $\theta(\Pi) \in \mathscr{R}(\widetilde{\mathrm{G}_1}, \omega_m)$ and using the persistence of Kudla [20], $\theta_m(\theta(\Pi)) \neq \{0\}$. We now assume that the dual pair $(\mathrm{G}_1, \mathrm{G}_1')$ is in the stable range (with $\mathrm{rk}(\mathrm{G}_1) \leq \mathrm{rk}(\mathrm{G}_1')$). As explained previously, the weights of the representations Π such that $\theta(\Pi) \neq \{0\}$ are well-known (see [17]). In [28], T. Przebinda proved that the formula given in equation (29) holds for $(\mathrm{G}_1, \mathrm{G}_1')$. In particular. the distribution character $\Theta_{\theta_m(\theta(\Pi))}$ is given, for all $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}_1'})$, by the following formula:

$$\Theta_{\theta_m(\theta(\Pi))}(\Psi) = \mathrm{K}_\Pi \sum_{i=1}^{2} \frac{1}{|\mathscr{W}(\mathrm{H_i})|} \int_{\widetilde{\mathrm{H}}_i^{\mathrm{reg}}} \overline{\Theta_{\theta(\Pi)}(\widetilde{h_i})} |D(\widetilde{h_i})|^2 \, \mathrm{Chc}_{\widetilde{h_i}}(\Psi) d\widetilde{h_i}.$$

where $\{\mathrm{H}_1, \mathrm{H}_2\}$ is the set of Cartan subgroups of G' (up to equivalence). The value of the character $\Theta_{\theta(\Pi)}$ on H_1 is obtained by the formula given in Proposition 6.4 (it can also be obtained using Enright's formula (see [4, Corollary 2.3]). The integral over $\widetilde{\mathrm{H}}_2^{\mathrm{reg}}$ is computed using [3, Theorem 0.9].

A. The oscillator semigroup for $\mathbf{U(1,1,\mathbb{C})}$

In this appendix, we would like to prove the equality given in equation 24 of the Proposition 6.1.

We recall here some well-known facts concerning complexifications. Let's V be a complex dimension n endowed with an antilinear involution c (i.e. $c^2 = 1$ and $c(\lambda v) = \bar{\lambda} c(v)$, $\lambda \in \mathbb{C}, v \in V$). Then, we have the decomposition:

$$V = \{v \in V, c(v) = v\} \oplus \{v \in V, c(v) = -v\} = \mathrm{Re(V,c)} \oplus \mathrm{Im(V,c)}. \tag{30}$$

Both $\mathrm{Re(V,c)}$ and $\mathrm{Im(V,c)}$ are vector spaces over $\mathbb{R}$ of dimension n. Moreover,

$$\mathrm{Re(V,c)} \ni \mathrm{v} \to \mathrm{iv} \in \mathrm{Im(V,c)}$$

is well-defined and an isomorphism of $\mathbb{R}$ vector spaces. We denote by $V_\mathbb{R}$ the vector space given in equation (30).

Example A.1. Let $V = \mathbb{C}^n$ and $c : V \to V$ given by $c(v) = \bar{v}$ the natural conjugation on $\mathbb{C}^n$. Then,

$$\mathrm{Re(V)} = \{\mathrm{x} \in \mathbb{C}^{\mathrm{n}}, \mathrm{x} = \bar{\mathrm{x}}\} = \mathbb{R}^{\mathrm{n}} \qquad \mathrm{Im(V)} = \{\mathrm{x} \in \mathbb{C}^{\mathrm{n}}, \mathrm{x} = -\bar{\mathrm{x}}\} = \mathrm{i}\mathbb{R}^{\mathrm{n}}.$$

On the vector space $V \oplus V$, we define the map $\tilde{c}$ given by

$$\tilde{c} : V \oplus V \ni (u, v) \to (c(v), c(u)) \in V \oplus V$$

is an antilinear involution. As in equation (30), we define the spaces $\mathrm{Re(V \oplus V, \tilde{c})}$ and $\mathrm{Im(V \oplus V, \tilde{c})}$. In particular, we have:

$$\mathrm{Re(V \oplus V, \tilde{c})} = \{(\mathrm{v}, \mathrm{c(v)}), \mathrm{v} \in \mathrm{V}\}.$$

Moreover, the map:

$$V_\mathbb{R} \ni v \to (v, c(v)) \in \mathrm{Re(V \oplus V, \tilde{c})} \subseteq \mathrm{V \oplus V} \tag{31}$$

is an isomorphism of real vector spaces. In particular, the complexification of $V_{\mathbb{R}}$, denoted by $(V_{\mathbb{R}})_{\mathbb{C}}$, is $V \oplus V$.

For all $(x,y) \in V \oplus V$, its "conjugate" $\tilde{c}(x,y)$ is equal to $(c(y), c(x))$. Moreover, there exists $a, b \in V$ such that

$$(x,y) = (a, c(a)) + (b, -c(b)) \in \mathrm{Re}(\mathrm{V} \oplus \mathrm{V}, \tilde{\mathrm{c}}) \oplus \mathrm{Im}(\mathrm{V} \oplus \mathrm{V}, \tilde{\mathrm{c}}).$$

More particularly, we have $a = \frac{u+c(v)}{2}$ and $b = \frac{u-c(v)}{2}$.

Let us now assume that the space V is endowed with an hermitian form $\mathrm{b_V}$ of signature (p,q) (i.e. there exists a basis $\mathscr{B}_V$ of V such that $\mathrm{F} = \mathrm{Mat}(\mathrm{b_V}, \mathscr{B}_\mathrm{V}) = \mathrm{Id_{p,q}}$). On the space $W = V_{\mathbb{R}}$, we have a natural symplectic form b given by $\mathrm{b} = \mathrm{Im}(\mathrm{b_V})$. We denote by $W_{\mathbb{C}}$ the complexification of W and by $\mathrm{b}_{\mathbb{C}}$ th corresponding symplectic form on $W_{\mathbb{C}}$. Then, using the identification given in equation (31), we get for all $w = (w_1, w_2) \in W_{\mathbb{C}} = V \oplus V$, we get:

$$\begin{aligned}
H(w,w) &= \mathrm{Im}\left(\mathrm{b_V}\left(\frac{\mathrm{w_1} + \mathrm{c(w_2)}}{2}, -\frac{\mathrm{i(w_1} - \mathrm{c(w_2))}}{2}\right)\right) \\
&= \frac{1}{4}\Big(\mathrm{Re(b_V(w_1,w_1))} - \mathrm{Re(b_V(w_1,c(w_2)))} + \mathrm{Re(b_V(c(w_2),w_1))} \\
&\qquad - \mathrm{Re(b_V(c(w_2),c(w_2)))}\Big) \\
&= \frac{1}{4}\Big(\mathrm{Re(b_V(w_1,w_1))} - \mathrm{Re(\overline{b_V(w_1,c(w_2))})} + \mathrm{Re(b_V(c(w_2),w_1))} \\
&\qquad - \mathrm{Re(b_V(c(w_2),c(w_2)))}\Big) \\
&= \frac{1}{4}\left(\mathrm{Re(b_V(w_1,w_1))} - \mathrm{Re(b_V(c(w_2),c(w_2)))}\right) \\
&= \frac{1}{4}\left(\mathrm{Re(b_V(w_1,w_1))} - \mathrm{Re(b_V(w_2,w_2))}\right)
\end{aligned}$$

For all $g \in \mathrm{Sp}(W_{\mathbb{C}})^{++}$, we get, according to equation (5), that:

$$\begin{aligned}
g \in \mathrm{G}^{++} &\Leftrightarrow H\left(g\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, g\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) < H\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) \qquad (\forall (w_1,w_2) \in V \oplus V) \\
&\Leftrightarrow \mathrm{Re(b_V(w_1,w_1))} - \mathrm{Re(b_V(w_2,w_2))} > \mathrm{Re(b_V(gw_1,gw_1))} - \mathrm{Re(b_V(g^*w_2,g^*w_2))} \\
&\Leftrightarrow \begin{cases} \mathrm{Re(b_V(w_1,w_1))} - \mathrm{Re(b_V(gw_1,gw_1))} & > 0 \\ \mathrm{Re(b_V(w_2,w_2))} - \mathrm{Re(b_V(g^*w_2,g^*w_2))} & < 0 \end{cases} \\
&\Leftrightarrow \begin{cases} \mathrm{b_V(w,w)} - \mathrm{b_V(gw,gw)} & > 0 \\ \mathrm{b_V(w,w)} - \mathrm{b_V(g^*w,g^*w)} & < 0 \end{cases} \qquad (\forall w \in \mathrm{Re(V,c)} \setminus \{0\}) \\
&\Leftrightarrow \mathrm{F} - g^* \, \mathrm{F} \, g > 0.
\end{aligned}$$

Example A.2. We assume that $V = \mathbb{C}^2$ and that the signature of $\mathrm{b_V}$ is $(1,1)$. The compact torus T is given by:

$$\mathrm{T} = \left\{ t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, t_1, t_2 \in \mathrm{U}(1, \mathbb{C}) \right\}.$$

and then, using the notations of Section 7,

$$\mathrm{F} - \bar{t}^t\,\mathrm{F}\,t < 0 \Leftrightarrow \mathrm{Id}_{1,1} - \begin{pmatrix} \bar{t_1} & 0 \\ 0 & \bar{t_2} \end{pmatrix} \mathrm{Id}_{1,1} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} > 0 \Leftrightarrow \begin{pmatrix} 1 - |t_1|^2 & 0 \\ 0 & |t_2|^2 - 1 \end{pmatrix} > 0,$$

i.e. $t \in \mathrm{T}^{++} \Leftrightarrow |t_1| < 1$ and $|t_2| > 1$.

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Received October 8, 2019
and in final form April 19, 2020