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Derivations of the Lie Algebra of Strictly Block Upper Triangular Matrices

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Abstract. Let $\mathcal{N}$ be the Lie algebra of all $n \times n$ strictly block upper triangular matrices over a field $\mathbb{F}$. Let $\mathrm{Der}(\mathcal{N})$ be Lie algebra of all derivations of $\mathcal{N}$. In this paper, we describe the elements and the structure of $\mathrm{Der}(\mathcal{N})$. We also determine the dimensions of component subalgebras of $\mathrm{Der}(\mathcal{N})$.

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1. Introduction

The main goal of this paper is to explicitly describe the derivations of the Lie algebra of strictly block upper triangular matrices over a field $\mathbb{F}$. A *derivation* of a Lie algebra $(\mathfrak{g}, [,])$ over a field (or more generally, over a ring) is a linear map $f\colon \mathfrak{g} \to \mathfrak{g}$ that satisfies the following Lie derivation property for all $X, Y \in \mathfrak{g}$:

$$f([X,Y]) = [f(X), Y] + [X, f(Y)]. \tag{1}$$

Let $\mathrm{Der}(\mathfrak{g})$ denote the set of all derivations of $\mathfrak{g}$, which itself forms a Lie algebra, called the *derivation algebra* of $\mathfrak{g}$, with the Lie bracket $[f,g] = f \circ g - g \circ f$ for all $f, g \in \mathrm{Der}(\mathfrak{g})$ [9, p.15].

The following elements of $\mathrm{End}(\mathfrak{g})$ are typical examples of derivations:

1. For $X \in \mathfrak{g}$, the linear map $\mathrm{ad}\, X$ defined by $\mathrm{ad}\, X(Y) = [X, Y]$ for all $Y \in \mathfrak{g}$ is called an *inner derivation;* the other derivations are called *outer derivations.* All inner derivations of $\mathfrak{g}$ form an ideal $\mathrm{Inn}(\mathfrak{g})$ of $\mathrm{Der}(\mathfrak{g})$.
2. Any linear map f that maps $\mathfrak{g}$ to the *center* $\mathrm{Z}(\mathfrak{g})$ of $\mathfrak{g}$ and maps $[\mathfrak{g}, \mathfrak{g}]$ to zero is a derivation, called a *central derivation.* All central derivations of $\mathfrak{g}$ form an ideal $\mathrm{Cen}(\mathfrak{g})$ of $\mathrm{Der}(\mathfrak{g})$.

The structures of a Lie algebra and its derivation algebra are closely linked. Numerous works have been done on this subject. Jacobson proved that when $\mathrm{char}(\mathbb{F}) = 0$, any Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ with a nonsingular derivation is nilpotent, and every nilpotent Lie algebra has a non-inner derivation [11]. Leger shown that when $\mathrm{char}(\mathbb{F}) = 0$, if $\mathrm{Z}(\mathfrak{g}) \neq 0$ and $\mathrm{Der}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$, then $\mathfrak{g}$ is not solvable and its radical is nilpotent [13]. Tôgô generalized Leger's result to determine Lie algebras $\mathfrak{g}$ such that

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$\mathrm{Cen}(\mathfrak{g}) \subset \mathrm{Inn}(\mathfrak{g})$, $\mathrm{Cen}(\mathfrak{g}) \supset \mathrm{Inn}(\mathfrak{g})$, or $\mathrm{Der}(\mathfrak{g}) = \mathrm{Cen}(\mathfrak{g}) + \mathrm{Inn}(\mathfrak{g})$, respectively [18]. It is well-known that any semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ admits only inner derivations and $\mathfrak{g} \cong \mathrm{Inn}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$ [9, 17]. Brice described the derivation algebra of the parabolic subalgebras of a reductive Lie algebra over an algebraically closed field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$ or over $\mathbb{R}$, and proved the zero-product determined property of such derivation algebras [4].

Ado-Iwasawa theorem states that every finite dimensional Lie algebra over a field can be realized as a matrix Lie algebra [1, 10]. In recent years, significant progress has been made in studying the derivations and generalized derivations of matrix Lie algebras over a field or a ring. Chen determined certain generalized derivations of a parabolic subalgebra of $\mathfrak{gl}(n, \mathbb{F})$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and $|\mathbb{F}| > n \geq 3$ [5]. Ghimire and Huang described the derivations and Lie triple derivations of the Lie algebra of dominant block upper triangular matrices over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ [8]. Let R be a commutative ring with identity. Cheung characterized proper Lie derivations and gave sufficient conditions for any Lie derivation of a triangular algebra over R to be proper [6]. Du and Wang investigated the Lie derivations of 2×2 block generalized matrix algebras over R [7]. Wang, Ou, and Yu described the derivations of intermediate Lie algebras between the Lie algebra of diagonal matrices and that of upper triangular matrices in $\mathfrak{gl}(n, R)$ [20]. Wang and Yu characterized all the derivations of parabolic subalgebras of $\mathfrak{gl}(n, R)$ [21]. Ou, Wang, and Yao described the derivations of the Lie algebra of strictly upper triangular matrices in $\mathfrak{gl}(n, R)$ [15]. More recently, Benkovič described the Lie derivations and Lie triple derivations of upper triangular matrix algebras over a unital algebra [3]. Some other results on the Lie triple derivations of certain matrix Lie algebras are given in [2, 12, 14, 19].

Fix a field $\mathbb{F}$. Let $M_{m,n}$ be the set of all $m \times n$ matrices over $\mathbb{F}$, and put $M_n = M_{n,n}$. Let $\mathcal{N}$ (resp. $\mathcal{B}$) denote the set of all strictly block upper triangular matrices (resp. block upper triangular matrices) in M_n relative to a given partition. Then $\mathcal{N}$ and $\mathcal{B}$ are Lie subalgebras of $\mathfrak{gl}(n, \mathbb{F})$, i.e. M_n with the standard Lie bracket. In this paper, we explicitly determine the derivations of $\mathcal{N}$ according to $\mathrm{char}(\mathbb{F})$:

1. when $\mathrm{char}(\mathbb{F}) \neq 2$, every derivation of $\mathcal{N}$ is a sum of the adjoint action of a block upper triangular matrix in $\mathcal{B}$, a central derivation, and two special linear maps (Theorem 2.1);
2. when $\mathrm{char}(\mathbb{F}) = 2$, every derivation of $\mathcal{N}$ is a sum of the adjoint action of a block upper triangular matrix in $\mathcal{B}$, a central derivation, and four special linear maps (Theorem 2.4).

The main motivation of this work comes from Ou, Wang and Yao's results on the derivations of the Lie algebra of strictly upper triangular matrices over a commutative ring R with identity [15]. Our work on $\mathrm{Der}(\mathcal{N})$ not only generalizes the main result of Ou, Wang and Yao over a field, but also use a new approach that is promising to find the derivations of other matrix Lie algebras with appropriate block forms. The essential tools are Lemmas 3.1–3.4, where four types of product preserving linear maps between matrix spaces are determined. They frequently occur in the derivations of matrix Lie algebras and matrix algebras. In exploring the $\mathrm{Der}(\mathcal{N})$ action on $\mathcal{N}$, we factor out the effects of the adjoint actions of block upper triangular matrices and those of central derivations of $\mathcal{N}$, to investigate the other derivations.

Section 2 gives the basic notations and characterizes the elements and the structure of $\mathrm{Der}(\mathcal{N})$ for both $\mathrm{char}(\mathbb{F}) \neq 2$ and $\mathrm{char}(\mathbb{F}) = 2$ cases. Section 3 determines four types of product preserving linear maps between matrix spaces that will play essential roles in finding the derivations of $\mathcal{N}$. Section 4 presents the other lemmas and proves Theorems 2.1, 2.2, 2.4, 2.5.

2. Main results

The derivations of the Lie algebra of strictly block upper triangular matrices will be determined in this section.

2.1. Notations. Let $[n] = \{1, 2, \cdots, n\}$. Fix a field $\mathbb{F}$. Let $M_{m,n}$ (resp. M_n) be the set of $m \times n$ (resp. $n \times n$) matrices over $\mathbb{F}$. Let $S^2(\mathbb{F}^n)$ be the set of $n \times n$ symmetric matrices over $\mathbb{F}$. Let I_n denote the identity matrix in M_n. The transpose of a matrix A is denoted by A^T. A $t \times t$ *block matrix form* in M_n is represented by a sequence $(n_1, n_2, \cdots, n_t)$, where $n_i \in \mathbb{Z}^+$ for $i \in [t]$ and $n_1 + \cdots + n_t = n$. Fixing a $t \times t$ block matrix form in M_n represented by a sequence $(n_1, n_2, \cdots, n_t)$, each $A \in M_n$ can be expressed as

$$A = \left[\, A_{i,j} \,\right]_{t\times t}$$

where the (i, j) block $A_{i,j} \in M_{n_i,n_j}$. The matrix A can also be expressed as

$$A = \sum_{(i,j)\in[t]\times[t]} A^{i,j}$$

such that each $A^{i,j} \in M_n$ has $A_{i,j}$ on the (i, j) block and 0's elsewhere. A is called

- *block upper triangular* if $A_{i,j} = 0$ for all $1 \leq j < i \leq t$,
- *strictly block upper triangular* if $A_{i,j} = 0$ for all $1 \leq j \leq i \leq t$,
- *block diagonal* if $A_{i,j} = 0$ for all $i \neq j$.

When A is not given in advance, $A^{i,j}$ and similar expressions may be used to express generic matrices in M_n with 0's outside of the (i, j) block.

Let $\mathcal{B}$ (resp. $\mathcal{N}$, $\mathcal{D}$) denote the set of all block upper triangular matrices (resp. strictly block upper triangular matrices, block diagonal matrices) in M_n. They are Lie subalgebras of the Lie algebra M_n with the standard bracket operation $[A, B] = AB - BA$ for $A, B \in M_n$. Moreover, $\mathcal{B} = \mathcal{D} \ltimes \mathcal{N}$, and $\mathcal{B}$ is the normalizer of $\mathcal{N}$ in M_n. Every $X \in \mathcal{B}$ induces a derivation of $\mathcal{N}$:

$$\mathrm{ad}_{\mathcal{N}} X : \mathcal{N} \to \mathcal{N}, \quad Y \mapsto [X, Y]. \tag{2}$$

For $i, j \in [t]$, let $M_n^{i,j}$ denote the set of matrices in M_n with 0's outside of the (i, j) block. Define the *block index set* of $\mathcal{N}$ as

$$\Gamma_{\mathcal{N}} = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i < j \leq t\}. \tag{3}$$

For $(i, j) \in \Gamma_{\mathcal{N}}$, denote $\mathcal{N}^{i,j} = M_n^{i,j}$. For $\Delta \subseteq \Gamma_{\mathcal{N}}$, denote

$$\mathcal{N}^{\Delta} = \bigoplus_{(i,j)\in\Delta} \mathcal{N}^{i,j}, \qquad \mathcal{N}^{\Delta^c} = \bigoplus_{(i,j)\in\Gamma_{\mathcal{N}}\setminus\Delta} \mathcal{N}^{i,j}. \tag{4}$$

Let $E_{p,q}^{i,j}$ denote the matrix in M_n that takes 1 on the (p, q) entry of the (i, j) block and 0's elsewhere for $(i, j) \in [t] \times [t]$ and $(p, q) \in [n_i] \times [n_j]$.

2.2. Derivations of $\mathcal{N}$ for $\mathrm{char}(\mathbb{F}) \neq 2$

We describe the elements and the structure of $\mathrm{Der}(\mathcal{N})$ when $\mathrm{char}(\mathbb{F}) \neq 2$. Obviously, $\mathrm{Der}(\mathcal{N}) = 0$ when $t = 1$, and $\mathrm{Der}(\mathcal{N}) = \mathrm{End}(\mathcal{N})$ when $t = 2$. The main results for $t \geq 3$ are given in Theorems 2.1 and 2.2 below. We will give the proofs of these theorems in section 4.

Theorem 2.1. *Suppose* $\mathrm{char}(\mathbb{F}) \neq 2$. *When $t \geq 3$, every derivation f of the Lie algebra $\mathcal{N}$ can be written (not uniquely) as*

$$f = \mathrm{ad}_{\mathcal{N}} X + \varphi + \phi_1 + \phi_2 \tag{5}$$

where the RHS components are determined below:

1. $X \in \mathcal{B}$.
2. φ *is a central derivation in*

$$\mathrm{Cen}(\mathcal{N}) = \{\varphi \in \mathrm{End}(\mathcal{N}) \mid \mathrm{Ker}\,\varphi \supseteq [\mathcal{N}, \mathcal{N}],\ \mathrm{Im}\,\varphi \subseteq \mathrm{Z}(\mathcal{N})\} \tag{6}$$

   where $[\mathcal{N}, \mathcal{N}] = \mathcal{N}^{\{(i,i+1) \mid i \in [t-1]\}^c}$ *and* $\mathrm{Z}(\mathcal{N}) = \mathcal{N}^{1,t}$.
3. $\phi_1 = 0$ *except for* $n_1 = 1$, *in which* $\phi_1 \in \mathrm{End}(\mathcal{N})$ *is uniquely determined by a 3-tensor* $[x_{i,j,k}]_{i,j,k} \in \mathbb{F}^{n_2 \times n_2 \times n_t}$, *such that each cross-section matrix* $X_k = \left[\, x_{i,j,k} \,\right]_{i,j} \in S^2(\mathbb{F}^{n_2})$ *for* $k \in [n_t]$, *and*

$$\begin{cases} \mathrm{Ker}(\phi_1) \supseteq \mathcal{N}^{\{(1,2)\}^c},\ \mathrm{Im}(\phi_1) \subseteq \mathcal{N}^{2,t}, \\ \phi_1(A^{1,2}) = \sum_{k=1}^{n_t} X_k^{2,2} (A^{1,2})^T E_{1,k}^{1,t} \qquad \text{for } A^{1,2} \in \mathcal{N}^{1,2}, \end{cases} \tag{7}$$

   where $X_k^{2,2} \in M_n^{2,2}$ *has* X_k *in the* $(2,2)$ *block.*
4. $\phi_2 = 0$ *except for* $n_t = 1$, *in which* $\phi_2 \in \mathrm{End}(\mathcal{N})$ *is uniquely determined by a 3-tensor* $[y_{i,j,k}]_{i,j,k} \in \mathbb{F}^{n_{t-1} \times n_{t-1} \times n_1}$, *such that each cross-section matrix* $Y_k = \left[\, y_{i,j,k} \,\right]_{i,j} \in S^2(\mathbb{F}^{n_{t-1}})$ *for* $k \in [n_1]$, *and*

$$\begin{cases} \mathrm{Ker}\,\phi_2 \supseteq \mathcal{N}^{\{(t-1,t)\}^c},\ \mathrm{Im}(\phi_2) \subseteq \mathcal{N}^{1,t-1}, \\ \phi_2(A^{t-1,t}) = \sum_{k=1}^{n_1} E_{k,1}^{1,t} (A^{t-1,t})^T Y_k^{t-1,t-1} \qquad \text{for } A^{t-1,t} \in \mathcal{N}^{t-1,t}, \end{cases} \tag{8}$$

   where $Y_k^{t-1,t-1} \in M_n^{t-1,t-1}$ *has* Y_k *in the* $(t-1, t-1)$ *block.*

All possible elements ϕ_1 in (7) (resp. ϕ_2 in (8)) form an abelian Lie subalgebra D_1 (resp. D_2) of $\mathrm{Der}(\mathcal{N})$:

$$\mathrm{D}_1 = \{\phi \in \mathrm{Der}(\mathcal{N}) \mid \mathrm{Ker}\,\phi \supseteq \mathcal{N}^{\{(1,2)\}^c}, \mathrm{Im}\,\phi \subseteq \mathcal{N}^{2,t}\}, \tag{9}$$

$$\mathrm{D}_2 = \{\phi \in \mathrm{Der}(\mathcal{N}) \mid \mathrm{Ker}\,\phi \supseteq \mathcal{N}^{\{(t-1,t)\}^c},\ \mathrm{Im}\,\phi \subseteq \mathcal{N}^{1,t-1}\}. \tag{10}$$

For any two subsets S_1, S_2 of a Lie algebra $\mathfrak{g}$, let $[S_1, S_2]$ denote the subspace of $\mathfrak{g}$ spanned by $\{[x, y] \mid x \in S_1,\ y \in S_2\}$.

It is known from [18, Lemma 2] that $\mathrm{Inn}(\mathcal{N}) \cap \mathrm{Cen}(\mathcal{N}) = \{\mathrm{ad}\,X \mid X \in \mathcal{N},\ [X, \mathcal{N}] \subseteq \mathrm{Z}(\mathcal{N})\}$ is nonzero when $\mathcal{N}$ is nonabelian nilpotent, that is, when the matrix block form of $\mathcal{N}$ has more than one nonzero block. Based on Theorem 2.1, $\mathrm{Der}(\mathcal{N})$ can be further elaborated below.

Theorem 2.2. *Suppose* $\mathrm{char}(\mathbb{F}) \neq 2$. *When* $t \geq 3$, $\mathrm{Der}(\mathcal{N})$ *can be decomposed as a vector space direct sum of Lie subalgebras:*

$$\mathrm{Der}(\mathcal{N}) = (\mathrm{ad}_{\mathcal{N}} \mathcal{B} + \mathrm{Cen}(\mathcal{N})) \oplus \mathrm{D}_1 \oplus \mathrm{D}_2 \tag{11}$$

where the RHS components satisfy the following:

1. $\mathrm{ad}_{\mathcal{N}} \mathcal{B}$ *contains the inner derivation ideal* $\mathrm{ad}\,\mathcal{N}$ *of* $\mathrm{Der}(\mathcal{N})$, $\mathrm{Cen}(\mathcal{N})$ *is the central ideal of* $\mathrm{Der}(\mathcal{N})$, *and* $\mathrm{ad}_{\mathcal{N}} \mathcal{B} \cap \mathrm{Cen}(\mathcal{N}) = \mathrm{ad}(\mathcal{N}^{1,t-1} + \mathcal{N}^{2,t})$.
2. *When* $n_1 \neq 1$, $\mathrm{D}_1 = 0$. *When* $n_1 = 1$, D_1 *is an abelian Lie subalgebra of* $\mathrm{Der}(\mathcal{N})$.
3. *When* $n_t \neq 1$, $\mathrm{D}_2 = 0$. *When* $n_t = 1$, D_2 *is an abelian Lie subalgebra of* $\mathrm{Der}(\mathcal{N})$.
4. *The following bracket relationships hold (all unlisted cases are zero):*

	$\mathrm{Cen}(\mathcal{N})$	$\mathrm{ad}_{\mathcal{N}} \mathcal{B}$	D_1	D_2
$\mathrm{Cen}(\mathcal{N})$	0	$\mathrm{Cen}(\mathcal{N})$	$\mathrm{ad}\,\mathcal{N}^{2,3}$ *if* $t = 3, n_1 = 1$	$\mathrm{ad}\,\mathcal{N}^{1,2}$ *if* $t = 3, n_t = 1$
$\mathrm{ad}_{\mathcal{N}} \mathcal{B}$	$\mathrm{Cen}(\mathcal{N})$	$\mathrm{ad}_{\mathcal{N}} \mathcal{B}_0$	$\mathrm{ad}\,\mathcal{N}^{2,t} \oplus \mathrm{D}_1$ *if* $n_1 = 1$	$\mathrm{ad}\,\mathcal{N}^{1,t-1} \oplus \mathrm{D}_2$ *if* $n_t = 1$
D_1	$\mathrm{ad}\,\mathcal{N}^{2,3}$ *if* $t = 3, n_1 = 1$	$\mathrm{ad}\,\mathcal{N}^{2,t} \oplus \mathrm{D}_1$ *if* $n_1 = 1$	0	$\mathrm{ad}\,M_n^{2,2}$ *if* $t = 3, n_1 = n_t = 1$
D_2	$\mathrm{ad}\,\mathcal{N}^{1,2}$ *if* $t = 3, n_t = 1$	$\mathrm{ad}\,\mathcal{N}^{1,t-1} \oplus \mathrm{D}_2$ *if* $n_t = 1$	$\mathrm{ad}\,M_n^{2,2}$ *if* $t = 3, n_1 = n_t = 1$	0

Table 1

where $\mathcal{B}_0 = [\mathcal{B}, \mathcal{B}]$ *is the Lie algebra of all block upper triangular matrices with trace zero on each diagonal block.*

5. *The dimensions of component subalgebras are listed below.*

subalgebra	**dimension**
$\mathrm{ad}_{\mathcal{N}} \mathcal{B}$	$\binom{n+1}{2} + \sum_{i=1}^{t} \binom{n_i}{2} - n_1 n_t - 1$
$\mathrm{Cen}(\mathcal{N})$	$n_1 n_t \left(\sum_{i=1}^{t-1} n_i n_{i+1}\right)$
$\mathrm{ad}_{\mathcal{N}} \mathcal{B} \cap \mathrm{Cen}(\mathcal{N})$	$n_1 n_{t-1} + n_2 n_t$
D_1	$\binom{n_2+1}{2} n_t$ *if* $n_1 = 1$; 0 *otherwise.*
D_2	$\binom{n_{t-1}+1}{2} n_1$ *if* $n_t = 1$; 0 *otherwise.*

(12)

2.3. Derivations of $\mathcal{N}$ for $\mathrm{char}(\mathbb{F}) = 2$

In this case, $\mathrm{Der}(\mathcal{N})$ has more different types of elements. Let us examine the following case.

Example 2.3. Suppose $\mathrm{char}(\mathbb{F}) = 2$, $n = 4$, $t = 4$, and $(n_1, n_2, n_3, n_4) = (1, 1, 1, 1)$. Let $E^{i,j}$ be the matrix in M_4 that has the only nonzero entry 1 in the (i, j) position. Then $\mathcal{N}$ has a basis $S = \{E^{1,2}, E^{1,3}, E^{1,4}, E^{2,3}, E^{2,4}, E^{3,4}\}$.

Choose $f \in \mathrm{End}(\mathcal{N})$ such that $f(E^{1,2}) = E^{3,4}$, $f(E^{1,3}) = E^{2,4}$, and $f(E) = 0$ for all other $E \in S$. It is easy to verify that for any $E, E' \in S$:

$$f([E,E']) = [f(E),E'] + [E,f(E')]. \tag{13}$$

Hence $f \in \mathrm{Der}(\mathcal{N})$. However, f can not be expressed by (5), since otherwise $f(E^{1,2}) \subseteq \mathrm{Span}(E^{1,2}, E^{1,3}, E^{1,4})$, which is a contradiction.

The above example will be explained by Lemmas 4.2 and 4.3. It can also be viewed as a special case of [15, Section 2(D)]. The following two theorems describe the elements and the structure of $\mathrm{Der}(\mathcal{N})$ for $\mathrm{char}(\mathbb{F}) = 2$.

Theorem 2.4. *Suppose* $\mathrm{char}(\mathbb{F}) = 2$. *When* $t \le 3$, $\mathrm{Der}(\mathcal{N})$ *is the same as in* $\mathrm{char}(\mathbb{F}) \neq 2$ *case. When* $t > 3$, *every* $f \in \mathrm{Der}(\mathcal{N})$ *can be decomposed (not uniquely) as*

$$f = \mathrm{ad}_{\mathcal{N}} X + \varphi + \phi_1 + \phi_2 + \phi_3 + \phi_4 \tag{14}$$

where $\mathrm{ad}_{\mathcal{N}} X$, φ, ϕ_1, ϕ_2 *are the same as in Theorem 2.1, and* ϕ_3 *and* ϕ_4 *are determined as follow:*

1. $\phi_3 = 0$ *except for* $n_1 = n_2 = 1$, *in which* $\phi_3 \in \mathrm{End}(\mathcal{N})$ *is determined by a constant matrix* $\phi_3(E^{1,2}_{1,1}) \in \mathcal{N}^{3,t}$ *such that*

$$\begin{cases} \mathrm{Ker}\,\phi_3 \supseteq \mathcal{N}^{\{(1,2),(1,3)\}^c}, \\ \phi_3(\mathcal{N}^{1,2}) \subseteq \mathcal{N}^{3,t}, \ \phi_3(\mathcal{N}^{1,3}) \subseteq \mathcal{N}^{2,t}, \\ \phi_3(A^{1,3}) = E^{2,1}_{1,1} A^{1,3} \phi_3(E^{1,2}_{1,1}) \qquad \text{for } A^{1,3} \in \mathcal{N}^{1,3}. \end{cases} \tag{15}$$

2. $\phi_4 = 0$ *except for* $n_t = n_{t-1} = 1$, *in which* $\phi_4 \in \mathrm{End}(\mathcal{N})$ *is determined by a constant matrix* $\phi_4(E^{t-1,t}_{1,1}) \in \mathcal{N}^{1,t-2}$ *such that*

$$\begin{cases} \mathrm{Ker}\,\phi_4 \supseteq \mathcal{N}^{\{(t-1,t),(t-2,t)\}^c}, \\ \phi_4(\mathcal{N}^{t-1,t}) \subseteq \mathcal{N}^{1,t-2}, \ \phi_4(\mathcal{N}^{t-2,t}) \subseteq \mathcal{N}^{1,t-1}, \\ \phi_4(A^{t-2,t}) = \phi_4(E^{t-1,t}_{1,1}) A^{t-2,t} E^{t,t-1}_{1,1} \qquad \text{for } A^{t-2,t} \in \mathcal{N}^{t-2,t}. \end{cases} \tag{16}$$

The structure of Lie algebra $\mathrm{Der}(\mathcal{N})$ is given in the next theorem. Let D_3 be the set of $\phi_3 \in \mathrm{End}(\mathcal{N})$ that satisfy (15), and D_4 the set of $\phi_4 \in \mathrm{End}(\mathcal{N})$ that satisfy (16).

Theorem 2.5. *Suppose* $\mathrm{char}(\mathbb{F}) = 2$. *When* $t \le 3$, $\mathrm{Der}(\mathcal{N})$ *is the same as in* $\mathrm{char}(\mathbb{F}) \neq 2$ *case. When* $t > 3$, $\mathrm{Der}(\mathcal{N})$ *is a vector space direct sum of Lie subalgebras:*

$$\mathrm{Der}(\mathcal{N}) = (\mathrm{ad}_{\mathcal{N}} \mathcal{B} + \mathrm{Cen}(\mathcal{N})) \oplus \mathrm{D}_1 \oplus \mathrm{D}_2 \oplus \mathrm{D}_3 \oplus \mathrm{D}_4 \tag{17}$$

where D_1 *and* D_2 *are the same as in Theorem 2.2, and the following claims hold:*

1. $\mathrm{D}_3 = 0$ *except for* $n_1 = n_2 = 1$, *in which* D_3 *is an abelian subalgebra of* $\mathrm{Der}(\mathcal{N})$ *of dimension* $n_3 n_t$.

2. $\mathrm{D}_4 = 0$ *except for* $n_t = n_{t-1} = 1$, *in which* D_4 *is an abelian subalgebra of* $\mathrm{Der}(\mathcal{N})$ *of dimension* $n_1 n_{t-2}$.

We will give the proofs of the above theorems in section 4.

3. Linear maps preserving matrix products

The Lie derivation property (1) over a matrix Lie algebra is closely relative to some matrix product preserving properties. Their relationships are much obvious when the Lie algebra consists of block matrices. Here we will determine linear maps that preserve four different types of matrix products. These maps play essential roles in exploring the derivations of $\mathcal{N}$ as well as other Lie algebras of block matrices. They will also be helpful in studying the derivations of matrix algebras, since every matrix algebra in M_n is conjugately isomorphic to a subalgebra of $\mathcal{B}$ (with an appropriate partition) determined by a set of linear equalities on the blocks [16, Theorem 1.1].

In Lemmas 3.1 – 3.4, let $E^{p,q}_{m\times n}$ denote the $m\times n$ matrix that has the only nonzero entry 1 in the (p,q) position.

Lemma 3.1. *If linear maps $\phi\colon M_{m,p}\to M_{m,q}$ and $\varphi\colon M_{n,p}\to M_{n,q}$ satisfy that*

$$\phi(AB)=A\varphi(B) \qquad \text{for all } A\in M_{m,n},\ B\in M_{n,p},$$

then there is $X\in M_{p,q}$ with $\phi(C)=CX$ for $C\in M_{m,p}$ and $\varphi(D)=DX$ for $D\in M_{n,p}$.

Proof. For any $j\in[n]$ and $B\in M_{n,p}$, we have $\phi(E^{1,j}_{m\times n}B)=E^{1,j}_{m\times n}\varphi(B)$. Let $R^1_{m,p}$ denote the subspace of $M_{m,p}$ consisting of matrices with 0's outside of the first row. Similarly for $R^1_{m,q}$. Then for every $j\in[n]$, $R^1_{m,p}=E^{1,j}_{m\times n}M_{n,p}$, so that

$$\phi(R^1_{m,p})=\phi(E^{1,j}_{m\times n}M_{n,p})=E^{1,j}_{m\times n}\varphi(M_{n,p})\subseteq R^1_{m,q}.$$

There exists an $X\in M_{p,q}$ such that the linear transformation $\phi|_{R^1_{m,p}}\colon R^1_{m,p}\to R^1_{m,q}$ can be expressed as

$$\phi|_{R^1_{m,p}}(T)=TX, \quad \text{for all } T\in R^1_{m,p}.$$

Then for every $B\in M_{n,p}$, we have $E^{1,j}_{m\times n}\varphi(B)=\phi(E^{1,j}_{m\times n}B)=E^{1,j}_{m\times n}BX$. Therefore, $\varphi(B)=BX$ for $B\in M_{n,p}$. Hence $\phi(AB)=A\varphi(B)=ABX$ for every $A\in M_{m,n}$ and $B\in M_{n,p}$. The linear combinations of all such AB form $M_{m,p}$. So $\phi(C)=CX$ for all $C\in M_{m,p}$. ■

Lemma 3.2. *If linear maps $\phi\colon M_{m,p}\to M_{n,p}$ and $\varphi\colon M_{m,q}\to M_{n,q}$ satisfy that*

$$\phi(BA)=\varphi(B)A \qquad \text{for all} \quad A\in M_{q,p},\ B\in M_{m,q},$$

then there is $X\in M_{n,m}$ with $\phi(C)=XC$ for $C\in M_{m,p}$ and $\varphi(D)=XD$ for $D\in M_{m,q}$.

Proof. The proof (omitted) is similar to that of Lemma 3.1. ■

Lemma 3.3. *If linear maps $\phi\colon M_{m,p}\to M_{m,q}$ and $\varphi\colon M_{q,n}\to M_{p,n}$ satisfy*

$$\phi(A)B=A\varphi(B) \qquad \text{for all} \quad A\in M_{m,p}, \quad B\in M_{q,n},$$

then there is $X\in M_{p,q}$ with $\phi(C)=CX$ for $C\in M_{m,p}$ and $\varphi(D)=XD$ for $D\in M_{q,n}$.

Proof. For any $j \in [p]$ and any $E^{k,l}_{q\times n} \in M_{q,n}$, we have

$$\phi(E^{1,j}_{m\times p})E^{k,l}_{q\times n} = E^{1,j}_{m\times p}\varphi(E^{k,l}_{q\times n}),$$

which shows that the only possibly nonzero row of $\phi(E^{1,j}_{m\times p})$ is the first. So ϕ maps the first row of $M_{m,p}$ to the first row of $M_{m,q}$. There exists $X \in M_{p,q}$ such that

$$E^{1,j}_{m\times p}\varphi(E^{k,l}_{q\times n}) = \phi(E^{1,j}_{m\times p})E^{k,l}_{q\times n} = E^{1,j}_{m\times p}XE^{k,l}_{q\times n}, \quad \text{for all } j \in [p], \quad E^{k,l}_{q\times n} \in M_{q,n}.$$

Hence $\varphi(E^{k,l}_{q\times n}) = XE^{k,l}_{q\times n}$ for all $E^{k,l}_{q\times n} \in M_{q,n}$. Therefore, $\varphi(B) = XB$ for $B \in M_{q,n}$. Now $\phi(A)B = AXB$ for $A \in M_{m,p}$ and $B \in M_{q,n}$, so that $\phi(A) = AX$ for $A \in M_{m,p}$. ∎

Lemma 3.4. *If linear maps $\phi : M_{p,q} \to M_{p,q}$, $\varphi : M_{q,r} \to M_{q,r}$, and $\psi : M_{p,r} \to M_{p,r}$ satisfy that*

$$\phi(A)B + A\varphi(B) = \psi(AB) \quad \text{for} \quad A \in M_{p,q}, \quad B \in M_{q,r}, \tag{18}$$

then there exist $X \in M_p, Y \in M_q, Z \in M_r$ such that

$$\phi(A) = XA - AY \qquad \text{for } A \in M_{p,q}, \tag{19}$$

$$\varphi(B) = YB - BZ \qquad \text{for } B \in M_{q,r}, \tag{20}$$

$$\psi(C) = XC - CZ \qquad \text{for } C \in M_{p,r}. \tag{21}$$

Moreover, any triple $(X', Y', Z') \in M_p \times M_q \times M_r$ satisfies (19), (20), *and* (21) *after replacing (X, Y, Z) by (X', Y', Z') if and only if there is $\lambda \in \mathbb{F}$ such that*

$$X' = X + \lambda I_p, \qquad Y' = Y + \lambda I_q, \qquad Z' = Z + \lambda I_r. \tag{22}$$

Proof. In this proof only, the (i, j) entry of a matrix B is expressed as $B_{i,j}$. For any $E^{n,j}_{p\times q} \in M_{p,q}$ and $E^{k,m}_{q\times r} \in M_{q,r}$,

$$\psi(E^{n,j}_{p\times q}E^{k,m}_{q\times r}) = \phi(E^{n,j}_{p\times q})E^{k,m}_{q\times r} + E^{n,j}_{p\times q}\varphi(E^{k,m}_{q\times r}). \tag{23}$$

There are two cases:

1. $j \neq k$: the LHS of (23) is zero and
$$\phi(E^{n,j}_{p\times q})E^{k,m}_{q\times r} = -E^{n,j}_{p\times q}\varphi(E^{k,m}_{q\times r}). \tag{24}$$

2. $j = k$: the LHS of (23) is $\psi(E^{n,m}_{p\times r})$, whose possibly nonzero entries are on row n and column m according to the RHS of (23):
$$\psi(E^{n,m}_{p\times r})_{i,m} = \phi(E^{n,k}_{p\times q})_{i,k} \qquad \text{for all } i \in [p] \setminus \{n\}, \tag{25}$$
$$\psi(E^{n,m}_{p\times r})_{n,l} = \varphi(E^{k,m}_{q\times r})_{k,l} \qquad \text{for all } l \in [r] \setminus \{m\}, \tag{26}$$
$$\psi(E^{n,m}_{p\times r})_{n,m} = \phi(E^{n,k}_{p\times q})_{n,k} + \varphi(E^{k,m}_{q\times r})_{k,m}. \tag{27}$$

   The (25) implies that $\psi(E^{n,m}_{p\times r})_{i,m} = \psi(E^{n,1}_{p\times r})_{i,1}$ when $i \neq n$. The (26) implies that $\psi(E^{n,m}_{p\times r})_{n,l} = \psi(E^{1,m}_{p\times r})_{1,l}$ when $l \neq m$. The (27) implies that
$$\psi(E^{n,m}_{p\times r})_{n,m} = \psi(E^{n,1}_{p\times r})_{n,1} + \psi(E^{1,m}_{p\times r})_{1,m} - \psi(E^{1,1}_{p\times r})_{1,1}.$$

Build the following matrices:

- $X_0 \in M_p$ with the (i,j) entry $\psi(E^{j,1}_{p\times r})_{i,1}$ for $i,j \in [p]$;
- $X = X_0 - \psi(E^{1,1}_{p\times r})_{1,1} I_p \in M_p$;
- $Z \in M_r$ with the (i,j) entry $-\psi(E^{1,i}_{p\times r})_{1,j}$ for $i,j \in [r]$.

Define a linear map $\psi' \colon M_{p,r} \to M_{p,r}$ by $\psi'(C) = XC - CZ$. A straightforward calculation together with the implications derived after (27) show that $\psi(E^{n,m}_{p\times r}) = \psi'(E^{n,m}_{p\times r})$ for all $n \in [p]$ and $m \in [r]$. Therefore, $\psi = \psi'$ and (21) holds.

Now for $A \in M_{p,q}$ and $B \in M_{q,r}$, by (18),

$$\phi(A)B + A\varphi(B) = \psi(AB) = XAB - ABZ \Longrightarrow (-\phi(A) + XA)B = A(BZ + \varphi(B)).$$

Applying Lemma 3.3 to $\phi_1 \colon M_{p,q} \to M_{p,q}$ defined by $\phi_1(A) = -\phi(A) + XA$ and $\varphi_1 \colon M_{q,r} \to M_{q,r}$ defined by $\varphi_1(B) = BZ + \varphi(B)$, we can find $Y \in M_q$ such that

$$\begin{aligned} -\phi(A) + XA &= \phi_1(A) = AY \qquad \text{for } A \in M_{p,q}, \\ BZ + \varphi(B) &= \varphi_1(B) = YB \qquad \text{for } B \in M_{q,r}, \end{aligned}$$

which lead to (19) and (20).

Finally, if $(X', Y', Z') \in M_p \times M_q \times M_r$ satisfies (19), (20), and (21) after replacing (X, Y, Z) by (X', Y', Z'), then for any $A \in M_{p,q}$, $\phi(A) = XA - AY = X'A - AY'$ so that

$$(X' - X)A = A(Y' - Y). \tag{28}$$

Let A go through all $E^{i,j}_{p\times q} \in M_{p,q}$ and compare both sides of (28). We will have $X' - X = \lambda I_p$ and $Y' - Y = \lambda I_q$ for some $\lambda \in \mathbb{F}$. Similarly, $Z' - Z = \lambda I_r$. So (22) is proved. ■

4. Proofs of the main results

The main goal of this section is to prove Theorems 2.1, 2.2, 2.4, 2.5. We always assume that $t \geq 3$ in the following discussion.

4.1. Derivation image locations

First we will give several auxiliary results on the image locations of $f(\mathcal{N}^{i,j})$ for $f \in \mathrm{Der}(\mathcal{N})$ and $\mathcal{N}^{i,j} \subseteq \mathcal{N}$. We will observe the following interesting fact: most nonzero blocks of $f(A^{i,j})$ for $A^{i,j} \in \mathcal{N}^{i,j}$ are located on the i-th block row and the j-th block column. It gives a hint that a generic derivation may be close to $\mathrm{ad}_{\mathcal{N}} X$ for some $X \in \mathcal{B}$.

The first lemma discusses the derivation image on $\mathcal{N}^{2,3}, \mathcal{N}^{3,4}, \cdots, \mathcal{N}^{t-2,t-1}$.

Lemma 4.1. *For $f \in \mathrm{Der}(\mathcal{N})$ and $1 < i < t-1$,*

$$f(\mathcal{N}^{i,i+1}) \subseteq \sum_{p=1}^{i-1} \mathcal{N}^{p,i+1} + \sum_{q=i+1}^{t} \mathcal{N}^{i,q} + \mathrm{Z}(\mathcal{N}). \tag{29}$$

In other words, matrices in $f(\mathcal{N}^{i,i+1})$ $(1 < i < t-1)$ have nonzero entries only on the block row i, the block column $(i+1)$, and the center $\mathrm{Z}(\mathcal{N}) = \mathcal{N}^{1,t}$.

Proof. For $A^{i,i+1} \in \mathcal{N}^{i,i+1}$, it suffices to prove that the (p,q) block $f(A^{i,i+1})^{p,q} = 0$ for $(p,q) \in \Gamma_{\mathcal{N}}$, $p \neq i$, $q \neq i+1$, and $(p,q) \neq (1,t)$. Either $p > 1$ or $q < t$. Without loss of generality, suppose $q < t$ (similarly for $p > 1$). Then for any $A^{q,t} \in \mathcal{N}^{q,t}$, $[A^{i,i+1}, A^{q,t}] = 0$ so that

$$0 = f([A^{i,i+1}, A^{q,t}])^{p,t} = [f(A^{i,i+1}), A^{q,t}]^{p,t} + [A^{i,i+1}, f(A^{q,t})]^{p,t} = f(A^{i,i+1})^{p,q} A^{q,t}.$$

Therefore $f(A^{i,i+1})^{p,q} = 0$. ■

Next we consider the derivation image on $\mathcal{N}^{1,2}$ and $\mathcal{N}^{t-1,t}$. The case $\operatorname{char}(\mathbb{F}) \neq 2$ is simpler in the following lemma.

Lemma 4.2. *Suppose $f \in \operatorname{Der}(\mathcal{N})$. Then*

$$f(\mathcal{N}^{1,2}) \subseteq \sum_{q=2}^{t} \mathcal{N}^{1,q} + \mathcal{N}^{2,t} + \mathcal{N}^{3,t}, \tag{30}$$

$$f(\mathcal{N}^{t-1,t}) \subseteq \sum_{p=1}^{t-1} \mathcal{N}^{p,t} + \mathcal{N}^{1,t-1} + \mathcal{N}^{1,t-2}. \tag{31}$$

Furthermore, if $\operatorname{char}(\mathbb{F}) \neq 2$ or $t = 3$, then $f(\mathcal{N}^{1,2})^{3,t} = 0$ and $f(\mathcal{N}^{t-1,t})^{1,t-2} = 0$.

Proof. The case $t = 3$ is obviously true. We now assume that $t \geq 4$.
To prove (30), we show that $f(A^{1,2})^{i,j} = 0$ for any $A^{1,2} \in \mathcal{N}^{1,2}$, $1 < i < j \leq t$ and $(i,j) \notin \{(2,t),(3,t)\}$. By assumption, either $i > 3$ or $j < t$.

1. Suppose $i > 3$. For any $A^{3,i} \in \mathcal{N}^{3,i}$,

$$0 = f([A^{1,2}, A^{3,i}])^{3,j} = [f(A^{1,2}), A^{3,i}]^{3,j} + [A^{1,2}, f(A^{3,i})]^{3,j} = -A^{3,i} f(A^{1,2})^{i,j},$$

which implies that $f(A^{1,2})^{i,j} = 0$.

2. Suppose $j < t$. By assumption $j > 2$. For any $A^{j,t} \in \mathcal{N}^{j,t}$, $[A^{1,2}, A^{j,t}] = 0$ so that

$$0 = f([A^{1,2}, A^{j,t}])^{i,t} = [f(A^{1,2}), A^{j,t}]^{i,t} + [A^{1,2}, f(A^{j,t})]^{i,t} = f(A^{1,2})^{i,j} A^{j,t}.$$

Therefore, $f(A^{1,2})^{i,j} = 0$.

Next, when $\operatorname{char}(\mathbb{F}) \neq 2$, we show that $f(A^{1,2})^{3,t} = 0$. For any $A^{2,3} \in \mathcal{N}^{2,3}$,

$$\begin{aligned} 0 &= f([A^{1,2}, [A^{1,2}, A^{2,3}]])^{1,t} \\ &= [f(A^{1,2}), [A^{1,2}, A^{2,3}]]^{1,t} + [A^{1,2}, [f(A^{1,2}), A^{2,3}]]^{1,t} + [A^{1,2}, [A^{1,2}, f(A^{2,3})]]^{1,t} \\ &= -2A^{1,2} A^{2,3} f(A^{1,2})^{3,t}. \end{aligned}$$

So $0 = A^{1,2} A^{2,3} f(A^{1,2})^{3,t}$. When $A^{1,2} = 0$, obviously $f(A^{1,2})^{3,t} = 0$. Otherwise, suppose row i of $A^{1,2}$ is nonzero; then row i of $A^{1,2}A^{2,3}$ for $A^{2,3} \in \mathcal{N}^{2,3}$ could be any vector in $\mathbb{F}^{1 \times n_3}$; hence $f(A^{1,2})^{3,t} = 0$.
The proofs of (31) and $f(\mathcal{N}^{t-1,t})^{1,t-2} = 0$ when $\operatorname{char}(\mathbb{F}) \neq 2$ are similar. ■

Now we consider the derivation image on the other $\mathcal{N}^{i,j}$.

Lemma 4.3. *For* $f \in \mathrm{Der}(\mathcal{N})$, $i,j \in [t]$ *and* $j > i+1$, *the image* $f(\mathcal{N}^{i,j})$ *satisfies:*

1. *If* $\mathrm{char}(\mathbb{F}) \neq 2$ *or* $t = 3$, *then*

$$f(\mathcal{N}^{i,j}) \subseteq \sum_{p=1}^{i-1} \mathcal{N}^{p,j} + \sum_{q=j}^{t} \mathcal{N}^{i,q}. \tag{32}$$

2. *If* $\mathrm{char}(\mathbb{F}) = 2$ *and* $t > 3$, *then* (32) *holds for* $(i,j) \notin \{(1,3),(t-2,t)\}$, *and*

$$f(\mathcal{N}^{1,3}) \subseteq \sum_{q=3}^{t} \mathcal{N}^{1,q} + \mathcal{N}^{2,t}, \tag{33}$$

$$f(\mathcal{N}^{t-2,t}) \subseteq \sum_{p=1}^{t-2} \mathcal{N}^{p,t} + \mathcal{N}^{1,t-1}. \tag{34}$$

Proof. First assume $\mathrm{char}(\mathbb{F}) \neq 2$. Let $j = i+k$, $k \geq 2$. We prove (32) by induction on k.

1. $k = 2$: $\mathcal{N}^{i,i+2} = \mathcal{N}^{i,i+1}\mathcal{N}^{i+1,i+2} = [\mathcal{N}^{i,i+1}, \mathcal{N}^{i+1,i+2}]$. For $A^{i,i+1} \in \mathcal{N}^{i,i+1}$ and $A^{i+1,i+2} \in \mathcal{N}^{i+1,i+2}$, according to Lemmas 4.1 and 4.2,

$$\begin{aligned} f([A^{i,i+1}, A^{i+1,i+2}]) &= [f(A^{i,i+1}), A^{i+1,i+2}] + [A^{i,i+1}, f(A^{i+1,i+2})] \\ &\in \mathcal{N}^{i,i+2} + \sum_{p=1}^{i-1} \mathcal{N}^{p,i+2} + \sum_{q=i+3}^{t} \mathcal{N}^{i,q}. \end{aligned} \tag{35}$$

   Hence $k = 2$ is done.

2. $k > 2$: Suppose (32) holds for all $\ell < k$. Now we have

$$\mathcal{N}^{i,i+k} = \mathcal{N}^{i,i+2}\mathcal{N}^{i+2,i+k} = [\mathcal{N}^{i,i+2}, \mathcal{N}^{i+2,i+k}].$$

   For any $A^{i,i+2} \in \mathcal{N}^{i,i+2}$ and $A^{i+2,i+k} \in \mathcal{N}^{i+2,i+k}$,

$$\begin{aligned} f([A^{i,i+2}, A^{i+2,i+k}]) &= [f(A^{i,i+2}), A^{i+2,i+k}] + [A^{i,i+2}, f(A^{i+2,i+k})] \\ &\in \mathcal{N}^{i,i+k} + \sum_{p=1}^{i-1} \mathcal{N}^{p,i+k} + \sum_{q=i+k+1}^{t} \mathcal{N}^{i,q} \end{aligned} \tag{36}$$

   where (36) is due to induction hypothesis, Lemmas 4.1 and 4.2. So (32) holds for k.

3. Overall, when $\mathrm{char}(\mathbb{F}) \neq 2$, we have proved (32) for all k.

Next assume $\mathrm{char}(\mathbb{F}) = 2$. By Lemma 4.2, (35) still holds except for $i = 1$ and $i = t-2$. When $i = 1$, by Lemmas 4.1 and 4.2, for any $A^{1,2} \in \mathcal{N}^{1,2}$, $A^{2,3} \in \mathcal{N}^{2,3}$:

$$\begin{aligned} f([A^{1,2}, A^{2,3}]) = [A^{1,2}, f(A^{2,3})] + [f(A^{1,2}), A^{2,3}] &\subseteq \sum_{q=3}^{t} \mathcal{N}^{1,q} + [\mathcal{N}^{2,t} + \mathcal{N}^{3,t}, A^{2,3}] \\ &\subseteq \sum_{q=3}^{t} \mathcal{N}^{1,q} + \mathcal{N}^{2,t}. \end{aligned}$$

We get (33). The (34) is proved similarly. The relation (36) still holds when $\mathrm{char}(\mathbb{F}) = 2$; it suffices to discuss the impact of (33) and (34) when $i = 1$ or $(i,k) = (t-4,4)$. We can use analogous induction process to prove (32) for $\mathrm{char}(\mathbb{F}) = 2$. ∎

4.2. Block relations of derivation images

The above lemmas determine all possibly nonzero blocks of $f(A^{i,j})$ for $f \in \mathrm{Der}(\mathcal{N})$ and $A^{i,j} \in \mathcal{N}^{i,j} \subseteq \mathcal{N}$. The next goal is to describe the f-images on these blocks.

The next two lemmas essentially show that the f-images on most blocks are identical with the corresponding images of $\mathrm{ad}_{\mathcal{N}} X$ for a certain block upper triangular matrix $X \in \mathcal{B}$.

Lemma 4.4. *Let $f \in \mathrm{Der}(\mathcal{N})$. Then there exist $X^{p,p} \in M_n^{p,p}$ and $X^{q,q} \in M_n^{q,q}$ for $p, q \in [t]$, such that*

$$f(A^{p,q})^{p,q} = X^{p,p}A^{p,q} - A^{p,q}X^{q,q} \qquad \text{for all } A^{p,q} \in \mathcal{N}^{p,q} \subseteq \mathcal{N}, \tag{37}$$

that is,

$$f(A^{p,q})^{p,q} = \left[\sum_{i=1}^{t} X^{i,i}, A^{p,q}\right]. \tag{38}$$

Proof. For any $1 \le p < q < r \le t$, $A^{p,q} \in \mathcal{N}^{p,q}$ and $A^{q,r} \in \mathcal{N}^{q,r}$,

$$f(A^{p,q}A^{q,r}) = f([A^{p,q}, A^{q,r}]) = [f(A^{p,q}), A^{q,r}] + [A^{p,q}, f(A^{q,r})]. \tag{39}$$

Comparing the (p,r) blocks on both sides, we have

$$f(A^{p,q}A^{q,r})^{p,r} = f(A^{p,q})^{p,q}A^{q,r} + A^{p,q}f(A^{q,r})^{q,r}. \tag{40}$$

Applying Lemma 3.4, there exist $X^{p,p} \in M_n^{p,p}$, $X^{q,q} \in M_n^{q,q}$, and $X^{r,r} \in M_n^{r,r}$, with

$$\begin{aligned} f(A^{p,q})^{p,q} &= X^{p,p}A^{p,q} - A^{p,q}X^{q,q} \qquad \text{for all } A^{p,q} \in \mathcal{N}^{p,q}, \\ f(A^{q,r})^{q,r} &= X^{q,q}A^{q,r} - A^{q,r}X^{r,r} \qquad \text{for all } A^{q,r} \in \mathcal{N}^{q,r}, \\ f(A^{p,r})^{p,r} &= X^{p,p}A^{p,r} - A^{p,r}X^{r,r} \qquad \text{for all } A^{p,r} \in \mathcal{N}^{p,r}. \end{aligned}$$

Let (p,q,r) go through the triples $(1,2,3), (2,3,4), \cdots, (t-2, t-1, t)$. By (22) in Lemma 3.4, if we fix the choice of $X^{1,1}$, then $X^{2,2}, X^{3,3}, \cdots, X^{t,t}$ will be uniquely determined. ∎

In the following lemma, denote the index set by

$$\Omega = \Gamma_{\mathcal{N}} \setminus \{(1, t-1), (1, t), (2, t)\}. \tag{41}$$

Lemma 4.5. *Let $f \in \mathrm{Der}(\mathcal{N})$. Then for any $(p,q) \in \Omega$, there exists $X^{p,q} \in \mathcal{N}^{p,q}$ such that*

$$f(A^{i,p})^{i,q} = -A^{i,p}X^{p,q} \quad \text{for all } \; A^{i,p} \in \mathcal{N}^{i,p} \subseteq \mathcal{N}, \tag{42}$$

$$f(A^{q,j})^{p,j} = X^{p,q}A^{q,j} \quad \text{for all } \; A^{q,j} \in \mathcal{N}^{q,j} \subseteq \mathcal{N}. \tag{43}$$

Proof. Fixing a $(p,q) \in \Omega$, we prove (42) and (43) by the following steps.

(1) We prove (43) for $(q,j) = (t-1, t)$. By (41), $(p,q) \ne (1, t-1)$. Hence $1 < p < t-1$. For $A^{t-1,t} \in \mathcal{N}^{t-1,t}$ and $A^{1,p} \in \mathcal{N}^{1,p}$, we have

$$\begin{aligned} 0 &= f([A^{1,p}, A^{t-1,t}]) = [f(A^{1,p}), A^{t-1,t}] + [A^{1,p}, f(A^{t-1,t})] \\ &= f(A^{1,p})A^{t-1,t} + A^{1,p}f(A^{t-1,t}). \end{aligned}$$

Comparing the $(1,t)$ block of the above equality, we get

$$-f(A^{1,p})^{1,t-1}A^{t-1,t} = A^{1,p}f(A^{t-1,t})^{p,t}.$$

Define $\phi\colon \mathcal{N}^{1,p} \to \mathcal{N}^{1,t-1}$ by $\phi(C) = -f(C)^{1,t-1}$ for $C \in \mathcal{N}^{1,p}$, and the map $\varphi : \mathcal{N}^{t-1,t} \to \mathcal{N}^{p,t}$ by $\varphi(D) = f(D)^{p,t}$ for $D \in \mathcal{N}^{t-1,t}$. Using Lemma 3.3, we can find $X^{p,t-1} \in \mathcal{N}^{p,t-1}$ such that $f(A^{t-1,t})^{p,t} = X^{p,t-1}A^{t-1,t}$ for all $A^{t-1,t} \in \mathcal{N}^{t-1,t}$. So (43) holds for $(q,j) = (t-1,t)$.

(2) Similarly, according to Lemma 3.3, (42) holds for $(i,p) = (1,2)$. In other words, when $2 < q < t$, there is $-Y^{2,q} \in \mathcal{N}^{2,q}$ such that $f(A^{1,2})^{1,q} = -A^{1,2}Y^{2,q}$ for all $A^{1,2} \in \mathcal{N}^{1,2}$.

(3) Now we prove (43) for $1 \le q < j \le t$ and $(q,j) \neq (t-1,t)$. We must have $q < t-1$. Given $j' \in [t]$ and $j' > j$, we have $\mathcal{N}^{q,j'} = \mathcal{N}^{q,j}\mathcal{N}^{j,j'} = [\mathcal{N}^{q,j}, \mathcal{N}^{j,j'}]$. Hence for any $A^{q,j} \in \mathcal{N}^{q,j}$ and $A^{j,j'} \in \mathcal{N}^{j,j'}$,

$$\begin{aligned} f(A^{q,j}A^{j,j'})^{p,j'} &= f([A^{q,j}, A^{j,j'}])^{p,j'} = [f(A^{q,j}), A^{j,j'}]^{p,j'} + [A^{q,j}, f(A^{j,j'})]^{p,j'} \\ &= f(A^{q,j})^{p,j}A^{j,j'}. \end{aligned}$$

Define $\phi_1\colon \mathcal{N}^{q,j'} \to \mathcal{N}^{p,j'}$ by $\phi_1(C) = f(C)^{p,j'}$ for $C \in \mathcal{N}^{q,j'}$, and $\varphi_1\colon \mathcal{N}^{q,j} \to \mathcal{N}^{p,j}$ by $\varphi_1(D) = f(D)^{p,j}$ for $D \in \mathcal{N}^{q,j}$. Applying Lemma 3.2, we can find $X^{p,q} \in \mathcal{N}^{p,q}$ such that $f(A^{q,j})^{p,j} = X^{p,q}A^{q,j}$ for all $A^{q,j} \in \mathcal{N}^{q,j}$ and $(q,j) \neq (t-1,t)$.

(4) Similarly, using Lemma 3.1, we can prove (42) for $(i,p) \neq (1,2)$. In other words, there exists $-Y^{p,q} \in \mathcal{N}^{p,q}$ such that $f(A^{i,p})^{i,q} = -A^{i,p}Y^{p,q}$ for all $A^{i,p} \in \mathcal{N}^{i,p}$ and $(i,p) \neq (1,2)$.

(5) Finally, when $i < p < q < j$, for any $A^{i,p} \in \mathcal{N}^{i,p}$ and $A^{q,j} \in \mathcal{N}^{q,j}$, we have $[A^{i,p}, A^{q,j}] = 0$, so that

$$\begin{aligned} 0 &= f([A^{i,p}, A^{q,j}])^{i,j} = [f(A^{i,p}), A^{q,j}]^{i,j} + [A^{i,p}, f(A^{q,j})]^{i,j} \\ &= f(A^{i,p})^{i,q}A^{q,j} + A^{i,p}f(A^{q,j})^{p,j} = -A^{i,p}Y^{p,q}A^{q,j} + A^{i,p}X^{p,q}A^{q,j}. \end{aligned}$$

Therefore, $X^{p,q} = Y^{p,q}$. The equalities (42) and (43) are proved. ■

According to Lemmas 4.1, 4.2, 4.3, the possibly nonzero blocks of f-images not described by (37), (42) and (43) are $f(\mathcal{N}^{i,i+1})^{1,t}$ for $i \in [t-1]$, $f(\mathcal{N}^{1,2})^{2,t}$, $f(\mathcal{N}^{t-1,t})^{1,t-1}$, and the following blocks when $\operatorname{char}(\mathbb{F}) = 2$:

$$f(\mathcal{N}^{1,2})^{3,t}, \qquad f(\mathcal{N}^{1,3})^{2,t}, \qquad f(\mathcal{N}^{t-1,t})^{1,t-2}, \qquad f(\mathcal{N}^{t-2,t})^{1,t-1}. \tag{44}$$

It turns out that $f(\mathcal{N}^{i,i+1})^{1,t}$ for $i \in [t-1]$ could be the image of any linear map due to the facts $\mathcal{N} = \bigoplus_{i=1}^{t-1} \mathcal{N}^{i,i+1} \oplus [\mathcal{N}, \mathcal{N}]$, $\mathrm{Z}(\mathcal{N}) = \mathcal{N}^{1,t}$, and the following classical result.

Lemma 4.6. *Let $\mathfrak{g}$ be a Lie algebra. Then any $f \in \operatorname{End}(\mathfrak{g})$ such that $\operatorname{Ker}(f) \supseteq [\mathfrak{g}, \mathfrak{g}]$ and $\operatorname{Im}(f) \subseteq \mathrm{Z}(\mathfrak{g})$ is in $\operatorname{Der}(\mathfrak{g})$.*

Proof. For any $A, B \in \mathfrak{g}$, we have $f([A,B]) = 0 = [f(A), B] + [A, f(B)]$. ■

The next two lemmas explicitly describe $f(\mathcal{N}^{1,2})^{2,t}$ and $f(\mathcal{N}^{t-1,t})^{1,t-1}$.

Lemma 4.7. *For $f \in \operatorname{Der}(\mathcal{N})$, $f(\mathcal{N}^{1,2})^{2,t}$ satisfies that:*

(1) *If $n_1 \ge 2$, then $f(\mathcal{N}^{1,2})^{2,t} = 0$.*

(2) If $n_1 = 1$, then $E^{1,2}_{1,i} f(E^{1,2}_{1,j})^{2,t} = E^{1,2}_{1,j} f(E^{1,2}_{1,i})^{2,t}$ *for any $i, j \in [n_2]$.* (45)

Moreover, if $n_1 = 1$, any $g \in \operatorname{End}(\mathcal{N})$ that satisfies $\operatorname{Ker} g \supseteq \mathcal{N}^{\{(1,2)\}^c}$, $\operatorname{Im} g \subseteq \mathcal{N}^{2,t}$, and (45), must be in $\operatorname{Der}(\mathcal{N})$.

We remark that (45) is equivalent to that row i of the $(2,t)$ block of $f(E^{1,2}_{1,j})$ equals row j of the $(2,t)$ block of $f(E^{1,2}_{1,i})$ for any $i,j\in[n_2]$.

Proof. Fix $f\in\mathrm{Der}(\mathcal{N})$. For any $A^{1,2},B^{1,2}\in\mathcal{N}^{1,2}$,

$$\begin{aligned}0 &= f([A^{1,2},B^{1,2}])^{1,t}=[f(A^{1,2}),B^{1,2}]^{1,t}+[A^{1,2},f(B^{1,2})]^{1,t}\\ &= -B^{1,2}f(A^{1,2})^{2,t}+A^{1,2}f(B^{1,2})^{2,t}.\end{aligned}$$

Therefore,
$$A^{1,2}f(B^{1,2})^{2,t}=B^{1,2}f(A^{1,2})^{2,t}. \tag{46}$$

In particular,
$$E^{1,2}_{i,j}f(E^{1,2}_{r,s})^{2,t}=E^{1,2}_{r,s}f(E^{1,2}_{i,j})^{2,t} \tag{47}$$

for any $i,r\in[n_1]$, $j,s\in[n_2]$, and we get (45). When $n_1\geq 2$, given any $E^{1,2}_{i,j}\in\mathcal{N}^{1,2}$, we can choose $r\in[n_1]\setminus\{i\}$. In (47), all rows except row i of the LHS are zero, but all rows except row r of the RHS are zero; hence both sides are zero. Since $s\in[n_2]$ is arbitrary, we have $f(E^{1,2}_{i,j})^{2,t}=0$. Therefore, $f(\mathcal{N}^{1,2})^{2,t}=0$. The last claim is straightforward to verify. ∎

Lemma 4.8. *For $f\in\mathrm{Der}(\mathcal{N})$, the image $f(\mathcal{N}^{t-1,t})^{1,t-1}$ satisfies the following properties:*

(1) *If $n_t\geq 2$, then $f(\mathcal{N}^{t-1,t})^{1,t-1}=0$.*

(2) *If $n_t=1$, then*

$$f(E^{t-1,t}_{j,1})^{1,t-1}E^{t-1,t}_{i,1}=f(E^{t-1,t}_{i,1})^{1,t-1}E^{t-1,t}_{j,1}\quad \text{for any } i,j\in[n_{t-1}]. \tag{48}$$

Moreover, when $n_t=1$, any $g\in\mathrm{End}(\mathcal{N})$ that satisfies $\mathrm{Ker}\,g\supseteq\mathcal{N}^{\{(t-1,t)\}^c}$, $\mathrm{Im}\,g\subseteq\mathcal{N}^{1,t-1}$, and (48), must be in $\mathrm{Der}(\mathcal{N})$.

Equation (48) is equivalent to that column i of the $(1,t\!-\!1)$ block of $f(E^{t-1,t}_{j,1})$ equals column j of the $(1,t\!-\!1)$ block of $f(E^{t-1,t}_{i,1})$ for any $i,j\in[n_{t-1}]$. The proof of Lemma 4.8 (omitted) is similar to that of Lemma 4.7.

Finally, when $\mathrm{char}(\mathbb{F})=2$, we shall determine $f(\mathcal{N}^{1,2})^{3,t}$, $f(\mathcal{N}^{1,3})^{2,t}$, $f(\mathcal{N}^{t-1,t})^{1,t-2}$, and $f(\mathcal{N}^{t-2,t})^{1,t-1}$. The following two lemmas explicitly describe their relationships.

Lemma 4.9. *Suppose $\mathrm{char}(\mathbb{F})=2$. Then every $f\in\mathrm{Der}(\mathcal{N})$ satisfies:*

(1) *If $n_1\geq 2$ or $n_2\geq 2$, then $f(\mathcal{N}^{1,2})^{3,t}=0$, $f(\mathcal{N}^{1,3})^{2,t}=0$.*

(2) *If $n_1=n_2=1$, then*
$$E^{1,2}_{1,1}f(E^{1,3}_{1,j})^{2,t}=E^{1,3}_{1,j}f(E^{1,2}_{1,1})^{3,t}\quad\text{for any } j\in[n_3]. \tag{49}$$

Moreover, when $n_1=n_2=1$, any $g\in\mathrm{End}(\mathcal{N})$ that satisfies $\mathrm{Ker}(g)\supseteq\mathcal{N}^{\{(1,2),(1,3)\}^c}$, $g(\mathcal{N}^{1,2})\subseteq\mathcal{N}^{3,t}$, $g(\mathcal{N}^{1,3})\subseteq\mathcal{N}^{2,t}$, and (49), must be in $\mathrm{Der}(\mathcal{N})$.

Proof. Suppose $f\in\mathrm{Der}(\mathcal{N})$. For any $E^{1,2}_{i,j}\in\mathcal{N}^{1,2}$ and $E^{1,3}_{r,s}\in\mathcal{N}^{1,3}$,

$$\begin{aligned}0 &= f([E^{1,2}_{i,j},E^{1,3}_{r,s}])^{1,t}=[E^{1,2}_{i,j},f(E^{1,3}_{r,s})]^{1,t}+[f(E^{1,2}_{i,j}),E^{1,3}_{r,s}]^{1,t}\\ &= E^{1,2}_{i,j}f(E^{1,3}_{r,s})^{2,t}-E^{1,3}_{r,s}f(E^{1,2}_{i,j})^{3,t}.\end{aligned}$$

Therefore,
$$E^{1,2}_{i,j}f(E^{1,3}_{r,s})^{2,t}=E^{1,3}_{r,s}f(E^{1,2}_{i,j})^{3,t}. \tag{50}$$

If $n_1\geq 2$, then for a fixed $E^{1,2}_{i,j}\in\mathcal{N}^{1,2}$, we can choose $r\in[n_1]\setminus\{i\}$. In (50), all rows except row i of the LHS are zero, but all rows except row r of the RHS are zero; so both sides must be zero.

Letting s go through all elements of $[n_3]$ on the right hand side of (50), we see that $f(E^{1,2}_{i,j})^{3,t} = 0$. Hence $f(\mathcal{N}^{1,2})^{3,t} = 0$. Similarly, $f(\mathcal{N}^{1,3})^{2,t} = 0$.

Now for any $A^{1,2} \in \mathcal{N}^{1,2}$ and $A^{2,3} \in \mathcal{N}^{2,3}$,

$$\begin{aligned} f(A^{1,2}A^{2,3})^{2,t} &= f([A^{1,2}, A^{2,3}])^{2,t} = [f(A^{1,2}), A^{2,3}]^{2,t} + [A^{1,2}, f(A^{2,3})]^{2,t} \\ &= -A^{2,3} f(A^{1,2})^{3,t}. \end{aligned} \tag{51}$$

For any $i \in [n_2]$, the space $\sum_{j=1}^{n_3} \mathcal{N}^{1,2} E^{2,3}_{i,j} = \mathcal{N}^{1,3}$. Then by (51),

$$f(\mathcal{N}^{1,3})^{2,t} = \sum_{j=1}^{n_3} f(\mathcal{N}^{1,2} E^{2,3}_{i,j})^{2,t} = -\sum_{j=1}^{n_3} E^{2,3}_{i,j} f(\mathcal{N}^{1,2})^{3,t},$$

which implies that the $(2,t)$ blocks of matrices in $f(\mathcal{N}^{1,3})^{2,t}$ have all zero rows except row i. If $n_2 \geq 2$, we can choose different i values and deduce that $f(\mathcal{N}^{1,3})^{2,t} = 0$, and thus $f(\mathcal{N}^{1,2})^{3,t} = 0$ by (51).

The remaining case is $n_1 = n_2 = 1$. Then (50) immediately leads to (49), which determines $f(\mathcal{N}^{1,3})^{2,t}$ in terms of a constant matrix $f(E^{1,2}_{1,1})^{3,t}$.

The last claim is straightforward to verify under the assumption $\mathrm{char}(\mathbb{F}) = 2$. ∎

Lemma 4.10. *Suppose* $\mathrm{char}(\mathbb{F}) = 2$. *Then every* $f \in \mathrm{Der}(\mathcal{N})$ *satisfies the following conditions:*

(1) *If* $n_t \geq 2$ *or* $n_{t-1} \geq 2$, *then* $f(\mathcal{N}^{t-1,t})^{1,t-2} = 0$, $f(\mathcal{N}^{t-2,t})^{1,t-1} = 0$.

(2) *If* $n_t = n_{t-1} = 1$, *then*

$$f(E^{t-2,t}_{j,1})^{1,t-1} E^{t-1,t}_{1,1} = f(E^{t-1,t}_{1,1})^{1,t-2} E^{t-2,t}_{j,1} \quad \textit{for any } j \in [n_{t-2}]. \tag{52}$$

Moreover, when $n_t = n_{t-1} = 1$, *any* $g \in \mathrm{End}(\mathcal{N})$ *satisfying* $\mathrm{Ker}(g) \supseteq \mathcal{N}^{\{(t-1,t),(t-2,t)\}^c}$, $g(\mathcal{N}^{t-1,t}) \subseteq \mathcal{N}^{1,t-2}$, $g(\mathcal{N}^{t-2,t}) \subseteq \mathcal{N}^{1,t-1}$, *and* (52), *must be in* $\mathrm{Der}(\mathcal{N})$.

The proof (omitted) is similar to that of Lemma 4.9.

4.3. Proofs of Theorems 2.1, 2.2, 2.4, 2.5

We are ready to prove the main structure theorems for $\mathrm{Der}(\mathcal{N})$.

Proof of Theorems 2.1 and 2.4. Let $X^{p,p}$ for $p \in [t]$ be chosen as in Lemma 4.4, and $X^{p,q}$ for $(p,q) \in \Omega$ be chosen as in Lemma 4.5. Define

$$X = \sum_{p=1}^{t} X^{p,p} + \sum_{(p,q)\in\Omega} X^{p,q} \;\in\; \mathcal{B}, \qquad f_0 = f - \mathrm{ad}_{\mathcal{N}} X \;\in\; \mathrm{Der}(\mathcal{N}). \tag{53}$$

Then (37), (42) and (43) imply that for $(k,l) \in \Gamma_{\mathcal{N}}$, $(p,q) \in \Omega$, $1 \leq i < p$ and $q < j \leq t$:

$$f_0(A^{k,l})^{k,l} = 0 \qquad \text{for all } A^{k,l} \in \mathcal{N}^{k,l}, \tag{54}$$

$$f_0(A^{i,p})^{i,q} = 0 \qquad \text{for all } A^{i,p} \in \mathcal{N}^{i,p}, \tag{55}$$

$$f_0(A^{q,j})^{p,j} = 0 \qquad \text{for all } A^{q,j} \in \mathcal{N}^{q,j}. \tag{56}$$

Therefore, by Lemmas 4.1, 4.2, 4.3, for $A = \sum_{(i,j)\in\Gamma_{\mathcal{N}}} A^{i,j} \in \mathcal{N}$, we have

$$f_0(A) = \sum_{(i,j)\in\Gamma_{\mathcal{N}}} \sum_{(p,q)\in\Gamma_{\mathcal{N}}} f_0(A^{i,j})^{p,q} = f(\sum_{i=1}^{t-1} A^{i,i+1})^{1,t} + f(A^{1,2})^{2,t} + f(A^{t-1,t})^{1,t-1} + \big(f(A^{1,2})^{3,t} + f(A^{1,3})^{2,t}\big) + \big(f(A^{t-1,t})^{1,t-2} + f(A^{t-2,t})^{1,t-1}\big). \tag{57}$$

So $f_0(A)$ is a sum of the following derivation images:

1. $\varphi(A) = f(\sum_{i=1}^{t-1} A^{i,i+1})^{1,t}$ defines a central derivation φ according to Lemma 4.6.
2. $\phi_1(A) = f(A^{1,2})^{2,t}$ defines a derivation ϕ_1 described in Lemma 4.7. In particular, $\phi_1(A) = 0$ except for $n_1 = 1$.
3. $\phi_2(A) = f(A^{t-1,t})^{1,t-1}$ defines a derivation ϕ_2 described in Lemma 4.8. In particular, $\phi_2(A) = 0$ except for $n_t = 1$.
4. $\phi_3(A) = f(A^{1,2})^{3,t} + f(A^{1,3})^{2,t}$ defines a derivation ϕ_3 described in Lemma 4.9. In particular, $\phi_3(A) = 0$ except for $\operatorname{char}(\mathbb{F}) = 2$ and $n_1 = n_2 = 1$.
5. $\phi_4(A) = f(A^{t-1,t})^{1,t-2} + f(A^{t-2,t})^{1,t-1}$ defines a derivation described in Lemma 4.10. In particular, $\phi_4(A) = 0$ except for $\operatorname{char}(\mathbb{F}) = 2$ and $n_{t-1} = n_t = 1$.

Therefore, we got (5) in Theorem 2.1 and (14) in Theorem 2.4.

It remains to verify the detailed formulas of $\phi_1, \phi_2, \phi_3, \phi_4$ in Theorems 2.1 and 2.4.

When $n_1 = 1$, $\phi_1(A) = f(A^{1,2})^{2,t} = \phi_1(A^{1,2})$ is a linear transformation from $\mathcal{N}^{1,2}$ to $\mathcal{N}^{2,t}$. Suppose that for $j \in [n_2]$,

$$\phi_1(E^{1,2}_{1,j}) = \sum_{i=1}^{n_2} \sum_{k=1}^{n_t} x_{i,j,k} E^{2,t}_{i,k}, \qquad x_{i,j,k} \in \mathbb{F}. \tag{58}$$

A direct computation shows that ϕ_1 can be expressed as

$$\phi_1(A^{1,2}) = \sum_{k=1}^{n_t} X^{2,2}_k (A^{1,2})^T E^{1,t}_{1,k}$$

where $X^{2,2}_k \in M^{2,2}_n$ has the $(2,2)$ block submatrix $X_k = \big[\, x_{i,j,k} \,\big]_{i,j}$. By (45) and (58),

$$E^{1,2}_{1,i}\phi_1(E^{1,2}_{1,j}) = \sum_{k=1}^{n_t} x_{i,j,k} E^{1,t}_{1,k} = E^{1,2}_{1,j}\phi_1(E^{1,2}_{1,i}) = \sum_{k=1}^{n_t} x_{j,i,k} E^{1,t}_{1,k}. \tag{59}$$

Therefore, $x_{i,j,k} = x_{j,i,k}$, so that $X_k \in S^2(\mathbb{F}^{n_2})$ for $k \in [n_t]$. Conversely, any $\phi_1 \in \operatorname{End}(\mathcal{N})$ satisfying (7) must be in $\operatorname{Der}(\mathcal{N})$. So Theorem 2.1 (3) is proved.

Theorem 2.1 (4) can be proved similarly.

When $\operatorname{char}(\mathbb{F}) = 2$ and $n_1 = n_2 = 1$, by (49), the linear transformation $\phi_3(A) = f(A^{1,2})^{3,t} + f(A^{1,3})^{2,t}$ satisfies that

$$\phi_3(A^{1,3}) = E^{2,1}_{1,1}E^{1,2}_{1,1}\phi_3(A^{1,3}) = E^{2,1}_{1,1}A^{1,3}\phi_3(E^{1,2}_{1,1}). \tag{60}$$

Therefore, ϕ_3 satisfies (15). Conversely, every $\phi_3 \in \operatorname{End}(\mathcal{N})$ that satisfies (15) is in $\operatorname{Der}(\mathcal{N})$. Hence Theorem 2.4 (1) is proved.

Theorem 2.4 (2) can be proved similarly. ∎

The proofs of Theorems 2.2 and 2.5 can be done by computation. Before proving Theorem 2.2, let us recall a classical result about derivations.

Lemma 4.11. *Let $\mathfrak{g}$ be a Lie algebra, $X \in \mathfrak{g}$, and $f \in \mathrm{Der}(\mathfrak{g})$. Then* $[f, \operatorname{ad} X] = \operatorname{ad} f(X)$.

Proof. For any $Y \in \mathfrak{g}$, we have

$$[f, \operatorname{ad} X](Y) = f([X,Y]) - [X, f(Y)] = [f(X), Y] = \operatorname{ad} f(X)(Y). \qquad \blacksquare$$

We use Lemma 4.11 in the way that for any subalgebra L of $\mathrm{Der}(\mathcal{N})$,

$$[L, \operatorname{ad}_{\mathcal{N}} \mathcal{B}] = [L, \operatorname{ad} \mathcal{N} + \operatorname{ad} \mathcal{D}] = \operatorname{ad} L(\mathcal{N}) + [L, \operatorname{ad} \mathcal{D}]. \tag{61}$$

Proof of Theorem 2.2. (5) gives the vector space sum:

$$\mathrm{Der}(\mathcal{N}) = (\operatorname{ad}_{\mathcal{N}} \mathcal{B} + \mathrm{Cen}(\mathcal{N})) + \mathrm{D}_1 + \mathrm{D}_2 .$$

To prove (11), it remains to show that if $\operatorname{ad} X + \varphi + \phi_1 + \phi_2 = 0$ for $X \in \mathcal{B}$, $\varphi \in \mathrm{Cen}(\mathcal{N})$, $\phi_1 \in \mathrm{D}_1$, $\phi_2 \in \mathrm{D}_2$, then $\phi_1 = 0$ and $\phi_2 = 0$. In this case, for any $A^{1,2} \in \mathcal{N}^{1,2}$,

$$0 = (\operatorname{ad} X + \varphi + \phi_1 + \phi_2)(A^{1,2})^{2,t} = \phi_1(A^{1,2})^{2,t}.$$

Hence $\phi_1 = 0$. Similarly, $\phi_2 = 0$. Therefore (11) is proved.

Consider $\operatorname{ad}_{\mathcal{N}} \mathcal{B} \cap \mathrm{Cen}(\mathcal{N})$. A matrix $X \in \mathcal{B}$ satisfies $\operatorname{ad}_{\mathcal{N}} X \in \mathrm{Cen}(\mathcal{N})$ if and only if $\operatorname{ad}_{\mathcal{N}} X(\mathcal{N}) \subseteq \mathcal{N}^{1,t}$, if and only if $X \in \mathcal{N}^{1,t-1} + \mathcal{N}^{2,t} + \mathcal{N}^{1,t}$. Therefore, $\operatorname{ad}_{\mathcal{N}} \mathcal{B} \cap \mathrm{Cen}(\mathcal{N}) = \operatorname{ad}(\mathcal{N}^{1,t-1} + \mathcal{N}^{2,t})$.

Theorem 2.2 (2) and (3) are obviously true, according to Theorem 2.1.

Next we determine all $[S,T]$ for $S, T \in \{\mathrm{Cen}(\mathcal{N}), \operatorname{ad}_{\mathcal{N}} \mathcal{B}, D_1, D_2\}$. If generic elements $\delta_1 \in S$ and $\delta_2 \in T$ satisfy that $\operatorname{Im} \delta_1 \subseteq \operatorname{Ker} \delta_2$ and $\operatorname{Im} \delta_2 \subseteq \operatorname{Ker} \delta_1$, then obviously $[S,T] = 0$. This includes all unlisted cases in Table 1 and the cases $S = T = \mathrm{Cen}(\mathcal{N})$, $S = T = \mathrm{D}_1$, and $S = T = \mathrm{D}_2$. The remaining cases in the upper triangular part of Table 1 are discussed below:

(1) $[\mathrm{Cen}(\mathcal{N}), \operatorname{ad}_{\mathcal{N}} \mathcal{B}]$: on one hand, $[\mathrm{Cen}(\mathcal{N}), \operatorname{ad}_{\mathcal{N}} \mathcal{B}] \subseteq \mathrm{Cen}(\mathcal{N})$ since $\mathrm{Cen}(\mathcal{N})$ is an ideal of $\mathrm{Der}(\mathcal{N})$. On the other hand, every $f \in \mathrm{Cen}(\mathcal{N})$ has $\operatorname{Ker} f \supseteq [\mathcal{N}, \mathcal{N}] = \mathcal{N}^{\{(i,i+1) | i \in [t-1]\}^c}$ and $\operatorname{Im} f \subseteq \mathcal{N}^{1,t}$. Construct $f_{i,i+1} \in \mathrm{End}(\mathcal{N})$ for $i \in [t-1]$ such that each $\operatorname{Ker} f_{i,i+1} \supseteq \mathcal{N}^{\{(i,i+1)\}^c}$ and $f_{i,i+1}|_{\mathcal{N}^{i,i+1}} = f|_{\mathcal{N}^{i,i+1}}$. Then $f = \sum_{i=1}^{t-1} f_{i,i+1}$ and each $f_{i,i+1} \in \mathrm{Cen}(\mathcal{N})$. Denote $I^{p,p} = (I_n)^{p,p}$, then

$$f_{1,2} = [f_{1,2}, \operatorname{ad}(-I^{2,2})]; \qquad f_{i,i+1} = [f_{i,i+1}, \operatorname{ad} I^{i,i}], \quad i = 2, 3, \cdots, t-1.$$

Therefore, $f \in [\mathrm{Cen}(\mathcal{N}), \operatorname{ad}_{\mathcal{N}} \mathcal{B}]$, and $[\mathrm{Cen}(\mathcal{N}), \operatorname{ad}_{\mathcal{N}} \mathcal{B}] = \mathrm{Cen}(\mathcal{N})$.

(2) $[\mathrm{Cen}(\mathcal{N}), \mathrm{D}_1]$, $t = 3$, $n_1 = 1$: given $\varphi \in \mathrm{Cen}(\mathcal{N})$ and $\phi \in \mathrm{D}_1$, we have $\operatorname{Im} \varphi \subseteq \mathcal{N}^{1,t} \subseteq \operatorname{Ker} \phi$ so that $[\varphi, \phi] = \varphi \circ \phi$. By $\operatorname{Ker}[\varphi, \phi] \supseteq \mathcal{N}^{\{(1,2)\}^c}$, $\operatorname{Im}[\varphi, \phi] \subseteq \mathcal{N}^{1,3}$, and $n_1 = 1$, we have $[\varphi, \phi] \in \operatorname{ad} \mathcal{N}^{2,3}$. So $[\mathrm{Cen}(\mathcal{N}), \mathrm{D}_1] \subseteq \operatorname{ad} \mathcal{N}^{2,3}$. On the other hand, given $f \in \operatorname{ad} \mathcal{N}^{2,3}$, we choose $\varphi, \phi \in \mathrm{End}(\mathcal{N})$ such that $\operatorname{Ker} \phi \supseteq \mathcal{N}^{\{(1,2)\}^c}$, $\operatorname{Ker} \varphi \supseteq \mathcal{N}^{\{(2,3)\}^c}$,

$$\phi(E^{1,2}_{1,q}) = E^{2,3}_{q,1}, \;\; \varphi(E^{2,3}_{q,1}) = f(E^{1,2}_{1,q}), \;\; \varphi(E^{2,3}_{q,r}) = 0, \quad \text{for } q \in [n_2], \; r \in [n_3] \setminus \{1\}.$$

Then $\varphi \in \mathrm{Cen}(\mathcal{N})$, $\phi \in \mathrm{D}_1$, and $[\varphi, \phi] = f$. Therefore, $[\mathrm{Cen}(\mathcal{N}), \mathrm{D}_1] = \operatorname{ad} \mathcal{N}^{2,3}$.

(3) $[\mathrm{Cen}(\mathcal{N}), \mathrm{D}_2]$, $t=3$, $n_t=1$: the analysis is similar to the previous one.

(4) $[\mathrm{ad}_{\mathcal{N}} \mathcal{B}, \mathrm{ad}_{\mathcal{N}} \mathcal{B}]$: for $X, Y \in \mathcal{B}$, we have $[\mathrm{ad}\, X, \mathrm{ad}\, Y] = \mathrm{ad}[X, Y]$ where $[X, Y]$ is in $\mathcal{B}_0 = [\mathcal{B}, \mathcal{B}]$ and all such $[X, Y]$ span $\mathcal{B}_0$. Hence $[\mathrm{ad}_{\mathcal{N}} \mathcal{B}, \mathrm{ad}_{\mathcal{N}} \mathcal{B}] = \mathrm{ad}_{\mathcal{N}} \mathcal{B}_0$.

(5) $[\mathrm{ad}_{\mathcal{N}} \mathcal{B}, \mathrm{D}_1]$, $n_1 = 1$: we have

$$[\mathrm{ad}_{\mathcal{N}} \mathcal{B}, \mathrm{D}_1] = [\mathrm{ad}\, \mathcal{N}, \mathrm{D}_1] + [\mathrm{ad}\, \mathcal{D}, \mathrm{D}_1].$$

By Lemma 4.11 and Lemma 4.7, $[\mathrm{ad}\, \mathcal{N}, \mathrm{D}_1] = \mathrm{ad}\, \mathcal{N}^{2,t}$. Next, every $f \in [\mathrm{ad}\, \mathcal{D}, \mathrm{D}_1]$ has $\mathrm{Ker}\, f \supseteq \mathcal{N}^{\{(1,2)\}^c}$ and $\mathrm{Im}\, f \subseteq \mathcal{N}^{2,t}$; therefore, $f \in \mathrm{D}_1$ in view of the decomposition (11). Moreover, since $n_1 = 1$, every $\phi \in \mathrm{D}_1$ can be written as

$$\phi = \phi \circ (\mathrm{ad}\, E_{1,1}^{1,1}) = [-\mathrm{ad}\, E_{1,1}^{1,1}, \phi] \in [\mathrm{ad}\, \mathcal{D}, \mathrm{D}_1].$$

Therefore, $[\mathrm{ad}\, \mathcal{D}, \mathrm{D}_1] = \mathrm{D}_1$, and $[\mathrm{ad}_{\mathcal{N}} \mathcal{B}, \mathrm{D}_1] = \mathrm{ad}\, \mathcal{N}^{2,t} \oplus \mathrm{D}_1$.

(6) $[\mathrm{ad}_{\mathcal{N}} \mathcal{B}, \mathrm{D}_2]$, $n_t = 1$: the analysis is similar to the previous case.

(7) $[\mathrm{D}_1, \mathrm{D}_2]$, $t = 3$, $n_1 = n_t = 1$: let $f \in \mathrm{D}_1$ and $g \in \mathrm{D}_2$. Then

$$[f, g](\mathcal{N}^{1,2}) \subseteq \mathcal{N}^{1,2}, \qquad [f, g](\mathcal{N}^{2,3}) \subseteq \mathcal{N}^{2,3}, \qquad [f, g](\mathcal{N}^{1,3}) = 0.$$

By Theorem 2.1 (3), there exists $X_1^{2,2} \in M_n^{2,2}$ with a symmetric $(2,2)$ block $X_1 = \left[x_{i,j,1} \right]_{i,j} \in S^2(\mathbb{F}^{n_2})$ such that $f(A^{1,2}) = X_1^{2,2} (A^{1,2})^T E_{1,1}^{1,3}$ for $A^{1,2} \in \mathcal{N}^{1,2}$; by Theorem 2.1 (4), there exists $Y_1^{2,2} \in M_n^{2,2}$ with a symmetric $(2,2)$ block $Y_1 = \left[y_{i,j,1} \right]_{i,j} \in S^2(\mathbb{F}^{n_2})$ such that $g(A^{2,3}) = E_{1,1}^{1,3} (A^{2,3})^T Y_1^{2,2}$ for $A^{2,3} \in \mathcal{N}^{2,3}$. So for $A^{1,2} \in \mathcal{N}^{1,2}$ and $A^{2,3} \in \mathcal{N}^{2,3}$,

$$\begin{aligned} [f, g](A^{1,2}) &= -gf(A^{1,2}) = -E_{1,1}^{1,3} (E_{1,1}^{1,3})^T A^{1,2} (X_1^{2,2})^T Y_1^{2,2} = -A^{1,2} X_1^{2,2} Y_1^{2,2}, \\ [f, g](A^{2,3}) &= fg(A^{2,3}) = X_1^{2,2} (Y_1^{2,2})^T A^{2,3} (E_{1,1}^{1,3})^T E_{1,1}^{1,3} = X_1^{2,2} Y_1^{2,2} A^{2,3}. \end{aligned}$$

So $[f, g] = \mathrm{ad}_{\mathcal{N}} (X_1^{2,2} Y_1^{2,2}) \in \mathrm{ad}_{\mathcal{N}} M_n^{2,2}$. Conversely, all products of two symmetric matrices in $M_n^{2,2}$ span $M_n^{2,2}$ since

$$E_{i,i}^{2,2} = E_{i,i}^{2,2} E_{i,i}^{2,2}, \qquad E_{i,j}^{2,2} = E_{i,i}^{2,2} (E_{i,j}^{2,2} + E_{j,i}^{2,2}) \quad \text{for } i \neq j,$$

Therefore $[\mathrm{D}_1, \mathrm{D}_2] = \mathrm{ad}_{\mathcal{N}} M_n^{2,2}$ when $t = 3$ and $n_1 = n_t = 1$.

Finally, we verify the table of dimensions in Theorem 2.2 (5). The centralizer of $\mathcal{N}$ in $\mathcal{B}$ is $\mathrm{Z}_{\mathcal{B}}(\mathcal{N}) = \mathcal{N}^{1,t} + \mathbb{F} I_n$, which is the kernel of $\mathrm{ad}_{\mathcal{N}} \colon \mathcal{B} \to \mathrm{Der}(\mathcal{N})$. So

$$\dim \mathrm{ad}_{\mathcal{N}} \mathcal{B} = \dim \mathcal{B} - \dim \mathrm{Z}_{\mathcal{B}}(\mathcal{N}) = \binom{n+1}{2} + \sum_{i=1}^{t} \binom{n_i}{2} - n_1 n_t - 1.$$

The ideal $\mathrm{Cen}(\mathcal{N})$ consists of all elements $f \in \mathrm{End}(\mathcal{N})$ with $\mathrm{Ker}\, f \supseteq [\mathcal{N}, \mathcal{N}]$ and $\mathrm{Im}\, f \subseteq \mathrm{Z}(\mathcal{N}) = \mathcal{N}^{1,t}$. So

$$\begin{aligned} \dim \mathrm{Cen}(\mathcal{N}) &= \dim(\mathcal{N}/[\mathcal{N}, \mathcal{N}]) \dim \mathrm{Z}(\mathcal{N}) = \dim(\sum_{i=1}^{t-1} \mathcal{N}^{i,i+1}) \dim \mathcal{N}^{1,t} \\ &= n_1 n_t (\sum_{i=1}^{t-1} n_i n_{i+1}). \end{aligned}$$

Obviously $\dim(\mathrm{ad}_{\mathcal{N}} \mathcal{B} \cap \mathrm{Cen}(\mathcal{N})) = \dim \mathrm{ad}(\mathcal{N}^{1,t-1} + \mathcal{N}^{2,t}) = n_1 n_{t-1} + n_2 n_t$.

When $n_1 = 1$, by Theorem 2.1 (3), $\dim D_1$ equals the dimension of the space of 3-tensors $\left[x_{i,j,k}\right]_{i,j,k} \in \mathbb{F}^{n_2 \times n_2 \times n_t}$ that have symmetric cross-section matrices $X_k = \left[x_{i,j,k}\right]_{i,j}$ for $k \in [n_t]$, so that $\dim D_1 = \binom{n_2+1}{2} n_t$. Similarly, when $n_t = 1$, $\dim \mathrm{D}_2 = \binom{n_{t-1}+1}{2} n_1$. ■

Proof of Theorem 2.5. The $t = 3$ case can be similarly treated as in $\mathrm{char}(\mathbb{F}) \neq 2$ situation. Now assume $t \geq 4$. (14) implies the vector space sum

$$\mathrm{Der}(\mathcal{N}) = (\mathrm{ad}_{\mathcal{N}} \mathcal{B} + \mathrm{Cen}(\mathcal{N})) + \mathrm{D}_1 + \mathrm{D}_2 + \mathrm{D}_3 + \mathrm{D}_4 .$$

To prove (17), it suffices to show that if $0 = \mathrm{ad}\, X + \varphi + \phi_1 + \phi_2 + \phi_3 + \phi_4$ for $X \in \mathcal{B}$, $\varphi \in \mathrm{Cen}(\mathcal{N})$, $\phi_i \in \mathrm{D}_i$ for $i = 1, 2, 3, 4$, then $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$. To prove $\phi_3 = 0$, for any $A^{1,2} \in \mathcal{N}^{1,2}$ and $B^{1,3} \in \mathcal{N}^{1,3}$,

$$\begin{aligned} 0 = (\mathrm{ad}\, X + \varphi + \phi_1 + \phi_2 + \phi_3 + \phi_4)(A^{1,2})^{3,t} &= \phi_3(A^{1,2})^{3,t}, \\ 0 = (\mathrm{ad}\, X + \varphi + \phi_1 + \phi_2 + \phi_3 + \phi_4)(B^{1,3})^{2,t} &= \phi_3(B^{1,3})^{2,t}. \end{aligned}$$

Therefore $\phi_3 = 0$. Similarly, we can prove $\phi_1 = \phi_2 = \phi_4 = 0$.
When $n_1 \neq 1$ or $n_2 \neq 1$, obviously $\mathrm{D}_3 = 0$. When $n_1 = n_2 = 1$, Theorem 2.4 (1) shows that D_3 is abelian, and that every $\phi_3 \in \mathrm{D}_3$ is uniquely determined by the constant matrix $\phi_3(E^{1,2}_{1,1}) \in \mathcal{N}^{3,t}$. Therefore $\dim \mathrm{D}_3 = n_3 n_t$ and Theorem 2.5 (1) is proved. Theorem 2.5 (2) can be similarly proved. ■

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