Journal of Lie Theory
Volume **30** (2020) 1061–1089

On Lie Algebras from Polynomial Poisson Structures

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Communicated by M. Schlichenmaier

Abstract. We consider a polynomial Poisson algebra $\mathcal{P}$ on $\mathbb{R}^{2n}$ $(n \geq 1)$ that is to say $\mathcal{P}$ consists only of polynomials in $\mathbb{R}^{2n}$. We manage the conditions on $\mathcal{P}$ in order to have: every derivation of $\mathcal{P}$ is a differential operator of order one which takes its coefficients in $\mathcal{P}$. Otherwise, this result may be not true. More, we have an analogous result for the derived ideal $[\mathcal{P}, \mathcal{P}]$ of $\mathcal{P}$. If $[\mathcal{P}, \mathcal{P}] = \mathcal{P}$, derivations of the normalizer $\mathfrak{N}$ of $\mathcal{P}$ are sum of derivations of $\mathcal{P}$ and non-local derivations of $\mathfrak{N}$. Without this last hypothesis on $[\mathcal{P}, \mathcal{P}]$, we can state a similar theorem about the normalizer of $[\mathcal{P}, \mathcal{P}]$. The first Chevalley-Eilenberg cohomology of these sub-algebras are computed. Moreover, some results from polynomial Hamiltonian vector fields Lie algebras on $\mathbb{R}^{2n}$ has been found out. A special intention to Lie sub-algebras of the polynomial Poisson algebra $\mathbb{R}(x, y)$ on $\mathbb{R}^2$ in which the Jacobian conjecture holds is given. We give a definition on a sub-Lie algebra of $\mathbb{R}(x, y)$ verifying the Jacobian conjecture and find that if it is different to $\mathbb{R}(x, y)$, it verifies the Jacobian conjecture.

Mathematics Subject Classification: Primary 17B66; secondary 53B15, 17B56.

Key Words: Lie algebras, polynomial Poisson structure, Jacobian conjecture, cohomology of Chevalley-Eilenberg, differential operators, non-local derivations.

1. Introduction and preliminaries

Let $n \geq 1$ be a natural integer and the usual Poisson structure $F(\mathbb{R}^{2n})$ on $\mathbb{R}^{2n}$, $F(\mathbb{R}^{2n})$ the corresponding Poisson Lie algebra on $\mathbb{R}^{2n}$ which is the ring of all smooth functions from $\mathbb{R}^{2n}$ to $\mathbb{R}$ with the Poisson bracket defined by

$$[R, Q] = \sum_{i=1}^{n} \left(\frac{\partial R}{\partial x^i} \frac{\partial Q}{\partial x^{n+i}} - \frac{\partial Q}{\partial x^{n+i}} \frac{\partial R}{\partial x^i} \right) \quad \text{for all} \quad R, Q \in F\left(\mathbb{R}^{2n}\right)$$

and $(x^i)_{1 \leq i \leq 2n}$ the canonical coordinates system of Darboux on $\mathbb{R}^{2n}$. Consider $\mathcal{P}$ a Lie sub-algebra of the Lie algebra $F(\mathbb{R}^{2n})$ consisting of polynomial functions from $\mathbb{R}^{2n}$ to $\mathbb{R}$. Explicitly $(\mathcal{P}, +, ., [,])$ where $+, [,]$ are inner laws on $\mathcal{P}$, the external law "." by $\mathbb{R}$ stabilizes $\mathcal{P}$. We can also verify the Jacobi identity on $\mathcal{P}$ and

$$\text{for all } O, Q, R \in \mathcal{P} \qquad [OQ, P] = [O, P]\, Q + [Q, P]\, O, \tag{1}$$

from that of $(F(\mathbb{R}^{2n}), +, ., [,])$. We mean by monomial of degree $i \in \mathbb{N}$ a monomial of degree i in the usual sense. Exceptionally, we set $\mathbb{R}(x^1, \ldots, x^{2n})$ the ring of polynomials on $\mathbb{R}^{2n}$ instead of $\mathbb{R}[x^1, \ldots, x^{2n}]$ to avoid confusion with the brackets of our considered Lie algebras. We denote by H_i for $i \in \mathbb{N} \cup \{-1\}$ the space spanned by all monomials of degree $i+1$ in $\mathbb{R}(x^1, \ldots, x^{2n})$ and $\mathcal{P}_i$ that of $\mathcal{P}$ so that $\mathcal{P} = \bigoplus_{i \geq -1} \mathcal{P}_i$.

ISSN 0949–5932 / \$2.50

We have $[\mathcal{P}_{-1}, \mathcal{P}_i] = \{0\}$ for all $i, j \neq -1$, $[\mathcal{P}_i, \mathcal{P}_j] \subset \mathcal{P}_{i+j-1}$. The Lie algebra $\mathcal{P}$ is mainly graded in this sense. When $n = 1$ and $\mathcal{P} = \mathbb{R}(x^1, x^2)$, it is the Lie algebra relative to the Jacobian conjecture [6] posed in the first time by Ott-Heinrich Keller (1939). After several studies, the conjecture says if $X, Y \in \mathbb{R}(x^1, x^2)$ such that $[X, Y] \in H_{-1} - \{0\}$, then the mapping (X, Y) is an injection from $\mathbb{R}^2$ to itself. It is clear that it defines an endomorphism $[X, .]$ of $\mathbb{R}(x^1, x^2)$ which is a derivation. In the first, we wish compute all derivations of $\mathbb{R}(x^1, x^2)$, more all those of $\mathcal{P}$ when $n \geq 1$. The result of [3] says that all derivations of the Lie algebra $F(\mathbb{R}^{2n})$ containing $\mathcal{P}$ are differential operators of order 1 on $F(\mathbb{R}^{2n})$. So it is natural to ask if we will have the same results as in $F(\mathbb{R}^{2n})$? Some facts that we will discuss in the beginning of the following section lead us to chose the following hypothesis: $\mathcal{P}$ contains the Lie algebra $\langle 1, x^1, \ldots, x^{2n} \rangle$ denoted by S. Then the centralizer of $\mathcal{P}$ is $\langle 1 \rangle$ and its normalizer is a subset of $\mathbb{R}(x^1, \ldots, x^{2n})$. Even if $S \subset \mathcal{P}$, $\mathcal{P}$ can admit a derivation which is not a differential operator. To avoid this situation, we state the following condition (H) to $\mathcal{P}$: for all $x \in \mathcal{P}_{u \geq 1}$, there is $(y, z) \in \mathcal{P}_{i \leq u} \times \mathcal{P}_{j \leq u}$ with $[y, z] = x$.

To make calculations easier we suppose that $\mathcal{P}_u$ for all $u \in \mathbb{N}$ is separated like a similar definition on a polynomial vector field in [8] unless special mention.

In other words, if $\sum_{i_1 + \cdots + i_{2n} = u+1} \alpha_{i_1, \ldots, i_{2n}} (x^1)^{i_1} \ldots (x^{2n})^{i_{2n}}$ is in $\mathcal{P}_u - \{0\}$ then each $\alpha_{i_1, \ldots, i_{2n}} (x^1)^{i_1} \ldots (x^{2n})^{i_{2n}} \in \mathcal{P}_u$ where these $i_l \in \{0, 1, \ldots, u\}$ and $\alpha_{i_1, \ldots, i_{2n}} \in \mathbb{R}$.

Along with these hypotheses, we get some interesting characteristics of $\mathcal{P}$. It is shown that the derived ideal $[\mathcal{P}, \mathcal{P}]$ of $\mathcal{P}$ and $\mathcal{P}$ differ in some elements on $\mathcal{P}_0$, otherwise they are equal. The normalizer $\mathfrak{N}$ of $\mathcal{P}$ contains $x^i x^{i+n}$ for all $1 \leq i \leq n$. In addition, every derivation of $\mathcal{P}$ is completely defined by its image on S and we give a necessary and sufficient condition on a differential operator of order 1 on $F(\mathbb{R}^{2n})$ to be a derivation of $\mathcal{P}$. Then, each derivation of $\mathcal{P}$ is a differential operator of order 1 with its coefficients on $\mathcal{P}$ where these coefficients are completely studied.

This derivation is inner on $\mathfrak{N}$ if and only if its value on 1 vanishes. If the derived ideal of $\mathcal{P}$ is $\mathcal{P}$ itself, every derivation of $\mathfrak{N}$ is a sum of a derivation of $\mathcal{P}$ and a non-local derivation of $\mathfrak{N}$ which is null on the derived ideal $[\mathfrak{N}, \mathfrak{N}]$ of $\mathfrak{N}$ and takes its value on $\mathbb{R}$. If $\mathcal{P}$ contains all $x^i x^{i+n}$ where $1 \leq i \leq n$, then $\mathfrak{N} = \mathcal{P}$. Therefore the first Chevalley-Eilenberg cohomology of $\mathcal{P}$, $H^1(\mathcal{P}) = (\mathfrak{N}/\mathcal{P}) \oplus \mathbb{R}$. That of $\mathfrak{N}$ is the following, $H^1(\mathfrak{N}) = \mathbb{R}^{k+1}$ when $[\mathcal{P}, \mathcal{P}] = \mathcal{P}$ where $k \in \{0, \ldots, n\}$ is the dimension of $\mathfrak{N} \ominus [\mathfrak{N}, \mathfrak{N}]$. We get analogous theorems as the above in $\mathcal{P}$ for $[\mathcal{P}, \mathcal{P}]$ which doesn't contain necessarily S.

In the second, we look at the applications of our results. It's natural to consider the Lie algebra $\mathcal{P} = \mathbb{R}(x^1, \ldots, x^{2n})$. We have $H^1(\mathbb{R}(x^1, \ldots, x^{2n})) = \mathbb{R}$ and the derived ideal of $\mathbb{R}(x^1, \ldots, x^{2n})$ is itself. More precisely $H_{t-1} = [H_i, H_j]$ when $t = i + j$ for $t \geq 0$ and $i, j \in \mathbb{N}$. It is well known that it exists a one to one correspondence with $F(\mathbb{R}^{2n})/H_{-1}$ and the Hamiltonian vector fields Lie algebra $\mathfrak{A}_w$ on $\mathbb{R}^{2n}$ with w the corresponding symplectic exact form, $H^1(\mathfrak{A}_w) = \mathbb{R}$ cf. [5]. Here our theorem affirms that it is always the case in term of cohomology when we write $\mathfrak{H} = \mathbb{R}(x^1, \ldots, x^{2n})/H_{-1} \subset \mathfrak{A}_w$ the Lie algebra of polynomial Hamiltonian vector fields on $\mathbb{R}^{2n}$ instead of $\mathfrak{A}_w$. More, we denote by $\mathfrak{A}$ the Lie algebra consisting of polynomial Hamiltonian vector fields on $\mathbb{R}^{2n}$, isomorphic in the same way as above one to $\mathcal{P}/H_{-1}$. A particular result when $n = 1$, the Lie algebra of triple derivations of $\mathcal{P}$ (resp. of $\mathfrak{A}$) equals to the Lie algebra of derivations of $\mathcal{P}$ (resp. $\mathfrak{A}$). More generally for $n \geq 1$, we obtain the following cohomological affirmations, $H^1(\mathfrak{A}) = (\mathfrak{N}/\mathcal{P}) \oplus \mathbb{R}$ and if we denote by N the normalizer of $\mathfrak{A}$ and N the normalizer of N, we use the results of [8] to have $H^1(\mathrm{N})$ is isomorphic to N/N. We will show similar results as above one in the previous section on the Lie algebras relative to derived ideal of $\mathcal{P}$. Particularly, if $n = 1$ and $\mathcal{P} \neq S$, $[\mathcal{P}, \mathcal{P}] = \mathcal{P}$, $H^1(\mathcal{P}) = \mathbb{R}$ and it follows that $H^1(\mathfrak{A}) = \mathbb{R}$. In [8], we computed derivations of the Lie algebra of polynomial vector fields on $\mathbb{R}^n$ containing all constant vector fields and the Euler vector field. Our theorems above explore the results about the derivations of that Lie algebra which doesn't contain all constant fields nor the Euler vector field.

Third, the Lie algebra $\mathbb{R}(x^1, \ldots, x^{2n})$ has a particular importance when $n = 1$ on the Jacobian conjecture as we said in the beginning of this introduction. In this paper we explore new approach on this conjecture basing on Lie algebras. In other words, our approach in finding the results and the results themselves are new in order to reply partially the Jacobian conjecture cf. [9, 6], because it is based only on the algebraic structures of $\mathbb{R}(x^1, x^2)$ and of all its Lie sub-algebras. By our theorem, if $X, Y \in \mathbb{R}(x^1, x^2)$ such that $[X, Y] \in \mathbb{R}^*$, then $[X, Y] = [X_0, Y_0]$ where (X_0, Y_0) verifies the Jacobian conjecture, X_0, Y_0 is respectively the part of X resp. of Y in H_0 and we can extend this result when $n > 1$ without Jacobian conjecture consideration.

We set the following definitions, a Lie sub-algebra $\mathfrak{T}$ of $\mathbb{R}(x^1, x^2)$ is admissible for the Jacobian conjecture (AJC) if there is $X, Y \in \mathfrak{T}$ such that $[X, Y] \in H^*_{-1}$, and $\mathfrak{T}$ verifies the Jacobian conjecture (JC) if for all $X, Y \in \mathfrak{T}$ such that $[X, Y] \in H^*_{-1}$, (X, Y) defines an injection on $\mathbb{R}^2$ into itself. In addition, $\mathfrak{T}$ satisfies (h) if for all $X \in \mathfrak{T}$, the part X_0 of X in H_0 is also in $\mathfrak{T}$. If $\mathfrak{T}$ follows (h), we find that it is (AJC) if and only if $S \subset \mathfrak{T}$. There are 20 types of (AJC) separated Lie sub-algebras of $\mathbb{R}(x^1, x^2)$, 3 of them verify (H), 11 of finite dimensional and 9 of infinite dimensional. The last Lie algebra in this list is $\mathbb{R}(x^1, x^2)$. We use these previous results in order to find that all Lie sub-algebras (AJC) following (h) (separated or not) of $\mathbb{R}(x^1, x^2)$ different to $\mathbb{R}(x^1, x^2)$ verify the Jacobian conjecture by Maple calculations. As a corollary, we have the same results for the Lie algebras (AJC) not necessarily under the (h) hypothesis but included in a Lie sub-algebra of $\mathbb{R}(x^1, x^2)$ different to $\mathbb{R}(x^1, x^2)$ verifying (h). Taking into account the above results, we can state that the rest of the work to solve the Jacobian Conjecture in $\mathbb{R}^2$ is to prove the following theorem:

Theorem 1.1. *Every Lie sub-algebra (AJC) of $\mathbb{R}(x^1, x^2)$ of dimension 3 which doesn't verify (h) and doesn't included in a Lie sub-algebra (AJC) satisfying (h) of $\mathbb{R}(x^1, x^2)$ different to $\mathbb{R}(x^1, x^2)$, verifies the Jacobian Conjecture (JC) where one of the vector space basis is 1 and one another has a degree at least 101.*

Throughout this paper the bracket operation on a Lie algebra is always noted by $[,]$.

2. Study of derivations of the Lie algebras relative to $\mathcal{P}$

Let $\mathfrak{S}$ be a Lie algebra which is a sub-algebra of the Lie algebra $\mathcal{S}$.

Definition 2.1. A *normalizer* of $\mathfrak{S}$ on $\mathcal{S}$ is the set $\{X \in \mathcal{S} / [X, \mathfrak{S}] \subset \mathfrak{S}\}$.

Definition 2.2. The *centralizer* of $\mathfrak{S}$ is $\{X \in \mathcal{S} / [X, \mathfrak{S}] = \{0\}\}$. The restriction of this centralizer on $\mathfrak{S}$ is the *center* of $\mathfrak{S}$.

Definition 2.3. A derivation of a Lie algebra $\mathfrak{S}$ is a linear mapping D from $\mathfrak{S}$ to itself such that for all $X, Y \in \mathfrak{S}$: $D[X,Y] = [DX, Y] + [X, DY]$.
A derivation is called *inner* in the normalizer of $\mathfrak{S}$ if it's a Lie derived L_X with respect to an element X of this normalizer. Shortly, it's inner if the previous $X \in \mathfrak{S}$.

Definition 2.4. A sub-set A of $\mathfrak{S}$ is called *ideal* of $\mathfrak{S}$ if $[A, \mathfrak{S}] \subset A$.

Let us take $\mathcal{P}$ having only the hypothesis in p.1 of our paper. If we take the Lie algebra generated by $1, xy$ on $\mathbb{R}^4$ with the Darboux coordinates system (x,z,y,t) and its derivation D defined by $D(1) = xy$ and $D(xy) = 1$, it is clear that it is not a differential operator. The normalizer coincides with the centralizer of this Lie algebra which is $\langle 1, xy, x^2y^2, \ldots, \rangle + \langle 1, xy, x^2y^2, \ldots, \rangle_{\mathfrak{D}}$ with $\mathfrak{D}$ the set of all functions in $F(\mathbb{R}^4)$ depending only on z, t. But we wish that these two last Lie algebras remain in $\mathbb{R}(x^1, \ldots, x^{2n})$ according to focus our study on polynomials.

These situations lead us to follow the hypothesis: In the following of the present section and in section 3, let $\mathcal{P}$ be a Lie sub-algebra of $\mathbb{R}(x^1, \ldots, x^{2n})$ described in the beginning of introduction's section which contains S unless special mention.

Proposition 2.5. *The centralizer of $\mathcal{P}$ is $H_{-1} = \mathbb{R}$ and then a characteristic ideal of $\mathcal{P}$, its normalizer $\mathfrak{N}$ is a sub-Lie algebra of $\mathbb{R}(x^1, \ldots, x^{2n})$.*

Proof. If X is in H_{-1}, $[X, \mathcal{P}] = \{0\}$ by the above graduation. Reciprocally if X is in the centralizer, $[X, x^i] = 0$ for all $i \in \{1, \ldots, 2n\}$. Then all derivatives of X with respect to all coordinates are null and $X \in \mathbb{R}$ because $\mathbb{R}^{2n}$ is connected. Recall that a characteristic ideal of a Lie algebra A is a sub-algebra of it, stabilized by all derivations of A. By the results of [1], H_{-1} is a characteristic ideal of $\mathcal{P}$. The result of the normalizer $\mathfrak{N}$ can be checked in the same way as that of the centralizer by integrations. That is to say, let X be in $\mathfrak{N}$ so that $[X, \mathcal{P}] \subset \mathcal{P}$. In a similar way than the previous, $\frac{\partial X}{\partial x^i}$ for $i \in \{1, \ldots, 2n\}$ is an element of $\mathcal{P}$. Thus it exists $t \in \mathbb{R}(x^1, \ldots, x^{2n})$ such that $X = t + P$ where P is a function independent on x^i. But $\frac{\partial X}{\partial x^{n+i}} \in \mathcal{P}$, so it exists a polynomial function K in $\mathbb{R}(x^1, \ldots, x^{2n})$ which doesn't depend on x^{n+i} such that $P = K + C$ where C is independent on x^i, x^{n+i}. Running i in $\{1, \ldots, n\}$, we have $C \in \mathbb{R}(x^1, \ldots, x^{2n})$ and $\mathcal{P}$ so. Then X is in $\mathbb{R}(x^1, \ldots, x^{2n})$ and $\mathfrak{N} \subset \mathbb{R}(x^1, \ldots, x^{2n})$. It is well known that normalizer of a Lie algebra is a Lie algebra, in this we achieve the proof. ∎
By our hypothesis, it is immediate that

Proposition 2.6. *The Lie algebra $\mathcal{P}$ is non-solvable and non-nilpotent.*

Then we can talk about derivations of $\mathcal{P}$ which are non trivial.
Even if $S \subset \mathcal{P}$, $\mathcal{P}$ can admit a derivation which is not a differential operator. Let us take $\mathbb{R}^2$ with (x, y) as coordinate system and $\mathcal{P} = \langle 1, x, y, x^2, x^3, \ldots \rangle$. The endomorphism D of $\mathcal{P}$ defined by $D(1) = 1$, $D(y) = y$, $D(x^s) = (1-s)x^s$ for all $s > 0$ is a derivation of $\mathcal{P}$ but is not a differential operator in the usual sense. That is to say, it's a differential operator of infinite order cf. [2], because the value of $D(x^s)$ for $s > 0$ depends on $\frac{\partial D\left(x^{s+1}\right)}{\partial x}$.
That is the reason in which we state that in the following of the section 2 and in the section 3, unless expressed mention, we suppose that $\mathcal{P}$ is such that for all $x \in \mathcal{P}_{k \geq 1}$, it exists $(y, z) \in \mathcal{P}_{i \leq k} \times \mathcal{P}_{j \leq k}$ such that $[y, z] = x$. This hypothesis will be denoted by (H).

Definition 2.7. A derived ideal of $\mathfrak{S}$ denoted by $[\mathfrak{S},\mathfrak{S}]$ is the Lie algebra spanned by all brackets of two elements in $\mathfrak{S}$.

Proposition 2.8. *The Lie algebra $\mathcal{P}=[\mathcal{P},\mathcal{P}]\oplus(\mathcal{P}_0\ominus[\mathcal{P},\mathcal{P}])$ where $[\mathcal{P},\mathcal{P}]$ the derived ideal of $\mathcal{P}$ and $\oplus$ is a direct sum of spaces.*

Proof. We know that $1=[x^i,x^{i+n}]\in[\mathcal{P},\mathcal{P}]$ and by (H) every $x\in\mathcal{P}_{k\geq 1}$ is in $[\mathcal{P},\mathcal{P}]$. It remains all x^i for $i=1,\dots,2n$ to be in $[\mathcal{P},\mathcal{P}]$. All $x^i, i=1,\dots,2n$ may not be in $[\mathcal{P},\mathcal{P}]$. ■

Remark 2.9. The following are the only conditions in which $(x^i,x^{i+n})\in[\mathcal{P},\mathcal{P}]^2$ if $1\leq i\leq n$:

(1) the existence of k,k' monomials of degree 2 which is not in $\langle x^ix^{i+n}\rangle$ where $x^ix^{i+n}+k, x^ix^{i+n}+k'\in\mathcal{P}$ such that

$$(x^i,x^{i+n})=([x^i,x^ix^{i+n}+k],-[x^{i+n},x^ix^{i+n}+k']).$$

(2) the existence of k,k' monomials of degree 2 which is not in $\langle x^ix^{t+n}\rangle$ ($t\neq i$, $1\leq t\leq n$) resp. $\langle x^{i+n}x^{t+n}\rangle$ with $(x^i,x^{i+n})=([x^t,x^ix^{t+n}+k],[x^t,x^{i+n}x^{t+n}+k'])$ where $x^ix^{t+n}+k, x^{i+n}x^{t+n}+k'\in\mathcal{P}$.

(3) the existence of k,k' monomials of degree 2 which is not in $\langle x^ix^t\rangle$ ($t\neq i$, $1\leq t\leq n$) resp. $\langle x^{i+n}x^t\rangle$ with $(x^i,x^{i+n})=(-[x^{t+n},x^ix^t+k],-[x^{t+n},x^{i+n}x^t+k'])$ where $x^ix^t+k, x^{i+n}x^t+k'\in\mathcal{P}$.

(4) the existence of k,k' monomials of degree 2 which is not in $\left\langle (x^i)^2\right\rangle$ ($1\leq i\leq n$) resp. $\left\langle (x^{i+n})^2\right\rangle$ with $(x^i,x^{i+n})=\frac{1}{2}\left(\left[(x^i)^2+k,x^{i+n}\right],\left[x^i,(x^{i+n})^2+k'\right]\right)$ where the following $(x^i)^2+k,(x^{i+n})^2+k'\in\mathcal{P}$. ■

In the following proposition, we suppose that $\mathcal{P}_1$ is separated. Moreover, we don't suppose $\mathcal{P}$ satisfying (H) in the following proposition but only the hypothesis every $x\in\mathcal{P}_{k>0}$ is in $[\mathcal{P},\mathcal{P}]$. The proposition which follows is a discussion in what $\mathcal{P}_1$ comes from $[\mathcal{P}_1,\mathcal{P}_1]$ or $[\mathcal{P}_2,\mathcal{P}_0]$.

Proposition 2.10. *We set $1\leq i\leq n$. If $x^ix^{i+n}\in\mathcal{P}$ and $\mathcal{P}$ verifies (H), then $(x^i)^2$ and $(x^{i+n})^2$ are in $\mathcal{P}$. But when $(x^i)^2$ (resp. $(x^{i+n})^2$) is in $\mathcal{P}$, then we have the previous result or it exists $1\leq l\leq n$ such that $x^lx^i, x^{l+n}x^i\in\mathcal{P}$ (resp. $x^lx^{i+n}, x^{l+n}x^{i+n}\in\mathcal{P}$).*

Proof. Suppose that $1\leq i\leq n$. In the first condition, if $x^ix^{i+n}\in\mathcal{P}_1$, by (H) and our hypothesis it exists $y,z\in\mathcal{P}_1$ such that $x^ix^{i+n}=[y,z]$. We can suppose that y,z are monomials with only one term by the separated hypothesis on $\mathcal{P}_1$. It is possible only if $y=(x^i)^2, z=(x^{i+n})^2$ up a multiplication by constants. If $(x^i)^2\in\mathcal{P}_1$, by (H) it exists $y,z\in\mathcal{P}_1$ such that $(x^i)^2=[y,z]$. As in the previous, we suppose that y,z are monomials with only one term. It is true only when $y=(x^i)^2, z=x^ix^{i+n}$ up a multiplication by constants or it exists $1\leq l\leq n$ such that $y=x^lx^i, z=x^lx^{i+n}$ where $l\neq i$. We proceed in a similar way when $(x^{i+n})^2\in\mathcal{P}_1$. ■

Remark 2.11. If $x^ix^{i+n}\in[\mathcal{P}_2,\mathcal{P}_0]$, then the following situations can be present:

1. $(x^i)^2x^{i+n}\in\mathcal{P}_2$ such that $x^ix^{i+n}=\frac{1}{2}\left[(x^i)^2x^{i+n},x^{i+n}\right]$ and $-\left[(x^i)^2x^{i+n},x^i\right]=(x^i)^2\in\mathcal{P}_1$,

2. $(x^{i+n})^2 x^i \in \mathcal{P}_2$ such that $x^i x^{i+n} = -\frac{1}{2}\left[(x^{i+n})^2 x^i, x^i\right]$ and $\left[(x^{i+n})^2 x^i, x^{i+n}\right] = (x^{i+n})^2 \in \mathcal{P}_1$,

3. $x^{t+n}x^i x^{i+n} \in \mathcal{P}_2$ such that $x^i x^{i+n} = -[x^{t+n}x^i x^{i+n}, x^t]$, then $x^i x^{t+n}$ is equal to $-[x^{t+n}x^i x^{i+n}, x^i]$ and $x^{t+n}x^{i+n} = [x^{t+n}x^i x^{i+n}, x^{i+n}]$ are in $\mathcal{P}_1$ where $t \neq i$, $1 \leq t \leq n$,

4. $x^t x^i x^{i+n} \in \mathcal{P}_2$ such that $x^i x^{i+n} = [x^t x^i x^{i+n}, x^{t+n}]$, then $x^i x^t$ is equal to $-[x^t x^i x^{i+n}, x^i]$ and $x^t x^{i+n} = [x^t x^i x^{i+n}, x^{i+n}]$ are in $\mathcal{P}_1$ where $t \neq i$, $1 \leq t \leq n$;

if $(x^i)^2$ respectively $(x^{i+n})^2$ is in $[\mathcal{P}_2, \mathcal{P}_0]$, then $(x^i)^3 \in \mathcal{P}_2$ such that $(x^i)^2 = \frac{1}{3}\left[(x^i)^3, x^{i+n}\right]$ respectively $(x^{i+n})^2 = -\frac{1}{3}\left[(x^{i+n})^3, x^i\right]$. ■

In the following of the present section and in section 3, $\mathcal{P}_{k\geq 1}$ are separated unless special mention. We insist in this hypothesis because there are examples in which it's not the case, for example $\mathcal{P} = \langle 1, x, y, x^2 + y^2 + 2xy\rangle$ where $\mathcal{P}$ is not separated. In the following, the expression $f\left(\hat{z}\right)$ means that f doesn't depend on z.

Proposition 2.12. *Here, the Lie algebra $\mathcal{P}$ may not verify (H). Let $\mathfrak{K}$ be the set $\{1 \leq i \leq n / x^i x^{i+n} \in \mathcal{P}\}$ and the sum of natural integers $i_1 + \cdots + i_{2n} \neq 0$. If $X = (x^1)^{i_1} \ldots (x^{2n})^{i_{2n}} \in \mathcal{P}$ such that it exists $k \in \mathfrak{K}$ where $i_k \neq i_{k+n}$, then $X \in [\mathcal{P}, \mathcal{P}]$. In the case that all $1 \leq k \leq n$, $i_k = i_{k+n}$ where it exists $1 \leq j \leq n$ such that $i_j > 1$, then $X \in [\mathcal{P}, \mathcal{P}]$, otherwise $X \in [\mathcal{P}, \mathcal{P}]$ if it exists $1 \leq j \leq n$ with*

$$f\left(x^1, ..., \overset{\wedge}{x^j}, ..., \overset{\wedge}{x^{j+n}}, ..., x^{2n}\right)\left(x^j\right)^2, g\left(x^1, ..., \overset{\wedge}{x^j}, ..., \overset{\wedge}{x^{j+n}}, ..., x^{2n}\right)\left(x^{j+n}\right)^2 \quad (2)$$

in $\mathcal{P}$, with $(fg)\left(x^1, \ldots, \overset{\wedge}{x^j}, \ldots, \overset{\wedge}{x^{j+n}}, \ldots, x^{2n}\right) = \frac{X}{x^j x^{j+n}}$ where if the expression of f has x^i (resp. x^{i+n}) then g doesn't depend on x^{i+n} (resp. x^i) for all $1 \leq i \leq n$, $i \neq j$ and conversely. In the conditions where for all $1 \leq k \leq n$, $i_k = i_{k+n}$, the bracket of X and $x^t x^{t+n}$ equals to 0 for all $1 \leq t \leq n$.

Proof. It is sufficient to check that in the first case $X = \left[X, \frac{1}{i_k - i_{k+n}} x^k x^{k+n}\right]$. In the second one, we do $\left[\ldots\left[\left[X, x^{j+n}\right], x^l\right]\ldots, x^t\right]$ to have $(x^j)^{i_j - 1}(x^{j+n})^{i_j} \in \mathcal{P}$ and $[X, x^j]$ to obtain $\frac{X}{x^{j+n}} \in \mathcal{P}$. Then

$$\left[\frac{X}{x^{j+n}}, \left(x^j\right)^{i_j-1}\left(x^{j+n}\right)^{i_j}\right] = \frac{X}{(2i_j - 1)\left(x^j\right)^{i_j}\left(x^{j+n}\right)^{i_j}}\left(x^j\right)^{2i_j-2}\left(x^{j+n}\right)^{2i_j-2} \in \mathcal{P}.$$

When $i_j > 1$, we can decrease the degree of $\frac{X}{(x^j)^{i_j}(x^{j+n})^{i_j}}\left(x^j\right)^{2i_j-2}\left(x^{j+n}\right)^{2i_j-2}$ in $\mathcal{P}$ to have $\frac{X}{(x^j)^{i_j}(x^{j+n})^{i_j}}\left(x^j\right)^{i_j}\left(x^{j+n}\right)^{i_j} \in \mathcal{P}$ by successive brackets with sufficient number of x^j and x^{j+n}. Thus our corresponding result. Otherwise, $1 \leq j \leq n$ is such that $i_j = i_{j+n} = t$ with $t = 0$ or 1 (it's not possible that every $i_j = i_{j+n} = 0$ by initial hypothesis $i_1 + \cdots + i_{2n} \neq 0$), the bracket of elements in (2) permits us to conclude. The final assertion is easy to compute. ■

The following Corollary is clear by Proposition 2.12:

Corollary 2.13. *If $\mathfrak{N}$ is the normalizer of $\mathcal{P}$ where $\mathcal{P}$ may not verify (H), then $x^i x^{i+n}$ with $i = 1, \ldots, n$ are in $\mathfrak{N}$.*

It is possible that $\mathfrak{N}$ doesn't satisfy (H) and doesn't contain all monomials of the form $x^i x^k$ with $1 \leq i, k \leq 2n$. Let's view it in the following example.

Example 2.14. In $\mathbb{R}^4$, with the coordinates system of Darboux (x, z, y, t). The Lie algebra of infinite dimension $\mathcal{P}$ is generated by $1, x, z, y, t, xt, xy, x^2, y^2, x^2t, x^2t^2$. By calculation, its normalizer $\mathfrak{N}$ coincides with $\mathcal{P} \oplus \langle zt \rangle$ which doesn't verify (H).

Let $\mathbb{P}_k$ be the part of $\mathcal{P}$ such that all its elements depend on x^k and x^{k+n} for a $k = 1, \ldots, 2n$ with the existence of $X \in \mathbb{P}_k$ such that its degree is 2.

Proposition 2.15. *If there is $k = 1, \ldots, n$ such that $\mathbb{P}_k$ exists, then $x^k x^{k+n} \in \mathcal{P}$ or it exists $t = 1, \ldots, n$ with $x^t x^{t+n} \in \mathcal{P}$.*

Proof. With this hypothesis and to simplify, there is $X \in \mathcal{P}$ such that its degree on x^k is more than or equal to 2 with a $1 \leq k \leq n$ or such degree is 1 and the degree of X is 2. So:

- $X = x^k x^{k+n}$, we have the trivial case.
- $X = \left(x^k\right)^2$, by (H) and separated hypothesis the only possible cases are $x^k x^{k+n} \in \mathcal{P}$ such that $\frac{1}{2}\left[\left(x^k\right)^2, x^k x^{k+n}\right] = \left(x^k\right)^2$ or it exists $t = 1, ..., n$ with $x^k x^t, x^k x^{t+n} \in \mathcal{P}$ such that $\left[x^k x^t, x^k x^{t+n}\right] = \left(x^k\right)^2$. If we have the second case, we must then have $x^k x^{k+n} \in \mathcal{P}$ or $x^t x^{t+n} \in \mathcal{P}$ such that $\left[x^k x^t, x^k x^{k+n}\right] = x^k x^t$ and $\left[x^k x^{t+n}, x^k x^{k+n}\right]$ equals to $x^k x^{t+n}$ or such that $\left[x^k x^t, x^t x^{t+n}\right] = x^k x^t$ and $-\left[x^k x^{t+n}, x^t x^{t+n}\right] = x^k x^{t+n}$ because of (H).
- $X = x^k x^t$ with $t \neq k, k+n$, by the same reasons as in the second part of the second bullet, $x^t x^{t+n}$ or $x^k x^{k+n}$ is in $\mathcal{P}$. ■

Lemma 2.16. *If we are in $\mathbb{R}^2$ with (x, y) as coordinate system and a Lie subalgebra $\mathfrak{T} \supset S$ of $\mathbb{R}(x, y)$ contains simultaneously x^3, y^2 or x^2, y^3, then $\mathfrak{T} = \mathbb{R}(x, y)$. Moreover, if $x^2 \in \mathfrak{T} \supset S$ and it exists $X \in \mathfrak{T}$ where the degree in y of X is ≥ 3, then $\mathfrak{T} = \mathbb{R}(x, y)$. The same result is obtained if we swap the place of x and y.*

Proof. The following calculations are up multiplication by a constant. If $\langle x^3, y^2 \rangle_{\mathbb{R}}$ is a subset of $\mathfrak{T}$ then $[x^3, y^2] = x^2 y$,

$$\underbrace{\left[x^2 y, \left[x^2 y, \left[\ldots \left[x^2 y, x^3\right]\right]\right]\right]}_{x^2 y \ i \text{ times}} = x^{i+3} \text{ for all } i \geq 1$$

are all in $\mathfrak{T}$. Thus $[x^i, y^2] = x^{i-1} y$ for all $i \geq 3$ and

$$\underbrace{\left[\ldots \left[\left[x^i y, y^2\right], y^2\right] \ldots, y^2\right]}_{y^2 \ j \text{ times}} = x^{i-j} y^{j+1}$$

for all $i \geq 3$ and for all $i \geq j \geq 1$ are element of $\mathfrak{T}$. But $S \subset \mathfrak{T}$ so all $y^j \in \mathfrak{T}$ where $j \geq 2$, and the bracket $[x^2, y^2]$ leads to $xy \in \mathfrak{T}$. All these situations conclude with $\mathfrak{T} = \mathbb{R}(x, y)$. The following assertion, $x^2 \in \mathfrak{T}$ containing S and there is a $X \in \mathfrak{T}$ where the degree in y of X is ≥ 3. By $S \subset \mathfrak{T}$,

$$X = y^3 \textstyle\sum_{l_3 \geq i \geq 0} \alpha_i^3 x^i + y^4 \sum_{l_4 \geq i \geq 0} \alpha_i^4 x^i + \cdots + y^m \sum_{l_m \geq i \geq 0} \alpha_i^m x^i \in \mathfrak{T}$$

where the $l_i \in \mathbb{N}$, $\alpha_i^j \in \mathbb{R}$ and m an integer more than or equal to 3 with $\alpha_{l_m}^m \neq 0$.

Let $W = \{j/\alpha^j_{l_j} \neq 0\}$, by successive brackets of X by y in $t = max\{l_j/j \in W\}$ times, we get $Y = \sum_{\{3\leq j\leq m/l_j=t\}} \alpha^j_{l_j}(l_j!)y^j \in \mathfrak{T}^*$. Reasonable successive brackets with x of this last expression give us $y^2 \in \mathfrak{T}$. Also, if $j_0 = max\{3 \leq j \leq m/l_j = t\}$, we do $\underbrace{[\ldots[Y,x],\ldots,x]}_{x \text{ for } (j_0-3)-\text{times}}$, and obtain by $y^2 \in \mathfrak{T}$ that $y^3 \in \mathfrak{T}$. By the above result, we have reached $\mathfrak{T} = \mathbb{R}(x,y)$. ■

Proposition 2.17. *If D a derivation of $\mathcal{P}$ then for all $u \in \mathcal{P}_{l\geq 0}$, $i,j = 1,\ldots,n$:*

$$D\left(\frac{\partial u}{\partial x^j}\right) = \frac{\partial D(u)}{\partial x^j} + \sum_{t=1}^{n}\left(\frac{\partial u}{\partial x^t}\frac{\partial D\left(x^{j+n}\right)}{\partial x^{t+n}} - \frac{\partial u}{\partial x^{t+n}}\frac{\partial D\left(x^{j+n}\right)}{\partial x^t}\right); \tag{3}$$

$$D\left(\frac{\partial u}{\partial x^{j+n}}\right) = \frac{\partial D(u)}{\partial x^{j+n}} - \sum_{t=1}^{n}\left(\frac{\partial u}{\partial x^t}\frac{\partial D\left(x^{j}\right)}{\partial x^{t+n}} - \frac{\partial u}{\partial x^{t+n}}\frac{\partial D\left(x^{j}\right)}{\partial x^t}\right); \tag{4}$$

$$\frac{\partial D\left(x^{i+n}\right)}{\partial x^j} = \frac{\partial D\left(x^{j+n}\right)}{\partial x^i},\ i \neq j; \tag{5}$$

$$\frac{\partial D\left(x^{i}\right)}{\partial x^{j+n}} = \frac{\partial D\left(x^{j}\right)}{\partial x^{i+n}},\ i \neq j; \tag{6}$$

$$\frac{\partial D\left(x^{i+n}\right)}{\partial x^{j+n}} = -\frac{\partial D\left(x^{j}\right)}{\partial x^i},\ i \neq j; \tag{7}$$

$$\frac{\partial D\left(x^{j}\right)}{\partial x^j} + \frac{\partial D\left(x^{j+n}\right)}{\partial x^{j+n}} = D(1). \tag{8}$$

Proof. Let $u \in \mathcal{P}_{l\geq 1}$ with $i,j = 1,\ldots,n$ and D a derivation of $\mathcal{P}$, we have $D\left[u,x^{j+n}\right] = \left[Du,x^{j+n}\right] + \left[u,Dx^{j+n}\right] = D\left(\frac{\partial u}{\partial x^j}\right)$. Then we have (3). We set x^j instead of x^{j+n} in the first equality and we get (4). We can use (3) replacing u by x^{i+n}, then we have (5) and we obtain (6) by analogue arguments. When $u = x^{i+n}$ in (4), we get (7). If we write $u = x^j$ in (3), we get (8). ■

In the following, we set for i between 1 and n and a derivation D of $\mathcal{P}$

$$\left(D(x^i) - D(1)(x^i)\right)\frac{\partial u}{\partial x^i} + \left(D(x^{i+n}) - D(1)(x^{i+n})\right)\frac{\partial u}{\partial x^{i+n}} \in \mathcal{P}\ \forall u \in \mathcal{P}. \tag{9}$$

and for i between 1 and $2n$: $\left(D(x^i) - D(1)(x^i)\right)\dfrac{\partial u}{\partial x^i} \in \mathcal{P}\ \forall u \in \mathcal{P}.$ (10)

Proposition 2.18. *For a $i = 1,\ldots,2n$, if $\mathbb{P}_i$ doesn't exist then* (10) *holds for i.*

Proof. If $\mathbb{P}_i$ doesn't exist with $i = 1,\ldots,2n$, then all elements of $\mathbb{P}$ depending on x^i, x^{i+n} are of degree less than or equal to 1. So these elements are in $\langle x^i, x^{i+n}\rangle_{\mathbb{R}}$. Then (10) is obtained by taking into account that $D(1) \in \mathbb{R}$ cf. Proposition 2.5 and $\frac{\partial u}{\partial x^i} \in \mathbb{R}$ for all $u \in \mathcal{P}$. ■

Theorem 2.19. *If there is a $i = 1,\ldots,n$ such that $\mathbb{P}_i$ exists and $x^i x^{i+n} \in \mathcal{P}$, then (9) is satisfied for i.*

Proof. It exists $\mathbb{P}_i$ with $i = 1,\ldots,n$ such that $x^i x^{i+n} \in \mathcal{P}$. So $\mathcal{P}$ is once again graded by $\mathcal{P} = \bigoplus_{t\in I}\mathcal{P}^t$ with $[x^i x^{i+n}, X] = tX$ with $t \in I \subset \mathbb{Z}$ with X a monomial in $\mathcal{P}$. Let us take u a monomial in $\mathcal{P}$. We have $\left[x^i x^{i+n}, x^k\right] = 0$ if $k \neq i, i+n$ and x^{i+n} if $k = i+n$, $-x^i$ if $k = i$. Then this second grading makes $\left(x^i, x^{i+n}, x^{k\neq i,i+n}\right) \in \mathcal{P}^{-1} \times \mathcal{P}^1 \times \mathcal{P}^0$. Because of this grading and the element $x^i x^{i+n}$, we can affirm by [4]'s results that D is of degree zero. It means $D\left(\mathcal{P}^t\right) \subset \mathcal{P}^t$ for each $t \in I$. So $\left(D\left(x^i\right), D\left(x^{i+n}\right), D\left(x^{k\neq i,i+n}\right)\right)$ belongs to $\mathcal{P}^{-1} \times \mathcal{P}^1 \times \mathcal{P}^0$. Then we have the following finite sums

$$D\left(x^i\right) = \sum_{w\geq t\geq 1} \alpha_t \left(x^i\right)^t \left(x^{i+n}\right)^{t-1} \in \mathcal{P},\ D\left(x^{i+n}\right) = \sum_{w\geq t\geq 1} \beta_t \left(x^i\right)^{t-1} \left(x^{i+n}\right)^t \in \mathcal{P} \tag{11}$$

where the α_t, β_t are polynomials independent on x^i, x^{i+n}, $w \in \mathbb{N}$ such that α_w or $\beta_w \neq 0$. We can write (8) and obtain

$$\alpha_1 + \beta_1 = D(1), \alpha_t = -\beta_t \forall t \geq 2. \tag{12}$$

Therefore $A = (D(x^i) - D(1)(x^i)) \frac{\partial u}{\partial x^i} + (D(x^{i+n}) - D(1)(x^{i+n})) \frac{\partial u}{\partial x^{i+n}}$ which is equal to $-D(1)x^i \frac{\partial u}{\partial x^i} + \sum_{t\geq 1} \frac{\alpha_t}{t} \left[u, \left(x^i\right)^t \left(x^{i+n}\right)^t\right]$. In the case $u = \left(x^i\right)^t \left(x^{i+n}\right)^t Q$ where Q is a monomial independent on x^i and x^{i+n} and $t \in \mathbb{N}$, then the polynomial $A = -D(1)x^i \frac{\partial u}{\partial x^i} \in \mathcal{P}$. Otherwise, $u = \left(x^i\right)^l \left(x^{i+n}\right)^s Q$ where Q is a monomial independent on x^i and x^{i+n} and $l, s \in \mathbb{N}$ with $l \neq s$. Let us remark that $-D(1)x^i \frac{\partial u}{\partial x^i} \in \mathcal{P}$. Without losing generaly, we set $l > s$ with $l > 1$. So we have to prove $\alpha_t Q \left(x^i\right)^{l+t-1} \left(x^{i+n}\right)^{s+t-1} \in \mathcal{P}$ for $t \geq 1$.

- If $\mathbb{P}_i \oplus \mathbb{R} \supset \mathbb{R}\left(x^i, x^{i+n}\right)$ then $\left(x^i\right)^t \left(x^{i+n}\right)^{t+1} \in \mathcal{P}$. But $\alpha_t x^i \in \mathcal{P}$, therefore we obtain $\left[\alpha_t x^i, \left(x^i\right)^t \left(x^{i+n}\right)^{t+1}\right] \in \mathcal{P}$ which leads to $\alpha_t \left(x^i\right)^t \left(x^{i+n}\right)^t \in \mathcal{P}$. Then $\left[\alpha_t \left(x^i\right)^t \left(x^{i+n}\right)^t, u\right] \in \mathcal{P}$ implies $C = \alpha_t Q \left(x^i\right)^{l+t-1} \left(x^{i+n}\right)^{s+t-1} \in \mathcal{P}$ for $t \geq 1$ if and only if $B = [\alpha_t, Q] \left(x^i\right)^{l+t} \left(x^{i+n}\right)^{s+t} \in \mathcal{P}$ by (1). But $Q \in \mathcal{P}$; then we have $[\alpha_t x^i, Q] = [\alpha_t, Q] x^i \in \mathcal{P}$. Moreover, $\left[[\alpha_t, Q] x^i, \left(x^i\right)^t \left(x^{i+n}\right)^{t+1}\right] \in \mathcal{P}$ leads to $[\alpha_t, Q] \left(x^i\right)^t \left(x^{i+n}\right)^t \in \mathcal{P}$. But $\mathbb{P}_i \oplus \mathbb{R} \supset \mathbb{R}\left(x^i, x^{i+n}\right)$, then $\left(x^i\right)^{l+1} \left(x^{i+n}\right)^{s+1} \in \mathcal{P}$ and

$$\left[[\alpha_t, Q] \left(x^i\right)^t \left(x^{i+n}\right)^t, \left(x^i\right)^{l+1} \left(x^{i+n}\right)^{s+1}\right] \in \mathcal{P} \quad \text{yields} \quad B \in \mathcal{P}.$$

- If $\mathbb{P}_i \oplus \mathbb{R} \subsetneq \mathbb{R}\left(x^i, x^{i+n}\right)$. Because $x^i x^{i+n} \in \mathcal{P}$, from Proposition 2.10, $\left(x^i\right)^2$ and $\left(x^{i+n}\right)^2$ are in $\mathcal{P}$. Again about the elements in $\mathcal{P}$, $\mathcal{P}$ doesn't contain $\left(x^i\right)^f \left(x^{i+n}\right)^j$ where $f + j \geq 3$ because of Lemma 2.16. So it remains two possibilities of elements depending on x^i, x^{i+n} to be checked, $u = Q\left(x^i\right)^2$ or $u = Q\left(x^{i+n}\right)^2$ (we consider only the first case, the second one can be done by symmetry). The degree on x^i of $D\left(x^i\right)$ is at most 1 and the degree on x^{i+n} of $D\left(x^{i+n}\right)$ is at most 1. It implies every $\alpha_{t\geq 2} = 0$ and $\beta_{t\geq 2} = 0$, $D\left(x^i\right) = \alpha_1 x^i$ and $D\left(x^{i+n}\right) = \beta_1 x^{i+n}$. Then $A = -D(1)x^i \frac{\partial u}{\partial x^i} + \alpha_1 \left[u, x^i x^{i+n}\right]$. Computing $\left[Q\left(x^i\right)^2, \left(x^{i+n}\right)^2\right] \in \mathcal{P}$ leads to

$$Q x^i x^{i+n} \in \mathcal{P} \tag{13}$$

and $\left[\alpha_1 x^i, Q\left(x^i\right)^2\right] = [\alpha_1, Q] \left(x^i\right)^3 = 0$ by the hypothesis $\mathbb{P}_i \oplus \mathbb{R} \subsetneq \mathbb{R}\left(x^i, x^{i+n}\right)$. Thus

$$[\alpha_1, Q] = 0. \tag{14}$$

Using (3) with $u = \left(x^i\right)^2$ and $j = i$, we have $\alpha_1 - \beta_1 = \delta_1$ where $D\left(\left(x^i\right)^2\right) = \delta_1 \left(x^i\right)^2$ because of the grading in the beginning of this proof, with δ_1 a polynomial independent on x^i, x^{i+n}. In addition to $\alpha_1 + \beta_1 = D(1)$, it yields $\delta_1 = 2\alpha_1 - D(1)$. It is known that $\delta_1\left(x^i\right)^2 \in \mathcal{P}$, then $\alpha_1\left(x^i\right)^2 \in \mathcal{P}$. We can compute $\left[\alpha_1\left(x^i\right)^2, Qx^ix^{i+n}\right] \in \mathcal{P}$, by (13) and (14) we have $\alpha_1 Q\left(x^i\right)^2 \in \mathcal{P}$. We conclude $A \in \mathcal{P}$ because $-D(1)x^i\frac{\partial u}{\partial x^i} \in \mathcal{P}$. ∎

Proposition 2.20. *If $x^tx^{t+n} \in \mathcal{P}$ then $D\left(x^tx^{t+n}\right) = 0$.*

Proof. By Proposition 2.10, $\left(x^t\right)^2$ and $\left(x^{t+n}\right)^2$ belong to $\mathcal{P}$. We denote the $D\left(\left(x^t\right)^2\right) = \delta\left(x^t\right)^2$ and $D\left(\left(x^{t+n}\right)^2\right) = \delta'\left(x^{t+n}\right)^2$ by the similar reason as in the second bullet of the proof of Theorem 2.19 where $\delta\left(\overbrace{x^t, x^{t+n}}^{\wedge}\right)$, $\delta'\left(\overbrace{x^t, x^{t+n}}^{\wedge}\right)$ are polynomials in $\mathbb{R}\left(x^1, \ldots, x^{2n}\right)$. We use (3) with $u = \left(x^t\right)^2$ and $j = t$ resp. (4) with $u = \left(x^{t+n}\right)^2$ and $j = t$, considering (8) where $j = t$ we get $2\alpha_1 = \delta + D(1)$ resp. $2\alpha_1 = -\delta' + D(1)$ with $D\left(x^t\right) = \alpha_1 x^t$ and $D\left(x^{t+n}\right) = \left(D(1) - \alpha_1\right)x^{t+n}$ by Theorem 2.19. Therefore $\delta = -\delta'$. Moreover, we have

$$D\left[\left(x^t\right)^2, \left(x^{t+n}\right)^2\right] = \left[D\left(\left(x^t\right)^2\right), \left(x^{t+n}\right)^2\right] + \left[\left(x^t\right)^2, D\left(\left(x^{t+n}\right)^2\right)\right].$$

It yields $D\left(x^tx^{t+n}\right) = \left(\delta + \delta'\right)x^tx^{t+n} = 0$. ∎

Theorem 2.21. *If some $i = 1, \ldots, n$ exists with $x^ix^{i+n} \notin \mathcal{P}$, but there is a $t = 1, \ldots, \hat{i}, \ldots, n$ with $x^tx^{t+n} \in \mathcal{P}$ following the second and third cases of the proof of Proposition* 2.15*, then we get* (9) *for i.*

Proof. There is $\mathbb{P}_i$ with $i = 1, \ldots, n$ such that $x^ix^{i+n} \notin \mathcal{P}$ but it exists $t = 1, \ldots, \hat{i}, \ldots, n$ with $x^tx^{t+n} \in \mathcal{P}$ by the two last bullets of the proof of Proposition 2.15 where $\left(x^i\right)^2, x^ix^t, x^ix^{t+n} \in \mathcal{P}$. In this proof, we denote this hypothesis by $(J)_{it}$.

For $l \in \mathbb{N}$, let $Q^l\left(\overbrace{x^i, x^{i+n}}^{\wedge}, x^tx^{t+n}\right)$, $Q^0\left(\overbrace{x^i, x^{i+n}}^{\wedge}, x^tx^{t+n}\right)$ followed by the polynomials

$U^l\left(\overbrace{x^i, x^{i+n}}^{\wedge}, x^tx^{t+n}\right)$, $R^l\left(\overbrace{x^i, x^{i+n}, x^t, x^{t+n}}^{\wedge}\right)$, $Q_1\left(\overbrace{x^i, x^{i+n}}^{\wedge}, x^tx^{t+n}\right)$, $R_l\left(\overbrace{x^i, x^{i+n}, x^t, x^{t+n}}^{\wedge}\right)$

be in $\mathbb{R}(x^1, \ldots, x^{2n})$. By analogous graduation as in the beginning of the proof of Theorem 2.19 from x^tx^{t+n} and by similar reasoning as in this proof, $D\left(x^i\right), D\left(x^{i+n}\right)$ are of degree zero. Therefore

$$D\left(x^i\right) = \sum_{l\geq 1} Q^l\left(x^i\right)^l + Q^0x^{i+n} + \sum_{l\geq 0} R^l\left(x^tx^{t+n}\right)^l,$$

$$D\left(x^{i+n}\right) = Q_1x^{i+n} + \sum_{l\geq 1} U^l\left(x^i\right)^l + \sum_{l\geq 0} R_l\left(x^tx^{t+n}\right)^l;$$

here we don't have $\left(x^{i+n}\right)^{l\geq 2} \in \mathcal{P}$ otherwise $x^ix^{i+n} \in \mathcal{P}$. So the possible element depending on x^{i+n} in $\mathcal{P}$ is cx^{i+n} where $c \in \mathbb{R}$ otherwise $\left(x^{i+n}\right)^2 \in \mathcal{P}$ like the statement on x^i. Then (10) is true for $i + n$ using the same arguments as in the proof of Proposition 2.18.

By (8), $Q^1 + Q_1 = D(1)$ and $Q^l = 0$ for all $l \geq 2$. By (6) and (5) when $j = t$ and Theorem 2.19, we have respectively $R^l = R_l = 0$ for all $l \geq 2$. Then

$$D\left(x^i\right) = Q^1x^i + Q^0x^{i+n} + \sum_{l=0,1} R^l\left(x^tx^{t+n}\right)^l,$$

and $$D\left(x^{i+n}\right)=Q_1x^{i+n}+\sum_{l\geq 1}U^l\left(x^i\right)^l+\sum_{l=0,1}R_l\left(x^tx^{t+n}\right)^l.$$

If we write

$$Q^1=\sum_{l\geq 0}Q^1_l\left(x^tx^{t+n}\right)^l,\ Q^0=\sum_{l\geq 0}Q^0_l\left(x^tx^{t+n}\right)^l,\ Q_1=\sum_{l\geq 0}q^l\left(x^tx^{t+n}\right)^l$$

and $U^l=\sum_{s\geq 1}U^l_s\left(x^tx^{t+n}\right)^s$ where all the coefficients on the sums are polynomials which don't depend on $x^i, x^{i+n}, x^t, x^{t+n}$ belonging to $\mathbb{R}(x^1,\ldots,x^{2n})$. Using (7) where j replaced by t respectively i replaced by t and j replaced by i, we have

$$Q^1_{l\geq 2}=Q^0_{l\geq 2}=0; q^{l\geq 2}=U^l_{s\geq 2}=0.$$

Moreover, $Q^1+Q_1=D(1)$ conducts us to $Q^1_1=-q^1$. Therefore

$$D\left(x^i\right)=\left(Q^1_0+Q^1_1x^tx^{t+n}\right)x^i+\left(Q^0_0+Q^0_1x^tx^{t+n}\right)x^{i+n}+\sum_{l=0,1}R^l\left(x^tx^{t+n}\right)^l, \quad (15)$$

$$D\left(x^{i+n}\right)=\left(D(1)-Q^1_0-Q^1_1x^tx^{t+n}\right)x^{i+n}+\sum_{l\geq 1}\left(x^i\right)^l\left(U^l_0+U^l_1x^tx^{t+n}\right)+ \\ +\sum_{l=0,1}R_l\left(x^tx^{t+n}\right)^l. \quad (16)$$

Let $D\left(x^ix^t\right)=\sum_{l\geq 1}\eta_l\left(x^t\right)^l\left(x^{t+n}\right)^{l-1}$, $D\left(x^ix^{t+n}\right)=\sum_{l\geq 1}\zeta_l\left(x^t\right)^l\left(x^{t+n}\right)^{l-1}$ by the same arguments as in (11) where each η_l,ζ_l are polynomials independent on x^t, x^{t+n}. By Theorem 2.19, $D\left(x^t\right)=\gamma x^t$ and $D\left(x^{t+n}\right)=\gamma' x^{t+n}$ where γ,γ' are polynomials which don't depend on x^t, x^{t+n}. We replace u by x^ix^t and j by t in (3), and obtain

$$D\left(x^i\right)=\sum_{l\geq 1}l\eta_l\left(x^t\right)^{l-1}\left(x^{t+n}\right)^{l-1}+\frac{\partial\gamma'}{\partial x^{i+n}}x^tx^{t+n}+\gamma' x^i. \quad (17)$$

Replacing u by x^ix^{t+n} and j by t in (3), then we get

$$D\left(x^i\right)=\sum_{l\geq 1}l\zeta_l\left(x^t\right)^{l-1}\left(x^{t+n}\right)^{l-1}-\frac{\partial\gamma}{\partial x^{i+n}}x^tx^{t+n}+\gamma x^i. \quad (18)$$

Because of $\frac{\partial\gamma'}{\partial x^{i+n}}x^tx^{t+n}+\gamma' x^i\in\mathcal{P}$, by separated hypothesis $\gamma' x^i\in\mathcal{P}$. So γ' doesn't depend on x^{i+n} because $x^ix^{i+n}\notin\mathcal{P}$, we have also $\gamma\left(\overbrace{x^{i+n}}^{\wedge}\right)$. Thus

$$D\left(x^i\right)=\sum_{l\geq 1}l\eta_l\left(x^t\right)^{l-1}\left(x^{t+n}\right)^{l-1}+\gamma' x^i, \quad (19)$$

and $$D\left(x^i\right)=\sum_{l\geq 1}l\zeta_l\left(x^t\right)^{l-1}\left(x^{t+n}\right)^{l-1}+\gamma x^i. \quad (20)$$

Then we replace i by t and j by i in (7) and we have

$$\frac{\partial\gamma'}{\partial x^{i+n}}x^{t+n}=-\left(Q^1_1x^{t+n}x^i+Q^0_1x^{t+n}x^{i+n}+R^1x^{t+n}\right),$$

which is null.

Therefore
$$D\left(x^{i}\right)=Q_{0}^{1}x^{i}+Q_{0}^{0}x^{i+n}+R^{0}. \tag{21}$$

Identifying (19) and (21) resp. (20) and (21), we get $\eta_{l\geq 2}$ resp. $\zeta_{l\geq 2}=0$ and
$$D\left(x^{i}\right)=\zeta_{1}+\gamma' x^{i}=\eta_{1}+\gamma x^{i}. \tag{22}$$

We write $D\left(\left(x^{i}\right)^{2}\right)=D\left[x^{i}x^{t},x^{i}x^{t+n}\right]=\left[\eta_{1}x^{t},x^{i}x^{t+n}\right]+\left[x^{i}x^{t},\zeta_{1}x^{t+n}\right]$. Therefore $D\left(\left(x^{i}\right)^{2}\right)=\left(\eta_{1}+\zeta_{1}\right)x^{i}+\left(\frac{\partial\zeta_{1}}{\partial x^{i+n}}-\frac{\partial\eta_{1}}{\partial x^{i+n}}\right)x^{t}x^{t+n}$. We replace u by $x^{i}x^{t}$ and j by i resp. u by $x^{i}x^{t+n}$ and j by i in (3), then we have $Q_{0}^{0}=\frac{\partial\eta_{1}}{\partial x^{i+n}}=\frac{\partial\zeta_{1}}{\partial x^{i+n}}=0$. So
$$D\left(\left(x^{i}\right)^{2}\right)=\left(\eta_{1}+\zeta_{1}\right)x^{i}. \tag{23}$$

By (22) and $\gamma+\gamma'=D(1)$, we have
$$\frac{\partial\eta_{1}}{\partial x^{i}}+\frac{\partial\zeta_{1}}{\partial x^{i}}=2Q_{0}^{1}-D(1). \tag{24}$$

We calculate (3), when $u=\left(x^{i}\right)^{2}$ and j replaced by i, leading to
$$D\left(x^{i}\right)=\frac{1}{2}\left(\left(\eta_{1}+\zeta_{1}\right)+\left(\frac{\partial\eta_{1}}{\partial x^{i}}+\frac{\partial\zeta_{1}}{\partial x^{i}}\right)x^{i}\right)+x^{i}\left(D(1)-Q_{0}^{1}\right).$$

By (24),
$$D\left(x^{i}\right)=\frac{1}{2}\left(\left(\eta_{1}+\zeta_{1}\right)+x^{i}\left(D(1)\right)\right). \tag{25}$$

Recall that now
$$D\left(x^{i}\right)=Q_{0}^{1}x^{i}+R^{0}. \tag{26}$$

We are looking for the constancy of Q_{0}^{1} and R^{0}. Only two situations are possible:

* If some $h=1,\ldots,\hat{i},\ldots,n$ exists satisfying $(J)_{ih}$, we have Q_{0}^{1},R^{0} independent on x^{h},x^{h+n}.
* If some $h\neq t$ exists with $h=1,\ldots,\hat{i},\ldots,n$ such that if a monomial $u\in\mathcal{P}$ depending on x^{i} or x^{i+n} or x^{t} or x^{t+n}, $\left[u,x^{h}\right]=\left[u,x^{h+n}\right]=0$. That is to say, it is impossible to have $x^{i}x^{h}\in\mathcal{P}$, $x^{i}x^{h+n}\in\mathcal{P}$, $x^{i+n}x^{h}\in\mathcal{P}$, $x^{i+n}x^{h+n}\in\mathcal{P}$, $x^{h}x^{t}\in\mathcal{P}$, $x^{h}x^{t+n}\in\mathcal{P}$, $x^{h+n}x^{t}\in\mathcal{P}$, $x^{h+n}x^{t+n}\in\mathcal{P}$. We call this hypothesis by $(I)_{ih}$. Then Q_{0}^{1} doesn't depend on x^{h},x^{h+n} if we refer to $Q_{0}^{1}x^{i}\in\mathcal{P}$.

We can conclude that Q_{0}^{1} is constant. Now we prove that R_{0} is a constant.
Identify (26) with (25), we get:

- If $\left(\eta_{1}+\zeta_{1}\right)\left(x^{i}\right)$, then $R^{0}=0$. In this case, $D\left(x^{i}\right)=Q_{0}^{1}x^{i}$, and (10) holds for i because $Q_{0}^{1}\in\mathbb{R}$.
- If $\left(\eta_{1}+\zeta_{1}\right)\left(\overset{\wedge}{x^{i}}\right)$ then $R^{0}=\frac{1}{2}\left(\eta_{1}+\zeta_{1}\right)$ and $Q_{0}^{1}=\frac{1}{2}D(1)$ by (24). The only case to check is the second $*$. But we have (23), then $R^{0}x^{i}\in\mathcal{P}$. This leads to $R^{0}\left(\overset{\wedge}{x^{h}},\overset{\wedge}{x^{h+n}}\right)$ because of $(I)_{ih}$.

Thus R^{0} is a constant, then we can conclude that (10) is true for x^{i}.
The final conclusion is then (9) holds for i. ∎

Theorem 2.22. *Equation* (9) *is true.*

Proof. For $i = 1, \ldots, 2n$, we have exactly three cases. The first is described by the hypothesis of Proposition 2.18. Considering Proposition 2.15, the second one is given by the hypothesis of Theorem 2.19 and the third by that of Theorem 2.21. Compounding the results of Proposition 2.18, Theorem 2.19 and Theorem 2.21, we achieve the proof. ■

In the following, $\mathbb{M}$ denotes a smooth manifold.

Definition 2.23. If $\mathfrak{S}$ is a Lie sub-algebra of the ring $F(\mathbb{M})$ of smooth real functions on $\mathbb{M}$, a derivation D of $\mathfrak{S}$ is called local if for every non-empty open set $\mathfrak{U}$ of $\mathbb{M}$ and $X \in \mathfrak{S}$ such that $X_{|\mathfrak{U}} \equiv 0$, we have $D(X)_{|\mathfrak{U}} \equiv 0$.

Definition 2.24. A differential operator on $F(\mathbb{M})$ is a $\mathbb{R}$-endomorphism of $F(\mathbb{M})$ which is local [7]. In this sense, the differential operator is of finite order.

Theorem 2.25. *A necessary and sufficient condition on a differential operator D which is equal to $\sum_{i=1}^{i=2n} X^i \frac{\partial(.)}{\partial x^i} + h\,(.)$ of order one of $F(\mathbb{R}^{2n})$ with $h \in F(\mathbb{R}^{2n})$ to be a derivation of $\mathcal{P}$ is:*

(a) $h = D(1)$ *is a constant.*

(b) *All its coefficients X^i where $1 \leq i \leq 2n$ are polynomials in $\mathcal{P}$ such that $\left(X^j \frac{\partial}{\partial x^j} + X^{j+n} \frac{\partial}{\partial x^{j+n}}\right)(\mathcal{P}) \subset \mathcal{P}$ for all $j = 1, \ldots, n$.*

(c) $\frac{\partial X^k}{\partial x^i} + \frac{\partial X^{i+n}}{\partial x^{k+n}} = -\delta_k^i D(1)$ *for all* $1 \leq i, k \leq n$.

(d) $\frac{\partial X^{k+n}}{\partial x^i} = \frac{\partial X^{i+n}}{\partial x^k}$ *with* $1 \leq k \leq n$ *and* $1 \leq i \leq n$.

(e) $\frac{\partial X^k}{\partial x^{i+n}} = \frac{\partial X^i}{\partial x^{k+n}}$ *with* $1 \leq k \leq n$ *where* $1 \leq i \leq n$.

Proof. We set $D = \sum_{i=1}^{i=2n} X^i \frac{\partial(.)}{\partial x^i} + h\,(.)$ a differential operator of order one on $F(\mathbb{R}^{2n})$ as cited above where $h \in \mathbb{R}$. Let $f, g \in F(\mathbb{R}^{2n})$, we compute

$$[Df, g] + [f, Dg] - D[f, g] \tag{27}$$

which is equal to

$$h[f,g] + \sum_{i=1,j=1}^{i=2n,j=n} \frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^{j+n}} - \frac{\partial X^i}{\partial x^{j+n}}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} - \frac{\partial X^i}{\partial x^j}\frac{\partial g}{\partial x^i}\frac{\partial f}{\partial x^{j+n}} + \frac{\partial X^i}{\partial x^{j+n}}\frac{\partial g}{\partial x^i}\frac{\partial f}{\partial x^j}. \tag{28}$$

Now, we check the necessity of our theorem. We suppose a derivation of $\mathcal{P}$, namely $D = \sum_{i=1}^{i=2n} X^i \frac{\partial(.)}{\partial x^i} + h\,(.)$ like in the previous with only $h \in F(\mathbb{R}^{2n})$. Because $\langle 1 \rangle$ is the centralizer of $\mathcal{P}$, D stabilizes $\langle 1 \rangle$ by Proposition 2.5. Therefore $D(1) = h$ is a real number and we have (a). It is $D(1) = h \in \mathbb{R}$ and $D(x^i) = X^i + hx^i \in \mathcal{P}$ for $i = 1, \ldots, 2n$. But every $hx^i \in S$, then $X^i \in \mathcal{P}$ such that for $j = 1, \ldots, n$, $\left(X^j \frac{\partial}{\partial x^j} + X^{j+n} \frac{\partial}{\partial x^{j+n}}\right)(\mathcal{P}) \subset \mathcal{P}$ by Theorem 2.22. So (b) is true. If we chose successively in (28) for $k = 1, \ldots, n$ and $i = 1, \ldots, n$: $f = x^k$ and $g = x^{i+n}$, $f = x^{k+n}$ and $g = x^{i+n}$, $f = x^k$ and $g = x^i$; we have respectively (c), (d) and (e) because the expression cf. (28) is null.

Conversely, (27) is always equal to (28) where we have (a). We remark that it is a differential operator of order strictly 1 on $(F(\mathbb{R}^{2n}))^2$. So its determination depends only on its value on $(x^k, x^i) \in \mathcal{P}^2$ where $k = 1, \dots, 2n; i = 1, \dots, 2n$, otherwise it is null. If we have the last three conditions, the function cf. (28) vanishes on these $(x^k, x^i) \in \mathcal{P}^2$. So D is a derivation of $F(\mathbb{R}^{2n})$. But $D_{|\mathcal{P}} \subset \mathcal{P}$ because of (b), D is a derivation of $\mathcal{P}$. ∎

Lemma 2.26. *A derivation D of $\mathcal{P}$ is null on $\mathcal{P}$ if and only if D is null on S.*

Proof. If a derivation D of $\mathcal{P}$ is null, then D is null on $S \subset \mathcal{P}$. Conversely, if D vanishes on S. We take $u \in \mathcal{P}_1$, $[u, x^i] = -\frac{\partial u}{\partial x^{i+n}}$ and $[u, x^{i+n}] = \frac{\partial u}{\partial x^i}$ for $i = 1, \dots, n$. Applying these relations to D with $D_{|\mathcal{P}_0} = 0$, we have $D(u) \in \mathbb{R}$. But (H) said, that there exists $y, z \in \mathcal{P}_1$ such that $[y, z] = u$; so we apply this last equality to D and we obtain:

$$D[y, z] = [Dy, z] + [y, Dz], \tag{29}$$

and then $D(u) = 0$. Now, $u \in \mathcal{P}_2$ in the same way as the previous one and using $D_{|\mathcal{P}_1} = 0$, we obtain $D(u) \in \mathbb{R}$. Adopting a similar equation as (29), we find $D(u) = 0$. By recurrence, if $u \in \mathcal{P}_k$ with $k \geq 2, D(u) = 0$. ∎

Theorem 2.27. *Every derivation of $\mathcal{P}$ is a differential operator of order one with coefficients in $\mathcal{P}$ described by Theorem 2.22. Generally, if P doesn't verify (H), if $\mathfrak{N}$ satisfies (H), every derivation of the normalizer $\mathfrak{N}$ of $\mathcal{P}$ is a differential operator of order one with coefficients in $\mathfrak{N}$ following Theorem 2.22.*

Proof. Let D be such a derivation, by hypothesis $D(1), D(x^i)$ for all i in $\{1, \dots, 2n\}$ are defined in $\mathcal{P}$. Inspiring to [3] p. 53, we set a linear mapping T such that for every $u \in \mathcal{P}$,

$$T(u) = D(u) - D(1)u - \sum_{i=1}^{2n} \left(D(x^i) - D(1)(x^i)\right) \frac{\partial u}{\partial x^i}.$$

It is clear that T is null on all elements of $\langle 1, x^1, \dots, x^{2n} \rangle \subset \mathcal{P}$. Now, the differential operator of order 1 $u \in F(\mathbb{R}^{2n}) \mapsto D(1)u + \sum_{i=1}^{2n} (D(x^i) - D(1)(x^i)) \frac{\partial u}{\partial x^i}$ is a derivation of $\mathcal{P}$ because of Theorem 2.22, (7), (8), (5), (6) in Proposition 2.17 and of Theorem 2.25. So T becomes a derivation of $\mathcal{P}$. By Lemma 2.26, we get $T = 0$ and

$$D(u) = D(1)u + \sum_{i=1}^{2n} \left(D(x^i) - D(1)(x^i)\right) \frac{\partial u}{\partial x^i} \ \forall u \in \mathcal{P}.$$

The Proposition 2.5 says that the elements of $\mathfrak{N}$ are polynomial on $\mathbb{R}^{2n}$. From the fact that $\mathfrak{N}$ verifies (H) and $\mathcal{P} \subset \mathfrak{N}$, we can apply the results on $\mathcal{P}$ to $\mathfrak{N}$ and we achieve the proof. ∎

Remark 2.28. The locality technique for a derivation D on $F(\mathbb{M})$ in [3] doesn't work in the proof of Theorem 2.27 because, if a polynomial is null on an open set of $\mathbb{R}^n$, it is null on $\mathbb{R}^n$ and it is a trivial situation for the locality of a derivation D of $\mathcal{P}$.

Theorem 2.29. *A derivation of $\mathcal{P}$ is inner in $\mathfrak{N}$ if and only if $D(1) = 0$.*

Proof. Let D be a derivation of $\mathcal{P}$, then by Theorem 2.27,

$$D(u \in \mathcal{P}) = D(1)u + \sum_{i=1}^{2n} \left(D(x^i) - D(1)(x^i)\right) \frac{\partial u}{\partial x^i}.$$

We will solve $L_X u = \sum_{1\leq i\leq n}\left(\frac{\partial X}{\partial x^i}\frac{\partial u}{\partial x^{i+n}} - \frac{\partial X}{\partial x^{i+n}}\frac{\partial u}{\partial x^i}\right) = D\,(u\in\mathcal{P})$, therefore

$\frac{\partial X}{\partial x^i} = D(x^{i+n}) - D(1)(x^{i+n})$ and $\frac{\partial X}{\partial x^{i+n}} = -D(x^i) + D(1)(x^i)$ with $X\in\mathfrak{N}$.

By the classical theorem of Frobenius on a system of partial differential equations and using (c), (d) and (e) of Theorem 2.25 a necessary and sufficient condition from which these identities are true is

$$\frac{\partial D(x^i)}{\partial x^i} = -\frac{\partial D(x^{i+n})}{\partial x^{i+n}} + 2D(1).$$

But D satisfies (c) of Theorem 2.25, giving $\frac{\partial D(x^i)}{\partial x^i} = -\frac{\partial D(x^{i+n})}{\partial x^{i+n}} + D(1)$ in the first part. Therefore, this condition is $D(1) = 0$. ■

We ask if $\mathcal{P}$ is not trivial with our hypothesis, we can answer it by some examples of $\mathcal{P}$.

Example 2.30. We can take $\mathcal{P}$ on $\mathbb{R}^{2n}$ with $n\geq 1$ generated by $1, x^1, \ldots, x^{2n}$, also that spanned by all monomials on $\mathbb{R}^{2n}$. Their normalizers are respectively the Lie algebra generated by all monomials of degree at most 2 and itself, they verify (H). We can affirm $[\mathcal{P},\mathcal{P}]\neq\mathcal{P}$ for the first Lie algebra and we see that $[\mathcal{P},\mathcal{P}]=\mathcal{P}$ for the second by Proposition 2.12. More, we are in $\mathbb{R}^4$ with system of Darboux coordinates (x,z,y,t) and let $\mathcal{P}$ be the Lie algebra of infinite dimension spanned by $1, x, z, y, t, x^2, xy, y^2, xy^2, x^2y, y^3$. Here $[\mathcal{P},\mathcal{P}]\neq\mathcal{P}$ because $z,t\notin[\mathcal{P},\mathcal{P}]$ and its normalizer is $\mathcal{P}\oplus\langle zt, z^2, t^2\rangle$ which satisfies (H). Then, every derivation of $\mathcal{P}$ (resp. of $\mathfrak{N}$) is like Theorem 2.27 said.

Theorem 2.31. *For all derivations D of $\mathfrak{N}$ such that $D\,(\mathcal{P}_0\ominus[\mathcal{P},\mathcal{P}])\subset\mathcal{P}$, we obtain D is a sum of a differential operator of order 1 where their coefficients are in $\mathcal{P}$ and a non local derivation which has its value in $\mathbb{R}$ and null on $[\mathfrak{N},\mathfrak{N}]$. Particularly, if $[\mathcal{P},\mathcal{P}]=\mathcal{P}$, we have the same result as above one and the normalizer of $\mathfrak{N}$ is itself.*

Proof. Let D be a derivation of $\mathfrak{N}$, $X\in\mathcal{P}$ and $Y\in\mathcal{P}$. We have $\mathcal{P}\subset\mathfrak{N}$, then $D\,[X,Y] = [DX,Y]+[X,DY]$. By definition of the normalizer, the above hypothesis and Proposition 2.8, if X,Y run into $\mathcal{P}$, we obtain $D\,(\mathcal{P})\subset\mathcal{P}$. Thus $D_{|\mathcal{P}} = D_1$ is a derivation of $\mathcal{P}$. By (H), $D_{|\mathcal{P}}$ is a differential operator of order one with its coefficients in $\mathcal{P}$ cf. first part of Theorem 2.27. Now, we take $X\in\mathfrak{N}$ and $Y\in\mathcal{P}$. We get again $D\,[X,Y]=[DX,Y]+[X,DY]$ where $[X,Y]\in\mathcal{P}$. So $D_1\,[X,Y] = [DX,Y]+[X,D_1Y]$ and by definition of the derivation D_1, we say $[DX - D_1X, Y] = 0$ for all $Y\in\mathcal{P}$. Then $DX - D_1X$ belongs to the centralizer of $\mathcal{P}$. By Proposition 2.5, $DX - D_1X$ is a constant and the derivation $D' = D - D_1$ is null on $[\mathfrak{N},\mathfrak{N}]$. In the situations of the present theorem, every derivation of the form $L_{X'}$ of $\mathfrak{N}$ is such that X' belongs to the normalizer of $\mathfrak{N}$. The above result says that this derivation is a differential operator of order strictly one where its coefficients are in $\mathcal{P}$. If we adapt the reasoning in Theorem 2.29, we have $D(1)=0$ and the corresponding $D'=0$. Then we have $X'\in\mathfrak{N}$, the normalizer of $\mathfrak{N}$ is $\mathfrak{N}$ itself. ■

Remark 2.32. Every x^ix^{i+n} is in $\mathfrak{N}$ for all $i=1,\ldots,n$ by Proposition 2.13, so all $x^i, x^{i+n}\in[\mathfrak{N},\mathfrak{N}]$ for $i=1,\ldots,n$. Then each element $X\in\mathfrak{N}$ such that $D'(X)$ is non null where D' is the derivation on the proof of the previous theorem, is outer of S.

Theorem 2.33. *We suppose that the derived ideal of $\mathcal{P}$ is different to $\mathcal{P}$. We consider $K' = \{1 \leq i \leq 2n / x^i \in \mathcal{P}_0 \cap [\mathcal{P},\mathcal{P}]\} \neq \varnothing$. Every derivation of $[\mathcal{P},\mathcal{P}]$ is a differential operator of order one which takes its coefficients on $[\mathcal{P},\mathcal{P}]$. Moreover, if we denote $\mathfrak{M}$ the normalizer of $[\mathcal{P},\mathcal{P}]$, then each derivation of $\mathfrak{M}$ is a sum of differential operator of order one taking its coefficients on $[\mathcal{P},\mathcal{P}]$ and an arbitrary endomorphism in $\mathfrak{M}$ where its image is in the centralizer of $[\mathcal{P},\mathcal{P}]$ vanishing on $[\mathfrak{M},\mathfrak{M}]$.*

Proof. In advance, we can say that $[\mathcal{P},\mathcal{P}] \neq \{0\}$ because $H_{-1} \subset [\mathcal{P},\mathcal{P}]$. Proposition 2.8 allows us to deal with $K' = \{i / x^i \in \mathcal{P}_0 \cap [\mathcal{P},\mathcal{P}]\}$. If $K' \neq \varnothing$, then $[\mathcal{P},\mathcal{P}]$ contains at least $\langle 1, x^t \rangle$ where $t \in K'$. For all $i \notin K'$, each element in $[\mathcal{P},\mathcal{P}]$ doesn't depend on x^i and $[\mathcal{P},\mathcal{P}] = \mathcal{P} \ominus \langle x^i, i \notin K' \rangle$. We can adapt the proof of Lemma 2.26 where S is replaced by $S' = \langle 1, x^i; i \in K' \rangle_{\mathbb{R}}$ and Theorem 2.27 to $[\mathcal{P},\mathcal{P}]$ in $\mathbb{R}^{2n}$ and every derivation D of $[\mathcal{P},\mathcal{P}]$ is of the form:

$$D(u) = D(1)u + \sum_{i \in K'} \left(D(x^i) - D(1)(x^i) \right) \frac{\partial u}{\partial x^i}, \quad \text{where } u \in [\mathcal{P},\mathcal{P}].$$

Next, we remark that the derived ideal of $[\mathcal{P},\mathcal{P}]$ is itself. Then a similar proof of that of Theorem 2.31 states the final assertion. ∎

Remark 2.34. The previous theorem permits us to consider a Lie algebra of polynomials on $\mathbb{R}^k$ where $k < 2n$. ∎

In the following, we suppose $K' \neq \varnothing$ and we split K' into a partition of subsets. Let

$$B = \left\{ i, i+n / (i, i+n) \in (K')^2, \text{ for } 1 \leq i \leq n \right\},$$
$$K = \{ i \in K' / i+n \notin K', \text{ for } 1 \leq i \leq n \}$$

and

$$L = \{ i+n \in K' / i \notin K', \text{ for } 1 \leq i \leq n \},$$
$$O = \{ i, i+n / i \notin K', i+n \notin K', \text{ for } 1 \leq i \leq n \}.$$

Then we obtain a partition of $K' = B \cup K \cup L$. In the following, we suppose $[\mathcal{P},\mathcal{P}] \neq H_{-1}$.

Proposition 2.35. *The set B is non-empty.*

Proof. If $\mathcal{P}$ is such that all $X \in \mathcal{P}$ are in $\mathcal{P}_{k \leq 0}$, then $[\mathcal{P},\mathcal{P}] = H_{-1}$. So it exists $X \in \mathcal{P}$ with $X \in \mathcal{P}_1$. By separated hypothesis, X is one of $(x^i)^2$, $(x^{i+n})^2$, $x^i x^{i+n}$, $x^i x^t$ where $1 \leq i \leq n$ and $t \neq i, i+n$. If $(x^i)^2 \in \mathcal{P}$, then there's a $l \in \{1, \ldots, n\}$ such that $x^i x^l$ and $x^i x^{l+n}$ in $\mathcal{P}$ from (H). Thus, $\left(x^l, x^{l+n}\right) \in [\mathcal{P},\mathcal{P}]^2$. In the same way we prove for $(x^{i+n})^2 \in \mathcal{P}$'s case. It is obvious that $x^i, x^{i+n} \in [\mathcal{P},\mathcal{P}]$ if $x^i x^{i+n} \in \mathcal{P}$. If $x^i x^t \in \mathcal{P}$, we have $x^i x^{i+n}$ or $x^t x^{t+n}$ or $x^t x^{t-n}$ in $\mathcal{P}$ because of (H). Therefore (x^i, x^{i+n}) or (x^t, x^{t+n}) or (x^t, x^{t-n}) in $[\mathcal{P},\mathcal{P}]^2$. ∎

Corollary 2.36. *If $n = 1$, then $[\mathcal{P},\mathcal{P}] = \mathcal{P}$ when $\mathcal{P} \neq S$.*

Proof. We can affirm this theorem using Proposition 2.35 and Proposition 2.8. ∎

In the following theorem, when the case where K or L is void, we cancel it. From the proof of Theorem 2.25, it's easy to check that:

Theorem 2.37. *A differential operator* $D=\sum_{i=1}^{i=2n} X^i \frac{\partial(.)}{\partial x^i}+h\,(.)$ *of order one of* $F\left(\mathbb{R}^{2n}\right)$ *with* $h\in F\left(\mathbb{R}^{2n}\right)$ *is a derivation of* $[\mathcal{P},\mathcal{P}]$ *if and only if:*

(1) $h=D(1)$ *is a constant.*

(2) *All its coefficients* X^i *where* $i\in K'$ *are polynomials in* $[\mathcal{P},\mathcal{P}]$ *such that* $\left(X^j\frac{\partial}{\partial x^j}+X^{j+n}\frac{\partial}{\partial x^{j+n}}\right)([\mathcal{P},\mathcal{P}])\subset[\mathcal{P},\mathcal{P}]$ *for all* $j=1,\dots,n$.

(3) $\frac{\partial X^k}{\partial x^i}+\frac{\partial X^{i+n}}{\partial x^{k+n}}=-\delta_k^i D\,(1)$ *for all* $1\leq i,k\leq n$ *where* $i,k\in B$.

(4) $\frac{\partial X^{k+n}}{\partial x^i}=\frac{\partial X^{i+n}}{\partial x^k}$ *with* $1\leq k\leq n$ *and* $1\leq i\leq n$ *when* $i,k\in B$.

(5) $\frac{\partial X^k}{\partial x^{i+n}}=\frac{\partial X^i}{\partial x^{k+n}}$ *with* $1\leq k\leq n$ *where* $1\leq i\leq n$ *such that* $i,k\in B$.

(6) $\frac{\partial X^i}{\partial x^k}=0$ *with* $1\leq k\leq 2n$ *where* $1\leq i\leq n$ *such that* $i\in K, k\in B\cup L$.

(7) $\frac{\partial X^{i+n}}{\partial x^k}=0$ *with* $1\leq k\leq 2n$ *and* $1\leq i\leq n$ *when* $i+n\in L, k\in B\cup K$.

We denote $\mathcal{M}$ the set of elements $X=X\left(x^i, i\in B\cup K\cup L\right)$ in $\mathfrak{M}$.

Theorem 2.38. *Let a derivation* D *of* $[\mathcal{P},\mathcal{P}]$. *If* $K\cup L\neq\varnothing$ *(resp.* $K\cup L=\varnothing$*),* D *is inner on* $\mathcal{M}$ *if and only if* $D\,(1)=0$ *and* $D(x^i)=0$ *for all* $i\in K\cup L$ *(resp.* $D\,(1)=0$*).*

Proof. We have to solve $D\,(u\in[\mathcal{P},\mathcal{P}])=L_X u$ with $X\in\mathcal{M}$.
Suppose $K\cup L\neq\varnothing$, by Theorem 2.33, we have $D\left(x^i\right)-D(1)x^i=-\frac{\partial X}{\partial x^{i+n}}$ for all $1\leq i\leq n$ with $i\in K$ and $D\left(x^{i+n}\right)-D(1)x^{i+n}=\frac{\partial X}{\partial x^i}$ for all $1\leq i\leq n$ with $i+n\in L$. We obtain both previous cases when $1\leq i\leq n$ in B. By Theorem 2.37 and classical Frobenius theorem, D is inner on $\mathcal{M}$ if and only if we have $D(1)=0$, $D\left(x^i\right)=0$ for all $i\in K\cup L$.

The proof of the last assertion is similar to the one of Theorem 2.29. ■

Example 2.39. Here we take again the last part of examples in Example 2.30, then $[\mathcal{P},\mathcal{P}]$ is the Lie algebra $\langle 1,x,y,x^2,xy,y^2,xy^2,x^2y,y^3\rangle$ which always satisfies (H). Next, we get the Lie algebra $\mathcal{P}$ in Example 2.14, where $[\mathcal{P},\mathcal{P}]$ coincides with the Lie algebra spanned by $1,x,y,t,xt,xy,x^2,y^2,x^2t,x^2t^2$ verifying (H). Every derivation of these Lie algebras is a differential operator of one order where its coefficients are in $[\mathcal{P},\mathcal{P}]$ as the previous theorem describes. Moreover, the Lie algebra $\mathcal{P}$ generated by $1,x,y,z,t,xt,xy,x^2,y^2,x^2t,x^2t^2$ coincides with $[\mathcal{P},\mathcal{P}]$ and has a normalizer $\mathfrak{N}$ equals to $\mathcal{P}\oplus\langle zt\rangle$. In addition, $\mathfrak{N}\ominus[\mathfrak{N},\mathfrak{N}]=\langle zt\rangle$. A linear mapping defined by D such that $D\,(zt)=1$ and D vanishing otherwise, is a non local derivation of $\mathfrak{N}$, which verifies Theorem 2.31.

Proposition 2.40. *If* $\mathcal{P}$ *contains all* $x^i x^{i+n}$ *with* $i=1,\dots,n$, *then* $\mathfrak{N}=\mathcal{P}$.

Proof. We know that $\mathcal{P}\subset\mathfrak{N}$. We will prove the converse in the following. The algebra $\mathfrak{N}$ is separated like $\mathcal{P}$ was, so let $X=\left(x^1\right)^{i_1}\dots\left(x^{2n}\right)^{i_{2n}}\in\mathfrak{N}$ of degree at least 2. In the first, we suppose it exists $1\leq j\leq n$ such that $i_j\neq i_{j+n}$ or all $j=1,\dots,n$, $i_j=i_{j+n}$ with existence of k such that $i_k>1$. If we have these hypotheses and use the definition of $\mathfrak{N}$, by Proposition 2.12 first and second situations, we have $X \in \mathcal{P}$. Now, we set that all non null $i_j = i_{j+n} = 1$ with $1 \le j \le n$ and the degree of X is at least 4 because all $x^i x^{i+n} \in \mathfrak{N} \cap \mathcal{P}$ cf. Corollary 2.13 and the case where all $i_j = 0$ is trivial. For simplification, we suppose $X = x^1 x^{1+n} x^2 x^{2+n} f$ where $f = (x^3)^{i_3} \dots (x^n)^{i_n} (x^{n+3})^{i_3} \dots (x^{2n})^{i_n}$ with all $i_{l \ge 3} \in \{0, 1\}$. By definition, $[[x^1 x^{1+n} x^2 x^{2+n} f, x^{1+n}], x^{2+n}] = f x^{1+n} x^{2+n} \in \mathcal{P}$ and we arrange our calculations to obtain $x^{1+n} x^{2+n} = Y \in \mathcal{P}$. Then $[x^1 x^{1+n} x^2 x^{2+n} f, Y] \in \mathcal{P}$. Because of the separated hypothesis of $\mathcal{P}$, $x^1 (x^{1+n})^2 x^{2+n} f = Q \in \mathcal{P}$. In the same way, we have $(x^1)^2 x^{1+n} x^{2+n} f \in \mathcal{P}$ when we obtain by successive brackets by monomials of degree 1 the $(x^1)^2 x^{1+n} x^{2+n} = Z \in \mathcal{P}$. Therefore, $[Z, Q] \in \mathcal{P}$ leads to $(x^1)^2 (x^{1+n})^2 (x^{2+n})^2 f \in \mathcal{P}$. By Proposition 2.10 we have $\left(x^2\right)^2 \in \mathcal{P}$ so that $\left[(x^2)^2, (x^1)^2 (x^{1+n})^2 (x^{2+n})^2 f\right] \in \mathcal{P}$, giving $(x^1)^2 (x^{1+n})^2 x^2 x^{2+n} f = W \in \mathcal{P}$. Thus $[[W, x^1], x^{1+n}]$ leads us to $X \in \mathcal{P}$. Thus $\mathfrak{N} \subset \mathcal{P}$ and we have $\mathfrak{N} = \mathcal{P}$. ∎

Proposition 2.41. *The space $\mathfrak{N} \ominus [\mathfrak{N}, \mathfrak{N}]$ is of finite dimensional where its element X is of the form $x^j x^{j+n}$ with $1 \le j \le n$. Moreover if $k = dim\,(\mathfrak{N} \ominus [\mathfrak{N}, \mathfrak{N}])$, then $0 \le k \le n$.*

Proof. We take $X = (x^1)^{i_1} \dots (x^{2n})^{i_{2n}} \in \mathfrak{N}$. By Corollary 2.13, every $x^i x^{i+n}$ for $1 \le i \le n$ are in $\mathfrak{N}$. If it exists $1 \le l \le n$ such that $i_l \ne i_{l+n}$, then $X = a\left[X, x^l x^{l+n}\right] \in [\mathfrak{N}, \mathfrak{N}]$ with $a \in \mathbb{R}^*$. If there is $1 \le j \le 2n$ such that $i_j > 1$ with for all $1 \le t \le n$, $i_t = i_{t+n}$, we can proceed like in the proof of Theorem 2.31 and we have $X \in [\mathfrak{N}, \mathfrak{N}]$. So X is such that all $i_j \in \{0, 1\}$. The last part of Proposition 2.40 says the degree of X is strictly less than 4. Then the space $\mathfrak{N} \ominus [\mathfrak{N}, \mathfrak{N}]$ is of finite dimensional and we have the special form of elements of $\mathfrak{N}$ cited above. Therefore, the dimension of $(\mathfrak{N} \ominus [\mathfrak{N}, \mathfrak{N}])$ is between 0 and n. ∎

Let U be the Lie algebra $\langle D \in Der\,(\mathcal{P})$, such that $D(1) \ne 0\rangle$ and V a sub-Lie algebra of U such that each $D \in U$ satisfies $D(1) = 0$. Remark now that if $D, D' \in Der\,(\mathcal{P})$ with $D(1)$ and $D'(1)$ non zero, $[D, D'](1) = 0$. It is easy to check that V is a non null ideal of U, then we can define the commutative Lie algebra ideal U/V. We consider the endomorphism $\psi : U \longrightarrow \mathbb{R}$ where $\psi(D) = D(1)$. We find that ψ is surjective , because if $a \in \mathbb{R}$, the differential operator $D = \sum_{t=1,\dots,2n} \frac{a}{2} x^t \frac{\partial(.)}{\partial x^t} - a\,(.)$ is a derivation of $\mathcal{P}$ by Theorem 2.25 such that $\psi\,(D) = a$. Moreover $Ker\,(\psi) = V$ then by isomorphism theorem, $U/V = \mathbb{R}$.

Theorem 2.42. *The first Chevalley-Eilenberg cohomology of $\mathcal{P}$, $H^1(\mathcal{P})$ is isomorphic to $(\mathfrak{N}/\mathcal{P}) \oplus \mathbb{R}$. When $\mathfrak{N} = \mathcal{P}$, $H^1(\mathcal{P}) \equiv \mathbb{R}$. Under the second hypothesis of Theorem 2.31, $H^1(\mathfrak{N}) = \mathbb{R}^{k+1}$.*

Proof. By definition, $H^1(\mathcal{P}) = Der\,(\mathcal{P}) / ad_{\mathcal{P}}$ where $Der\,(\mathcal{P})$ is the Lie algebra of derivations of $\mathcal{P}$ and $ad_{\mathcal{P}}$ is the one of inner derivations $L_X = [X, .]$ $(X \in \mathcal{P})$ of $\mathcal{P}$. By Theorem 2.27, all derivations of $\mathcal{P}$ are differential operators of order one with coefficients in $\mathcal{P}$. Because of Theorem 2.29, $H^1(\mathcal{P})$ is isomorphic to $\mathfrak{N}/\mathcal{P} \oplus U/V$ and to $\mathfrak{N}/\mathcal{P} \oplus \mathbb{R}$ by our previous computation. The mapping $\varphi : Der\,(\mathcal{P}) \longrightarrow \mathbb{R}$ by $\varphi\,(D) = D(1)$ is a surjective endomorphism in the same way as for ψ in the assertion just before the present theorem where $Ker\,(\varphi) = ad_{\mathfrak{N}}$ by Theorem 2.29. Then $Der\,(\mathcal{P}) / ad_{\mathfrak{N}} = \mathbb{R}$. Particularly, if $\mathfrak{N} = \mathcal{P}$, $H^1(\mathcal{P}) = \mathbb{R}$. By Theorem

2.31, $Der\left(\mathfrak{N}\right) = Der_{loc}\left(\mathfrak{N}\right) \oplus Der_{nloc}\left(\mathfrak{N}\right)$ where $Der_{loc}\left(\mathfrak{N}\right)$ the Lie algebra of local derivations of $\mathfrak{N}$, $Der_{nloc}\left(\mathfrak{N}\right)$ that of non local derivations of $\mathfrak{N}$. The mapping $\phi : Der_{loc}\left(\mathfrak{N}\right) \longrightarrow \mathbb{R}$ is linear surjective with $\phi\left(D\right) = D(1)$. Now, $Ker\left(\phi\right) = ad_{\mathfrak{N}}$ via Theorem 2.29 and Theorem 2.31. Then, $Der_{loc}\left(\mathfrak{N}\right)/ad_{\mathfrak{N}} = \mathbb{R}$. The fact in Proposition 2.41 says $k = dim\left(\mathfrak{N} \ominus [\mathfrak{N}, \mathfrak{N}]\right)$ is finite. Because of the results of Theorem 2.31 about non local derivations of $\mathfrak{N}$, $Der_{nloc}\left(\mathfrak{N}\right) = Der\left(\mathfrak{N}\right)/ad_{\mathfrak{N}}$ is isomorphic to $\mathbb{R}^k$. Moreover $[Der_{loc}\left(\mathfrak{N}\right), Der_{nloc}\left(\mathfrak{N}\right)] = \{0\}$, then $H^1\left(\mathfrak{N}\right)$ is isomorphic to $\mathbb{R} \otimes \mathbb{R}^k = \mathbb{R}^{k+1}$. ■

Example 2.43. We can take $\mathbb{R}^4$ with the Darboux coordinate system (x, z, y, t) and $\mathcal{P} = \langle 1, x, y, z, t\rangle$. Its normalizer $\mathfrak{N}$ is $\mathcal{P} + \langle x^2, y^2, z^2, t^2, xy, xz, xt, yz, yt, zt\rangle$ verifying (H) but $[\mathcal{P}, \mathcal{P}] \neq \mathcal{P}$ where the normalizer of $\mathfrak{N}$ is itself. By our theorems, $H^1\left(\mathcal{P}\right) = \langle x^2, y^2, z^2, t^2, xy, xz, xt, yz, yt, zt\rangle \oplus \mathbb{R}$ and $H^1\left(\mathfrak{N}\right) = \mathbb{R}$.

Let us take an illustrative example for the cohomology of the normalizer $\mathfrak{N}$ of $\mathcal{P} = [\mathcal{P}, \mathcal{P}]$,

Example 2.44. If we refer to the last example of $\mathcal{P}$ in Example 2.39, we have $H^1\left(\mathfrak{N}\right) = \mathbb{R}^2$ by the above theorem.

If $K \cup L \neq \varnothing$, we denote by $\mathfrak{Z}_i = \langle 1, x^i\rangle$ for $i \in K \cup L$ and $\mathfrak{Z}$ the direct product of all $\mathfrak{Z}_i$ that is to say $\mathfrak{Z} = \otimes \mathfrak{Z}_{i \in K \cup L}$ which is isomorphic to $\mathbb{R}^{card(K \cup L)}$. We set the sequence of sets $\left(\eta_i\right)_i$ with $\eta_i = \mathfrak{C}$ for all $i \in \mathbb{N}$ where $\mathfrak{C}$ is the centralizer of $[\mathcal{P}, \mathcal{P}]$ on $\mathbb{R}^{2n}$. We find $\mathfrak{C} = \{f \in F\left(\mathbb{R}^{2n}\right)/f$ depends only on x^i where $i \in K \cup L \cup O$ $\}$.

Theorem 2.45. *The first space of Chevalley-Eilenberg cohomology of* $[\mathcal{P}, \mathcal{P}]$ *is isomorphic to* $\left(\mathcal{M}/[\mathcal{P}, \mathcal{P}]\right) \oplus \mathbb{R}^{card(K \cup L)+1}$. *When* $dim\left(\mathfrak{M} \ominus [\mathfrak{M}, \mathfrak{M}]\right)$ *is countable infinite resp. equals to* $q \in \mathbb{N}$, *the* $H^1\left(\mathfrak{M}\right) = \mathbb{R} \otimes \left(\eta_i\right)_i$ *resp. is isomorphic to* $\mathbb{R} \otimes \mathfrak{C}^q$ *when* $q > 0$, *to* $\mathbb{R}$ *if* $q = 0$.

Proof. The linear mapping $\psi : Der\left([\mathcal{P}, \mathcal{P}]\right) \longrightarrow \mathbb{R} \times \mathfrak{Z}$ with $\psi(D)$ is equal to $(D(1), D(x^i)_{i \in K \cup L})$ is well-defined surjective mapping. That is to say, it is clear that if for $i \in K \cup L$ with $\alpha^i \in \mathfrak{Z}_i$, then the linear mapping D_i defined by $D_i\left(x^i\right) = \alpha^i$ and null otherwise, is a derivation on $[\mathcal{P}, \mathcal{P}]$ by Theorem 2.37. Therefore, if $\left(a, (\alpha^i)_{i \in K \cup L}\right)$ is in $\mathbb{R} \times \mathfrak{Z}$, the derivation $D = \sum_{t \in B} \frac{a}{2} x^t \frac{\partial(.)}{\partial x^t} - a\left(.\right) + \sum_{i \in K \cup L} D_i$ is such that $\psi\left(D\right) = \left(a, (\alpha^i)_{i \in K \cup L}\right)$. Combining these arguments, we find ψ is surjective. In addition, ψ has $Ker\left(\psi\right)$ isomorphic to $\mathcal{M} \supset [\mathcal{P}, \mathcal{P}]$ by Theorem 2.38. So we use Theorem 2.38 with the above affirmations and isomorphism theorem on ψ to find: $H^1[\mathcal{P}, \mathcal{P}] = Der[\mathcal{P}, \mathcal{P}]/ad_{[\mathcal{P},\mathcal{P}]}$ is $\left(\mathcal{M}/[\mathcal{P}, \mathcal{P}]\right) \oplus \left(Der[\mathcal{P}, \mathcal{P}]/ad_{\mathcal{M}}\right) = \left(\mathcal{M}/[\mathcal{P}, \mathcal{P}]\right) \oplus \mathbb{R}^{card(K \cup L)+1}$. The rest of the assertions follows from the technical adopted in the proof of Theorem 2.42, our affirmations just before the present theorem and Theorem 2.33. ■

Example 2.46. We get the usual coordinate system (x, z, y, t) on $\mathbb{R}^4$ and the Lie algebra $\mathcal{P}$ generated by $1, x, y, z, t, x^2, y^2, xy$. We have $K \cup L = \varnothing$ and $O \neq \varnothing$. The derived ideal of $\mathcal{P}$ is $\langle 1, x, y, x^2, y^2, xy\rangle$. By our results, the normalizer $\mathfrak{M}$ of $[\mathcal{P}, \mathcal{P}]$ is $\left(\mathcal{A} = \{f \in F\left(\mathbb{R}^4\right)/f = f\left(z, t\right)\} \ominus \langle 1\rangle_{\mathbb{R}}\right) \oplus [\mathcal{P}, \mathcal{P}]$. Then $H^1\left([\mathcal{P}, \mathcal{P}]\right) = \mathcal{A} \oplus \mathbb{R}$. We find that $\mathfrak{M} \ominus [\mathfrak{M}, \mathfrak{M}] = \{0\}$ and we can write $H^1\left(\mathfrak{M}\right) = \mathbb{R}$.

Example 2.47. In $\mathbb{R}^4$ with Darboux coordinates (x,z,y,t), we can consider $\mathcal{P}$ the Lie algebra generated by $1,x,z,y,t,x^2,y^2,xy,xt,yt,t^2$. We obtain $H^1(\mathcal{P}) = \langle zt\rangle_{\mathbb{R}} \oplus \mathbb{R}$. It is remarkable that $L \neq \varnothing, K = \varnothing$ and $[\mathcal{P},\mathcal{P}] = \mathcal{P} \ominus \langle z\rangle_{\mathbb{R}}$ verifying (H). The normalizer of $\mathcal{P}$ doesn't verify (H) and $[\mathcal{P},\mathcal{P}] \neq \mathcal{P}$. Here $\mathcal{M}$ is $[\mathcal{P},\mathcal{P}] \oplus (\mathcal{B} = \{f \in F(\mathbb{R}^4) / f = f(t)\} \ominus \langle 1,t,t^2\rangle_{\mathbb{R}})$, then $H^1([\mathcal{P},\mathcal{P}]) = \mathcal{B} \oplus \mathbb{R}^2$. And $\mathfrak{M}$ is $\mathcal{P} \oplus \langle zt\rangle_{\mathbb{R}} \cup \mathcal{B}$ so $H^1(\mathfrak{M}) = \mathbb{R}$ because $\mathfrak{M} \ominus [\mathfrak{M},\mathfrak{M}] = \{0\}$.

3. Applications to the derivations of the polynomial Poisson Lie algebra and of Lie algebras of polynomial Hamiltonian vector fields

Proposition 3.1. *We have*

$$\left(\mathbb{R}\left(x^1,x^2,\ldots,x^{2n}\right),+,.,[,]\right) = \bigoplus_{i\geq -1} H_i,\ [H_{-1},H_i] = \{0\}$$

for all $i,j \neq -1$, $[H_i,H_j]$ is equal to H_{i+j-1}. The derived ideal of $\mathbb{R}(x^1,x^2,\ldots,x^{2n})$ and its normalizer are $\mathbb{R}(x^1,x^2,\ldots,x^{2n})$, the first Chevalley-Eilenberg cohomology $H^1(\mathbb{R}(x^1,x^2,\ldots,x^{2n})) = \mathbb{R}$.

Proof. Let $u = 1,\ldots,n$, to simplify, we set only here $x^u = x, x^{u+n} = y$. It is obvious that $[H_{-1},H_i] = \{0\}$ for all $i \geq -1$. Let $i,j \neq -1$, naturally $[H_i,H_j] \subset H_{i+j-1}$. First, we take $X \in H_{i+j-1}$ and we can proceed on the form of X such that:

For $i+j-1>0$ with $i,j\neq -1$,

- $xy^{i+j-1} = \frac{1}{2(j+1)}\left[x^2y^{i-1},y^{j+1}\right]$ for $i>0$,
- $xy^{j-1} = \frac{1}{j}\left[x,xy^j\right]$ because $j>1$ in this case.

For $i+j-2>0$ with $i,j\neq -1$,

- $x^2y^{i+j-2} = \frac{1}{3(j+1)}\left[x^3y^{i-2},y^{j+1}\right]$ for $i>1$,
- $x^2y^{j-2} = \frac{1}{j-1}\left[x,x^2y^{j-1}\right]$ because $j>2$ in this case,
- $x^2y^{j-1} = \frac{1}{2j}\left[x^2,xy^j\right]$ because $j>1$ in this case.

For $i+j-3>0$ with $i,j\neq -1$,

- $x^3y^{i+j-3} = \frac{1}{4(j+1)}\left[x^4y^{i-3},y^{j+1}\right]$ for $i>2$,
- $x^3y^{j-3} = \frac{1}{j-2}\left[x,x^3y^{j-2}\right]$ because $j>3$ in this case,
- $x^3y^{j-2} = \frac{1}{2(j-1)}\left[x^2,x^2y^{j-1}\right]$ because $j>2$ in this case,
- $x^3y^{j-1} = \frac{1}{3j}\left[x^3,xy^j\right]$ because $j>1$ in this case.

In general, if $t>1$ with $i+j-t>0$ and $i,j\neq -1$,

- $x^ty^{i+j-t} = \frac{1}{(t+1)(j+1)}\left[x^{t+1}y^{i-t},y^{j+1}\right]$ for $i>t-1$,
- $x^ty^{j-t} = \frac{1}{j-t+1}\left[x,x^ty^{j-t+1}\right]$ because $j>t$ in this case,
- $x^ty^{j-t+1} = \frac{1}{2(j-t+2)}\left[x^2,x^{t-1}y^{j-t+2}\right]$ because $j>t-1$ in this case,
- $\ldots,$
- $x^ty^{j-1} = \frac{1}{tj}\left[x^t,xy^j\right]$ because $j>1$ in this case.

If $X = x^{i+j}$ respectively y^{i+j} with $i+j > 1$, $X = \frac{1}{i+j+1}\left[x^{i+j+1}, y\right]$ respectively $X = \frac{1}{i+j+1}\left[y^{i+j+1}, -x\right]$.

For $i+j=1$ with $i,j \neq -1$, we have i=1 and $j=0$ or $i=1$ and $j=0$. It suffices to do $x = \frac{1}{2}\left[x^2, y\right]$ and $y = \frac{1}{2}\left[y^2, -x\right]$.

When $i+j=0$ where $i,j \neq -1$, then $i=j=0$ and $X \in \mathbb{R}$ is such that $X = [Xx, y]$.

Second, if $X = x^{\alpha_1}\left(x^2\right)^{\alpha_2}\ldots y^{\alpha_{1+n}}\ldots\left(x^{2+n}\right)^{\alpha_{2+n}}\ldots\left(x^{2n}\right)^{\alpha_{2n}}$ is a monomial of degree $i+j$ where $i+j=\sum_{l=1}^{2n}\alpha_l > 0$. That is to say $X \in H_{i+j-1}$. To simplify the proof of X to be in $[H_i, H_j]$, we set $k=1$. Now, we suppose then $i,j \neq -1$, (i,j) is such that if $i=0$ resp. $j=0$, then $j>0$ resp. $i>0$ in the following otherwise they are trivial cases. We have the following remaining cases:

(a) $\alpha_1 > 0, \alpha_{1+n} > 0$ with $i,j>0$ and $j \leq \alpha_{1+n}$ denoting by $g = \frac{X}{x^{\alpha_1}y^{\alpha_{1+n}}}$, then $X = \frac{1}{(\alpha_1+1)(j+1)}\left[x^{\alpha_1+1}gy^{\alpha_{1+n}-j}, y^{j+1}\right]$.

(b) $\alpha_1 > 0, \alpha_{1+n} > 0$ with $i,j>0$ and $j > \alpha_{1+n}$. There is a sequence of integers $k_l \leq \alpha_l, l = 1, \ldots, \widehat{1+n}, 2+n, \ldots, 2n$ such that $j = \alpha_{1+n} + \sum_l k_l$. Thus X is the bracket of
$$x^{\alpha_1-k_1+1}\left(x^2\right)^{\alpha_2-k_2}\ldots\hat{y}\left(x^{2+n}\right)^{\alpha_{2+n}-k_{2+n}}\ldots\left(x^{2n}\right)^{\alpha_{2n}-k_{2n}}$$
and
$$x^{k_1}\left(x^2\right)^{k_2}\ldots y^{\alpha_{1+n}+1}\left(x^{2+n}\right)^{k_{2+n}}\ldots\left(x^{2n}\right)^{k_{2n}}$$
up to a multiplication by a constant.

(c) $\alpha_1 = 0$ and $\alpha_{1+n} > 0$ with $i,j>0$, we remark that the methods (a) and (b) are again applicable to the present situation.

(d) $\alpha_1 > 0$ and $\alpha_{1+n} = 0$ with $i,j>0$, we proceed like in (c) by swapping places between x and y in (a) and (b).

(e) $\alpha_1 = \alpha_{1+n} = 0$ with $i=0$, then $X = [x, yX]$.

(f) $\alpha_1 = \alpha_{1+n} = 0$ with $j=0$, then $X = [xX, y]$.

(g) $\alpha_1 = \alpha_{1+n} = 0$ with $i,j>0$ and if it exists $1 \leq t \leq n$ such that X depends on x^t and x^{t+n}, then we return to the cases (a) and (b) with a similar method.

(h) $\alpha_1 = \alpha_{1+n} = 0$ with $i,j>0$ and if there is no $1 \leq t \leq n$ such that X depends on x^t and x^{t+n}, then we return to the situations (c) and (d) with a similar method.

We can conclude that for all $X \in H_{i+j-1}$, $X \in [H_i, H_j]$. In our proof, if $X \in H_{t-1}$, we can choose arbitrarily i,j such that $i+j=t$ and $X \in [H_i, H_j]$. Thus we have $H_{t-1} \subset [H_i, H_j]$, but it's obvious that $[H_i, H_j] \subset H_{t-1}$, then $H_{t-1} = [H_i, H_j]$. It leads to the fact that the derived ideal of $\mathbb{R}\left(x^1, \ldots, x^{2n}\right)$ is $\mathbb{R}\left(x^1, \ldots, x^{2n}\right)$ by another method than in Corollary 2.36. The situation in Proposition 2.5 permits us to say that the normalizer of $\mathbb{R}\left(x^1, \ldots, x^{2n}\right)$ is itself. By Theorem 2.42, we have the last affirmation. ∎

Proposition 3.2. *If $u, l > -1$ then*
$$[H_{-1} + \cdots + H_u, H_{-1} + \cdots + H_l] = H_{-1} + \cdots + H_{u+l-1}.$$

Proof. It is sufficient to apply Proposition 3.1 knowing that the bracket is linear and we can conclude. ∎

Definition 3.3. A triple derivation of a Lie algebra $\mathcal{A}$ is an endomorphism f of $\mathcal{A}$ in which we have for all $X, Y, Z \in \mathcal{A}$,

$$f\,[X,[Y,Z]] = [fX,[Y,Z]] + [X,[fY,Z]] + [X,[Y,fZ]]\,.$$

Proposition 3.4. *If the derived ideal of $\mathcal{P}$ is itself, then the set of 3-derivations of $\mathcal{P}$ is equal to one of derivations of $\mathcal{P}$. Particularly, the set of 3-derivations of $\mathbb{R}\left(x^1,\ldots,x^{2n}\right)/H_{-1}$ equals to the set of derivations of $\mathbb{R}\left(x^1,\ldots,x^{2n}\right)/H_{-1}$.*

Proof. We know that H_{-1} is a characteristic ideal of $\mathcal{P}$, then we can consider the Lie algebra $\mathcal{P}/H_{-1}$. By $[\mathcal{P},\mathcal{P}] = \mathcal{P}$, we have $[\mathcal{P}/H_{-1}, \mathcal{P}/H_{-1}] = \mathcal{P}/H_{-1}$. Because H_{-1} is the centralizer of $\mathcal{P}$ cf. Proposition 2.5, $\mathcal{P}/H_{-1}$ will be centerless. By [10], every triple derivation of $\mathcal{P}/H_{-1}$ is a derivation of $\mathcal{P}/H_{-1}$. Conversely, it is clear that every derivation of $\mathcal{P}/H_{-1}$ is a triple derivation of $\mathcal{P}/H_{-1}$. We can now take $\mathcal{P} = \mathbb{R}\left(x^1,...,x^{2n}\right)$. By Theorem 3.1 $\left[\mathbb{R}\left(x^1,...,x^{2n}\right), \mathbb{R}\left(x^1,...,x^{2n}\right)\right] = \mathbb{R}\left(x^1,...,x^{2n}\right)$. We can now apply the previous result for $\mathbb{R}\left(x^1,\ldots,x^{2n}\right)$ to obtain the last assertion of our proposition. ∎

It is immediate to state the following by this previous proposition and Corollary 2.36,

Corollary 3.5. *When $n = 1$, the Lie algebra of all 3-derivations of $\mathcal{P} \neq S$ coincides with the Lie algebra of all derivations of $\mathcal{P}$, in particular that of $\mathcal{P}/H_{-1}$ has analogous situation.*

Proposition 3.6. *If a Lie sub-algebra $\mathcal{P}$ of $\mathbb{R}(x,y)$ verifying (H) is different to S, $H^1(\mathcal{P}) = \mathbb{R}$.*

Proof. If $\mathcal{P}$ follows these hypotheses, $[\mathcal{P},\mathcal{P}] = \mathcal{P}$ cf. Corollary 2.36. By Proposition 4.6, $\mathcal{P}$ is in $\left\{\langle 1,x,y,xy,x^2,y^2\rangle_{\mathbb{R}}, \mathbb{R}(x,y)\right\}$. Then by simple calculations, the normalizer of $\mathcal{P}$ is $\mathcal{P}$ itself. Thus $H^1(\mathcal{P}) = \mathbb{R}$ by Theorem 2.42. ∎

In this paragraph, we deal with Hamiltonian vector fields on $\mathbb{R}^{2n}$. It is well known that the set of Hamiltonian vector fields $\mathcal{H}$ on $\mathbb{R}^{2n}$ is a Lie algebra. That is to say, the mapping Y from $F\left(\mathbb{R}^{2n}\right)$ to $\mathcal{H}$ where for $f \in F\left(\mathbb{R}^{2n}\right)$ assigns to an unique Y_f such that $Y_f = Y^i \frac{\partial}{\partial x^i}$ with $Y^{i+n} = \frac{\partial f}{\partial x^i}, Y^i = -\frac{\partial f}{\partial x^{i+n}}$ for all $1 \leq i \leq n$, is an homomorphism of Lie algebras. The set $\mathfrak{P}$ of polynomial vector fields on $\mathbb{R}^{2n}$ is a Lie algebra cf. [8]. Let $\mathfrak{H}$ be the set of polynomial Hamiltonian vector fields on $\mathbb{R}^{2n}$, then $\mathfrak{H} = \mathcal{H} \cap \mathfrak{P}$ is a Lie algebra.

Proposition 3.7. *The Lie algebra $\mathcal{P}/H_{-1}$ is isomorphic to a Lie sub-algebra $\mathfrak{A}$ of $\mathfrak{H}$, particularly $\mathbb{R}\left(x^1,\ldots,x^{2n}\right)/H_{-1}$ is isomorphic to $\mathfrak{H}$.*

Proof. We denote again the restricted homomorphism $Y : \mathbb{R}\left(x^1,\ldots,x^{2n}\right) \longrightarrow \mathcal{H}$. It is clear that if f is polynomial, then Y_f is simultaneously a polynomial vector field in $\mathfrak{P}$ and an Hamiltonian vector field of $\mathbb{R}^{2n}$, then Y_f is in the Lie algebra $\mathfrak{H}$. Thus $Y(\mathcal{P})$ is a Lie sub-algebra of $\mathfrak{H}$ denoted by $\mathfrak{A}$. We set again $Y : \mathcal{P} \longrightarrow \mathfrak{A}$ which is a surjective homomorphism with $Ker(Y) = H_{-1}$. By isomorphism theorem of Lie algebras, $\bar{Y} : \mathcal{P}/H_{-1} \longrightarrow \mathfrak{A}$ is an isomorphism of Lie algebras. Then, the last affirmation is true. ∎

Thus, we have the following properties of $\mathfrak{A}$ by those of $\mathcal{P}$: $\mathfrak{A}$ is a Lie sub-algebra of $\mathfrak{H}$ containing all $\frac{\partial}{\partial x^i}$ for $i = 1, \ldots, 2n$ verifying $\mathfrak{A} = \bigoplus_{i=-1} \mathfrak{A}_i$ where $\mathfrak{A}_i$ is the part of $\mathfrak{A}$ of all monomials vector fields of degree $i \in \mathbb{N} \cup \{-1\}$ and for all $X \in \mathfrak{A}_{i\geq 0}$, it exists $Y \in \mathfrak{A}_{u\leq i}$, $Z \in \mathfrak{A}_{l\leq i}$ such that $X = [Y, Z]$. But $\mathfrak{A}$ is not necessarily separated.

Theorem 3.8. *The cohomology $H^1(\mathfrak{A}) = (\mathfrak{N}/\mathcal{P}) \oplus \mathbb{R}$, every derivation of the normalizer* N *of* $\mathfrak{A}$ *is inner with respect to the normalizer* N *of* N, $H^1(\mathrm{N})$ *is isomorphic to* N/N *and an analogous assertion as in* N *to the normalizer of* N *can be stated. Particularly* $H^1(\mathfrak{H}) = \mathbb{R}$ *and when* $n = 1$ *with* $\mathfrak{A} \neq \left\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right\rangle$, $H^1(\mathfrak{A}) = \mathbb{R}$.

Proof. By isomorphism and the fact that H_{-1} is a characteristic ideal of $\mathcal{P}$ cf. Proposition 2.5, each derivation D of $\mathfrak{A}$ is such that it exists a derivation D' of $\mathcal{P}$ with $D(.) = D'(.) + H_{-1}$ and conversely. Then by Theorem 2.42, we prove the first part of our theorem. We know that $\langle E \rangle$ is always a subset of N because for every $X \in \mathfrak{A}_i$, $[E, X] = iX$, where $E = \sum_{1\leq i\leq 2n} x^i \frac{\partial}{\partial x^i}$ is the Euler vector field on $\mathbb{R}^{2n}$. Then N contains all constant vector fields $\frac{\partial}{\partial x^i}$, $i = 1, \ldots, 2n$ and E. It is easy to find by normalizer's definition that $\mathrm{N} \subset \mathfrak{P}$. By the results of [8], all derivations of N are inner in the normalizer N of N with $H^1(\mathrm{N})$ is isomorphic to N divided by N. The affirmation about N just before the latter is clear in the same way as in N. It is easy by Proposition 3.1 to find the result on $H^1(\mathfrak{H})$ and by Proposition 3.6 with the previous result to check the last assertion. ■

Example 3.9. Let $\mathfrak{A}$ be the Lie algebra of vector fields on $\mathbb{R}^4$ with coordinate system (x, z, y, t), $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t} \right\rangle$. If we chose the usual Darboux coordinate system on $\mathbb{R}^4$, we state that it is isomorphic to $\langle 1, x, z, y, t \rangle / (H_{-1})$. By our theorem, $H^1(\mathfrak{A}) = \langle x^2, y^2, z^2, t^2, xy, xz, xt, yz, yt, zt \rangle \oplus \mathbb{R}$ and $H^1(\mathrm{N}) = 0$ cf. [8] because the normalizer N of $\mathfrak{A}$ is the Lie algebra of affine vector fields on $\mathbb{R}^4$.

Remark 3.10. We can remark that the normalizer of $\mathfrak{A}$ is not in general isomorphic to the one of $\mathcal{P}/(H_{-1})$. We can take the case where $\mathfrak{A}$ is the corresponding Lie algebra in the Example 2.43. The normalizer of $\mathcal{P}/(H_{-1})$ coincides with $(\mathcal{P} + \langle x^2, y^2, z^2, t^2, xy, xz, xt, yz, yt, zt \rangle) / (H_{-1})$ of dimension 14, but that of the normalizer of $\mathfrak{A}$ in the Lie algebra of vector fields on $\mathbb{R}^4$, that is to say the Lie algebra of affine vector fields on $\mathbb{R}^4$ is of dimension 20. ■

It is easy to see that H_{-1} is the center of $[\mathcal{P}, \mathcal{P}]$ and then a characteristic ideal of $[\mathcal{P}, \mathcal{P}]$ cf. [1]. By theorem of isomorphism of Lie algebras, $\bar{Y} : [\mathcal{P}, \mathcal{P}] / H_{-1} \longrightarrow \mathfrak{B}$ is an isomorphism of Lie algebras where $\mathfrak{B}$ is its image on $\mathfrak{H}$. Then, the consideration of $[\mathcal{P}, \mathcal{P}]$ conducts us to the following equivalent conditions as those stating Theorem 2.45. Thus $B \neq \varnothing$ says if we have $\frac{\partial}{\partial x^i} \in \mathfrak{B}$, then $\frac{\partial}{\partial x^{i+n}} \in \mathfrak{B}$ or $\frac{\partial}{\partial x^{i-n}} \in \mathfrak{B}$ if respectively $1 \leq i \leq n$ or $n + 1 \leq i \leq 2n$. The one $K \neq \varnothing$ (resp. $L \neq \varnothing$) means if we have $\frac{\partial}{\partial x^i} \in \mathfrak{B}$, then $\frac{\partial}{\partial x^{i+n}} \notin \mathfrak{B}$ (resp. $\frac{\partial}{\partial x^{i-n}} \notin \mathfrak{B}$) for $1 \leq i \leq n$ (resp. $n + 1 \leq i \leq 2n$). Therefore, by Theorem 2.45 we state the following.

Theorem 3.11. *The* $H^1(\mathfrak{B})$ *is isomorphic to* $(\mathcal{M}/[\mathcal{P}, \mathcal{P}]) \oplus \mathbb{R}^{card(K\cup L)+1}$.

Theorem 3.12. *If we denote* M *the normalizer of* $\mathfrak{B}$ *in* $\mathfrak{P}$ *and* M *the normalizer of* M *in* $\mathfrak{P}$, *all derivations of* M *are inner on* M, $H^1(M) = \mathrm{M}/M$ *and we can state a similar result on* M.

Proof. By the Jacobi identity, always the normalizer of $[\mathcal{P},\mathcal{P}]$ contains $\mathcal{P}$. Then all constant vector fields in $\mathbb{R}^{2n}$ are in M. Because all elements of $\mathfrak{B}$ are polynomial vector fields, then the Euler vector field is in M. We can conclude in the same way as in the proof of Theorem 3.8. The last affirmation can be proved remarking that $\mathrm{M} \subset \mathfrak{P}$ and using the same arguments as above on M. ■

Remark 3.13. The normalizer of $[\mathcal{P},\mathcal{P}]$ can be outer of $\mathfrak{P}$. Let us take the first part of Example 2.39 when $[\mathcal{P},\mathcal{P}]$ is the Lie algebra $\langle 1,x,y,x^2,xy,y^2,xy^2,x^2y,y^3\rangle$ on $\mathbb{R}^4$. The corresponding $\mathfrak{B}$ is generated by $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial y}$, $y^2\frac{\partial}{\partial y}-2xy\frac{\partial}{\partial x}$, $2xy\frac{\partial}{\partial y}-x^2\frac{\partial}{\partial x}$, $y^2\frac{\partial}{\partial x}$ in $\mathbb{R}^4$ has a non-polynomial vector field $e^z\frac{\partial}{\partial t}$ in its normalizer because $O \neq \varnothing$, it justifies our choice in Theorem 3.12.

Example 3.14. We consider the usual coordinate system (x,z,y,t) on $\mathbb{R}^4$ and $\mathcal{P}$ the Lie algebra generated by $1,x,y,z,t,x^2,y^2,xy$ like in Example 2.46. By calculation, the corresponding Lie algebra $\mathfrak{B}$ is $\left\langle \frac{\partial}{\partial x},\frac{\partial}{\partial y},y\frac{\partial}{\partial x},x\frac{\partial}{\partial y},y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}\right\rangle$ and $H^1(\mathfrak{B}) = \mathcal{A}\oplus\mathbb{R}$. The part M of the normalizer of $\mathfrak{B}$ in $\mathfrak{P}$ is the following Lie algebra $\left\langle \frac{\partial}{\partial x},\frac{\partial}{\partial y},y\frac{\partial}{\partial x},x\frac{\partial}{\partial y},y\frac{\partial}{\partial y},x\frac{\partial}{\partial x}\right\rangle \oplus \left(\left\langle \frac{\partial}{\partial z},\frac{\partial}{\partial t}\right\rangle_{\mathbb{R}(z,t)}\right)$. The normalizer of M is itself and by our theorem, $H^1(M) = \{0\}$.

Example 3.15. If we take $\mathfrak{B}$ corresponding to the $[\mathcal{P},\mathcal{P}]$ of Example 2.47, it is generated by $\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z},y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x},y\frac{\partial}{\partial x},x\frac{\partial}{\partial y},t\frac{\partial}{\partial y}-x\frac{\partial}{\partial z},t\frac{\partial}{\partial x}+y\frac{\partial}{\partial z},t\frac{\partial}{\partial z}$. By our result, $H^1(\mathfrak{B}) = \mathbb{R}^2\otimes\mathcal{B}$ and M is

$$\mathfrak{B}\oplus\left\langle E,\frac{\partial}{\partial t},t^2\frac{\partial}{\partial z},t^3\frac{\partial}{\partial z},\dots\right\rangle_{\mathbb{R}}.$$

Moreover, $H^1(M) = \{0\}$.

4. The Jacobian Conjecture on the Lie sub-algebras of $\mathbb{R}(x,y)$

The Jacobian Conjecture in $\mathbb{R}^2$ is still an open question. That is to say if a polynomial map $(x,y)\in\mathbb{R}^2 \mapsto (X(x,y),Y(x,y))$ has Jacobian in $\mathbb{R}^*$, then map (X,Y) is invertible (and have a polynomial inverse) which is equivalent to (X,Y) is an injection. In [6], there is a partial answer saying that if the degrees of X and of Y are all no more than 100, the conjecture is true. Some little propositions about Jacobian conjecture are the following. In this section, $\mathcal{P}$ is not supposed to be separated unless special mention.

Proposition 4.1. *If we have $[X,Y]\in H_{-1}$ for $X,Y\in\mathbb{R}(x^1,\dots,x^{2n})$ where we denote X_0,Y_0 the respective part of X and Y on H_0, then $[X,Y]=[X_0,Y_0]$.*

We can directly deduce from the previous proposition the following

Corollary 4.2. *In $\mathbb{R}^2$ of coordinate system (x,y), if we state $[X,Y]\in H^*_{-1}$ for $X,Y\in\mathbb{R}(x,y)$ where X_0,Y_0 the respective part of X and Y on H_0 with $X_0=\alpha x+\beta y$, $Y_0=\alpha' x+\beta' y$ and $\alpha,\alpha',\beta,\beta'\in\mathbb{R}$, then $\alpha\beta'-\beta\alpha'=[X,Y]$ and (X_0,Y_0) realizes the Jacobian conjecture.*

Definition 4.3. A Lie sub-algebra $\mathfrak{T}$ of $\mathbb{R}(x^1,x^2)$ is *admissible* for the Jacobian conjecture if it exists $X,Y\in\mathfrak{T}$ such that $[X,Y]\in H^*_{-1}$.

Definition 4.4. A Lie sub-algebra $\mathfrak{T}$ of $\mathbb{R}(x^1, x^2)$ *verifies the Jacobian conjecture* if for all $X, Y \in \mathfrak{T}$ with $[X, Y] \in H_{-1}^*$ we have that $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by (X, Y) is an injection.

The following theorem gives a necessary and sufficient condition for a Lie sub-algebra of $\mathbb{R}(x^1, x^2)$ to be admissible for Jacobian conjecture.

We suppose in the rest of this section unless special mention that a Lie sub-algebras $\mathfrak{T}$ of $\mathbb{R}(x^1, x^2)$ is such that if $X \in \mathfrak{T}$, the part of X in H_0 is in $\mathfrak{T}$ also. This last hypothesis will be denoted by (h).

Theorem 4.5. *A Lie sub-algebras $\mathfrak{T}$ of $\mathbb{R}(x^1, x^2)$ is admissible for Jacobian conjecture if and only if $S \subset \mathfrak{T}$.*

Proof. If $S \subset \mathfrak{T}$ then $1, x^1, x^2 \in \mathfrak{T}$ and $[x^1, x^2] = 1$ with (x^1, x^2) realizes the Jacobian conjecture cf. Corollary 4.2. Therefore $\mathfrak{T}$ is admissible for Jacobian conjecture. Conversely, if $\mathfrak{T}$ is admissible for Jacobian conjecture, there is $X, Y \in \mathfrak{T}$ such that $[X, Y] \in H_{-1}^*$. By Corollary 4.2 and (h), it exists $X_0, Y_0 \in H_0 \cap \mathfrak{T}$ such that $[X, Y] = [X_0, Y_0]$ and (X_0, Y_0) satisfies the Jacobian conjecture. In this case it exists $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\langle 1, X_0 = \alpha x^1 + \beta x^2, Y_0 = \gamma x^1 + \delta x^2 \rangle \subset \mathfrak{T}$ with $\alpha\delta - \beta\gamma = [X_0, Y_0]$. Thus, $x^1, x^2 \in \mathfrak{T}$ by linearity independence of X_0 and Y_0. It achieves our proof. ■

When $n = 1$, we denote $x^1 = x$ and $x^2 = y$ in the following and (AJC) means admissible for Jacobian conjecture.

Proposition 4.6. *The Lie separated sub-algebras of $\mathbb{R}(x, y)$ (AJC) are of the following types:*

$$\langle 1, x, y \rangle, \left\langle 1, x, y, (x)^2 \right\rangle, \left\langle 1, x, y, (y)^2 \right\rangle, \langle 1, x, y, xy \rangle, \left\langle 1, x, y, (x)^2, xy \right\rangle, \left\langle 1, x, y, (y)^2, xy \right\rangle,$$

$$A_0 = \left\langle 1, x, y, (x)^2, (y)^2, xy \right\rangle, A_1 = \left\langle 1, x, y, y^2, y^3, \dots, y^t \right\rangle,$$

$$A_2 = \left\langle 1, x, y, xy, y^2, y^3, \dots, y^t \right\rangle, A_3 = \left\langle 1, x, y, x^2, x^3, \dots, x^t \right\rangle,$$

$$A_4 = \left\langle 1, x, y, xy, x^2, x^3, \dots, x^t \right\rangle \text{ for } t \geq 3 \text{ of finite dimension and}$$

$$A_5 = \left\langle 1, x, y, y^2, y^3, \dots \right\rangle, A_6 = \left\langle 1, x, y, xy, y^2, y^3, \dots \right\rangle,$$

$$A_7 = \left\langle 1, x, y, xy, xy^2, y^2, y^3, \dots \right\rangle, A_8 = \left\langle 1, x, y, x^2, x^3, \dots \right\rangle,$$

$$A_9 = \left\langle 1, x, y, xy, x^2, x^3, \dots \right\rangle, A_{10} = \left\langle 1, x, y, xy, x^2y, x^2, x^3, \dots \right\rangle,$$

$$A_{11} = \left\langle 1, x, y, xy, xy^2, xy^3, \dots, y^2, y^3, \dots \right\rangle,$$

$$A_{12} = \left\langle 1, x, y, xy, x^2y, x^3y, \dots, x^2, x^3, \dots \right\rangle,$$

$\mathbb{R}(x, y)$ of infinite dimension. The $1^{st}, 7^{th}, 20^{th}$ of the above Lie algebras verify (H).

Proof. Let $\mathfrak{T}$ be a such Lie sub-algebra of $\mathbb{R}(x, y)$, be separated and (AJC). Then $\langle 1, x, y \rangle \subset \mathfrak{T}$ by Theorem 4.5. It is clear that the first seven Lie algebras verify all these hypotheses. Following Lemma 2.16, if $\mathfrak{T}$ doesn't contain simultaneously x^2, y^3 or x^3, y^2, $\mathfrak{T} \neq \mathbb{R}(x, y)$. Without the case of the seven previous Lie algebras, first we can consider $y^2 \in \mathfrak{T}$ then $x^3 \notin \mathfrak{T}$. So for $t \geq 3$, $\mathfrak{T}$ is A_1 or A_2 or A_i for $i = 5, 6, 7, 11$. Similar situations when $x^2 \in \mathfrak{T}$ with $y^3 \notin \mathfrak{T}$, $\mathfrak{T}$ is A_3 or A_4 or A_i for $i = 8, 9, 10, 12$. The rest of the cases is $x^2, y^3 \in \mathfrak{T}$ or $x^3, y^2 \in \mathfrak{T}$, by Lemma 2.16 $\mathfrak{T} = \mathbb{R}(x, y)$. The last assertion is easy to check. ■

Theorem 4.7. *The Lie algebra A_{11} verifies the Jacobian conjecture.*

Proof. Let a, b and all Greek letters are real number. To simplify, let

$$X = a + \alpha x + \beta y + \sum_{s\geq i\geq 1} \lambda_i x y^i + \sum_{s\geq i\geq 2} \alpha_i y^i$$

and

$$Y = b + \gamma x + \delta y + \sum_{s\geq i\geq 1} \eta_i x y^i + \sum_{s\geq i\geq 2} \beta_i y^i$$

in A_{11} such that $[X, Y] = 1$ and s a natural integer. By Corollary 4.2, $\alpha\delta - \beta\gamma = 1$ and by identification $\alpha\eta_1 - \gamma\lambda_1 = 0$, $\alpha\eta_2 - \gamma\lambda_2 = 0$, $\delta\lambda_1 - \beta\eta_1 + 2\alpha\beta_2 - 2\gamma\alpha_2 = 0$, and for all $i \geq 2$

$$(i+1)\alpha\beta_{i+1} - \beta\eta_i + \delta\lambda_i - (i+1)\gamma\alpha_{i+1} + \sum_{\substack{t+l=i+1\\ t\geq 1,\ l\geq 2}} l\lambda_t\beta_l - \sum_{t+l=i+1, t\geq 2, l\geq 1} t\alpha_t\eta_l = 0,$$

$$(i+1)\alpha\eta_{i+1} - (i+1)\gamma\lambda_{i+1} + \sum_{\substack{t+l=i+1\\ t,l\geq 1}} (t-l)\lambda_l\eta_t = 0.$$

We solve this system of equations with Maple following the condition that X and Y are polynomials, then we can take

$$\alpha_{2s} = \alpha_{2s-1} = \cdots = \alpha_{s+1} = \beta_{2s} = \beta_{2s-1} = \cdots = \beta_{s+1} = 0$$

and

$$\lambda_{2s} = \lambda_{2s-1} = \cdots = \lambda_{s+1} = \eta_{2s} = \eta_{2s-1} = \cdots = \eta_{s+1} = 0$$

such that

$$X = a + \alpha x + \beta y + \sum_{2s\geq i\geq 1} \lambda_i x y^i + \sum_{2s\geq i\geq 2} \alpha_i y^i$$

and

$$Y = b + \gamma x + \delta y + \sum_{2s\geq i\geq 1} \eta_i x y^i + + \sum_{2s\geq i\geq 2} \beta_i y^i.$$

We find the following results, if $\gamma \neq 0$, then α, δ, β_i are arbitrary for $2 \leq i \leq s$ and

$$\beta = \frac{\alpha\delta - 1}{\gamma},\ \eta_i = \lambda_i = 0\ \forall 1 \leq i \leq s,\ \alpha_i = \frac{\alpha\beta_i}{\gamma}\ \forall\ 2 \leq i \leq s$$

or if $\gamma = 0$ then $\alpha \neq 0$ and α, β, α_i for $2 \leq i \leq s$ are arbitrary,

$$\delta = \frac{1}{\alpha},\ \eta_i = \lambda_i = 0\ \forall\ 1 \leq i \leq s,\ \beta_i = 0\ \forall\ 2 \leq i \leq s.$$

Then we have two corresponding types of possibilities:

$$\left(X = a + \alpha x + \frac{\alpha\delta - 1}{\gamma} y + \frac{\alpha}{\gamma} \sum_{s\geq i\geq 2} \beta_i y^i, Y = b + \gamma x + \delta y + \sum_{s\geq i\geq 2} \beta_i y^i\right) \tag{30}$$

or

$$\left(X = a + \alpha x + \beta y + \sum_{s\geq i\geq 2} \alpha_i y^i, Y = b + \frac{1}{\alpha} y\right). \tag{31}$$

Let $(x', y') \in \mathbb{R}^2$ such that:

$$\left(X = a + \alpha x' + \frac{\alpha\delta - 1}{\gamma} y' + \frac{\alpha}{\gamma} \sum_{s \geq i \geq 2} \beta_i y'^i, Y = b + \gamma x' + \delta y' + \sum_{s \geq i \geq 2} \beta_i y'^i\right)$$

or

$$\left(X = a + \alpha x' + \beta y' + \sum_{s \geq i \geq 2} \alpha_i y'^i, Y = b + \frac{1}{\alpha} y'\right).$$

Resolving the linear system of equations in $X' = x' - x, Y' = y' - y$:

$$\begin{cases} \alpha X' + \dfrac{\alpha\delta - 1}{\gamma} Y' = \dfrac{\alpha}{\gamma} \displaystyle\sum_{s \geq i \geq 2} \beta_i \left(y^i - y'^i\right) \\ \gamma X' + \delta Y' = \displaystyle\sum_{s \geq i \geq 2} \beta_i \left(y^i - y'^i\right) \end{cases}$$

or

$$\begin{cases} \alpha X' + \beta Y' = \displaystyle\sum_{s \geq i \geq 2} \alpha_i \left(y^i - y'^i\right) \\ \dfrac{1}{\alpha} Y' = 0 \end{cases}$$

we find respectively $\left(X' = \dfrac{1}{\gamma} \displaystyle\sum_{s \geq i \geq 2} \beta_i \left(y^i - y'^i\right), Y' = 0\right)$

and

$$\left(X' = \frac{1}{\alpha} \sum_{s \geq i \geq 2} \alpha_i \left(y^i - y'^i\right) - \beta Y', \frac{1}{\alpha} Y' = 0\right).$$

Thus all cases lead us to $X' = Y' = 0$. We can swap the values of X and Y and we have a similar result. Hence, the mapping defined by (X, Y) is an injection and A_{11} verifies the Jacobian conjecture. ∎

Theorem 4.8. *Each Lie sub-algebra $\mathfrak{T}$ verifying (h) of $\mathbb{R}(x, y)$ admissible for Jacobian conjecture different to $\mathbb{R}(x, y)$ verifies the Jacobian conjecture.*

Proof. First, suppose $\mathfrak{T} \neq \mathbb{R}(x, y)$ separated (AJC) verifying (h). By Proposition 4.6 it is one of the 19 types of the above Lie algebras. If all degrees of elements of $\mathfrak{T}$ are no more than 2, we can use a theorem of [6] or [9] to conclude. Otherwise, $\mathfrak{T} \neq \mathbb{R}(x, y)$ is one of A_i for $i = 1, \ldots, 12$. We remark that for a given $t \geq 3$, $A_1 \subset A_2 \subset A_6 \subset A_7 \subset A_{11}$, $A_5 \subset A_6 \subset A_7 \subset A_{11}$ and $A_3 \subset A_4 \subset A_9 \subset A_{10} \subset A_{12}$, $A_8 \subset A_9 \subset A_{10} \subset A_{12}$. Therefore, it is sufficient to verify the Jacobian conjecture for A_{11} and A_{12} in order that all A_i for all $i = 1, \ldots, 12$ satisfy the Jacobian conjecture. But the methods to find this result are similar for A_{11} and A_{12}, then we will check only the case of A_{11}. By Theorem 4.7, A_{11} satisfies the Jacobian conjecture.

Second, $\mathfrak{T} \neq \mathbb{R}(x, y)$ is non-separated (AJC) verifying (h). If the degree of all elements in $\mathfrak{T}$ is no more than 2, we can conclude by the same way as in the first case. Otherwise, it is sufficient to check the following:

(1) Suppose a Lie algebra $\mathfrak{T}$ contains strictly A_{11}. By Theorem 4.5 applied to A_{11}, $x, y \in \mathfrak{T}$. But we have

$$X = x^2 \sum_{l_1 \geq i \geq 0} \alpha_i^1 y^i + x^3 \sum_{l_2 \geq i \geq 0} \alpha_i^2 y^i + \cdots + + x^m \sum_{l_m \geq i \geq 0} \alpha_i^{m-1} y^i \in \mathfrak{T}$$

where the $l_i \in \mathbb{N}$, $\alpha_i^j \in \mathbb{R}$ and m an integer more than or equal to 2 with $\alpha_{l_{m-1}}^{m-1} \neq 0$. Let $R = \{j/\alpha_{l_{j-1}}^{j-1} \neq 0\}$, by successive brackets of X by x in $t = max\{l_j/j \in R\}$ times, we get $\sum_{\{2\leq j\leq m/l_j=t\}} \alpha_{l_{j-1}}^{j-1}(l_{j-1}!)x^j \in \mathfrak{T}$ with $j \geq 2$. Successive brackets with y of this last expression give us $x^2 \in \mathfrak{T}$. We denote by (T) this last result. Then, knowing $y^3 \in \mathfrak{T}$, Lemma 2.16 leads to $\mathfrak{T} = \mathbb{R}(x, y)$ and we find a contradiction.
(2) Suppose a Lie algebra $\mathfrak{T}$ containing strictly A_{12}. By symmetry, we reason in the same way when we replace the above A_{11} by A_{12}.

(3) Suppose a Lie algebra $\mathfrak{T}$ doesn't contain strictly A_{11} nor A_{12}. Denote the vector space $\mathfrak{T}^{sep} = \langle X \in \mathfrak{T}/\exists i, j \geq 0 \text{ with } X = \langle x^i y^j\rangle_{\mathbb{R}}\rangle_{\mathbb{R}}$ and consider the vector space $\mathfrak{T}/\mathfrak{T}^{sep} = \mathfrak{T}^{nonsep}$ in order to have $\mathfrak{T} \cong \mathfrak{T}^{sep} \oplus \mathfrak{T}^{nonsep}$. Then we have the following cases:

- $*$ For all $X \in \mathfrak{T}^{nonsep}$, the degree of X relative to x is < 2:
- • If all $X \in \mathfrak{T}^{nonsep}$, the degree of X relative to y is < 2. Then $X \in \langle xy\rangle_{\mathbb{R}}$ because $S \subset \mathfrak{T}^{sep}$, a contradiction.
- • If it exists $Y \in \mathfrak{T}^{nonsep}$, the degree of Y relative to y is ≥ 2. So $Y \in A_{11}$, otherwise we have the same situation as in the first •. So we obtain $\mathfrak{T}^{nonsep} \subset A_{11}$. Therefore the only possibility is that $\mathfrak{T}$ is in A_{11}. But A_{11} verifies the Jacobian conjecture, then $\mathfrak{T}$ also.
- $*$ It exists a $X \in \mathfrak{T}^{nonsep}$ where the degree on x is ≥ 2, otherwise if the degree on x is < 2, $X \in A_{11}$. Then, like in (T), $x^2 \in \mathfrak{T}$. By Lemma 2.16, the degree in y of each element of $\mathfrak{T}$ is < 3. Therefore,

$$X = x^2 \textstyle\sum_{2\geq i\geq 1} \alpha_i^1 y^i + x^3 \sum_{2\geq i\geq 0} \alpha_i^2 y^i + \cdots + x^m \sum_{2\geq i\geq 0} \alpha_i^{m-1} y^i \in \mathfrak{T}$$

  where $\alpha_i^j \in \mathbb{R}$ and m an integer more than or equal to 2. Then we have the following situations:
- • If the degree on x of X is 2 and the degree on y of X is 2, then the only non null coefficients of X are $\alpha_1^1, \alpha_2^1 \neq 0$. So $[[X, x^2], x]$ leads to $x^3 \in \mathfrak{T}$, then by Lemma 2.16, $\mathfrak{T} = \mathbb{R}(x, y)$ a contradiction.
- • If the degree on x of X is 2 and the degree on y of X is ≤ 1, impossible because $X \in \mathfrak{T}^{nonsep}$.
- • If the degree on x of X is > 2 and the degree on y of X is 2, then by Lemma 2.16, $\mathfrak{T} = \mathbb{R}(x, y)$ conducts to a contradiction.
- • If the degree on x of X is > 2 and the degree on y of X is ≤ 1, then $X \in A_{12}$. By the fact that $x^3 \in \mathfrak{T}$ by the last part of the proof of Lemma 2.16, by the same Lemma every $Y \in \mathfrak{T}$ has a degree on y strictly less than 2.

We can conclude that:

(a) It exists $X \in \mathfrak{T}^{nonsep}$ is such that its degree on x is > 2 and its degree on y is ≤ 1.

(b) It exists $X \in \mathfrak{T}^{nonsep}$ is such that its degree on x is ≤ 1 and its degree on y is ≤ 1.

(c) Every $X \in \mathfrak{T}^{sep}$ is such that its degree on y is ≤ 1.

So the only compatible cases on $X \in \mathfrak{T}$ are (a), (b) and (c), therefore $\mathfrak{T} \subset A_{12}$. But A_{12} satisfies the Jacobian conjecture then $\mathfrak{T}$ also.

Thus all non-separated Lie sub-algebras verifying (h) of $\mathbb{R}(x,y)$ different to $\mathbb{R}(x,y)$ verify the Jacobian conjecture. ■

According the results of Theorem 4.8, it's clear that:

Corollary 4.9. *A Lie sub-algebra of $\mathbb{R}(x,y)$ different to $\mathbb{R}(x,y)$ (AJC) not necessary verifying (h) but included in one of Lie sub-algebras of $\mathbb{R}(x,y)$ verifying (h) satisfies the Jacobian conjecture.*

Hence, we can say that if the following theorem would be proved, the Jacobian conjecture in $\mathbb{R}^2$ is also proved.

Theorem 4.10. *Each Lie sub-algebra (AJC) $\mathcal{P}$ of $\mathbb{R}(x,y)$ which doesn't verify (h) of dimension 3 which is not in a one of Lie sub-algebras (AJC) satisfying (h) different to $\mathbb{R}(x,y)$, satisfies the Jacobian conjecture (JC).*

The Lie algebra in Theorem 4.10 contains only the three generators $X, Y \in \mathcal{P}$ such that the degree of X or of Y is at least 101 and $[X,Y] = a \in \mathbb{R}^*$ with $\mathcal{P} = \langle 1, X, Y\rangle_{\mathbb{R}}$.

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Received June 26, 2019
and in final form May 5, 2020