Journal of Lie Theory
Volume **30** (2020) 1047–1060

Classification of Nilpotent Lie Superalgebras of Multiplier-Rank Less Than or Equal to 2

Wende Liu* **and Yanling Zhang†**

Communicated by A. Fialowski

Abstract. We introduce the notion of (super-)multiplier-ranks for Lie superalgeras and classify all the finite-dimensional nilpotent Lie superalgebras of multiplier-rank ≤ 2 over an algebraically closed field of characteristic zero. In the process, we also determine the multipliers of Heisenberg Lie superalgebras.

Mathematics Subject Classification: 17B05, 17B30, 17B56.

Key Words: Lie superalgebra, multipliers, super-multiplier-rank.

1. Introduction

As is well known, the notion of multipliers and covers for a group arose from Schur's work on projective representations of groups. Analogous to the group theory case, for a finite-dimensional Lie algebra L over a field, a cover is a central extension of the maximal possible dimension of L with a kernel contained in the derived algebra of L and the corresponding kernel is a (Schur) multiplier of L. For a finite-dimensional Lie algebra, there exist uniquely a cover and a multiplier up to Lie algebra isomorphism, respectively. A typical fact is that the multiplier of a finite-dimensional Lie algebra L is isomorphic to the second cohomology group of L with coefficients in the 1-dimensional trivial module [1]. The study on multipliers of Lie algebras began in 1990's (see [3, 9], for example) and the theory has seen a fruitful development (see [2, 5, 6, 7, 11, 14, 17], for example). Among the literatures, a main work is finding an upper bound for the multiplier dimension for a finite-dimensional nilpotent Lie algebra and classifying finite-dimensional nilpotent Lie algebras under certain conditions in terms of multipliers (see [2, 6, 7, 11, 14, 17], for example).

The notion of multipliers for Lie algebras may be naturally generalized to the Lie superalgebra case. In this paper, we first establish several lemmas for Lie superalgebras, which are parallel to the ones in non-super case. Then we introduce the notions of (super-)multiplier-ranks and (super-)derived-ranks, which are analogous to the two corresponding invariants in Lie algebra case. Our main result is classifying all the nilpotent Lie superalgebras of multiplier-rank ≤ 2. As a byproduct, we also determine the multipliers of Heisenberg Lie superalgebras.

* Supported by the NSF of China (11701158).

† Corresponding author.

ISSN 0949–5932 / \$2.50

2. Basics

In this paper, all (linear) superspaces and superalgebras are over an algebraically closed field $\mathbb{F}$ of characteristic zero. Let $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ be the abelian group of order 2 and $V = V_{\bar{0}} \oplus V_{\bar{1}}$ a superspace. For a homogeneous element x in V, write $|x|$ for the parity of x. The symbol $|x|$ implies that x has been assumed to be a homogeneous element. In $\mathbb{Z} \times \mathbb{Z}$, we define a partial order as follows:

$$(m, n) \leq (k, l) \Longleftrightarrow m \leq k, n \leq l.$$

For $m, n \in \mathbb{Z}$, we write $|(m, n)| = m + n$. We also view $\mathbb{Z} \times \mathbb{Z}$ as the additive group in the usual way. Write $\mathrm{sdim}V$ for the superdimension of a superspace V and $\dim V$ for the dimension of V as an ordinary linear space. Note that $\dim V = |\mathrm{sdim}V|$. Let Π be the parity functor of superspaces. Then

$$\mathrm{sdim}V + \mathrm{sdim}\Pi(V) = (\dim V, \dim V).$$

Moreover, if W is a subsuperspace of a superspace V, then

$$\mathrm{sdim}V/W = \mathrm{sdim}V - \mathrm{sdim}W.$$

In this paper, we write $\mathrm{Ab}(m, n)$ for the abelian Lie superalgebra of superdimension (m, n). As in the Lie algebra case [3, p. 4302], we introduce the following definition.

Definition 2.1. Let L be a finite-dimensional Lie superalgebra. A Lie superalgebra pair (K, M) is called a *defining pair* of L provided that $L \cong K/M$ and $M \subset \mathrm{Z}(K) \cap K^2$, where $\mathrm{Z}(K)$ is the center of K and $K^2 := [K, K]$ is the derived subalgebra of K. A defining pair (K, M) of L is said to be *maximal* if among all the defining pairs of L, M is of a maximal superdimension. In the case (K, M) being a maximal defining pair of L, we also call K a cover and M a (Schur) multiplier of L.

Definition 2.1 makes sense, since one may check as in Lie algebra case (see [3]) that for a finite-dimensional Lie superalgebra, covers and multipliers always exist and they are unique up to Lie superalgebra isomorphism, respectively. Write $\mathcal{C}(L)$ and $\mathcal{M}(L)$ for the cover and multiplier of Lie superalgebra L.

As in Lie algebra case [9], we will give an upper bound for the superdimension of the multiplier of a Lie superalgebra. To that aim, we first establish the following lemmas.

Lemma 2.2. *Let L be a Lie superalgebra with* $\mathrm{sdim}L/\mathrm{Z}(L) = (m, n)$. *Then*

$$\mathrm{sdim}L^2 \leq \left(\frac{1}{2}m(m-1) + \frac{1}{2}n(n+1), mn\right).$$

Proof. Straightforward. ∎

Lemma 2.3. *Let L be a Lie superalgebra of* $\mathrm{sdim}L = (s, t)$. *Then*

$$\mathrm{sdim}\mathcal{M}(L) \leq \left(\frac{1}{2}s(s-1) + \frac{1}{2}t(t+1), st\right).$$

Proof. This is a direct consequence of Lemma 2.2. ∎

We should note that a non-super version of Lemmas 2.2 and 2.3 has been given in [10, Theorems 3.1 and 3.4]. For a Lie superalgebra L of superdimension (s,t), we define the *super-multiplier-rank* of L to be the number pair

$$\mathrm{smr}(L) = \left(\frac{1}{2}s(s-1) + \frac{1}{2}t(t+1), st\right) - \mathrm{sdim}\mathcal{M}(L)$$

and the *multiplier-rank* of L to be $\mathrm{mr}(L) = |\mathrm{smr}(L)|$. By Lemma 2.3, we have $\mathrm{smr}(L) \geq (0,0)$.

As in the Lie algebra case [2, Lemma 4 and Theorem 1], using the notion of free presentations for Lie superalgebras, one may prove the following two lemmas.

Lemma 2.4. *Let L be a finite-dimensional Lie superalgebra. Then $L^2 \cap \mathrm{Z}(L)$ is a homomorphic image of $\mathcal{M}(L/\mathrm{Z}(L))$.*

Lemma 2.5. *Let A and B be finite-dimensional Lie superalgebras. Then*

$$\mathrm{sdim}\mathcal{M}(A \oplus B) = \mathrm{sdim}\mathcal{M}(A) + \mathrm{sdim}\mathcal{M}(B) + \mathrm{sdim}(A/A^2 \otimes B/B^2).$$

3. Multiplier-rank 0 nilpotent Lie superalgebras

The multiplier-rank 0 case was also considered in [10, Theorem 3.5], where the multiplier was described in terms of non-super dimensions. For completeness, we give a proof, which is also somewhat different from the one in [10, Theorem 3.5].

Proposition 3.1. *Let L be a finite-dimensional Lie superalgebra. Then* $\mathrm{smr}(L) = (0,0)$ *if and only if L is abelian.*

Proof. Let L be of superdimension (m,n). Suppose L is abelian and let H be a superspace with a homogeneous basis $\{u_i, x_{k,l}, z_{s,t} \mid v_j, y_{p,q}\}$, where

$$1 \leq i \leq m,\ 1 \leq k < l \leq m,\ 1 \leq s \leq t \leq n,\ 1 \leq j \leq n,\ 1 \leq p \leq m,\ 1 \leq q \leq n$$

and $|u_i| = |x_{k,l}| = |z_{s,t}| = \bar{0}$, $|v_j| = |y_{p,q}| = \bar{1}$. Then H becomes a Lie superalgebra by letting

$$[u_k, u_l] = x_{k,l},\ [u_p, v_q] = y_{p,q},\ [v_s, v_t] = z_{s,t}$$

and the other brackets of basis elements vanish. Clearly, $L \cong H/H^2$. Since $\mathrm{Z}(H) = H^2$, one sees that $H^2 \subset \mathrm{Z}(H) \cap H^2$. Hence, (H, H^2) is a defining pair of L and $\mathrm{smr}(L) = (0,0)$.

Conversely, suppose $\mathrm{smr}(L) = (0,0)$. Let (K, M) be a maximal defining pair of L and suppose $\mathrm{sdim}K/\mathrm{Z}(K) = (k,l)$. Since $M \subset \mathrm{Z}(K)$ and $M \subset K^2$, it follows from Lemma 2.2 that $M = K^2$ and $L \cong K/M$ is abelian. ∎

4. Multiplier-rank 1 nilpotent Lie superalgebras

In this section, suppose L is a finite-dimensional non-abelian nilpotent Lie superalgebra with $\mathrm{sdim}L/\mathrm{Z}(L) = (m,n)$. Write $\mathrm{Z}_2(L)$ for the ideal of L such that $\mathrm{Z}_2(L)/\mathrm{Z}(L) = \mathrm{Z}(L/\mathrm{Z}(L))$. Suppose $z \in \mathrm{Z}_2(L)\backslash\mathrm{Z}(L)$ is a homogeneous element. Then $[L,z] \subset \mathrm{Z}(L)$ is an ideal of L. For convenience, write $\lambda(z) = \mathrm{sdim}[L,z]$ and

$$\mu(z) = \mathrm{sdim}((L/[L,z])/\mathrm{Z}(L/[L,z])). \tag{1}$$

Lemma 4.1. *Suppose $z \in \mathrm{Z}_2(L)\backslash \mathrm{Z}(L)$.*

(1) *If $|z| = \bar{0}$, then $\lambda(z) \le (m-1,n)$ and $\mu(z) \le (m-1,n)$.*

(2) *If $|z| = \bar{1}$, then $\mu(z) \le (m,n-1)$.*

Proof. (1) In this case, we have $L/\mathrm{Z}_L(z) \cong [L,z]$ as superspaces. Since $z \notin \mathrm{Z}(L)$, we have $\mathrm{Z}(L) \subsetneq \mathrm{Z}_L(z)$ and $\mathrm{sdim}\mathrm{Z}(L) + (1,0) \le \mathrm{sdim}\mathrm{Z}_L(z)$. Hence $\lambda(z) \le (m-1,n)$. Note that $z + [L,z] \in \mathrm{Z}(L/[L,z])$ and $z + [L,z] \notin \mathrm{Z}(L)/[L,z]$. Then

$$\mathrm{Z}(L)/[L,z] \subsetneq \mathrm{Z}(L/[L,z]) \ \text{ and } \ \mathrm{sdim}\mathrm{Z}(L)/[L,z] + (1,0) \le \mathrm{sdim}\mathrm{Z}(L/[L,z]).$$

By eq. (1), we have $\mu(z) \le (m-1,n)$.

(2) The proof is similar to the one of (1). ■

Recall that $\mathrm{sdim}L/\mathrm{Z}(L) = (m,n)$. Define the *super-derived-rank* of Lie superalgebra L to be

$$\mathrm{sdr}(L) = \left(\frac{1}{2}m(m-1) + \frac{1}{2}n(n+1), mn\right) - \mathrm{sdim}L^2$$

and the *derived-rank* to be $\mathrm{dr}(L) = |\mathrm{sdr}(L)|$.

It follows from Lemma 2.2 that $\mathrm{sdr}(L) \ge (0,0)$. For our purpose, we will first determine all the nilpotent Lie superalgebras L with $\mathrm{dr}(L) \le 1$. Let $z \in \mathrm{Z}_2(L)\backslash \mathrm{Z}(L)$. Suppose $|z| = \bar{1}$. Then

$$f : L \longrightarrow [L,z], \quad x \longmapsto [x,z]$$

is an odd linear epimorphism. Then we have the following superspace isomorphism: $\Pi\left(L/\mathrm{Z}_L(z)\right) \cong [L,z]$.

Lemma 4.2. *The following statements are true:*

(1) *If the center of $L/\mathrm{Z}(L)$ has a nonzero even part, then*

$$\mathrm{sdim}(L/\mathrm{Z}(L))^2 \le \mathrm{sdr}(L) + (1,0).$$

(2) *If the center of $L/\mathrm{Z}(L)$ has a nonzero odd part, then*

$$\mathrm{sdim}(L/\mathrm{Z}(L))^2 \le (\mathrm{dr}(L), \mathrm{dr}(L)) - \mathrm{sdr}(L).$$

Proof. Suppose z is a homogeneous element in $\mathrm{Z}_2(L)\backslash \mathrm{Z}(L)$ and $\mu(z) = (b_1, b_2)$. By Lemma 2.2, we have

$$\mathrm{sdim}L^2 \le \left(\frac{1}{2}b_1(b_1-1) + \frac{1}{2}b_2(b_2+1), b_1b_2\right) + \mathrm{sdim}[L,z]. \tag{2}$$

(1) Suppose $|z| = \bar{0}$. By Lemma 4.1, we have $\mu(z) \le (m-1,n)$ and it follows from eq. (2) that

$$\lambda(z) \ge (m,n) - (\mathrm{sdr}(L) + (1,0)). \tag{3}$$

Clearly, $\mathrm{sdim}L/(L^2 + \mathrm{Z}(L)) \ge \mathrm{sdim}L/\mathrm{Z}_L(z)$ and it follows from eq. (3) that

$$\mathrm{sdim}(L/\mathrm{Z}(L))^2 \le \mathrm{sdr}(L) + (1,0).$$

(2) Suppose $|z| = \bar{1}$. By Lemma 4.1, we have $\mu(z) \le (m,n-1)$ and then by eq. (2), we have $\lambda(z) \ge (n,m) - \mathrm{sdr}(L)$.

As in (1), we have

$$\mathrm{sdim}(L/\mathrm{Z}(L))^2 \leq \mathrm{sdim}L/\mathrm{Z}(L) - \mathrm{sdim}\Pi[L,z] \leq (\mathrm{dr}(L),\mathrm{dr}(L)) - \mathrm{sdr}(L). \qquad \blacksquare$$

Lemma 4.3. *Suppose* $z \in \mathrm{Z}_2(L)\backslash \mathrm{Z}(L)$, $|z| = \bar{0}$ *and* $\mathrm{sdim}(L/\mathrm{Z}(L))^2 = \mathrm{sdr}(L)+(1,0)$. *Then* $\mathrm{Z}_L(z) = L^2 + \mathrm{Z}(L)$ *and* $\lambda(z) = (m,n) - (\mathrm{sdr}(L) + (1,0))$.

Proof. By eq. (3), we have

$$\mathrm{sdim}(L/\mathrm{Z}_L(z)) = \lambda(z) \geq (m,n) - (\mathrm{sdr}(L) + (1,0)).$$

Hence

$$\begin{aligned}\mathrm{sdr}(L) + (1,0) &= \mathrm{sdim}(L/\mathrm{Z}(L)) - \mathrm{sdim}L/(L^2 + \mathrm{Z}(L)) \\ &\leq \mathrm{sdim}(L/\mathrm{Z}(L)) - \mathrm{sdim}(L/\mathrm{Z}_L(z)) \\ &\leq \mathrm{sdr}(L) + (1,0).\end{aligned}$$

Therefore, $\mathrm{sdim}\mathrm{Z}_L(z) = \mathrm{sdim}\,(L^2 + \mathrm{Z}(L))$. Then $L^2 + \mathrm{Z}(L) = \mathrm{Z}_L(z)$ and

$$\lambda(z) = (m,n) - (\mathrm{sdr}(L) + (1,0)).$$

The proof is complete. $\blacksquare$

Recall that a finite-dimensional Lie superalgebra $\mathfrak{g}$ is called a *Heisenberg Lie superalgebra* provided that $\mathfrak{g}^2 = \mathrm{Z}(\mathfrak{g})$ and $\mathrm{sdim}\mathrm{Z}(\mathfrak{g}) = (1,0)$ or (0,1). Heisenberg Lie superalgebras consist of two types according to the parity of the central elements (see [16]). Suppose $\mathfrak{g}$ is a Heisenberg Lie superalgebra with $\mathrm{Z}(\mathfrak{g}) = \mathbb{F}z$.

(1) If $|z| = \bar{0}$, then $\mathfrak{g}$ has a homogeneous basis (called a standard basis)

$$\{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p, z \mid w_1, w_2, \ldots, w_q\},$$

where $|u_i| = |v_j| = |z| = \bar{0}$, $|w_k| = \bar{1}$; $i = 1, \ldots, p$, $j = 1, \cdots, p$, $k = 1, \ldots, q$, and the multiplication is given by $[u_i, v_i] = -[v_i, u_i] = z$, $[w_k, w_k] = z$ and the other brackets of basis elements vanishing. Denote by $H(p,q)$ the Heisenberg Lie superalgebra $\mathfrak{g}$ of even center, where $p + q \geq 1$.

(2) If $|z| = \bar{1}$, then $\mathfrak{g}$ has a homogeneous basis (called a standard basis)

$$\{u_1, u_2, \ldots, u_k \mid z, w_1, w_2, \ldots, w_k\},$$

where $|u_i| = \bar{0}$, $|w_j| = |z| = \bar{1}$; $i = 1, \ldots, m$, $j = 1, \ldots, k$, and the multiplication is given by

$$[u_i, w_i] = -[w_i, u_i] = z$$

and the other brackets of basis elements vanishing. We write $H(k)$ for the Heisenberg Lie superalgebra $\mathfrak{g}$ of odd center, where $k \geq 1$.

Proposition 4.4. *Let* $H(p,q)$ *be a Heisenberg Lie superalgebra of even center. Then*

$$\mathrm{sdim}\mathcal{M}(H(p,q)) = \begin{cases} (2p^2 - p + \frac{1}{2}q^2 + \frac{1}{2}q - 1, 2pq), & p + q \geq 2 \\ (0,0), & p = 0, q = 1 \\ (2,0), & p = 1, q = 0. \end{cases}$$

Proof. We only consider the case $p=0$, $q=1$, while the remaining cases may be argued as in [10, Theorem 4.3]. Suppose (K,M) is a defining pair of $H(0,1)$. Then $K/M\cong H(0,1)$ and K/M has a standard basis $\{a+M\mid b+M\}$, where $a,b\in K$ with $|a|=\bar{0}$, $|b|=\bar{1}$. Then $[b,b]\equiv a \pmod{M}$. Since $[[b,b],b]=0$, one sees that $[a,b]=0$. It follows that K^2 is 1-dimensional and not contained in M. Since $M\subset K^2$, we have $M=0$. The proof is complete. ■

The multipliers for Heisenberg Lie superalgebras of odd center were determined in [8].

Proposition 4.5. *Let $H(k)$ be a Heisenberg Lie superalgebra of odd center. Then*

$$\mathrm{sdim}\mathcal{M}(H(k))=\begin{cases}(k^2,k^2-1), & k\geq 2\\ (1,1), & k=1.\end{cases}$$

Lemma 4.6. *If* $\mathrm{sdr}(L)=(0,0)$, *then* $L/\mathrm{Z}(L)$ *is either abelian or isomorphic to* $H(1,0)$.

Proof. If $\mathrm{Z}_2(L)/\mathrm{Z}(L)$ has a nonzero odd part, then by Lemma 4.2(2), we have $\mathrm{sdim}(L/\mathrm{Z}(L))^2=(0,0)$ and hence $L/\mathrm{Z}(L)$ is abelian.

Suppose the odd part of $\mathrm{Z}_2(L)/\mathrm{Z}(L)$ is zero. Following Lemma 4.2(1) we have $\mathrm{sdim}(L/\mathrm{Z}(L))^2=(0,0)$ or $(1,0)$. If $\mathrm{sdim}(L/\mathrm{Z}(L))^2=(0,0)$, then $L/\mathrm{Z}(L)$ is abelian. So we suppose $\mathrm{sdim}(L/\mathrm{Z}(L))^2=(1,0)$. Then $(L/\mathrm{Z}(L))^2\subset \mathrm{Z}(L/\mathrm{Z}(L))$.

We claim that $\mathrm{sdim}\mathrm{Z}(L/\mathrm{Z}(L))=(1,0)$. If not, there are $x,y\in\mathrm{Z}_2(L)$ such that $x+\mathrm{Z}(L)$ and $y+\mathrm{Z}(L)$ are linearly independent in $\mathrm{Z}(L/\mathrm{Z}(L))$. For any even element $x\in \mathrm{Z}_2(L)\backslash\mathrm{Z}(L)$, by Lemma 4.1(1), we have $\mu(x)\leq(m-1,n)$. Since $\mathrm{sdr}(L)=(0,0)$, $(L/[L,x])^2=L^2/[L,x]$ and $\mu(x)\leq(m-1,n)$, it follows from Lemma 2.2 that $\lambda(x)\geq(m-1,n)$. By Lemma 4.1(1), we also have $\lambda(x)\leq(m-1,n)$. Therefore, $\lambda(x)=(m-1,n)$. Then, since $[L,x]\cong L/\mathrm{Z}_L(x)$, we have

$$\mathrm{sdim}\ \mathrm{Z}_L(x)/\mathrm{Z}(L)=\mathrm{sdim}(L/\mathrm{Z}(L))-\lambda(x)=(1,0).$$

Let $y\in\mathrm{Z}_2(L)\backslash\mathrm{Z}(L)$ be even. Then we have $\mathrm{Z}(L)\subset\mathrm{Z}_L(x)\cap\mathrm{Z}_L(y)\subset\mathrm{Z}_L(x)$.
If $\mathrm{Z}(L)=\mathrm{Z}_L(x)\cap\mathrm{Z}_L(y)$, since $L^2\subset\mathrm{Z}_L(x)\cap\mathrm{Z}_L(y)$, we have $L^2\subset\mathrm{Z}(L)$ and $\mathrm{sdim}(L/\mathrm{Z}(L))^2=(0,0)$, a contrdiction. Hence $\mathrm{Z}_L(x)\cap\mathrm{Z}_L(y)=\mathrm{Z}_L(y)$ and then $\mathrm{Z}_L(x)=\mathrm{Z}_L(y)$. Since $\mathrm{sdim}\mathrm{Z}_L(x)/\mathrm{Z}(L)=(1,0)$, one sees that $x+\mathrm{Z}(L)$ and $y+\mathrm{Z}(L)$ are linearly dependent. Hence $\mathrm{sdim}\mathrm{Z}(L/\mathrm{Z}(L))=(1,0)$ and $L/\mathrm{Z}(L)$ is a Heisenberg Lie superalgebra of even center. Suppose $L/\mathrm{Z}(L)\cong H(p,q)$, where $p+q\geq 1$. Assume that $p+q\geq 2$. By Proposition 4.4 and Lemma 2.4, we have

$$\mathrm{sdim}L^2\leq\mathrm{sdim}(L/\mathrm{Z}(L))^2+\mathrm{sdim}\mathcal{M}(L/\mathrm{Z}(L)). \tag{4}$$

Then $\mathrm{sdim}L^2<(2p^2+p+\frac{1}{2}q^2+\frac{1}{2}q,2pq+q)$. However, since $\mathrm{sdr}(L)=(0,0)$, we get

$$\mathrm{sdim}L^2=(2p^2+p+\frac{1}{2}q^2+\frac{1}{2}q,2pq+q),$$

a contradiction. Assume now that $p=0$ and $q=1$. Following Proposition 4.4, we have $\mathrm{sdim}\mathcal{M}(H(0,1))=(0,0)$. In consequence of eq. (4) and $\mathrm{sdr}(L)=(0,0)$, we have $(1,\ 1)=\mathrm{sdim}L^2\leq(1,0)$, a contradiction. Summarizing, $L/\mathrm{Z}(L)$ is abelian or isomorphic to $H(1,0)$. ■

Lemma 4.7. *Suppose $L/\mathrm{Z}(L)$ is a Heisenberg Lie superalgebra.*

(1) *If* $\mathrm{sdr}(L)=(1,0)$, *then* $L/\mathrm{Z}(L)\cong H(1,0)$.

(2) *If* $\mathrm{sdr}(L)=(0,1)$, *then* $L/\mathrm{Z}(L)\cong H(0,1)$.

Proof. (1) Suppose $L/\mathrm{Z}(L)=H(p,q)$, where $p+q\geq 1$. Assume that $p+q\geq 2$. Then by eq. (4) and Proposition 4.4, for $p\neq 0$, we have

$$\begin{aligned}\mathrm{sdim}L^2 &< \mathrm{sdim}L/\mathrm{Z}(L)-(1,0)+\mathrm{sdim}\mathcal{M}(L/\mathrm{Z}(L))\\ &=(2p^2+p+\frac{1}{2}q^2+\frac{1}{2}q-1,2pq+q),\end{aligned}$$

and for $p=0$, similarly, we have $\mathrm{sdim}L^2<(2p^2+p+\frac{1}{2}q^2+\frac{1}{2}q,2pq+q-1)$. However, since $\mathrm{sdr}(L)=(1,0)$, we have $\mathrm{sdim}L^2=\left(2p^2+p+\frac{1}{2}q^2+\frac{1}{2}q-1,2pq+q\right)$, a contradiction. Assume now that $p=0$ and $q=1$. Following Proposition 4.4, $\mathrm{sdim}\mathcal{M}(H(0,1))=(0,0)$. Then by $\mathrm{sdr}(L)=(1,0)$ and eq. (4), we obtain that $(0,1)=\mathrm{sdim}L^2\leq(1,0)$, a contradiction.

Suppose $L/\mathrm{Z}(L)=H(k)$, where $k\geq 1$. Assume that $k>1$. By Proposition 4.5, we have $\mathrm{sdim}\mathcal{M}(L/\mathrm{Z}(L))=(k^2,k^2-1)$. Then by eq. (4), we get

$$\mathrm{sdim}L^2<\mathrm{sdim}L/\mathrm{Z}(L)-(0,1)+\mathrm{sdim}\mathcal{M}(L/\mathrm{Z}(L))=(k^2+k,k^2+k-1).$$

However, since $\mathrm{sdr}(L)=(1,0)$, we have $\mathrm{sdim}L^2=(k^2+k,k^2+k)$, a contradiction. Assume that $k=1$. By Proposition 4.5, we have $\mathrm{sdim}\mathcal{M}(H(1))=(1,1)$. Then by $\mathrm{sdr}(L)=(1,0)$ and (4), we have

$$(2,2)=\mathrm{sdim}L^2<\mathrm{sdim}L/\mathrm{Z}(L)-(0,1)+\mathrm{sdim}\mathcal{M}(L/\mathrm{Z}(L))=(2,2),$$

a contradiction. Summarizing, we have $L/\mathrm{Z}(L)=H(1,0)$.

(2) Suppose $L/\mathrm{Z}(L)=H(p,q)$, where $p+q\geq 1$. Assume that $p+q\geq 2$. Then by eq. (4) and Proposition 4.4, we have

$$\mathrm{sdim}L^2\leq\left(2p^2-p+\frac{1}{2}q^2+\frac{1}{2}q,2pq\right).$$

Since $\mathrm{sdr}(L)=(0,1)$, it follows that $p+q\leq 1$, a contradiction. Assume that $p=1,q=0$. Since $\mathrm{sdr}(L)=(0,1)$, we have $\mathrm{sdim}L^2=(3,-1)$, contradicting the assumption that $\mathrm{sdim}L^2>(0,0)$.

Suppose $L/\mathrm{Z}(L)=H(k)$, where $k\geq 1$. Assume that $k>1$. Then by eq. (4) and Proposition 4.5, we have

$$\mathrm{sdim}L^2<\mathrm{sdim}L/\mathrm{Z}(L)-(0,1)+\mathrm{sdim}\mathcal{M}(L/\mathrm{Z}(L))=(k^2+k,k^2+k-1).$$

However, since $\mathrm{sdr}(L)=(0,1)$, we have $\mathrm{sdim}L^2=(k^2+k+1,k^2+k-1)$, a contradiction. Assume that $k=1$. By Proposition 4.5, we have $\mathrm{sdim}\mathcal{M}(H(1))=(1,1)$. Since $\mathrm{sdr}(L)=(0,1)$, it follows from eq. (4) that

$$(3,1)=\mathrm{sdim}L^2<\mathrm{sdim}L/\mathrm{Z}(L)+\mathrm{sdim}\mathcal{M}(L/\mathrm{Z}(L))=(2,3),$$

a contradiction. Summarizing, we have $L/\mathrm{Z}(L)=H(0,1)$. ∎

The following proposition is analogues to [2, Theorem 3].

Proposition 4.8. *Suppose L is a non-abelian nilpotent Lie superalgebra. Then*
(1) $\mathrm{smr}L \neq (0,1)$.
(2) $\mathrm{smr}(L) = (1,0)$ *if and only if* $L \cong H(1,0)$.

Proof. Write $\mathrm{sdim}L = (s,t)$.

(1) Assume conversely that $\mathrm{smr}(L) = (0,1)$. By Proposition 3.1, L is not abelian. Let (K,M) be a maximal defining pair of L. Since $\mathrm{smr}(L) = (0,1)$, we have

$$\mathrm{sdim}M = \Big(\frac{1}{2}s(s-1) + \frac{1}{2}t(t+1), st\Big) - (0,1). \tag{5}$$

We now claim that $M = \mathrm{Z}(K)$. If not, since $M \subsetneq \mathrm{Z}(K)$, we have in consequence $\mathrm{sdim}(K/\mathrm{Z}(K)) < \mathrm{sdim}K/M = (s,t)$. Therefore $\mathrm{sdim}(K/\mathrm{Z}(K)) \leq (s-1,t)$ or $\mathrm{sdim}(K/\mathrm{Z}(K)) \leq (s,t-1)$. Suppose $\mathrm{sdim}(K/\mathrm{Z}(K)) \leq (s-1,t)$. Then by Lemma 2.2, we have

$$\mathrm{sdim}K^2 \leq \Big(\frac{1}{2}(s-1)(s-2) + \frac{1}{2}t(t+1), (s-1)t\Big).$$

Since $M \subset K^2$, by eq. (5) we have $\mathrm{sdim}L \leq (1,1)$. Since L is not abelian, we must have $\mathrm{sdim}L = (1,1)$. It is easy to deduce that $L \cong H(0,1)$. Consequently, $\mathrm{smr}(L) = (1,1)$, contradicting the assumption that $\mathrm{smr}(L) = (0,1)$.

Suppose $\mathrm{sdim}K/\mathrm{Z}(K) \leq (s,t-1)$. Then by Lemma 2.2 and eq. (5), we have $\mathrm{sdim}L \leq (1,0)$, contradicting the assumption that L is not abelian. Hence $M = \mathrm{Z}(K)$ and $\mathrm{sdim}K/\mathrm{Z}(K) = (s,t)$. Since L is not abelian, we have $M \subsetneq K^2$. So we have $\mathrm{sdr}(K) = (0,0)$. By Lemma 4.6, $L \cong K/\mathrm{Z}(K)$ is abelian or $H(1,0)$. Then $\mathrm{smr}(L) = (0,0)$ or $(1,0)$, a contradiction.

(2) Suppose $L \cong H(1,0)$. By Proposition 4.4, we have $\mathrm{sdim}\mathcal{M}(L) = (2,0)$ and hence $\mathrm{smr}(L) = (1,0)$.

Suppose $\mathrm{smr}(L) = (1,0)$. By Proposition 3.1, L is not abelian. Let (K,M) be a maximal defining pair of L. As in (1), we have $M = \mathrm{Z}(K)$. Since L is not abelian, we have $M \subsetneq K^2$. It follows that $\mathrm{sdr}(K) = (0,0)$. By Lemma 4.6, we have $L \cong K/\mathrm{Z}(K) = H(1,0)$. ∎

5. Multiplier-rank 2 nilpotent Lie superalgebras

In this section, suppose L is a finite-dimensional non-abelian nilpotent Lie superalgebra and $\mathrm{sdim}L/\mathrm{Z}(L) = (m,n)$. Let us establish several technical lemmas.

Lemma 5.1. *Suppose that $z \in \mathrm{Z}_2(L)\backslash\mathrm{Z}(L)$ is an even element and that $\lambda(z) = (m,n) - (\mathrm{sdr}(L) + (1,0))$. Then $\mu(z) = (m-1,n)$. Moreover, $L/[L,z]/\mathrm{Z}(L/[L,z])$ is either $\mathrm{Ab}(m-1,n)$ or $H(1,0)$.*

Proof. Suppose $\mu(z) = (b_1,b_2)$. By Lemma 4.1, we have $\mu(z) \leq (m-1,n)$. Then, by Lemma 2.2, we have

$$\Big(\tfrac{1}{2}m(m-1) + \frac{1}{2}n(n+1), mn\Big) - \mathrm{sdr}(L) = \mathrm{sdim}L^2$$
$$\leq \Big(\tfrac{1}{2}b_1(b_1-1) + \tfrac{1}{2}b_2(b_2+1), b_1b_2\Big) + \lambda(z) \leq \Big(\tfrac{1}{2}m(m-1) + \tfrac{1}{2}n(n+1), mn\Big) - \mathrm{sdr}(L).$$

Therefore, $\mu(z) = (m-1, n)$ and $\mathrm{sdr}(L/[L,z]) = (0,0)$. Our lemma follows from Lemma 4.6. ∎

Lemma 5.2. *Let* $\mathrm{sdr}(L) = (1,0)$*. Then* $L/\mathrm{Z}(L)$ *is isomorphic to one of the following Lie superalgebras:*

(1) *an abelian Lie superalgebra;*

(2) $H(1,0)$*;*

(3) $H(1,0) \oplus \mathrm{Ab}(1,0)$*;*

(4) *the Lie algebra with basis* $\{x, y, z, t\}$ *and multiplication given by*

$$[x,y] = -[y,x] = z, \ [x,z] = -[z,x] = t$$

and the other brackets of basis elements vanishing.

Proof. Our argument is divided into two parts.
(1) Suppose $\mathrm{Z}_2(L)/\mathrm{Z}(L)$ has a nonzero odd part. Then by Lemma 4.2(2), we have $\mathrm{sdim}(L/\mathrm{Z}(L))^2 = (0,0)$ or $(0,1)$. If $\mathrm{sdim}(L/\mathrm{Z}(L))^2 = (0,0)$, then $L/\mathrm{Z}(L)$ is abelian. Thus we suppose $\mathrm{sdim}(L/\mathrm{Z}(L))^2 = (0,1)$. Then $(L/\mathrm{Z}(L))^2 \subset \mathrm{Z}(L/\mathrm{Z}(L))$. If $\mathrm{sdim}\mathrm{Z}(L/\mathrm{Z}(L)) = (0,1)$, then $L/\mathrm{Z}(L)$ is a Heisenberg Lie superalgebra of odd center. Then by Lemma 4.7(1), we have $\mathrm{sdr}(L) \neq (1,0)$, contradicting the assumption.

Then we can assume that $\mathrm{sdim}\mathrm{Z}(L/\mathrm{Z}(L)) = (k, l+1) > (0,1)$. Suppose S is a subsuperspace of $L/\mathrm{Z}(L)$ such that $(L/\mathrm{Z}(L))^2 \oplus S = \mathrm{Z}(L/\mathrm{Z}(L))$. Suppose T is a subsuperspace such that $((L/\mathrm{Z}(L))^2 \oplus S) \oplus T = L/\mathrm{Z}(L)$. Write $H = (L/\mathrm{Z}(L))^2 + T$.

Then $H \lhd L/\mathrm{Z}(L)$ and it is easy to deduce that $H^2 = (L/\mathrm{Z}(L))^2 = \mathrm{Z}(H)$. Hence $H \cong H(p)$ for some $p \geq 1$. Then $\mathrm{sdim}S = (k,l)$ and $(m,n) - (k,l) = (p, p+1)$.

Assume that $p > 1$. Since $\mathrm{sdr}(L) = (1,0)$, by Proposition 4.5, eq. (4) and Lemma 2.5, we have

$$\begin{aligned}\left(\tfrac{1}{2}m(m-1) + \tfrac{1}{2}n(n+1), mn\right) - (1,0) &= \mathrm{sdim}L^2 \leq \mathrm{sdim}\mathcal{M}\left(L/\mathrm{Z}(L)\right) + (0,1) \\ &= \left(\tfrac{1}{2}k(k-1) + \tfrac{1}{2}l(l+1), kl\right) + \left(p^2, p^2-1\right) + (pk+pl, pk+pl) + (0,1).\end{aligned}$$

Substituting $m = p+k$ and $n = p+1+l$, one may obtain that $p+l \leq 0$, contradicting the assumption that $p+l > 0$.
Assume that $p = 1$. As in the case $p > 1$, one may obtain that $l \leq -1$, contradicting the assumption that $l \geq 0$.

(2) Suppose the odd part of $\mathrm{Z}_2(L)/\mathrm{Z}(L)$ is zero. By Lemma 4.2(1), we have $\mathrm{sdim}(L/\mathrm{Z}(L))^2 = (0,0)$, $(1,0)$ or $(2,0)$. If $\mathrm{sdim}(L/\mathrm{Z}(L))^2 = (0,0)$, then $L/\mathrm{Z}(L)$ is abelian. Suppose that $\mathrm{sdim}(L/\mathrm{Z}(L))^2 = (1,0)$. If $\mathrm{sdim}\mathrm{Z}(L/\mathrm{Z}(L)) = (1,0)$, then by Lemma 4.7(1), we have $L/\mathrm{Z}(L) \cong H(1,0)$. Since $\mathrm{sdim}(L/\mathrm{Z}(L))^2 = (1,0)$ and hence $(L/\mathrm{Z}(L))^2 \subset \mathrm{Z}(L/\mathrm{Z}(L))$, we can assume that $\mathrm{sdim}\mathrm{Z}(L/\mathrm{Z}(L)) = (k+1, 0) > (1,0)$.

Let S be a subsuperspace of $L/\mathrm{Z}(L)$ such that $(L/\mathrm{Z}(L))^2 \oplus S = \mathrm{Z}(L/\mathrm{Z}(L))$. Suppose H is a subsuperspace containing $(L/\mathrm{Z}(L))^2$ such that $H \oplus S = L/\mathrm{Z}(L)$.

Then H is a subsuperalgebra of $L/\mathrm{Z}(L)$ and $H^2 = (L/\mathrm{Z}(L))^2 = \mathrm{Z}(H)$. Since $\mathrm{sdim}(L/\mathrm{Z}(L))^2 = (1,0)$, we have $H \cong H(p,q)$, where $p+q \geq 1$. Then $\mathrm{sdim}S = (k,0)$ and $(m-k, n) = (2p+1, q)$.

Assume that $p+q \geq 2$. Since $\mathrm{sdr}(L)=(1,0)$, by Proposition 4.4, eq. (4) and Lemma 2.5, we have

$$\begin{aligned}\left(\tfrac{1}{2}m(m-1)+\tfrac{1}{2}n(n+1),mn\right)-(1,0)&=\mathrm{sdim}L^2 \leq \mathrm{sdim}\mathcal{M}(L/\mathrm{Z}(L))+(1,0)\\ &=\left(\tfrac{1}{2}k(k-1),0\right)+\left(2p^2-p+\tfrac{1}{2}q^2+\tfrac{1}{2}q-1,2pq\right)+(2pk,kq)+(1,0).\end{aligned}$$

Substituting $m=2p+1+k$ and $n=q$, one may obtain that $p+q=0$, contradicting the assumption that $p+q \geq 2$.

Assume that $p=1$ and $q=0$. As in the case $p+q \geq 2$, one may obtain that $k=1$. Hence $L/\mathrm{Z}(L) \cong H(1,0) \oplus \mathrm{Ab}(1,0)$.

Assume that $p=0$ and $q=1$. As in the case $p+q \geq 2$, one may obtain that $1 \leq 0$, a contradiction.

Now suppose $\mathrm{sdim}(L/\mathrm{Z}(L))^2=(2,0)$. For any even element $x \in \mathrm{Z}_2(L)\backslash\mathrm{Z}(L)$, by Lemma 4.3, we have that $L^2+\mathrm{Z}(L)=\mathrm{Z}_L(x)$ and $\lambda(x)=(m-2,n)$. Since $x \in \mathrm{Z}_L(x)=L^2+\mathrm{Z}(L)$, we have $\mathrm{Z}_2(L) \subset \mathrm{Z}_L(x)=L^2+\mathrm{Z}(L)$. Let us show that $\mathrm{sdim}\mathrm{Z}(L/\mathrm{Z}(L))=(1,0)$. If not, we have $\mathrm{sdim}\mathrm{Z}_2(L)/\mathrm{Z}(L)=(2,0)$, since

$$\mathrm{sdim}\mathrm{Z}_2(L)/\mathrm{Z}(L) \leq \mathrm{sdim}\mathrm{Z}_L(x)/\mathrm{Z}(L)=(2,0).$$

Then $\mathrm{Z}_L(x)=\mathrm{Z}_2(L)$ for all $x \in \mathrm{Z}_2(L)\backslash\mathrm{Z}(L)$. Since $\langle x,\mathrm{Z}(L)\rangle/[L,x] \subset \mathrm{Z}(L/[L,x])$, by Lemma 5.1, we have

$$\begin{aligned}(m-1,n)&=\mathrm{sdim}(L/\mathrm{Z}(L))/(\langle x,\mathrm{Z}(L)\rangle/\mathrm{Z}(L))\\ &\geq \mathrm{sdim}(L/[L,x])/\mathrm{Z}(L/[L,x])=(m-1,n).\end{aligned}$$

Then we have the following Lie superalgebra isomorphism:

$$(L/\mathrm{Z}(L))/(\langle x,\mathrm{Z}(L)\rangle/\mathrm{Z}(L)) \cong (L/[L,x])/\mathrm{Z}(L/[L,x]).$$

Then by Lemma 5.1, $L/\mathrm{Z}(L)/\langle x,\mathrm{Z}(L)\rangle/\mathrm{Z}(L)$ is abelian or isomorphic to $H(1,0)$. However, since $\mathrm{sdim}(L/\mathrm{Z}(L))^2=(2,0)$, one sees that $L/\mathrm{Z}(L)/\langle x,\mathrm{Z}(L)\rangle/\mathrm{Z}(L)$ is not abelian. Thus $L/\mathrm{Z}(L)/\langle x,\mathrm{Z}(L)\rangle/\mathrm{Z}(L) \cong H(1,0)$. Then it is routine to deduce that $\mathrm{sdim}\mathrm{Z}(L/\mathrm{Z}(L))=(1,0)$, a contradiction.

Suppose $\mathrm{sdim}\mathrm{Z}(L/\mathrm{Z}(L))=(1,0)$. Let x and $\mathrm{Z}(L)$ generate $\mathrm{Z}_2(L)$. By Lemma 4.3, we have $L^2+\mathrm{Z}(L)=\mathrm{Z}_L(x) \supsetneq \mathrm{Z}_2(L)$, since $\mathrm{sdim}\mathrm{Z}_L(x)/\mathrm{Z}(L)=(2,0)$. By Lemma 5.1, we have $\mu(x)=(m-1,n)$. Clearly, $\mathrm{Z}_2(L)/[L,x] \subset \mathrm{Z}(L/[L,x])$. Then

$$\begin{aligned}(m-1,n)&=\mathrm{sdim}L/\mathrm{Z}(L)/\mathrm{Z}_2(L)/\mathrm{Z}(L)\\ &\geq \mathrm{sdim}L/[L,x]/\left(\mathrm{Z}\left(L/[L,x]\right)\right)=(m-1,n).\end{aligned}$$

Therefore, $\mathrm{Z}(L/[L,x])=\mathrm{Z}_2(L)/[L,x]$. By Lemma 5.1, we have

$$L/\mathrm{Z}(L)/\mathrm{Z}(L/\mathrm{Z}(L)) \cong L/\mathrm{Z}_2(L) \cong L/[L,x]/\mathrm{Z}(L/[L,x])$$

is isomorphic to $H(1,0)$, since $L/\mathrm{Z}_2(L)$ is not abelian. Hence $L/\mathrm{Z}(L)$ is isomorphic to the Lie algebra in (4). ∎

Lemma 5.3. *Let* $\mathrm{sdr}(L)=(0,1)$. *Then* $L/\mathrm{Z}(L)$ *is isomorphic to one of the following Lie superalgebras:*

(1) *An abelian Lie superalgebra;*

(2) $H(0,1)$*;*

(3) $H(1,0)\oplus \mathrm{Ab}(0,1)$.

Proof. Since $\mathrm{sdr}(L)=(0,1)$, by Lemma 4.2, we have $\mathrm{sdim}(L/\mathrm{Z}(L))^2\le(1,1)$. If $\mathrm{sdim}(L/\mathrm{Z}(L))^2=(0,0)$, then $L/\mathrm{Z}(L)$ is abelian.

Suppose $\mathrm{sdim}(L/\mathrm{Z}(L))^2=(1,0)$. If $\mathrm{sdim}\mathrm{Z}(L/\mathrm{Z}(L))=(1,0)$, then by Lemma 4.7, we have $L/\mathrm{Z}(L)\cong H(0,1)$. Suppose $\mathrm{sdim}\mathrm{Z}(L/\mathrm{Z}(L))=(k+1,l)>(1,0)$. Since $\mathrm{sdim}(L/\mathrm{Z}(L))^2=(1,0)$, we have $(L/\mathrm{Z}(L))^2\subset \mathrm{Z}(L/\mathrm{Z}(L))$. Let S be a subsuperspace of $L/\mathrm{Z}(L)$ such that $(L/\mathrm{Z}(L))^2\oplus S=\mathrm{Z}(L/\mathrm{Z}(L))$. Suppose H is a subsuperspace containing $(L/\mathrm{Z}(L))^2$ such that $H\oplus S=L/\mathrm{Z}(L)$. Then H is a subsuperalgebra of $L/\mathrm{Z}(L)$ and $H^2=(L/\mathrm{Z}(L))^2=\mathrm{Z}(H)$. Since $\mathrm{sdim}(L/\mathrm{Z}(L))^2=(1,0)$, we have $H\cong H(p,q)$, where $p+q\ge 1$. Then $\mathrm{sdim}S=(k,l)$ and $(m-k,n-l)=(2p+1,q)$.

Assume that $p+q\ge 2$. Since $\mathrm{sdr}(L)=(0,1)$, by Proposition 4.4, eq. (4) and Lemma 2.5, we have

$$\begin{aligned}\Big(\tfrac12 m(m-1)+\tfrac12 n(n+1),mn\Big)-(0,1)&=\mathrm{sdim}L^2\le \mathrm{sdim}\mathcal{M}(L/\mathrm{Z}(L))+(1,0)\\ &=\Big(\tfrac12 k(k-1)+\tfrac12 l(l+1),kl\Big)+\Big(2p^2-p+\tfrac12 q^2+\tfrac12 q-1,2pq\Big)\\ &\qquad+(2pk+ql,\ kq+2pl)+(1,0).\end{aligned}$$

Substituting $m=2p+1+k$ and $n=q+l$, one may obtain that $p+q\le 1$, contradicting the assumption that $p+q\ge 2$.

Assume that $p=1$, $q=0$. As in the case $p+q\ge 2$, one gets $k=0$ and $l=1$. Hence $L/\mathrm{Z}(L)\cong H(1,0)\oplus\mathrm{Ab}(0,1)$.

Assume that $p=0$, $q=1$. As in the case $p+q\ge 2$, one may obtain that $k+l=0$, contradicting the assumption that $k+l\ge 1$.

Assume that $\mathrm{sdim}(L/\mathrm{Z}(L))^2=(0,1)$. Then $(L/\mathrm{Z}(L))^2\subset\mathrm{Z}(L/\mathrm{Z}(L))$, contradicting Lemma 4.2(2).

Suppose $\mathrm{sdim}(L/\mathrm{Z}(L))^2=(1,1)$. Suppose the odd part of $\mathrm{Z}_2(L)/\mathrm{Z}(L)$ is zero. For any even element $x\in \mathrm{Z}_2(L)\backslash \mathrm{Z}(L)$, by Lemma 4.3, $L^2+\mathrm{Z}(L)=\mathrm{Z}_L(x)$ and $\lambda(x)=(m-1,n-1)$. Hence $\mathrm{sdim}\mathrm{Z}_L(x)/\mathrm{Z}(L)=(1,1)$. Now $x\in\mathrm{Z}_L(x)$ for all $x\in\mathrm{Z}_2(L)\backslash\mathrm{Z}(L)$. Hence $\mathrm{Z}_2(L)\subsetneq \mathrm{Z}_L(x)$ for all $x\in \mathrm{Z}_2(L)\backslash\mathrm{Z}(L)$. We claim that $L/\mathrm{Z}_2(L)$ is not abelian. If not, since $\mathrm{Z}(L)\subset\mathrm{Z}_2(L)$, we have $\mathrm{Z}_L(x)\subset \mathrm{Z}_2(L)$, contradicting the assumption that $\mathrm{Z}_2(L)\subsetneq\mathrm{Z}_L(x)$. By Lemma 4.2(2), we have $\mathrm{sdim}\mathrm{Z}_2(L)/\mathrm{Z}(L)=(1,0)$. Suppose x and $\mathrm{Z}(L)$ generate $\mathrm{Z}_2(L)$. By Lemma 5.1, $\mu(x)=(m-1,n)$. Clearly, $\mathrm{Z}_2(L)/[L,x]\subset \mathrm{Z}(L/[L,x])$. Note that

$$\begin{aligned}(m-1,n)&=\mathrm{sdim}L/\mathrm{Z}(L)/\mathrm{Z}_2(L)/\mathrm{Z}(L)\\ &\ge \mathrm{sdim}L/[L,x]/(\mathrm{Z}(L/[L,x]))=(m-1,n).\end{aligned}$$

We have $\mathrm{Z}(L/[L,x])=\mathrm{Z}_2(L)/[L,x]$. Then by Lemma 5.1, we have that

$$L/\mathrm{Z}(L)/\mathrm{Z}(L/\mathrm{Z}(L))\cong L/\mathrm{Z}_2(L)\cong L/[L,x]/\mathrm{Z}(L/[L,x])$$

is isomorphic to $H(1,0)$, since $L/\mathrm{Z}_2(L)$ is not abelian. Hence $\mathrm{sdim}L/\mathrm{Z}(L) = (4,0)$, contradicting the assumption that $\mathrm{sdim}(L/\mathrm{Z}(L))^2 = (1,1)$. ■

A Lie superalgebra L is called *capable* if there is a Lie superalgebra H such that $L \cong H/\mathrm{Z}(H)$. As in Lie algebra case [9, Theorem 21], if L is capable and (K,M) is a maximal defining pair of L, then $M = \mathrm{Z}(K)$.

Lemma 5.4. *Let L be a non-capable, nilpotent, non-abelian Lie superalgebra of superdimension (s,t). Then $(s-1,t) < \mathrm{smr}(L)$ or $(t,s) < \mathrm{smr}(L)$.*

Proof. Let (K,M) be a maximal defining pair of L. Since L is not abelian, we have $L \cong K/M$ and $M \subsetneq K^2$. Since L is not capable, we have $M \subsetneq \mathrm{Z}(K)$ and $\mathrm{sdim}K/\mathrm{Z}(K) \leq (s-1,t)$ or $(s,t-1)$. Since $M \subsetneq K^2$, by Lemma 2.2, one may easily obtain that $\mathrm{smr}(L) > (s-1,t)$ or $\mathrm{smr}(L) > (t,s)$. ■

Lemma 5.5. *Let L be a capable, nilpotent, non-abelian Lie superalgebra. Then* $\mathrm{sdr}(\mathcal{C}(L)) < \mathrm{smr}(L)$.

Proof. Let (K,M) be a maximal defining pair of L. We have $L \cong K/M$ and $M \subset \mathrm{Z}(K) \cap K^2$. Since L is capable, we have $M = \mathrm{Z}(K)$. Since L is not abelian, we have $M \subsetneq K^2$. It follows that $\mathrm{sdr}(\mathcal{C}(L)) < \mathrm{smr}(L)$. ■

Proposition 5.6. *Let L be a finite-dimentional, non-abelian, nilpotent Lie superalgebra. Then*

(1) $\mathrm{smr}(L) \neq (0,2)$.

(2) $\mathrm{smr}L = (2,0)$ *if and only if* $L \cong H(1,0) \oplus \mathrm{Ab}(1,0)$.

(3) $\mathrm{smr}(L) = (1,1)$ *if and only if L is isomorphic to one of the following Lie superalgebras:*

  (3.1) $H(1,0) \oplus \mathrm{Ab}(0,1)$;

  (3.2) $H(0,1)$.

Proof. Suppose $\mathrm{sdim}L = (s,t)$.

(1) Assume conversely that $\mathrm{smr}(L) = (0,2)$. Then by Proposition 3.1, L is not abelian. First suppose L is not capable. Then by Lemma 5.4, we have

$$(s,t) < \mathrm{smr}(L) + (1,0) \quad \text{or} \quad (t,s) < \mathrm{smr}(L).$$

Since L is nilpotent and not abelian, we must have $\mathrm{sdim}L = (1,1)$. It is easy to deduce that $L \cong H(0,1)$. Consequently, $\mathrm{smr}(L) = (1,1)$, contradicting the assumption that $\mathrm{smr}(L) = (0,2)$.

Next suppose L is capable and K is a cover of L. Then we have $L \cong K/\mathrm{Z}(K)$. By Lemma 5.5, we have $\mathrm{sdr}(K) < \mathrm{smr}(L) = (0,2)$. If $\mathrm{sdr}(K) = (0,0)$, then by Lemma 4.6, L is either abelian, a contradiction, or $L = H(1,0)$, which yields $\mathrm{smr}(L) = (1,0)$, also a contradiction. Hence $\mathrm{sdr}(K) = (0,1)$. Therefore, $L \cong K/\mathrm{Z}(K)$ is one of the Lie superalgebras listed in Lemma 5.3. Then $\mathrm{smr}(L) = (0,0)$, $(1,1)$ or $(1,1)$, which is impossible. Hence, $\mathrm{smr}(L) \neq (0,2)$.

(2) Suppose $L = H(1,0) \oplus \mathrm{Ab}(1,0)$. Then by Proposition 3.1, 4.4 and Lemma 2.5, one may compute $\mathrm{sdim}\mathcal{M}(L) = (4,0)$. Then, since $\mathrm{sdim}L = (4,0)$, we have $\mathrm{smr}(L) = (2,0)$.

Conversely, suppose $\operatorname{smr}(L) = (2,0)$. By Proposition 3.1, L is not abelian. First suppose L is not capable. Then by Lemma 5.4, we have $(s,t) < \operatorname{smr}(L) + (1,0)$ or $(t,s) < \operatorname{smr}(L)$, contradicting the assumption that L is nilpotent and not abelian.

Next suppose L is capable and K is a cover of L. Then we have $L \cong K/\mathrm{Z}(K)$. By Lemma 5.5, we have $\operatorname{sdr}(K) < \operatorname{smr}(L) = (2,0)$. If $\operatorname{sdr}(K) = (0,0)$, then by Lemma 4.6, either L is abelian, a contradiction, or $L = H(1,0)$, which yields $\operatorname{smr}(L) = (1,0)$, also a contradiction. Hence $\operatorname{sdr}(K) = (1,0)$. Therefore, $L \cong K/\mathrm{Z}(K)$ is one of the Lie superalgebras listed in Lemma 5.2. A direct verification shows that $L \cong H(1,0) \oplus \mathrm{Ab}(1,0)$.

(3) Suppose $L = H(1,0) \oplus \mathrm{Ab}(0,1)$. By Propositions 4.4, 3.1 and Lemma 2.5, one may compute $\operatorname{sdim}\mathcal{M}(L) = (3,2)$. Then, since $(s,t) = (3,1)$, we have $\operatorname{smr}(L) = (1,1)$.

Suppose $L = H(0,1)$. Then by Proposition 4.4, we have $\operatorname{sdim}\mathcal{M}(L) = (0,0)$. Since $(s,t) = (1,1)$, we have $\operatorname{smr}(L) = (1,1)$.

Conversely, suppose $\operatorname{smr}(L) = (1,1)$. By Proposition 3.1, L is not abelian. First suppose L is not capable. Then by Lemma 5.4, we have $(s,t) < \operatorname{smr}(L) + (1,0)$ or $(t,s) < \operatorname{smr}(L)$. Since L is nilpotent and not abelian, we must have $\operatorname{sdim}L = (1,1)$. Then it is easy to deduce that $L \cong H(0,1)$.

Next suppose L is capable and K is a cover of L. Then we have $L \cong K/\mathrm{Z}(K)$. By Lemma 5.5, we have $\operatorname{sdr}(K) < \operatorname{smr}(L) = (1,1)$. If $\operatorname{sdr}(K) = (0,0)$, then either L is abelian, a contradiction, or $L = H(1,0)$, which yields $\operatorname{smr}(L) = (1,0)$, also a contradiction. If $\operatorname{sdr}(K) = (1,0)$, then $L \cong K/\mathrm{Z}(K)$ is one of the Lie superalgebras listed in Lemma 5.2, and then $\operatorname{smr}(L) = (0,0), (1,0), (2,0)$ or $(4,0)$, a contradiction.

Suppose $\operatorname{sdr}(K) = (0,1)$. Then $L \cong K/\mathrm{Z}(K)$ is one of the Lie superalgebras listed in Lemma 5.3. A direct verification shows that $L \cong H(1,0) \oplus \mathrm{Ab}(0,1)$ or $H(0,1)$. ■

Recall that $\mathrm{Ab}(m,n)$ denotes the abelian Lie superalgebra of superdimension (m,n) and $H(m,n)$ is the $(2m+1,n)$-dimensional Heisenberg Lie superalgebras of even center. We compose Propositions 3.1, 4.8, 5.6 to formulate the following classification theorem.

Theorem 5.7. *Up to isomorphism, all the finite-dimensional nilpotent Lie superalgebras L of multiplier-rank ≤ 2 are listed below:*

$\operatorname{smr}(L)$	L
$(0,0)$	any abelian Lie superalgebra
$(1,0)$	$H(1,0)$
$(2,0)$	$H(1,0) \oplus \mathrm{Ab}(1,0)$
$(1,1)$	$H(1,0) \oplus \mathrm{Ab}(0,1)$
$(1,1)$	$H(0,1)$

References

[1] P. Batten: *Multipliers and Covers of Lie Algebras*, Ph.D. Thesis, North Carolina State University, Raleigh (1993).

[2] P. Batten, K. Moneyhun, E. Stitzinger: *On characterizing nilpotent Lie algebras by their multipliers*, Comm. Algebra 24 (1996) 4319–4330.

[3] P. Batten, E. Stitzinger: *On covers of Lie algebras*, Comm. Algebra 24 (1996) 4301–4317.

[4] L. R. Bosko: *On Schur multipliers of Lie algebras and groups of maximal class*, Int. J. Algebra Comput. 20 (2010) 807–821.

[5] M. Eshrati, F. Saeedi, H. Darabi: *On the multiplier of nilpotent n-Lie algebras*, J. Algebra 450 (2016) 162–172.

[6] P. Hardy: *On characterizing nilpotent Lie algebras by their multipliers III*, Comm. Algebra 33 (2005) 4205–4210.

[7] P. Hardy, E. Stitzinger: *On characterizing nilpotent Lie algebras by their multipliers* $t(L) = 3, 4, 5, 6$, Comm. Algebra 26 (1998) 3527–3539.

[8] X.-X. Miao, W.-D. Liu: *Multipliers, covers and stem extensions for Lie superalgebras*, arXiv: 1808.02991 (2018).

[9] K. Moneyhun: *Isoclinisms in Lie algebras*, Algebras Groups Geometry 11 (1994) 9–22.

[10] S. Nayak: *Multipliers of nilpotent Lie superalgebras*, Comm. Algebra 47 (2019) 689–705.

[11] P. Niroomand: *On dimension of the Schur multipliers of nilpotent Lie algebras*, Cent. Eur. J. Math. 9 (2011) 57–64.

[12] P. Niroomand, F. Johari: *The structure, capability and the Schur multiplier of generalized Heisenberg Lie algebras*, J. Algebra 505 (2018) 482–489.

[13] P. Niroomand, F. Johari, M. Parvizi: *On the capability and Schur multiplier of nilpotent Lie algebra of class two*, Proc. Amer. Math. Soc. 144 (2016) 4157–4168.

[14] P. Niroomand, F. G. Russo: *A note on the Schur multiplier of a nilpotent Lie algebra*, Comm. Algebra 39 (2011) 1293–1297.

[15] Z. Riyahi, A. R. Salemkar: *A remark on the Schur multiplier of nilpotent Lie algebras*, J. Algebra 438 (2015) 1–6.

[16] M. C. Rodríguez-Vallarte, G. Salgado and O. A. Sánchez-Valenzuela: *Heisenberg Lie superalgebras and their invariant superorthogonal and supersymplectic forms*, J. Algebra 332 (2011) 71–86.

[17] F. Saeedi, H. Arabyani, P. Niroomand: *On dimension of Schur multiplier of nilpotent Lie algebras II*, Asian-Eur. J. Math. 10 (2017) 1750076 (8 pages).

[18] A. R. Salemkar, B. Edalatzadeh: *On covers of perfect Lie algebras*, Algebra Colloq. 18 (2011) 419–427.

[19] B. Yankosky: *On the multiplier of a Lie algebra*, J. Lie Theory 13 (2003) 1–6.

Wende Liu, School of Mathematics and Statistics, Hainan Normal University, Haikou 571158, P. R. China; wendeliu@ustc.edu.cn.

Yanling Zhang, School of Mathematical Sciences, Harbin Normal University, Harbin 150025, P. R. China; 741488940@qq.com.

Received April 8, 2019
and in final form May 6, 2020