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Holomorphic Multiplier Representations for Bounded Homogeneous Domains

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Abstract. We study the unitarizations in the spaces of holomorphic sections of equivariant holomorphic line bundles over a bounded homogeneous domain under the action of the identity component of an algebraic group acting transitively on the domain. The classification of all such unitary representations is accomplished in this paper. As an application, we give an explicit description of the classification for a specific five-dimensional non-symmetric bounded homogeneous domain.

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1. Introduction

Unitary representations realized in spaces of holomorphic sections of equivariant holomorphic line bundles appear in various areas of the representation theory of Lie groups. For instance, we can recall the Borel-Weil theory for compact Lie groups, the holomorphic discrete series representations of Hermitian Lie groups, and the Bargmann-Fock representation of the Heisenberg group. In this paper, we complete classification of such representations over bounded homogeneous domains. Let $\mathcal{D} \subset \mathbb{C}^N$ be a domain which is biholomorphic to a bounded domain. Suppose that a Lie group acts holomorphically, transitively, and effectively on $\mathcal{D}$ and the Lie group is isomorphic to the identity component of a real linear algebraic group. Let G be a covering group of the group.

Example 1.1. (i) $G =$ any covering group of the identity component $\mathrm{Aut}_{hol}(\mathcal{D})^0$ of the holomorphic automorphism group $\mathrm{Aut}_{hol}(\mathcal{D})$ of $\mathcal{D}$ (see [14, Theorem 3.2]).

(ii) $\mathcal{D} =$ any bounded symmetric domain, and $G =$ any covering group of the identity component of a parabolic subgroup of $\mathrm{Aut}_{hol}(\mathcal{D})^0$. ∎

For a G-equivariant holomorphic line bundle L over $\mathcal{D}$, let $\Gamma^{hol}(L)$ denote the space of holomorphic sections of L, and let τ_L be the natural representation of G on $\Gamma^{hol}(L)$ given by

$$\tau_L(g)s(z) = gs(g^{-1}z) \quad (g \in G, s \in \Gamma^{hol}(L), z \in \mathcal{D}),$$

where we denote the actions of G on the base space $\mathcal{D}$ and the total space L by $\mathcal{D} \ni z \mapsto gz \in \mathcal{D}$ and $L \ni \tilde{z} \mapsto g\tilde{z} \in L$.

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Let us consider all G-equivariant holomorphic line bundles L over $\mathcal{D}$ and the following basic questions:

(Q1) When are the representations τ_L of G unitarizable?

(Q2) Which unitarizations are equivalent as unitary representations of G?

Here we make precise the class of representations we study.

Definition 1.2. We say that the representation τ_L of G is *unitarizable* if there exists a nonzero Hilbert space $\mathcal{H} \subset \Gamma^{hol}(L)$ satisfying the following conditions:

(i) the inclusion map $\iota\colon \mathcal{H} \hookrightarrow \Gamma^{hol}(L)$ is continuous with respect to the open compact topology of $\Gamma^{hol}(L)$,

(ii) $\tau_L(g)\mathcal{H} \subset \mathcal{H}\,(g \in G)$ and $\|\tau_L(g)s\|_{\mathcal{H}} = \|s\|_{\mathcal{H}} \quad (g \in G, s \in \mathcal{H})$.

In this case, we call the subrepresentation $(\tau_L, \mathcal{H})$ a *unitarization* of the representation $(\tau_L, \Gamma^{hol}(L))$ of G. ■

A Hilbert space $\mathcal{H}$ satisfying the condition (i) is a reproducing kernel Hilbert space, and the notion of a unitarization in Definition 1.2 is closely related to the coherent state representations in [20, 21]. We note that a Hilbert space giving a unitarization of τ_L is unique if it exists. In particular, any unitarization is irreducible (see [13, 16, 19]). In this paper, we shall give a complete answer to the questions (Q1) and (Q2) by reducing the problems to the case where G is real split solvable.

We note that (analytic continuation of) the holomorphic discrete series of scalar type is a special case of the representation we consider. For now we assume that $\mathcal{D}$ is an irreducible bounded symmetric domain and G is the universal covering group of $\mathrm{Aut}_{hol}(\mathcal{D})^0$. Let $J\colon G \times \mathcal{D} \ni (g, z) \mapsto J(g, z) \in \mathbb{C}^\times$ be the complex Jacobian. For $\gamma \in \mathbb{C}$, we define a power of the Jacobian $J^{-\gamma}\colon G \times \mathcal{D} \to \mathbb{C}^\times$ by analytic continuation. By using this function, we define a G-equivariant holomorphic line bundle over $\mathcal{D}$. We consider the action of G on the trivial bundle $\mathcal{D} \times \mathbb{C}$ given by

$$\mathcal{D} \times \mathbb{C} \ni (z, \zeta) \mapsto (gz, J^{-\gamma}(g, z)\zeta) \in \mathcal{D} \times \mathbb{C} \quad (g \in G, z \in \mathcal{D}, \zeta \in \mathbb{C}).$$

We denote this G-equivariant holomorphic line bundle by L_γ, and put $\tau_\gamma = \tau_{L_\gamma}$. Harish-Chandra [9, 8] showed that τ_γ is unitarizable for $\gamma \in \mathbb{R}$ sufficiently large, and the unitarizabilities of τ_γ have been completely determined by Vergne and Rossi [27] and Wallach [28]. Hence the answer to (Q1) is already known in this case. Indeed, every G-equivariant holomorphic line bundle over $\mathcal{D}$ is isomorphic to L_γ for some $\gamma \in \mathbb{C}$. In this case, the theory of the highest weight representations gives an answer to (Q2), i.e. for $\gamma, \gamma' \in \mathbb{C}$, if τ_γ and $\tau_{\gamma'}$ are unitarizable and the unitarizations are equivalent as unitary representations of G, then we have $\gamma = \gamma'$.

In our setting, an Iwasawa subgroup (i.e., a maximal connected real split solvable subgroup) B of G acts on $\mathcal{D}$ simply transitively (see [22, Chapter 4, Theorem 4.7]). As we will see in Section 2, we can reduce the question (Q1) for G to the question for B.

Theorem 1.3 (see Theorem 2.5). *Let L be a G-equivariant holomorphic line bundle over $\mathcal{D}$ which has a G invariant Hermitian metric, and let $\mathcal{H} \subset \Gamma^{hol}(L)$ be a reproducing kernel Hilbert space. Suppose that $\tau_L(b)\mathcal{H} \subset \mathcal{H}$ for all $b \in B$ and $\|\tau_L(b)s\|_{\mathcal{H}} = \|s\|_{\mathcal{H}}$ for all $b \in B$ and $s \in \Gamma^{hol}(L)$. Then we have $\tau_L(g)\mathcal{H} \subset \mathcal{H}$ for all $g \in G$ and $\|\tau_L(g)s\|_{\mathcal{H}} = \|s\|_{\mathcal{H}}$ for all $g \in G$ and $s \in \Gamma^{hol}(L)$. Namely, the unitarizability as the representation of B implies the one as the representation of G.*

We fix a reference point $p \in \mathcal{D}$, and let K be the isotropy subgroup of G at p. Concerning the question (Q2), we obtain

Theorem 1.4 (see Theorem 9.5). *Let L and L' be G-equivariant holomorphic line bundles over $\mathcal{D}$. Suppose that Hilbert spaces $\mathcal{H} \subset \Gamma^{hol}(L)$ and $\mathcal{H}' \subset \Gamma^{hol}(L')$ give unitarizations of representations τ_L and $\tau_{L'}$, respectively. Then the unitarizations $(\tau_L, \mathcal{H})$ and $(\tau_{L'}, \mathcal{H}')$ are equivalent as unitary representations of G if and only if they are equivalent as unitary representations of B and the actions of K on the fibers of L and L' over the point p coincide.*

Here we remark that Ishi [13] gave a classification of unitarizations in the case $G = B$, and hence Theorems 1.3 and 1.4 together with the result complete the classification of unitarizations. As a byproduct of the discussions of this paper, we obtain the following theorem.

Theorem 1.5 (see Corollary 7.8). *Let L and L' be G-equivariant holomorphic line bundles over $\mathcal{D}$. Suppose that the actions of K on the fibers of L and L' over the point p coincide. Then L and L' are isomorphic as K-equivariant holomorphic line bundles.*

Let us explain the organization of this paper. In Section 2, we see that problems about equivariant holomorphic line bundles over $\mathcal{D}$ are equivalent to the problems about holomorphic multipliers. Then we prove Theorem 1.3. In Section 3, we review the theory of normal j-algebras and Siegel domains, which are useful tools for the explicit description of group actions on bounded homogeneous domains. In Sections 4, 5, 6, and 7, we work toward studying the actions of K on G-equivariant holomorphic line bundles over $\mathcal{D}$ and proving a result (Theorem 7.7), from which Theorem 1.5 follows. This result follows from the purely algebraic result (Proposition 6.3), which we aim at in Sections 5 and 6. In Section 4, we review some explicit descriptions of complete holomorphic vector fields on a Siegel domain. In Section 5, we disscuss properties of gradations of Lie algebras consist of the complete holomorphic vector fields. In Section 6, we study the isotropy representation. Using results about B-equivariant holomorphic line bundles in [13, 24], we prove Theorem 1.5 in Section 7. In Section 8, we review a construction of an intertwining operator between two unitarizations of holomorphic multiplier representations of B. In Section 9, we prove Theorem 1.4 by using the intertwining operator. In Section 10, we concentrate on a specific five-dimensional non-symmetric domain $\mathcal{D}_5$ and $\mathrm{Aut}_{hol}(\mathcal{D}_5)^0$-equivariant holomorphic line bundles L over $\mathcal{D}_5$, and as an application of Theorems 1.4 and 1.5, we classify all unitarizations of the representations τ_L.

2. Unitarizability

Throughout this paper, for a Lie group, we denote its Lie algebra by the corresponding Fraktur small letter. Let $\mathcal{D}_0$ be a domain in $\mathbb{C}^N$, and let G_0 be a Lie group acting holomorphically on $\mathcal{D}_0$.

Definition 2.1. A smooth function $m\colon G_0 \times \mathcal{D}_0 \to \mathbb{C}^\times$ is called a *multiplier* if the following cocycle condition is satisfied:

$$m(gg', z) = m(g, g'z)m(g', z) \quad (g, g' \in G_0, z \in \mathcal{D}_0),$$

where we denote the action of G_0 on $\mathcal{D}_0$ by $\mathcal{D}_0 \ni z \mapsto gz \in \mathcal{D}_0$. Moreover, a multiplier m is called a *holomorphic multiplier* if $m(g,z)$ is holomorphic in $z \in \mathcal{D}_0$. ■

Remark 2.2. Let $\mathcal{G} = \{m \colon G_0 \times \mathcal{D}_0 \to \mathbb{C}^\times; m \text{ is a holomorphic multiplier}\}$. Pointwise multiplication of holomorphic multipliers gives $\mathcal{G}$ the natural structure of a group. We write the product of two elements m, m' of $\mathcal{G}$ as mm'. ■

For a holomorphic multiplier $m \colon G_0 \times \mathcal{D}_0 \to \mathbb{C}^\times$, we define a G_0-equivariant holomorphic line bundle over $\mathcal{D}_0$. Let us consider the action of G_0 on the trivial bundle $\mathcal{D}_0 \times \mathbb{C}$ given by

$$\mathcal{D}_0 \times \mathbb{C} \ni (z, \zeta) \mapsto (gz, m(g,z)\zeta) \in \mathcal{D}_0 \times \mathbb{C} \quad (g \in G_0, z \in \mathcal{D}_0, \zeta \in \mathbb{C}). \tag{1}$$

We denote this G_0-equivariant holomorphic line bundle by L_m.
As we will see in Section 3, the domain $\mathcal{D}$ is biholomorphic to a Siegel domain. Thus $\mathcal{D}$ is a Stein manifold (see [1]). By the Oka-Grauert principle, every G-equivariant holomorphic line bundle over $\mathcal{D}$ is isomorphic as a G-equivariant holomorphic line bundle to L_m with some holomorphic multiplier $m \colon G \times \mathcal{D} \to \mathbb{C}^\times$. Let $\mathcal{O}(\mathcal{D})$ be the space of holomorphic functions on $\mathcal{D}$. Then the representation τ_{L_m} gives the following multiplier representation T_m on $\mathcal{O}(\mathcal{D})$:

$$T_m(g)f(z) = m(g^{-1}, z)^{-1} f(g^{-1}z) \quad (g \in G, f \in \mathcal{O}(\mathcal{D}), z \in \mathcal{D}).$$

Next we discuss representations T_m and their unitarizations. Let us reduce the question (Q1) for G to the question for B (see Theorem 2.5). The next lemma plays a key role for the reduction.

Lemma 2.3 ([13, Lemma 5], [18, Lemma 3.3]). *Let $\mathcal{H} \subset \mathcal{O}(\mathcal{D})$ be a Hilbert space with a reproducing kernel $\mathcal{K} \colon \mathcal{D} \times \mathcal{D} \to \mathbb{C}$, and let $m \colon G \times \mathcal{D} \to \mathbb{C}^\times$ be a holomorphic multiplier. Then $\mathcal{H}$ gives a unitarization of T_m if and only if we have*

$$\mathcal{K}(gz, gw) = m(g,z)\mathcal{K}(z,w)\overline{m(g,w)} \quad (z, w \in \mathcal{D}, g \in G). \tag{2}$$

We get the following lemma by the decomposition $G = BK$.

Lemma 2.4. *Let $m \colon G \times \mathcal{D} \to \mathbb{C}^\times$ be a (not necessarily holomorphic) multiplier. If a function f on $\mathcal{D}$ satisfies*

$$m(k,p)f(p) = f(p) \quad (k \in K) \tag{3}$$

and

$$f(bz) = m(b,z)f(z) \quad (b \in B, z \in \mathcal{D}), \tag{4}$$

then we have

$$f(gz) = m(g,z)f(z) \quad (g \in G, z \in \mathcal{D}). \tag{5}$$

Proof. We consider the G-equivariant line bundle $\mathcal{D} \times \mathbb{C}$, and regard f as a section of the line bundle. Then (4) means that the section f is B-invariant under the action of B, and (5) means that the section f is G-invariant under the action of G. Therefore, since B acts on $\mathcal{D}$ transitively, (4) and (5) are equivalent. ■

The following theorem is just an application of the previous two lemmas.

Theorem 2.5. *Let $m\colon G \times \mathcal{D} \to \mathbb{C}^\times$ be a holomorphic multiplier such that $|m(k,p)| = 1$ for all $k \in K$, and let $\mathcal{H} \subset \mathcal{O}(\mathcal{D})$ be a reproducing kernel Hilbert space. Suppose that the representation T_m satisfies $T_m(b)\mathcal{H} \subset \mathcal{H}\,(b \in B)$ and $\|T_m(b)f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}\,(b \in B, f \in \mathcal{H})$. Then we have $T_m(g)\mathcal{H} \subset \mathcal{H}\,(g \in G)$ and $\|T_m(g)f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}\,(g \in G, f \in \mathcal{H})$.*

Proof. Let $\mathcal{K}\colon \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ be the reproducing kernel of $\mathcal{H}$. By Lemma 2.3, for all $g \in B$, we have

$$\mathcal{K}(gz, gz) = m(g,z)\mathcal{K}(z,z)\overline{m(g,z)} \quad (z \in \mathcal{D}). \tag{6}$$

Let $\mathcal{K}^d$ be the function on $\mathcal{D}$ given by $\mathcal{K}^d(z) = \mathcal{K}(z,z)\,(z \in \mathcal{D})$, and $\tilde{m}\colon G\times\mathcal{D} \to \mathbb{C}^\times$ be the multiplier defined by

$$\tilde{m}(g,z) = |m(g,z)|^2 \quad (g \in G, z \in \mathcal{D}).$$

Applying Lemma 2.4 to $\tilde{m}$ and $\mathcal{K}^d$, we see that (6) holds for all $g \in G$. By the analytic continuation, the equation

$$\mathcal{K}(gz, gw) = m(g,z)\mathcal{K}(z,w)\overline{m(g,w)} \quad (g \in G, z, w \in \mathcal{D})$$

holds. This proves the result by Lemma 2.3. ∎

3. Normal j-algebra

Reviewing the theory of normal j-algebras in [2, 7, 5, 23, 24], we see in this section that the domain $\mathcal{D}$ is biholomorphic to a Siegel domain.

For $X \in \mathfrak{aut}_{hol}(\mathcal{D})$, let $X^\#$ denote the vector field on $\mathcal{D}$ given by

$$X^\#(z) = \left.\frac{d}{dt}\right|_{t=0} e^{tX}z \quad (z \in \mathcal{D}). \tag{7}$$

We fix a B-invariant Kähler metric $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathcal{D}$ such that $\langle\langle j_0X, j_0Y\rangle\rangle = \langle\langle X,Y\rangle\rangle$ for all vector fields X,Y over $\mathcal{D}$, where j_0 denotes the complex structure on $\mathcal{D}$ induced from the one of $\mathbb{C}^N$. For example, $\langle\langle\cdot,\cdot\rangle\rangle$ may be the Bergman metric on $\mathcal{D}$, which is a Kähler metric and $\mathrm{Aut}_{hol}(\mathcal{D})$-invariant (cf. for example [17, Chapter 9, §6]), or if $\mathcal{D}$ is contained in a complex domain $\hat{\mathcal{D}}$ of larger dimension as B-submanifold, then we can take $\langle\langle\cdot,\cdot\rangle\rangle$ as the restriction of the Bergman metric of $\hat{\mathcal{D}}$ to $\mathcal{D}$. Let j be the complex structure on $\mathfrak{b}$ given by

$$(jX)^\#(p) = j_0X^\#(p) \quad (X \in \mathfrak{b}),$$

and let $\langle\cdot,\cdot\rangle$ be the inner product on $\mathfrak{b}$ given by

$$\langle X,Y\rangle = \langle\langle X^\#, Y^\#\rangle\rangle(p) \quad (X,Y \in \mathfrak{b}). \tag{8}$$

Now $\mathfrak{b}$ is a real split solvable Lie algebra with the equality

$$[X,Y] + j[jX,Y] + j[X,jY] = [jX,jY] \quad (X,Y \in \mathfrak{b}), \tag{9}$$

and it is known [7] that there exists a linear form $\omega \in \mathfrak{b}^*$ such that

$$\langle X,Y\rangle = \omega([jX,Y]) \quad (X,Y \in \mathfrak{b}).$$

Thus the triple $(\mathfrak{b}, j, \omega)$ is a normal j-algebra. Let $\mathfrak{a}$ be the orthogonal complement to $[\mathfrak{b}, \mathfrak{b}]$ in $\mathfrak{b}$ relative to the inner product $\langle \cdot, \cdot \rangle$. Then $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{b}$. For $\alpha \in \mathfrak{a}^*$, let $\mathfrak{b}_\alpha$ be the root space associated to α given by

$$\mathfrak{b}_\alpha = \{X \in \mathfrak{b}; [A, X] = \alpha(A)X \ (A \in \mathfrak{a})\}.$$

Theorem 3.1 (Piatetski-Shapiro, [23, Chapter 2, Sections 3 and 5])**.** *For a suitable basis $A_1, \cdots, A_r$ of $\mathfrak{a}$, the following assertions hold: if we put $E_k = -jA_k$, then we have $[A_k, E_l] = \delta_{k,l} E_l \ (1 \le k, l \le r)$, if we denote the dual basis of $A_1, \cdots, A_r$ by $\alpha_1, \cdots, \alpha_r \in \mathfrak{a}^*$, then we have*

$$\mathfrak{b} = \mathfrak{b}(0) \oplus \mathfrak{b}(1/2) \oplus \mathfrak{b}(1),$$

where
$$\mathfrak{b}(0) = \mathfrak{a} \oplus \bigoplus_{1 \le k < l \le r} \mathfrak{b}_{(\alpha_l - \alpha_k)/2}, \quad \mathfrak{b}(1/2) = \bigoplus_{1 \le k \le r} \mathfrak{b}_{\alpha_k/2},$$
$$\mathfrak{b}(1) = \bigoplus_{1 \le k \le r} \mathfrak{b}_{\alpha_k} \oplus \bigoplus_{1 \le k < l \le r} \mathfrak{b}_{(\alpha_l + \alpha_k)/2},$$

and the equalities $\mathfrak{b}_{\alpha_k} = \mathbb{R}E_k$, $j\mathfrak{b}_{(\alpha_l - \alpha_k)/2} = \mathfrak{b}_{(\alpha_l + \alpha_k)/2}$, and $j\mathfrak{b}_{\alpha_k/2} = \mathfrak{b}_{\alpha_k/2}$ hold. We have the relation

$$[\mathfrak{b}(\gamma), \mathfrak{b}(\gamma')] \subset \mathfrak{b}(\gamma + \gamma') \quad (\gamma, \gamma' = 0, 1/2, 1),$$

where we put $\mathfrak{b}(\gamma) = 0$ for $\gamma > 1$.

Following [23, Chapter 2, Section 5], we introduce the Siegel domain $\mathcal{D}(\Omega, Q)$ on which the group B acts simply transitively as affine automorphisms as follows. Put

$$E = E_1 + \cdots + E_r.$$

Let $B(0)$ be the Lie subgroup of B with Lie algebra $\mathfrak{b}(0)$. We know that $B(0)$ acts on $\mathfrak{b}(1)$ by adjoint action. Let Ω be the $B(0)$-orbit through E. Then $\Omega \subset \mathfrak{b}(1)$ is an open convex cone containing no straight lines, and $B(0)$ acts on Ω simply transitively. Let $Q\colon (\mathfrak{b}(1/2), j) \times (\mathfrak{b}(1/2), j) \to \mathfrak{b}(1)_{\mathbb{C}}$ be the sesquilinear map defined by

$$Q(V, V') = \frac{1}{4}([jV, V'] + i[V, V']) \quad (V, V' \in \mathfrak{b}(1/2)).$$

Then Q is an Ω-positive Hermitian map, that is,

$$Q(V, V) \in \overline{\Omega} \setminus \{0\} \quad (V \in \mathfrak{b}(1/2) \setminus \{0\}),$$

where $\overline{\Omega}$ denotes the closure of Ω. Let

$$\mathcal{D}(\Omega, Q) = \{(U, V) \in \mathfrak{b}(1)_{\mathbb{C}} \oplus \mathfrak{b}(1/2); \operatorname{Im} U - Q(V, V) \in \Omega\}.$$

The subgroup $B(0)$ acts on $\mathcal{D}(\Omega, Q)$ by

$$t_0(U, V) = (\operatorname{Ad}(t_0)U, \operatorname{Ad}(t_0)V) \quad (t_0 \in B(0), (U, V) \in \mathcal{D}(\Omega, Q)),$$

and for $U_0 \in \mathfrak{b}(1)$ and $V_0 \in \mathfrak{b}(1/2)$, the element $\exp(U_0 + V_0)$ of B acts on $\mathcal{D}(\Omega, Q)$ by

$$\exp(U_0 + V_0)(U, V) = (U + U_0 + 2iQ(V, V_0) + iQ(V_0, V_0), V + V_0), \ (U, V) \in \mathcal{D}(\Omega, Q). \quad (10)$$

We define a Cayley transform $\mathcal{C}\colon \mathcal{D}(\Omega, Q) \to \mathcal{D}$ by $\mathcal{C}(b(iE, 0)) = bp \ (b \in B)$. Then the map $\mathcal{C}$ is a biholomorphic.

Remark 3.2. (i) By D'Atri [2], the decomposition

$$[\mathfrak{b},\mathfrak{b}] = \bigoplus_{1\le k<l\le r} \mathfrak{b}_{(\alpha_l-\alpha_k)/2} \oplus \bigoplus_{1\le k\le r} \mathfrak{b}_{\alpha_k/2} \oplus \bigoplus_{1\le k\le r} \mathfrak{b}_{\alpha_k} \oplus \bigoplus_{1\le k<l\le r} \mathfrak{b}_{(\alpha_l+\alpha_k)/2}$$

is orthogonal with respect to $\langle\cdot,\cdot\rangle$.

(ii) The number $r=\dim\mathfrak{a}$ is called the rank of $\mathfrak{b}$. ■

4. Complete holomorphic vector fields on $\mathcal{D}(\Omega, Q)$

In this section, first we review some explicit descriptions of complete holomorphic vector fields on $\mathcal{D}(\Omega,Q)$. After that, we see some technical lemmas.

Let $\mathfrak{X}$ be the space of complete holomorphic vector fields on $\mathcal{D}(\Omega,Q)$. It is well known that the following map is bijective:

$$\mathfrak{aut}_{hol}(\mathcal{D}(\Omega,Q)) \ni X \mapsto X^{\#} \in \mathfrak{X}.$$

For $U_0\in\mathfrak{b}(1)$, let ∂_{U_0} be the holomorphic vector field on $\mathcal{D}(\Omega,Q)$ given by

$$\partial_{U_0}(U,V) = (U_0,0) \quad ((U,V)\in\mathcal{D}(\Omega,Q)).$$

Here for every $(U,V)\in\mathcal{D}(\Omega,Q)$, we identify the tangent space $T_{(U,V)}\mathcal{D}(\Omega,Q)$ with $\mathfrak{b}(1)_{\mathbb{C}}\oplus\mathfrak{b}(1/2)$, and we consider a vector filed $X\in\mathfrak{X}$ as a $(\mathfrak{b}(1)_{\mathbb{C}}\oplus\mathfrak{b}(1/2))$-valued function. For $V_0\in\mathfrak{b}(1/2)$, let $\tilde{\partial}_{V_0}$ be the holomorphic vector field on $\mathcal{D}(\Omega,Q)$ given by

$$\tilde{\partial}_{V_0}(U,V) = (2iQ(V,V_0),V_0) \quad ((U,V)\in\mathcal{D}(\Omega,Q)).$$

For complex endomorphisms $\mathcal{A}\in\mathfrak{gl}(\mathfrak{b}(1)_{\mathbb{C}})$ and $\mathcal{B}\in\mathfrak{gl}(\mathfrak{b}(1/2))$, let $X_{\mathcal{A},\mathcal{B}}$ be the holomorphic vector field on $\mathcal{D}(\Omega,Q)$ given by

$$X_{\mathcal{A},\mathcal{B}}(U,V) = (\mathcal{A}U,\mathcal{B}V) \quad ((U,V)\in\mathcal{D}(\Omega,Q)).$$

We say $\mathcal{B}\in\mathfrak{gl}(\mathfrak{b}(1/2))$ is associated with $\mathcal{A}\in\mathfrak{gl}(\mathfrak{b}(1)_{\mathbb{C}})$ if the equality

$$\mathcal{A}Q(V,V') = Q(\mathcal{B}V,V') + Q(V,\mathcal{B}V') \quad (V,V'\in\mathfrak{b}(1/2))$$

holds. Let ∂ be the infinitesimal generator of the one-parameter subgroup $\mathcal{D}(\Omega,Q)\ni(U,V)\mapsto(e^tU,e^{t/2}V)\in\mathcal{D}(\Omega,Q)$ $(t\in\mathbb{R})$. Then we have

$$\partial = X_{\mathrm{id}_{\mathfrak{b}(1)_{\mathbb{C}}},\frac{1}{2}\mathrm{id}_{\mathfrak{b}(1/2)}}.$$

For $\gamma\in\mathbb{R}$, we put $$\mathfrak{X}(\gamma) = \{X\in\mathfrak{X};[\partial,X]=\gamma X\}.$$

Let $$G(\Omega) = \{A\in GL(\mathfrak{b}(1)); A\Omega=\Omega\}.$$

Theorem 4.1 (Kaup, Matsushima, Ochiai, [15, Theorems 4 and 5])**.** *The Lie algebra $\mathfrak{X}$ has the following gradation:*

$$\mathfrak{X} = \mathfrak{X}(-1)\oplus\mathfrak{X}(-1/2)\oplus\mathfrak{X}(0)\oplus\mathfrak{X}(1/2)\oplus\mathfrak{X}(1).$$

One has $\mathfrak{X}(-1)=\{\partial_{U_0};U_0\in\mathfrak{b}(1)\}$, $\mathfrak{X}(-1/2)=\{\tilde{\partial}_{V_0};V_0\in\mathfrak{b}(1/2)\}$, *and*

$$\mathfrak{X}(0) = \{X_{\mathcal{A},\mathcal{B}};\ \mathcal{A}\in\mathfrak{g}(\Omega),\ \mathcal{B}\in\mathfrak{gl}(\mathfrak{b}(1/2)),\ \mathcal{B} \text{ is associated with } \mathcal{A}\}.$$

We denote by $\mathcal{D}(\Omega)$ the tube domain $\{U \in \mathfrak{b}(1)_{\mathbb{C}}; \operatorname{Im} U \in \Omega\}$, and for $\mathcal{A} \in \mathfrak{gl}(\mathfrak{b}(1)_{\mathbb{C}})$, let $X_{\mathcal{A}}$ be the holomorphic vector field on $\mathcal{D}(\Omega)$ given by $X_{\mathcal{A}}(U) = \mathcal{A}U$ $(U \in \mathcal{D}(\Omega))$. We have the following formulas (see [25, Chapter V, §1]):

$$[X_{\mathcal{A},\mathcal{B}}, \partial_{U_0}] = -\partial_{\mathcal{A}U_0}, \tag{11}$$

$$[X_{\mathcal{A},\mathcal{B}}, \tilde{\partial}_{V_0}] = -\tilde{\partial}_{\mathcal{B}V_0}, \tag{12}$$

$$[X_{\mathcal{A},\mathcal{B}}, X_{\mathcal{A}',\mathcal{B}'}] = -X_{[\mathcal{A},\mathcal{A}'],[\mathcal{B},\mathcal{B}']}. \tag{13}$$

Next we see explicit descriptions of $\mathfrak{X}(1/2)$ and $\mathfrak{X}(1)$. For $U', U'' \in \mathfrak{b}(1)$, let $\overline{U' + iU''} = U' - iU''$.

Proposition 4.2 (Satake, [25, Chapter V, Proposition 2.1]). *Every element of* $\mathfrak{X}(1/2)$ *can uniquely be written as*

$$Y_{\Phi,c}(U, V) = (2iQ(V, \Phi(\overline{U})), \Phi(U) + c(V, V)) \quad ((U, V) \in \mathcal{D}(\Omega, Q)) \tag{14}$$

where $\Phi\colon \mathfrak{b}(1)_{\mathbb{C}} \to \mathfrak{b}(1/2)$ *is a* $\mathbb{C}$*-linear map and* $c\colon \mathfrak{b}(1/2) \times \mathfrak{b}(1/2) \to \mathfrak{b}(1/2)$ *is a symmetric* $\mathbb{C}$*-bilinear map, satisfying the following conditions:*

(Y1) *for each* $V_0 \in \mathfrak{b}(1/2)$*, the linear map* $\Phi_{V_0}\colon \mathfrak{b}(1) \ni U \mapsto \operatorname{Im} Q(\Phi(U), V_0) \in \mathfrak{b}(1)$ *belongs to* $\mathfrak{g}(\Omega)$*,*

(Y2) $Q(c(V', V'), V) = 2iQ(V', \Phi(Q(V, V')))$ $\quad (V, V' \in \mathfrak{b}(1/2))$.

Conversely, for any pair (Φ, c) *satisfying* (Y1) *and* (Y2)*, the vector field* $Y_{\Phi,c}$ *given by* (14) *belongs to* $\mathfrak{X}(1/2)$*.*

As we shall see later, every vector field $Y_{\Phi,c}$ is uniquely determined by the vector $\Phi(E) \in \mathfrak{b}(1/2)$, so that $Y_{\Phi,c}$ will be also written as Y_{Φ}.

Proposition 4.3 (Satake, [25, Chapter V, Proposition 2.2]). *Every element of* $\mathfrak{X}(1)$ *can uniquely be written as*

$$Z_{a,b}(U, V) = (a(U, U), b(U, V)) \quad ((U, V) \in \mathcal{D}(\Omega, Q)) \tag{15}$$

with a symmetric $\mathbb{R}$*-bilinear map* $a\colon \mathfrak{b}(1) \times \mathfrak{b}(1) \to \mathfrak{b}(1)$ *(which we extend to a* $\mathbb{C}$*-bilinear map* $a\colon \mathfrak{b}(1)_{\mathbb{C}} \times \mathfrak{b}(1)_{\mathbb{C}} \to \mathfrak{b}(1)_{\mathbb{C}}$*), and a* $\mathbb{C}$*-bilinear map* $b\colon \mathfrak{b}(1)_{\mathbb{C}} \times \mathfrak{b}(1/2) \to \mathfrak{b}(1/2)$ *which satisfy the following conditions:*

(Z1) *for each* $U_0 \in \mathfrak{b}(1)$*, the linear map* $\mathcal{A}_{U_0}\colon \mathfrak{b}(1) \ni U \mapsto a(U_0, U) \in \mathfrak{b}(1)$ *belongs to* $\mathfrak{g}(\Omega)$*,*

(Z2) *for any* $U_0 \in \mathfrak{b}(1)$*, the linear map* $\mathcal{B}_{U_0}\colon \mathfrak{b}(1/2) \ni V \mapsto \frac{1}{2} b(U_0, V) \in \mathfrak{b}(1/2)$ *is associated with* $\mathcal{A}_{U_0}$*, and* $\operatorname{Im} \operatorname{tr} \mathcal{B}_{U_0} = 0$*,*

(Z3) *for any* $V, V' \in \mathfrak{b}(1/2)$*, the linear map* $\mathfrak{b}(1) \ni U \mapsto \operatorname{Im} Q(b(U, V), V') \in \mathfrak{b}(1)$ *belongs to* $\mathfrak{g}(\Omega)$*,*

(Z4) $Q(b(Q(V'', V'), V''), V) = Q(V'', b(Q(V, V''), V'))$ $\quad (V, V', V'' \in \mathfrak{b}(1/2))$.

Conversely, for any pair (a, b) *satisfying* (Z1)*,* (Z2)*,* (Z3)*, and* (Z4)*, the vector field* $Z_{a,b}$ *given by* (15) *belongs to* $\mathfrak{X}(1)$*.*

We have the following formulas (see [25, Chapter V, §2]):

$$[\partial_{U_0}, Y_{\Phi,c}] = \tilde{\partial}_{\Phi(U_0)}, \tag{16}$$

$$[\tilde{\partial}_{V_0}, Y_{\Phi,c}] = X_{\mathcal{A},\mathcal{B}}, \tag{17}$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by $\mathcal{A} = 4\Phi_{V_0}$ and

$$\mathcal{B} \colon \mathfrak{b}(1/2) \ni V \mapsto 2j\Phi(Q(V, V_0)) + 2c(V_0, V) \in \mathfrak{b}(1/2).$$

Moreover we have
$$[\partial_{U_0}, Z_{a,b}] = 2X_{\mathcal{A}_{U_0}, \mathcal{B}_{U_0}}. \tag{18}$$

Next we see a description of the subalgebra: $\mathfrak{X}_{(iE,0)} = \{X \in \mathfrak{X}; X(iE, 0) = 0\}$. Let ∂' be the element of $\mathfrak{X}(0)$ given by $\partial'(U, V) = (0, jV)$, $(U, V) \in \mathcal{D}(\Omega, Q)$, and let $\psi_E \colon \mathfrak{X}(1/2) \to \mathfrak{X}(-1/2)$, and $\varphi_E \colon \mathfrak{X}(1) \to \mathfrak{X}(-1)$ be linear maps given by

$$\psi_E = \mathrm{ad}(\partial')\mathrm{ad}(\partial_E)|_{\mathfrak{X}(1/2)}, \quad \text{and} \quad \varphi_E = \frac{1}{2}\mathrm{ad}(\partial_E)^2|_{\mathfrak{X}(1)}, \quad \text{respectively.}$$

Put $\mathfrak{m} = \{X + \psi_E(X); X \in \mathfrak{X}(1/2)\}$ and $\mathfrak{m}' = \{X + \varphi_E(X); X \in \mathfrak{X}(1)\}$.

Theorem 4.4 (Kaup, Matsushima, Ochiai, [15, Theorem 6])**.**

$$\mathfrak{X}_{(iE,0)} = (\mathfrak{X}_{(iE,0)} \cap \mathfrak{X}(0)) + \mathfrak{m}' + \mathfrak{m}.$$

We note that φ_E and ψ_E are injective (see [25, p. 211−212]), and by [25, p. 215], we have

$$\psi_E(Y_\Phi) = \tilde{\partial}_{-j\Phi(E)}. \tag{19}$$

Next let us prove some results, which will be used later. We obtain the following two lemmas through straightforward computations.

Lemma 4.5. *Let $X_{\mathcal{A},\mathcal{B}} \in \mathfrak{X}(0)$, and let $Y_{\Phi,c} \in \mathfrak{X}(1/2)$. We define a $\mathbb{C}$-linear map $\Phi' \colon \mathfrak{b}(1)_\mathbb{C} \to \mathfrak{b}(1/2)$ and a $\mathbb{C}$-bilinear map $c' \colon \mathfrak{b}(1/2) \times \mathfrak{b}(1/2) \to \mathfrak{b}(1/2)$ by $Y_{\Phi',c'} = [Y_\Phi, X_{\mathcal{A},\mathcal{B}}]$. Then the following conditions hold:*

(i) $\Phi'(U) = -\Phi(\mathcal{A}U) + \mathcal{B}\Phi(U) \quad (U \in \mathfrak{b}(1)_\mathbb{C})$,

(ii) $c'(V, V) = \mathcal{B}c(V, V) - 2c(\mathcal{B}V, V) \quad (V \in \mathfrak{b}(1/2))$.

Lemma 4.6. *Let $Z_{a,b} \in \mathfrak{X}(1)$, and let $X_{\mathcal{A},\mathcal{B}} \in \mathfrak{X}(0)$. We define a $\mathbb{R}$-bilinear map $a' \colon \mathfrak{b}(1) \times \mathfrak{b}(1) \to \mathfrak{b}(1)$ and a $\mathbb{C}$-bilinear map $b' \colon \mathfrak{b}(1)_\mathbb{C} \times \mathfrak{b}(1/2) \to \mathfrak{b}(1/2)$ by $Z_{a',b'} = [Z_{a,b}, X_{\mathcal{A},\mathcal{B}}]$. Then the following conditions hold:*

(i) $a'(U, U) = \mathcal{A}a(U, U) - 2a(\mathcal{A}U, U) \quad (U \in \mathfrak{b}(1)_\mathbb{C})$,

(ii) $b'(U, V) = \mathcal{B}b(U, V) - b(\mathcal{A}U, V) - b(U, \mathcal{B}V) \quad (U \in \mathfrak{b}(1)_\mathbb{C}, V \in \mathfrak{b}(1/2))$.

For a subspace $\mathcal{W} \subset \mathfrak{aut}_{hol}(\mathcal{D}(\Omega, Q))$, let $\mathcal{W}^\# = \{X^\#; X \in \mathcal{W}\} \subset \mathfrak{X}$. Since

$$T^\# = X_{\mathrm{ad}(T)|_{\mathfrak{b}(1)_\mathbb{C}}, \mathrm{ad}(T)|_{\mathfrak{b}(1/2)}} \quad (T \in \mathfrak{b}(0)),$$

we have $\partial = (jE)^\# \in \mathfrak{b}^\#$. By (10), we also have

$$U^\# = \partial_U \quad (U \in \mathfrak{b}(1)), \quad V^\# = \tilde{\partial}_V \quad (V \in \mathfrak{b}(1/2)).$$

Note that there is a natural action of G on $\mathcal{D}(\Omega, Q)$ which is given by the transfer of the action of G on $\mathcal{D}$ by means of the biholomorphism $\mathcal{C}$. We also have the B-invariant metric on $\mathcal{D}(\Omega, Q)$ which is the transfer of the metric $\langle\langle \cdot, \cdot \rangle\rangle$ on $\mathcal{D}$.

To simplify the notation, we denote the metric on $\mathcal{D}(\Omega, Q)$ by the same symbol. We denote by ∇ the Levi-Civita connection on $(\mathcal{D}(\Omega, Q), \langle\langle\cdot,\cdot\rangle\rangle)$. We define a map $\tilde{\nabla}\colon \mathfrak{b}\times\mathfrak{b}\ni(X,Y)\mapsto\tilde{\nabla}_X Y\in\mathfrak{b}$ by

$$\left.\frac{d}{dt}\right|_{t=0}\exp(t\tilde{\nabla}_X Y)(iE,0) = (\nabla_{X^\#}Y^\#)(iE,0).$$

Then we have

$$-2\langle\tilde{\nabla}_X Y, Z\rangle = \langle[X,Y],Z\rangle - \langle[Z,X],Y\rangle - \langle X,[Z,Y]\rangle \quad (X,Y,Z\in\mathfrak{b}). \tag{20}$$

We see from the above equation that $\tilde{\nabla}_X Y = \tilde{\nabla}_Y X$ for all $X, Y \in \mathfrak{b}(1)$. Furthermore, recalling Remark 3.2(i), we obtain the following lemma by a mechanical calculation using (20).

Lemma 4.7. *Let $1\le k\le r$, and let $X\in\mathfrak{b}(1)$. Then* $\tilde{\nabla}_{E_k}X = j\mathrm{ad}(A_k)X$.

Lemma 4.8. *Let $\mathcal{A}\in\mathfrak{g}(\Omega)$, and let $1\le k\le l\le r$. Then we have*

$$\mathcal{A}(\mathfrak{b}_{(\alpha_l+\alpha_k)/2})\subset\bigoplus_{1\le m\le r}\mathfrak{b}_{(\alpha_k+\alpha_m)/2}+\bigoplus_{1\le m\le r}\mathfrak{b}_{(\alpha_l+\alpha_m)/2}.$$

Proof. First let us consider the case of $l=k$. We show that

$$\mathcal{A}E_k\in\bigoplus_{1\le m\le r}\mathfrak{b}_{(\alpha_k+\alpha_m)/2}. \tag{21}$$

The Lie subgroup of B with Lie algebra $\mathfrak{b}(0)\oplus\mathfrak{b}(1)$ is an Iwasawa subgroup of $\mathrm{Aut}_{hol}(\mathcal{D}(\Omega))$, and we consider the action of the subgroup on the tube domain $\mathcal{D}(\Omega)$ in this proof. We have

$$\mathfrak{g}(\Omega)=\mathfrak{g}_E(\Omega)\oplus\{\mathrm{ad}(X)|_{\mathfrak{b}(1)}; X\in\mathfrak{b}(0)\},$$

where $\mathfrak{g}_E(\Omega)$ is the Lie algebra of the Lie group $G_E(\Omega)=\{A\in G(\Omega); AE=E\}$. The result for $\mathcal{A}=\mathrm{ad}(X)|_{\mathfrak{b}(1)}$ with $X\in\mathfrak{b}(0)$ follows from (9). Let $\mathcal{A}\in\mathfrak{g}_E(\Omega)$. By (11), we have $[X_{\mathcal{A}}, E_k^\#]=-(\mathcal{A}E_k)^\#$. Let $\langle\langle\cdot,\cdot\rangle\rangle'$ be the Bergman metric on $\mathcal{D}(\Omega)$, and let ∇' denote the connection on $(\mathcal{D}(\Omega),\langle\langle\cdot,\cdot\rangle\rangle')$. Then we also have the map $\tilde{\nabla}'\colon(\mathfrak{b}(0)\oplus\mathfrak{b}(1))\times(\mathfrak{b}(0)\oplus\mathfrak{b}(1))\ni(X,Y)\mapsto\tilde{\nabla}'_X Y\in\mathfrak{b}(0)\oplus\mathfrak{b}(1)$ which is defined by

$$\left.\frac{d}{dt}\right|_{t=0}\exp(t\tilde{\nabla}'_X Y)iE=(\nabla'_{X^\#}Y^\#)(iE).$$

Let $1\le m\le r$ and $m\ne k$. Since $X_{\mathcal{A}}$ generates isometries of $\mathcal{D}(\Omega)$, we have

$$[X_{\mathcal{A}},\nabla'_{E_m^\#}E_k^\#]=-\nabla'_{(\mathcal{A}E_m)^\#}E_k^\#-\nabla'_{E_m^\#}(\mathcal{A}E_k)^\#. \tag{22}$$

By Lemma 4.7, we have $(\nabla'_{E_m^\#}E_k^\#)(iE)=0$. By looking at the value of (22) at $iE\in\mathcal{D}(\Omega)$, we have

$$0=\tilde{\nabla}'_{\mathcal{A}E_m}E_k+\tilde{\nabla}'_{E_m}\mathcal{A}E_k=\tilde{\nabla}'_{E_k}\mathcal{A}E_m+\tilde{\nabla}'_{E_m}\mathcal{A}E_k. \tag{23}$$

We remark that equation (23) can be seen from [3] and [4]. By Lemma 4.7, we have $[A_k,\mathcal{A}E_m]=-[A_m,\mathcal{A}E_k]\in\mathfrak{b}$. Thus we obtain $[A_m,\mathcal{A}E_k]\in\mathfrak{b}_{(\alpha_m+\alpha_k)/2}$ $(m\ne k)$, and (21) follows.

Next, let us consider the case of $k<l$. Let $\mathcal{A}\in\mathfrak{g}(\Omega)$. By (21), we have

$$\begin{aligned}\mathcal{A}(\mathfrak{b}_{(\alpha_l+\alpha_k)/2}) &\subset \sum_{X\in\mathfrak{b}_{(\alpha_l-\alpha_k)/2}} \mathcal{A}\,\mathrm{ad}(X)(\mathbb{R}E_k)\\ &\subset \sum_{\mathcal{A}'\in\mathfrak{g}(\Omega)} \mathcal{A}'(\mathbb{R}E_k)+\sum_{X\in\mathfrak{b}_{(\alpha_l-\alpha_k)/2}} \mathrm{ad}(X)\mathcal{A}(\mathbb{R}E_k)\\ &\subset \bigoplus_{1\le m\le r}\mathfrak{b}_{(\alpha_k+\alpha_m)/2}+\bigoplus_{1\le m\le r}\mathfrak{b}_{(\alpha_l+\alpha_m)/2},\end{aligned}$$

and the proof is complete. ■

Lemma 4.9. *Let $Y_\Phi\in\mathfrak{X}(1/2)$. Then the followings hold:*

(i) $\Phi(E_k)\in\mathfrak{b}_{\alpha_k/2}\quad(1\le k\le r)$,

(ii) $[A_k^{\#},[A_l^{\#},Y_\Phi]]=0\quad(1\le k\le r, 1\le l\le r, k\ne l)$.

Proof. (i) By Lemma 4.8, we have $\Phi(E_k)\in\mathfrak{b}_{\alpha_k/2}$ because

$$[V,\Phi(E_k)]=-4\Phi_V(E_k)\in\bigoplus_{1\le m\le r}\mathfrak{b}_{(\alpha_k+\alpha_m)/2}\quad(V\in\mathfrak{b}(1/2)).$$

(ii) By (i) and Lemma 4.5, it is straightforward to verify (ii). ■

5. Gradation and subalgebras of $\mathfrak{X}$

The purpose of this section is to extend the next lemma to certain subalgebras of $\mathfrak{X}$.

Lemma 5.1. *Let $\partial_{U_0}\in\mathfrak{X}(-1)$. If $[\partial_{U_0},\mathfrak{X}(1)]=\{0\}$, then we have $[\partial_{U_0},\mathfrak{X}(1/2)]=\{0\}$.*

Proof. Let $Y_\Phi\in\mathfrak{X}(1/2)$. Then $Y_{j\Phi}\in\mathfrak{X}(1/2)$ by Proposition 4.2. We put $Z_{a,b}=[Y_\Phi,Y_{j\Phi}]$. Then the equality

$$a(U,U)=4Q(\Phi(U),\Phi(U))\quad(U\in\mathfrak{b}(1))$$

holds (see [25, Chapter V, Lemma 2.5]). Then $0=[\partial_{U_0},Z_{a,b}]=2X_{\mathcal{A}_{U_0},\mathcal{B}_{U_0}}$ follows from (18). Hence

$$0=\mathcal{A}_{U_0}(U_0)=a(U_0,U_0)=4Q(\Phi(U_0),\Phi(U_0)).$$

By the Ω-positivity of Q, we get $\Phi(U_0)=0$. Using (16), we get

$$[\partial_{U_0},Y_\Phi]=\tilde{\partial}_{\Phi(U_0)}=0.$$ ■

In the above proof, the fact that $\mathfrak{X}(1/2)$ has a complex structure is crucial. We extend this result to certain subalgebras of $\mathfrak{X}$ in Proposition 5.8 below. Proposition 5.8 is also crucial to the proof of Proposition 5.10, which is the main result of this section. Let us consider the case $\dim\mathfrak{b}(1)=1$.

Proposition 5.2. *Assume that* $\dim\mathfrak{b}(1)=1$ *and that $\mathfrak{f}$ is a subalgebra of $\mathfrak{X}$ which contains $\mathfrak{X}(-1/2)$ and ∂. Then for any $Y_\Phi\in\mathfrak{f}$, we have $Y_{j\Phi}\in\mathfrak{f}$.*

Proof. Define $\mathbb{C}$-linear maps $\mathcal{A}\in\mathfrak{gl}(\mathfrak{b}(1)_{\mathbb{C}})$, $\mathcal{B}\in\mathfrak{gl}(\mathfrak{b}(1/2))$, and $\Phi'\colon\mathfrak{b}(1)_{\mathbb{C}}\to\mathfrak{b}(1/2)$:

$$X_{\mathcal{A},\mathcal{B}}=[\tilde{\partial}_{\Phi(E)},Y_\Phi],\quad Y_{\Phi'}=[Y_\Phi,X_{\mathcal{A},\mathcal{B}}].$$

By assumption, we have $Y_{\Phi'} \in \mathfrak{f}$. Since ψ_E is injective, it is enough to show that $\Phi'(E) = Cj\Phi(E)$ with some constant $C \in \mathbb{R}$. By (17), we have $\mathcal{A}E = 0$. Since $\mathfrak{b}(1) = \mathbb{R}E$, we can define a Hermitian form q on $\mathfrak{b}(1/2)$ by

$$Q(V, V') = q(V, V')E \quad (V, V' \in \mathfrak{b}(1/2)).$$

Put $V_0 = \Phi(E)$. By (Y2), we obtain $c(V_0, V_0) = 2q(V_0, V_0)jV_0$. Using (17), we are lead to $\mathcal{B}V_0 = 6q(V_0, V_0)jV_0$. Finally, thanks to Lemma 4.5 (i), we obtain $\Phi'(E) = 6q(V_0, V_0)j\Phi(E)$. ■

We shall extend Proposition 5.2 to the cases $\dim \mathfrak{b}(1) \geq 2$ by induction. The main idea is to decompose $\mathfrak{X}$ into subalgebras consist of complete holomorphic vector fields on complex domains of smaller dimensions. We assume that $r \geq 2$. We define a subalgebra $\check{\mathfrak{b}} \subset \mathfrak{b}$ by

$$\check{\mathfrak{b}} = \{X \in \mathfrak{b}; [X, A_1] = [X, E_1] = 0\}.$$

Put $\check{\mathfrak{b}}(\gamma) = \check{\mathfrak{b}} \cap \mathfrak{b}(\gamma)$ for $\gamma = 0, 1/2, 1$. Then $\check{\mathfrak{b}}$ is a normal j-algebra of rank $r - 1$.

Define
$$\check{\Omega} = \exp(\check{\mathfrak{b}}(0))(E_2 + \cdots + E_r),$$

$$\mathcal{D}(\check{\Omega}, \check{Q}) = \{(U, V) \in \check{\mathfrak{b}}(1)_{\mathbb{C}} \oplus \check{\mathfrak{b}}(1/2); \operatorname{Im} U - Q(V, V) \in \check{\Omega}\}.$$

According to Lemma 5.3 below, the following equality holds:

$$iE_1 + \mathcal{D}(\check{\Omega}, \check{Q}) = \mathcal{D}(\Omega, Q) \cap (iE_1 + \check{\mathfrak{b}}(1)_{\mathbb{C}} \oplus \check{\mathfrak{b}}(1/2)). \tag{24}$$

Lemma 5.3. $\check{\Omega} + E_1 = \Omega \cap (\check{\mathfrak{b}}(1) + E_1)$.

Proof. This follows from [11, Proposition 2.5]. ■

For a subalgebra $\mathfrak{f} \subset \mathfrak{X}$, we put

$$\mathfrak{f}(\gamma) = \{X \in \mathfrak{f}; [\partial, X] = \gamma X\} \ (\gamma \in \mathbb{R}), \text{ and } \check{\mathfrak{f}} = \{X \in \mathfrak{f}; [X, A_1^{\#}] = [X, E_1^{\#}] = 0\}.$$

Now we shall associate a complete holomorphic vector field $X \in \check{\mathfrak{X}}$ to a vector field on $\mathcal{D}(\check{\Omega}, \check{Q})$.

Lemma 5.4. (i) *Let* $Y_\Phi \in \check{\mathfrak{X}}(1/2)$. *Then*

$$Y_\Phi(iE_1 + U, V) \in \check{\mathfrak{b}}(1)_{\mathbb{C}} \oplus \check{\mathfrak{b}}(1/2) \quad (U \in \check{\mathfrak{b}}(1)_{\mathbb{C}}, V \in \check{\mathfrak{b}}(1/2)).$$

(ii) *An element* Y_Φ *of* $\mathfrak{X}(1/2)$ *belongs to* $\check{\mathfrak{X}}(1/2)$ *if and only if* $[Y_\Phi, A_1^{\#}] = 0$.

Proof. (i) By Lemma 4.5 (i), we have $\Phi([A_1, U]) = [A_1, \Phi(U)]$, $U \in \mathfrak{b}(1)_{\mathbb{C}}$, which implies

$$\Phi(E_1) = 0 \quad \text{and} \quad \Phi(U) \in \check{\mathfrak{b}}(1/2) \quad (U \in \check{\mathfrak{b}}(1)_{\mathbb{C}}). \tag{25}$$

However, from Lemma 4.5 (ii), it follows that $c(V, V) \in \check{\mathfrak{b}}(1/2)\,(V \in \check{\mathfrak{b}}(1/2))$, where c is given by $Y_\Phi = Y_{\Phi,c}$. Hence the result follows immediately from (14).

(ii) Let $Y_\Phi \in \mathfrak{X}(1/2)$, and suppose that $[Y_\Phi, A_1^{\#}] = 0$. Then we see from (16) and (25) that $[E_1^{\#}, Y_\Phi] = (\Phi(E_1))^{\#} = 0$. This proves (ii). ■

Lemma 5.5. *Let* $Z_{a,b} \in \check{\mathfrak{X}}(1)$. *Then*

$$Z_{a,b}(iE_1 + U, V) \in \check{\mathfrak{b}}(1)_{\mathbb{C}} \oplus \check{\mathfrak{b}}(1/2) \quad (U \in \check{\mathfrak{b}}(1)_{\mathbb{C}}, V \in \check{\mathfrak{b}}(1/2)).$$

Proof. By Lemma 4.6 (i), for $U \in \mathfrak{b}(1)_{\mathbb{C}}$ and $V \in \mathfrak{b}(1/2)$, we have

$$[A_1, a(U,U)] = 2a([A_1, U], U), \tag{26}$$

which implies $a(iE_1, iE_1) = 0$ and $a(U,U) \in \check{\mathfrak{b}}(1)_{\mathbb{C}} \quad (U \in \check{\mathfrak{b}}(1)_{\mathbb{C}})$. (27)

Let $U \in \check{\mathfrak{b}}(1)_{\mathbb{C}}$. From (26) and (27), it follows that

$$2[A_1, a(iE_1, U)] = [A_1, a(iE_1 + U, iE_1 + U)] = 2a(iE_1, U).$$

Thus $a(iE_1, U) \in (\mathfrak{b}_{\alpha_1})_{\mathbb{C}}$. By Lemma 4.8, we have also

$$a(E_1, U) = \mathcal{A}_{E_1}(U) \in \left(\sum_{2 \le l \le r, 1 \le k \le r} \mathfrak{b}_{(\alpha_l + \alpha_k)/2} \right)_{\mathbb{C}}.$$

Thus $a(iE_1, U) = 0$. On the other hand, from Lemma 4.6 (ii), it follows that

$$[A_1, b(U,V)] = b([A_1, U], V) + b(U, [A_1, V]) \quad (U \in \mathfrak{b}(1)_{\mathbb{C}}, V \in \mathfrak{b}(1/2)),$$

which implies $b(iE_1, V) = 0$ and $b(U,V) \in \check{\mathfrak{b}}(1/2)$ $(U \in \check{\mathfrak{b}}(1)_{\mathbb{C}},\ V \in \check{\mathfrak{b}}(1/2))$. Hence the result follows immediately from (15). ∎

Let $\mathfrak{X}_s$ be the space of complete holomorphic vector fields on $\mathcal{D}(\check{\Omega}, \check{Q})$, and let ∂_s be the element of $\mathfrak{X}_s$ given by $\partial_s(U,V) = (U, 1/2V)\,(U \in \check{\mathfrak{b}}(1)_{\mathbb{C}}, V \in \check{\mathfrak{b}}(1/2))$. Put $\mathfrak{X}_s(\gamma) = \{X \in \mathfrak{X}_s; [\partial_s, X] = \gamma X\}\,(\gamma \in \mathbb{R})$.

Lemma 5.6. *Let* $\gamma \in \{-1, -1/2, 0, 1/2, 1\}$, *and let* $X \in \check{\mathfrak{X}}(\gamma)$. *Then we have*

$$X|_{iE_1 + \check{\mathfrak{b}}(1)_{\mathbb{C}} \oplus \check{\mathfrak{b}}(1/2)} \in \check{\mathfrak{b}}(1)_{\mathbb{C}} \oplus \check{\mathfrak{b}}(1/2).$$

If we regard $\mathcal{D}(\check{\Omega}, \check{Q})$ *as a complex submanifold of* $\mathcal{D}(\Omega, Q)$ *by means of* (24), *then we have* $X|_{\mathcal{D}(\check{\Omega}, \check{Q})} \in \mathfrak{X}_s(\gamma)$.

Proof. The results for $\gamma = -1$ and $\gamma = -1/2$ follow from (11) and (12). Let $X_{\mathcal{A},\mathcal{B}} \in \mathfrak{X}(0)$, and suppose $[X_{\mathcal{A},\mathcal{B}}, A_1^{\#}] = 0$. Then we see from (13) that $\mathcal{A}U \in \check{\mathfrak{b}}(1)_{\mathbb{C}}, \mathcal{B}V \in \check{\mathfrak{b}}(1/2)$ for all $U \in \check{\mathfrak{b}}(1)_{\mathbb{C}}$ and $V \in \check{\mathfrak{b}}(1/2)$. Now suppose $[X_{\mathcal{A},\mathcal{B}}, E_1^{\#}] = 0$. Then $\mathcal{A}E_1 = 0$ by (11). We have

$$X_{\mathcal{A},\mathcal{B}}(iE_1 + U, V) = (\mathcal{A}U, \mathcal{B}V) \quad (U \in \check{\mathfrak{b}}(1)_{\mathbb{C}}, V \in \check{\mathfrak{b}}(1/2)),$$

which shows the assertion for $\gamma = 0$. Let $Y_\Phi \in \check{\mathfrak{X}}(1/2)$. By Lemma 5.4 and (24), it follows that $Y_\Phi|_{\mathcal{D}(\check{\Omega}, \check{Q})} \in \mathfrak{X}_s(1/2)$. This proves the assertion for $\gamma = 1/2$. By Lemma 5.5 and the same argument as above, we obtain the result for $\gamma = 1$. ∎

Remark 5.7. By Lemma 5.6, for any $X \in \check{\mathfrak{X}}$, we have $X|_{\mathcal{D}(\check{\Omega}, \check{Q})} \in \mathfrak{X}_s$, and the map

$$\check{\mathfrak{X}} \ni X \mapsto X|_{\mathcal{D}(\check{\Omega}, \check{Q})} \in \mathfrak{X}_s$$

defines a Lie algebra homomorphism. Since ψ_E is injective and since we have $(iE, 0) \in iE_1 + \check{\mathfrak{b}}(1)_{\mathbb{C}} \oplus \check{\mathfrak{b}}(1/2)$, the map $\check{\mathfrak{X}}(1/2) \ni Y_\Phi \mapsto Y_\Phi|_{\mathcal{D}(\check{\Omega}, \check{Q})} \in \mathfrak{X}_s(1/2)$ is injective. ∎

Proposition 5.8. *Let $\mathfrak{f} \subset \mathfrak{X}$ be a subalgebra which contains $\mathfrak{X}(-1/2)$ and $\mathfrak{a}^{\#}$. Then for any $Y_\Phi \in \mathfrak{f}$, one has $Y_{j\Phi} \in \mathfrak{f}$.*

Proof. We show the assertion by induction on $r \geq 1$. For the case $r = 1$, we have shown the assertion in Proposition 5.2. Let $r \geq 2$. We define a $\mathbb{C}$-linear map $\Phi' \colon \mathfrak{b}(1)_{\mathbb{C}} \to \mathfrak{b}(1/2)$ by $Y_{\Phi'} = [Y_\Phi, A_1^{\#}]$. By Lemma 4.9 (ii), we have $[Y_{\Phi'}, A_k^{\#}] = 0$ $(2 \leq k \leq r)$. Thus

$$[Y_{\Phi+2\Phi'}, A_1^{\#}] = Y_{\Phi'} + [Y_{2\Phi'}, \partial] = 0.$$

From Lemma 5.4, it follows that $Y_{\Phi+2\Phi'} \in \check{\mathfrak{f}}$. We denote by R the Lie algebra homomorphism in Remark 5.7. Now we can apply the inductive hypothesis to the domain $\mathcal{D}(\check{\Omega}, \check{Q})$ and the subalgebra $R(\check{\mathfrak{f}}) \subset \mathfrak{X}_s$. Then we see from Remark 5.7 that $Y_{j(\Phi+2\Phi')} \in \check{\mathfrak{f}}$. Thus in order to show that $Y_{j\Phi} \in \mathfrak{f}$, it suffices to show that

$$Y_{j\Phi'} \in \mathfrak{f}. \tag{28}$$

Next we prove (28). Let $c' \colon \mathfrak{b}(1/2) \times \mathfrak{b}(1/2) \to \mathfrak{b}(1/2)$ be the $\mathbb{C}$-bilinear map such that $Y_{\Phi'} = Y_{\Phi',c'}$. We define $\mathbb{C}$-linear maps $\Phi'' \colon \mathfrak{b}(1)_{\mathbb{C}} \to \mathfrak{b}(1/2)$, $\mathcal{A} \in \mathfrak{gl}(\mathfrak{b}(1)_{\mathbb{C}})$, and $\mathcal{B} \in \mathfrak{gl}(\mathfrak{b}(1/2))$ by

$$X_{\mathcal{A},\mathcal{B}} = [\tilde{\partial}_{\Phi'(E)}, Y_{\Phi'}], \text{ and } Y_{\Phi''} = [Y_{\Phi'}, X_{\mathcal{A},\mathcal{B}}].$$

By assumption $Y_{\Phi''} \in \mathfrak{f}$. We define a Hermitian form q on $\mathfrak{b}_{\alpha_1/2}$ by

$$Q(V, V') = q(V, V')E_1 \quad (V, V' \in \mathfrak{b}_{\alpha_1/2}).$$

Let $V_0 = \Phi'(E)$. By the same argument as for Proposition 5.2, it follows that $\Phi''(E) = 6q(V_0, V_0)j\Phi'(E)$, and (28) holds (see Remark 5.9). ■

Remark 5.9. In the above proof, we have $V_0 \in \mathfrak{b}_{\alpha_1/2}$ by Lemma 4.5 (i) and Lemma 4.9 (i). Hence $c'(V_0, V_0) \in \mathfrak{b}_{\alpha_1/2}$ by (Y2). ■

Proposition 5.10. *Let $\mathfrak{f} \subset \mathfrak{X}$ be a subalgebra which contains $\mathfrak{X}(-1/2)$ and $\mathfrak{a}^{\#}$, and let $\partial_{U_0} \in \mathfrak{X}(-1)$. If $[\partial_{U_0}, \mathfrak{f}(1)] = 0$, then one has $[\partial_{U_0}, \mathfrak{f}(1/2)] = 0$.*

Proof. For $\gamma = 1/2, 1$, we replace $\mathfrak{X}(\gamma)$ in the proof of Lemma 5.1 by $\mathfrak{f}(\gamma)$. To complete the proof, it is enough to show that $Y_{j\Phi} \in \mathfrak{f}(1/2)$. Hence the result follows from Proposition 5.8. ■

6. Isotropy representation

From now on, we assume that $\langle\langle \cdot, \cdot \rangle\rangle$ is the Bergman metric on $\mathcal{D}$, and let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathfrak{b}$ given by (8). We consider the Hermitian inner product $(\cdot, \cdot)_{\mathfrak{b}}$ on $\mathfrak{b}$ such that $\operatorname{Re}(X, Y)_{\mathfrak{b}} = \langle X, Y \rangle$ for all $X, Y \in \mathfrak{b}$, and we regard $\mathfrak{b}$ as a complex Hilbert space. Recall that G is a covering group of a connected Lie group which acts holomorphically, transitively, and effectively on $\mathcal{D}(\Omega, Q)$. Let π denote the covering homomorphism. Let us now think of the action of $\pi(G)$ on $\mathcal{D}(\Omega, Q)$. Identifying $T_{(iE,0)}(\mathcal{D}(\Omega, Q))$ with $\mathfrak{b}$ via the correspondence $T_{(iE,0)}(\mathcal{D}(\Omega, Q)) \ni X^{\#}(iE, 0) \leftrightarrow X \in \mathfrak{b}$, we consider the isotropy representation $\rho \colon \pi(K) \to GL(\mathfrak{b})$ at $(iE, 0)$. Then ρ is an unitary representation of $\pi(K)$ on the Hilbert space $\mathfrak{b}$. Let $1 \colon \pi(K) \to \mathbb{C}^{\times}$ be the trivial representation, and let $\mathfrak{b}_1$ be the 1-component of $\mathfrak{b}$. Then we have $\mathfrak{b}_1 = \{X \in \mathfrak{b}; [X, \mathfrak{k}] \subset \mathfrak{k}\}$.

Lemma 6.1. *For any* $X \in \mathfrak{b}_1$, *we have* $[X, \mathfrak{k}] = \{0\}$.

Proof. Let $X \in \mathfrak{b}_1$. The linear map $\mathrm{ad}(X)\colon \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues (see [22, p. 161]). On the other hand, we have $\mathrm{ad}(X)([\mathfrak{k},\mathfrak{k}]) \subset [\mathfrak{k},\mathfrak{k}]$, and $\mathrm{ad}(X)|_{[\mathfrak{k},\mathfrak{k}]}$ has only pure imaginary eigenvalues and is diagonalizable. Hence $\mathrm{ad}(X)|_{[\mathfrak{k},\mathfrak{k}]} = 0$. Let $Z_{\mathfrak{k}}$ be the center of $\mathfrak{k}$, and let $Z^0_{\pi(K)}$ be the Lie subgroup of $\pi(K)$ with Lie algebra $Z_{\mathfrak{k}}$. Since the inner automorphisms of $\pi(G)$ defined by e^{tX} $(t \in \mathbb{R})$ induce isomorphisms of the Lie group $Z^0_{\pi(K)}$, which is isomorphic to a torus, it follows that $\mathrm{ad}(X)|_{Z_{\mathfrak{k}}} = 0$. This completes the proof. ■

Put $\mathfrak{f} = \mathfrak{g}^{\#}$. For $\gamma \in \{-1, -1/2, 0, 1/2, 1\}$, let $\mathfrak{g}(\gamma) \subset \mathfrak{g}$ be the subspace given by $\mathfrak{g}(\gamma)^{\#} = \mathfrak{f}(\gamma)$. Then we have $\mathfrak{g}(\gamma) = \{X \in \mathfrak{g}; \mathrm{ad}(jE)X = -\gamma X\}$ and also $\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(-1/2) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1/2) \oplus \mathfrak{g}(1)$. Note that

$$\mathfrak{b}(1) = \mathfrak{g}(-1), \quad \mathfrak{b}(1/2) = \mathfrak{g}(-1/2), \quad \mathfrak{b}(0) \subset \mathfrak{g}(0).$$

Let $\mathfrak{n}, \mathfrak{n}' \subset \mathfrak{g}$ be the subspaces given by $\mathfrak{n}^{\#} = \mathfrak{m}$ and $\mathfrak{n}'^{\#} = \mathfrak{m}'$. Then we have

$$(\mathfrak{n} \cap \mathfrak{g})^{\#} = \mathfrak{m} \cap \mathfrak{f} = \{X + \psi_E(X); X \in \mathfrak{f}(1/2)\}$$

and

$$(\mathfrak{n}' \cap \mathfrak{g})^{\#} = \mathfrak{m}' \cap \mathfrak{f} = \{X + \varphi_E(X); X \in \mathfrak{f}(1)\}.$$

By Theorem 4.4, we also have $\mathfrak{k} = (\mathfrak{k} \cap \mathfrak{g}(0)) \oplus (\mathfrak{n} \cap \mathfrak{g}) \oplus (\mathfrak{n}' \cap \mathfrak{g})$. From now on, for $X \in \mathfrak{g}$ and $\gamma \in \{-1, -1/2, 0, 1/2, 1\}$, let X_γ denote the projection of X on $\mathfrak{g}(\gamma)$.

Proposition 6.2. *The subalgebra* $\mathfrak{b}_1 \subset \mathfrak{b}$ *is* $\mathrm{ad}(jE)$*-invariant.*

Proof. Let $X \in \mathfrak{b}_1$. We see from Lemma 6.1 that $[Y, X]_\gamma = 0$ for all $Y \in \mathfrak{k}$ and $\gamma \in \{-1, -1/2, 0, 1/2, 1\}$.
For $Y \in \mathfrak{k} \cap \mathfrak{g}(0)$, we have $\mathrm{ad}(Y)\mathrm{ad}(jE)(X) = \frac{1}{2}[Y, X]_{-1/2} + [Y, X]_{-1} = 0$.
For $Y \in \mathfrak{n}' \cap \mathfrak{g}$, we have $\mathrm{ad}(Y)\mathrm{ad}(jE)(X) = \frac{1}{2}[Y, X]_{1/2} + [Y, X]_0 = 0$.
Since $[Y, X]_0 = [Y_1, X_{-1}]$, we see from Proposition 5.10 that $[Y', X_{-1}] = 0$ where $Y' \in \mathfrak{g}(1/2)$. Thus, for $Y \in \mathfrak{n} \cap \mathfrak{g}$, we have

$$\mathrm{ad}(Y)\mathrm{ad}(jE)(X) = \frac{1}{2}[Y, X]_{-1} + \frac{1}{2}[Y, X]_0 + [Y_{1/2}, X_{-1}] = 0.$$

This completes the proof. ■

Let $\mathfrak{b}_1^{\perp}$ be the orthogonal complement of $\mathfrak{b}_1$ in $\mathfrak{b}$ relative to $(\cdot,\cdot)_{\mathfrak{b}}$.

Proposition 6.3. *The subspace* $\mathfrak{b}_1^{\perp} \subset \mathfrak{b}$ *is* $\mathrm{ad}(jE)$*-invariant.*

Proof. By Remark 3.2(i), for $X \in \mathfrak{b}_1^{\perp}$ and $Y \in \mathfrak{b}_1$, we have

$$\langle \mathrm{ad}(jE)X, Y\rangle = \langle X, \mathrm{ad}(jE)Y\rangle,$$

which is equal to 0 by Proposition 6.2. This implies that the assertion holds. ■

7. Actions of K on G-equivariant holomorphic line bundles

Let $\mathfrak{g}_- \subset \mathfrak{g}_{\mathbb{C}}$ be the complex subalgebra defined by

$$\mathfrak{g}_- = \left\{ Z = X + iY \in \mathfrak{g}_{\mathbb{C}}; \left.\frac{d}{dt}\right|_{t=0} e^{tX}p + i\left.\frac{d}{dt}\right|_{t=0} e^{tY}p \in T^{0,1}_p\mathcal{D} \right\}, \tag{29}$$

where $T^{0,1}_p\mathcal{D}$ denotes the antiholomorphic tangent vector space at p.

Set $\mathfrak{b}_- = \mathfrak{g}_- \cap \mathfrak{b}_{\mathbb{C}}$.

Proposition 7.1 (Rossi and Vergne, [24, Proposition 4.21]). *Let $\tau\colon \mathfrak{b} \to \mathfrak{b}_-$ be the $\mathbb{R}$-linear map defined by*

$$\tau(U+V+T) = (V+ijV)/2 + T + ijT \quad (U \in \mathfrak{b}(1), V \in \mathfrak{b}(1/2), T \in \mathfrak{b}(0)).$$

Then τ is a Lie algebra homomorphism, and if τ is extended to a $\mathbb{C}$-linear map $\tau\colon \mathfrak{b}_{\mathbb{C}} \to \mathfrak{b}_{\mathbb{C}}$, then we have $\tau|_{\mathfrak{b}_-} = \mathrm{id}_{\mathfrak{b}_-}$.

For a complex representation $\theta\colon \mathfrak{g}_- \to \mathbb{C}$, let χ^θ be the function on B defined by

$$\chi^\theta(\exp X) = e^{\theta\circ\tau(X)} \quad (X \in \mathfrak{b}). \tag{30}$$

Theorem 7.2. *Let $\theta\colon \mathfrak{g}_- \to \mathbb{C}$ be a complex representation, and suppose that $\theta(\mathfrak{k}) = 0$. Extend the representation $d\chi^\theta\colon \mathfrak{b} \to \mathbb{C}$ of $\mathfrak{b}$ to a linear map $d\chi^\theta\colon \mathfrak{g} \to \mathbb{C}$ by the zero-extension along with the decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{k}$. Then $d\chi^\theta\colon \mathfrak{g} \to \mathbb{C}$ defines a representation of $\mathfrak{g}$.*

Proof. For $\zeta \in \mathbb{C}$ and $W \in \mathfrak{k}$, we put $\mathfrak{b}(W,\zeta) = \{X \in \mathfrak{b}; d\rho(W)X = \zeta X\}$. For $\gamma \in \mathbb{R}\backslash\{0\}$, $W \in \mathfrak{k}$, and $X \in \mathfrak{b}(W, i\gamma)$, we have

$$0 = \theta([W, X+ijX]) = \theta(d\rho(W)X + id\rho(W)(jX)) = -\gamma i\theta(X+ijX).$$

Thus $\theta(X+ijX) = 0$. Since $\mathfrak{b}_1^\perp = \sum_{W\in\mathfrak{k},\gamma\in\mathbb{R}\backslash\{0\}} \mathfrak{b}(W, i\gamma)$, we obtain

$$\theta(X+ijX) = 0 \quad (X \in \mathfrak{b}_1^\perp).$$

For $X \in \mathfrak{b}_1^\perp$, we have $X_{-1/2}, X_0 \in \mathfrak{b}_1^\perp$ by Proposition 6.3. Thus

$$d\chi^\theta(X) = \theta(\tau(X)) = \theta((X_{-1/2} + ijX_{-1/2})/2 + X_0 + ijX_0) = 0 \quad (X \in \mathfrak{b}_1^\perp).$$

We see from the above equality that $d\chi^\theta([\mathfrak{b},\mathfrak{k}]) = 0$. Now let $X, X' \in \mathfrak{g}$, and write $X = Y+W, X' = Y'+W'$ with $Y, Y' \in \mathfrak{b}$ and $W, W' \in \mathfrak{k}$. Then we have

$$d\chi^\theta([X,X']) = d\chi^\theta([Y,Y']) = [d\chi^\theta(Y), d\chi^\theta(Y')] = [d\chi^\theta(X), d\chi^\theta(X')].$$

This completes the proof. ∎

Lemma 7.3 ([13, Lemma 1]). *Let $\mathcal{D}_0$ be a domain in $\mathbb{C}^N$, and let G_0 be a Lie group which acts holomorphically on $\mathcal{D}_0$. Let $m, m'\colon G_0 \times \mathcal{D}_0 \to \mathbb{C}^\times$ be holomorphic multipliers. Then L_m and $L_{m'}$ are isomorphic as G_0-equivariant holomorphic line bundles if and only if there exists a holomorphic function $f\colon \mathcal{D}_0 \to \mathbb{C}^\times$ such that*

$$m'(g,z) = f(gz)m(g,z)f(z)^{-1} \quad (g \in G_0, z \in \mathcal{D}_0). \tag{31}$$

Definition 7.4. We say that two holomorphic multipliers $m, m'\colon G_0 \times \mathcal{D}_0 \to \mathbb{C}^\times$ are G_0-*equivalent* if they satisfy (31) with some holomorphic function f.

For a holomorphic multiplier $M\colon G \times \mathcal{D}(\Omega, Q) \to \mathbb{C}^\times$, let $\theta_M\colon \mathfrak{g}_- \to \mathbb{C}$ be the complex representation of $\mathfrak{g}_-$ given by

$$\theta_M(Z) = \frac{d}{dt}\bigg|_{t=0} M(e^{tX}, (iE,0)) + i\,\frac{d}{dt}\bigg|_{t=0} M(e^{tY}, (iE,0)) \quad (Z = X+iY \in \mathfrak{g}_-).$$

Theorem 7.5 (Ishi, [13, Theorem 12]). *Let $M\colon G \times \mathcal{D}(\Omega, Q) \to \mathbb{C}^\times$ be a holomorphic multiplier. Then the holomorphic multiplier*

$$B \times \mathcal{D}(\Omega, Q) \ni (b, (U, V)) \mapsto \chi^{\theta_M}(b) \in \mathbb{C}^\times$$

is B-equivalent to M.

Lemma 7.6. *For a holomorphic multiplier $M\colon G \times \mathcal{D}(\Omega, Q) \to \mathbb{C}^\times$, there exists a unique holomorphic multiplier $M_0\colon G \times \mathcal{D}(\Omega, Q) \to \mathbb{C}^\times$ which is G-equivalent to M and satisfies the following conditions:*

$$M_0(k, (iE, 0)) = M(k, (iE, 0)) \quad (k \in K), \tag{32}$$

$$M_0(b, (U, V)) = \chi^{\theta_M}(b) \quad (b \in B). \tag{33}$$

Proof. By Lemma 7.3 and Theorem 7.5, there exists a holomorphic function $f\colon \mathcal{D}(\Omega, Q) \to \mathbb{C}^\times$ such that the equality

$$\chi^{\theta_M}(b) = f(b(U, V))M(b, (U, V))f(U, V)^{-1} \quad (b \in B, (U, V) \in \mathcal{D}(\Omega, Q))$$

holds. We define a holomorphic multiplier $M_0\colon G \times \mathcal{D}(\Omega, Q) \to \mathbb{C}^\times$ by

$$M_0(g, (U, V)) = f(g(U, V))M(g, (U, V))f(U, V)^{-1} \quad (g \in G, (U, V) \in \mathcal{D}(\Omega, Q)).$$

It is immediate that M satisfies the conditions. ■

From now on, for a holomorphic multiplier $M\colon G \times \mathcal{D}(\Omega, Q) \to \mathbb{C}^\times$, let M_0 denote the holomorphic multiplier which satisfies all conditions in the previous lemma.

Theorem 7.7. *Let $M, M'\colon G \times \mathcal{D}(\Omega, Q) \to \mathbb{C}^\times$ be holomorphic multipliers. Suppose that $M(k, (iE, 0)) = M'(k, (iE, 0))$ for all $k \in K$. Then $M_0(k, (U, V)) = M_0'(k, (U, V))$ for all $k \in K$ and $(U, V) \in \mathcal{D}(\Omega, Q)$.*

Proof. By Theorem 7.2, the representation $d\chi^{\theta_M - \theta_{M'}}\colon \mathfrak{b} \to \mathbb{C}$ of $\mathfrak{b}$ extends to a representation of $\mathfrak{g}$, and this representation lifts to a group representation $\chi\colon G \to \mathbb{C}^\times$. Now χ defines a holomorphic multiplier $\chi\colon G \times \mathcal{D}(\Omega, Q) \to \mathbb{C}^\times$, and we see from (32) and (33) that

$$M_0 M_0'^{-1} \chi^{-1}(k, (iE, 0)) = 1 \quad (k \in K),$$

and
$$M_0 M_0'^{-1} \chi^{-1}(b, (U, V)) = 1 \quad (b \in B, (U, V) \in \mathcal{D}(\Omega, Q)).$$

By the same arguments in Lemma 2.4, we have

$$M_0 M_0'^{-1} \chi^{-1}(g, (U, V)) = 1 \quad (g \in G, (U, V) \in \mathcal{D}(\Omega, Q)).$$

This proves the result. ■

Under the assumption of Theorem 7.7, we see that the identity map $\mathrm{id}\colon L_{M_0} \to L_{M_0'}$ is K-equivariant. Thus we obtain the following corollary.

Corollary 7.8. *Let L and L' be G-equivariant holomorphic line bundles over $\mathcal{D}$. Suppose that the actions of K on the fibers of L and L' over the point p coincide. Then L and L' are isomorphic as K-equivariant holomorphic line bundles.*

8. Unitary equivalences among representations of B

In this section, we first associate a holomorphic multiplier $M\colon G\times\mathcal{D}(\Omega,Q)\to\mathbb{C}^\times$ to a linear form on $\mathfrak{g}$. Next we review the construction of an intertwining operator between two unitarizations of holomorphic multiplier representations of B studied in [10, 12] in a form convenient to us by using linear forms on $\mathfrak{g}$.

We shall deal with unitarizable representations T_M with holomorphic multipliers $M\colon G\times\mathcal{D}(\Omega,Q)\to\mathbb{C}^\times$. Given such a representation let us consider the reproducing kernel of the unitarization $(T_M,\mathcal{H})$. Let $\mathcal{K}\colon\mathcal{D}(\Omega,Q)\times\mathcal{D}(\Omega,Q)\to\mathbb{C}$ be the reproducing kernel of $\mathcal{H}$. Put $\mathcal{K}_{(iE,0)}=\mathcal{K}(\cdot,(iE,0))\in\mathcal{H}$. For $g\in G$ and $f\in\mathcal{H}$, we have

$$(f,T_M(g)\mathcal{K}_{(iE,0)})_{\mathcal{H}}=(T_M(g^{-1})f,\mathcal{K}_{(iE,0)})_{\mathcal{H}}=T_M(g^{-1})f((iE,0)).$$

The right hand side of the above equation is a C^ω-function of $g\in G$. Hence $\mathcal{K}_{(iE,0)}$ is a C^ω-vector of the representation $(T_M,\mathcal{H})$. By the definition of the holomorphic multiplier, if the representation dT_M of $\mathfrak{g}$ is extended to a complex representation, then we have

$$dT_M(\overline{Z})\mathcal{K}_{(iE,0)}=-\overline{\theta_M(Z)}\mathcal{K}_{(iE,0)}\quad(Z\in\mathfrak{g}_-),\tag{34}$$

where $\overline{X+iY}=X-iY$ for $X,Y\in\mathfrak{g}$. Let $J\colon\mathcal{H}^\infty\ni f\mapsto J_f\in\mathfrak{g}^*$ be the moment map of $(T_M,\mathcal{H})$ defined by

$$\langle X,J_f\rangle=\frac{1}{i}\frac{(dT_M(X)f,f)_{\mathcal{H}}}{(f,f)_{\mathcal{H}}}\quad(X\in\mathfrak{g}).$$

We put $\xi=J_{\mathcal{K}(\cdot,(iE,0))}\in\mathfrak{g}^*$. Then by (34), if we extend ξ to a complex-linear form on $\mathfrak{g}_{\mathbb{C}}$, we have

$$\theta_M(Z)=i\xi(Z)\quad(Z\in\mathfrak{g}_-).$$

From now on, a one-dimensional complex representation of $\mathfrak{g}_-$ which is given by the restriction of $i\xi\,(\xi\in\mathfrak{g}^*)$ will be written as $i\xi$ in this section and the next. For such a $\xi\in\mathfrak{g}^*$ with $T_{\chi^{i\xi}}$ unitarizable, let $(T_{\chi^{i\xi}},\mathcal{H}_\xi)$ denote the unitarization of the representation $T_{\chi^{i\xi}}$ of B, and let $\mathcal{K}^\xi$ be the reproducing kernel of $\mathcal{H}_\xi$. Let us now review the construction of an intertwining operator between such unitary representations $(T_{\chi^{i\xi}},\mathcal{H}_\xi)$ and $(T_{\chi^{i\xi'}},\mathcal{H}_{\xi'})$ of B. The results presented in this section will be used in the next section to determine the image of $\mathcal{K}^\xi$ under the intertwining operator. Define an action of $B(0)$ on $\mathfrak{b}(1)^*$ by

$$\langle U,t_0\ell\rangle=\langle t_0^{-1}U,\ell\rangle\quad(U\in\mathfrak{b}(1),t_0\in B(0),\ell\in\mathfrak{b}(1)^*).$$

Theorem 8.1 (Ishi, [10]). *There exist a unique $B(0)$-orbit $\mathcal{O}_\xi^*\subset\mathfrak{b}(1)^*$ and a unique measure $d\nu_\xi$ on $\mathcal{O}_\xi^*$ such that*

$$d\nu_\xi(t_0\ell)=|\chi^{i\xi}(t_0)|^2d\nu_\xi(\ell)\quad(\ell\in\mathcal{O}_\xi^*,t_0\in B(0)),$$

$$\int_{\mathcal{O}_\xi^*}e^{-\langle U,\ell\rangle}\,d\nu_\xi(\ell)<\infty\ \textit{for all}\ U\in\Omega.$$

If $T_{\chi^{i\xi}}$ and $T_{\chi^{i\xi'}}$ define equivalent unitarizations, then $\mathcal{O}_\xi^*=\mathcal{O}_{\xi'}^*$.

In [10], $\mathcal{O}^*_\xi$ and $d\nu_\xi$ are written as $\mathcal{O}^*_\varepsilon$ and $d\mathcal{R}^*_{\mathrm{Re}\, s^*}$, respectively. The dual cone $\Omega^* \subset \mathfrak{b}(1)^*$ of Ω is defined by

$$\Omega^* = \{\ell \in \mathfrak{b}(1)^*; \langle U, \ell\rangle > 0 \text{ for all } U \in \overline{\Omega}\backslash\{0\}\}.$$

For $\ell \in \overline{\Omega^*}$, let Q_ℓ be the Hermitian form on $\mathfrak{b}(1/2)$ given by

$$Q_\ell(V, V') = \langle 2Q(V, V'), \ell\rangle \quad (V, V' \in \mathfrak{b}(1/2)),$$

where we extend ℓ to a complex-linear form on $\mathfrak{b}(1)_\mathbb{C}$. Then Q_ℓ is positive definite. For $\ell \in \overline{\Omega^*}$, let
$$N_\ell = \{V \in \mathfrak{b}(1/2); Q_\ell(V, V) = 0\},$$

and let $\mathcal{F}_\ell$ be the space of holomorphic functions F on $\mathfrak{b}(1/2)$ such that

(i) $F(V + V') = F(V)$ for all $V \in \mathfrak{b}(1/2)$ and $V' \in N_\ell$,

(ii) $\|F\|^2_{\mathcal{F}_\ell} = \int_{\mathfrak{b}(1/2)/N_\ell} |F(V)|^2 e^{-Q_\ell(V,V)}\, d\mu_\ell([V]) < \infty$,

where $[V] = V + N_\ell\, (V \in \mathfrak{b}(1/2))$ and $d\mu_\ell$ denotes the Lebesgue measure on $\mathfrak{b}(1/2)/N_\ell$ normalized in such a way that

$$\int_{\mathfrak{b}(1/2)/N_\ell} e^{-Q_\ell(V,V)}\, d\mu_\ell([V]) = 1.$$

Let $\mathcal{L}_\xi$ be the function space consists of all equivalence classes of measurable functions f on $\mathcal{O}^*_\xi \times \mathfrak{b}(1/2)$ such that

(i) $f(\ell, \cdot) \in \mathcal{F}_\ell$ for almost all $\ell \in \mathcal{O}^*_\xi$ with respect to the measure $d\nu_\xi$,

(ii) $\|f\|^2_{\mathcal{L}_\xi} = \int_{\mathcal{O}^*_\xi} \|f(\ell, \cdot)\|^2_{\mathcal{F}_\ell} d\nu_\xi(\ell) < \infty$.

Theorem 8.2 (Ishi, [10, Theorem 4.10])**.** *The map $\phi_\xi \colon \mathcal{L}_\xi \to \mathcal{H}_\xi$ defined by*

$$\phi_\xi f(U, V) = \int_{\mathcal{O}^*_\xi} e^{i\langle U,\ell\rangle} f(\ell, V)\, d\nu_\xi(\ell) \quad ((U, V) \in \mathcal{D}(\Omega, Q))$$

gives a Hilbert space isomorphism.

We define a unitary representation $\check{T}_{\chi^{i\xi}}$ of B on $\mathcal{L}_\xi$ by

$$\phi_\xi \check{T}_{\chi^{i\xi}}(b) f = T_{\chi^{i\xi}}(b)\phi_\xi f \quad (b \in B, f \in \mathcal{L}_\xi).$$

For $(U_0, V_0) \in \mathcal{D}(\Omega, Q)$, let

$$k^\xi_{(U_0,V_0)}(\ell, V) = e^{-i\langle \overline{U_0}, \ell\rangle} e^{Q_\ell(V,V_0)} \quad ((\ell, V) \in \mathcal{O}^*_\xi \times \mathfrak{b}(1/2)).$$

Then $k^\xi_{(U_0,V_0)} \in \mathcal{L}_\xi$, and we have the following equalities (see [10, p. 450]):

$$\phi_\xi f(U, V) = (f | k^\xi_{(U,V)})_{\mathcal{L}_\xi} \quad ((U, V) \in \mathcal{D}(\Omega, Q), f \in \mathcal{L}_\xi), \tag{35}$$

$$(k^\xi_{(U',V')} | k^\xi_{(U,V)})_{\mathcal{L}_\xi} = \mathcal{K}^\xi((U, V), (U', V')) \quad ((U, V), (U', V') \in \mathcal{D}(\Omega, Q)). \tag{36}$$

From now on, we assume that two unitarizations of $T_{\chi^{i\xi}}$ and $T_{\chi^{i\xi'}}$ are equivalent as unitary representations of B. As in [12, p. 541], we fix a function $\Upsilon \neq 0$ on $\mathcal{O}^*_\xi$, which is also a function on $\mathcal{O}^*_{\xi'}$, such that

$$\Upsilon(t_0\ell) = \overline{\chi^{i\xi}(t_0)\chi^{-i\xi'}(t_0)}\Upsilon(\ell) \quad (t_0 \in B(0), \ell \in \mathcal{O}^*_\xi).$$

Proposition 8.3 (Ishi, [12, Proposition 4.5]). *There exists a nonzero constant C such that the following map $\check{\Psi}_{\xi,\xi'}\colon \mathcal{L}_\xi \to \mathcal{L}_{\xi'}$ gives the intertwining operator between the unitary representations $(\check{T}_{\chi^{i\xi}}, \mathcal{L}_\xi)$ and $(\check{T}_{\chi^{i\xi'}}, \mathcal{L}_{\xi'})$ of B:*

$$\check{\Psi}_{\xi,\xi'} f(\ell, V) = C\Upsilon(\ell) f(\ell,\, V) \quad (f \in \mathcal{L}_\xi, (\ell, V) \in \mathcal{O}_\xi^* \times \mathfrak{b}(1/2)).$$

Let Δ_ξ and $\Delta_{\xi,\xi'}$ be the functions on Ω defined by

$$\Delta_\xi(t_0 E) = |\chi^{i\xi}(t_0)|^2, \quad \Delta_{\xi,\xi'}(t_0 E) = \overline{\chi^{i\xi}(t_0)\chi^{-i\xi'}(t_0)} \quad (t_0 \in B(0)).$$

Proposition 8.4 (Ishi, [10, Corollary 2.5 and Proposition 4.6]). *The two functions Δ_ξ and $\Delta_{\xi,\xi'}$ extend to holomorphic functions on $\Omega + i\mathfrak{b}(1)$. For $(U, V), (U', V') \in \mathcal{D}(\Omega, Q)$ we have*

$$\mathcal{K}^\xi((U, V), (U', V')) = \Delta_\xi\left(\frac{U - \overline{U'}}{i} - 2Q(V, V')\right).$$

9. Unitary equivalences among representations of G

In this section, we study unitary equivalences among unitarizations of holomorphic multiplier representations by using Theorem 7.7 and the intertwining operator between two irreducible unitary representations of B. Suppose that holomorphic multipliers $M, M'\colon G \times \mathcal{D}(\Omega, Q) \to \mathbb{C}^\times$ satisfy

$$\theta_M(Z) = i\xi(Z), \quad \theta_{M'}(Z) = i\xi'(Z) \quad (Z \in \mathfrak{g}_-),$$

where ξ and ξ' are linear forms on $\mathfrak{g}$ in the previous section. Now the unitarizations $(T_{M_0}, \mathcal{H}_\xi)$ and $(T_{M_0'}, \mathcal{H}_{\xi'})$ are equivalent as unitary representations of B. By Schur's lemma and the decomposition $G = BK$, the unitarizations are equivalent as unitary representations of G if and only if the intertwining operator between representations $(T_{M_0}|_B, \mathcal{H}_\xi)$ and $(T_{M_0'}|_B, \mathcal{H}_{\xi'})$ preserves the actions of K. From this point of view, we get the equation (38) in Proposition 9.3 below, which determines whether the unitarizations are equivalent as unitary representations of G. The equation (38) together with the results in Section 7 yields Theorem 9.5, which gives an answer to the question (Q2).

Let $\Psi_{\xi,\xi'} = \phi_{\xi'}\check{\Psi}_{\xi,\xi'}\phi_\xi^{-1}$.

Lemma 9.1. *There exists a nonzero constant C' such that*

$$\Psi_{\xi,\xi'}\mathcal{K}^\xi_{(iE,0)}(U, V) = C'\mathcal{K}^{\xi'}_{(iE,0)}(U, V)\Delta_{\xi,\xi'}\left(\frac{U - \overline{iE}}{i}\right) \quad ((U, V) \in \mathcal{D}(\Omega, Q)).$$

Proof. Put $\mathcal{K}' = \Psi_{\xi,\xi'}\mathcal{K}^\xi_{(iE,0)}$. By (35) and (36), we have $\mathcal{K}' = \phi_{\xi'}\check{\Psi}_{\xi,\xi'}k^\xi_{(iE,0)}$. Hence we have

$$\mathcal{K}'(U, V) = C\int_{\mathcal{O}^*_{\xi'}} e^{i\langle U - \overline{iE}, \ell\rangle}\Upsilon(\ell)\, d\nu_{\xi'}(\ell) \quad ((U, V) \in \mathcal{D}(\Omega, Q)).$$

For $(iU_0, V) \in \mathcal{D}(\Omega, Q)$ with $U_0 \in \Omega$, it follows that $U_0 + E \in \Omega$ from $U_0 - Q(V, V) \in \Omega$ and $Q(V, V) \in \overline{\Omega}$ (see Remark 9.2). Thus there exists $t_0 \in B(0)$ such that $t_0 E = U_0 + E$.

Then
$$\mathcal{K}'(iU_0, V) = C\int_{\mathcal{O}^*_{\xi'}} e^{-\langle t_0E,\ell\rangle}\Upsilon(\ell)\,d\nu_{\xi'}(\ell)$$
$$= C|\chi^{i\xi'}(t_0)|^2\overline{\chi^{i\xi}(t_0)\chi^{-i\xi'}(t_0)}\int_{\mathcal{O}^*_{\xi'}} e^{-\langle E,\ell\rangle}\Upsilon(\ell)\,d\nu_{\xi'}(\ell)$$
$$= C'\Delta_{\xi'}\left(\frac{iU_0+iE}{i}\right)\Delta_{\xi,\xi'}\left(\frac{iU_0+iE}{i}\right),$$

where we put $C' = C\int_{\mathcal{O}^*_{\xi'}} e^{-\langle E,\ell\rangle}\Upsilon(\ell)\,d\nu_{\xi'}(\ell)$. By the analytic continuation and Proposition 8.4, we obtain the results. ∎

Remark 9.2. Let Int: $P(\mathfrak{b}(1)) \to P(\mathfrak{b}(1))$ denote the interior operator. Then we have $\{U+U'; U\in\Omega \text{ and } U'\in\overline{\Omega}\}\subset \mathrm{Int}(\overline{\Omega}) = \mathrm{Int}(\Omega) = \Omega$, where the first equality holds because $\Omega\subset\mathfrak{b}(1)$ is a convex set. ∎

If the unitarizations $(T_{M_0}, \mathcal{H}_\xi)$ and $(T_{M_0'}, \mathcal{H}_{\xi'})$ are equivalent as unitary representations of G, then the equality

$$T_{M_0'}(k)\Psi_{\xi,\xi'}\mathcal{K}^{\xi}_{(iE,0)} = \Psi_{\xi,\xi'}T_{M_0}(k)\mathcal{K}^{\xi}_{(iE,0)} \quad (k\in K) \tag{37}$$

holds. The converse is also true as we shall see in the next proposition. In what follows, we put $(U(g), V(g)) = g(U,V)$ for $g\in G$ and $(U,V)\in\mathcal{D}(\Omega,Q)$.

Proposition 9.3. *The following are equivalent:*

(i) *the unitarizations of T_M and $T_{M'}$ are equivalent as unitary representations of the group G,*

(ii) (37) *holds,*

(iii) *the following equality holds for $k\in K$, $(U,V)\in\mathcal{D}(\Omega,Q)$:*

$$\Delta_{\xi,\xi'}\left(\frac{U(k^{-1})-\overline{iE}}{i}\right) = MM'^{-1}(k,(iE,0))\Delta_{\xi,\xi'}\left(\frac{U-\overline{iE}}{i}\right) \tag{38}$$

Proof. First we show that (i) and (ii) are equivalent. Thanks to the remark preceding Proposition 9.3, it is enough to show that (ii) implies (i). We suppose that (37) holds. By the decomposition $G = BK = KB$, we have

$$\Psi_{\xi,\xi'}T_{M_0}(k)T_{M_0}(b)\mathcal{K}^{\xi}_{(iE,0)} = T_{M_0'}(k)\Psi_{\xi,\xi'}T_{M_0}(b)\mathcal{K}^{\xi}_{(iE,0)} \quad (b\in B, k\in K).$$

Now $\Psi_{\xi,\xi'}$ is continuous, and the subspace of $\mathcal{H}_\xi$ generated by $T_{M_0}(b)\mathcal{K}^{\xi}_{(iE,0)}$ $(b\in B)$ is dense in $\mathcal{H}_\xi$. Thus

$$\Psi_{\xi,\xi'}T_{M_0}(k)f = T_{M_0'}(k)\Psi_{\xi,\xi'}f \quad (k\in K, f\in\mathcal{H}_\xi),$$

which implies $(T_{M_0}, \mathcal{H}_\xi)$ and $(T_{M_0'}, \mathcal{H}_{\xi'})$ are equivalent as unitary representations of G. Thus (i) follows. Next we show that (ii) and (iii) are equivalent. By Lemma 9.1 and (2), for $k\in K$ and $(U,V)\in\mathcal{D}(\Omega,Q)$, we have

$$C'^{-1}(T_{M_0'}(k)\Psi_{\xi,\xi'}\mathcal{K}^{\xi}_{(iE,0)})(U,V) = M'(k,(iE,0))\mathcal{K}^{\xi'}_{(iE,0)}(U,V)\Delta_{\xi,\xi'}\left(\frac{U(k^{-1})-\overline{iE}}{i}\right).$$

We also have

$$C'^{-1}(\Psi_{\xi,\xi'}T_{M_0}(k)\mathcal{K}^{\xi}_{(iE,0)})(U,V) = C'^{-1}M(k,(iE,0))(\Psi_{\xi,\xi'}\mathcal{K}^{\xi}_{(iE,0)})(U,V)$$
$$= M(k,(iE,0))\mathcal{K}^{\xi'}_{(iE,0)}(U,V)\Delta_{\xi,\xi'}\left(\frac{U-\overline{iE}}{i}\right).$$

Thus (37) holds if and only if (38) holds. ■

We see one formula on the function $\Delta_{\xi,\xi'}$. We have

$$\Delta_{\xi,\xi'}\left(\tfrac{U(b)-\overline{U'(b)}}{i} - 2Q(V(b),V'(b))\right) = \overline{\chi^{i\xi-i\xi'}(b)}\Delta_{\xi,\xi'}\left(\tfrac{U-\overline{U'}}{i} - 2Q(V,V')\right) \tag{39}$$

where $b\in B,\ (U,V),(U',V')\in\mathcal{D}(\Omega,Q)$.

Proposition 9.4. *The unitarizations of T_M and $T_{M'}$ are equivalent as unitary representations of G if and only if*

$$M(k,(iE,0)) = M'(k,(iE,0)) \quad (k\in K).$$

Proof. First we show the 'only if' part. Putting $(U,V)=(iE,0)$ in (38), we obtain $M(k,(iE,0)) = M'(k,(iE,0))$ for all $k\in K$. Second we show the 'if' part. Suppose that $M(k,(iE,0)) = M'(k,(iE,0))$ for all $k\in K$. Then (39) gives

$$\Delta_{\xi,\xi'}\left(\tfrac{U(b)-\overline{U(b)}}{i} - 2Q(V(b),V(b))\right) = \overline{\chi^{i\xi-i\xi'}(b)}\Delta_{\xi,\xi'}\left(\tfrac{U-\overline{U}}{i} - 2Q(V,V)\right)$$
$$= \overline{M_0M_0'^{-1}(b,(U,V))}\Delta_{\xi,\xi'}\left(\tfrac{U-\overline{U}}{i} - 2Q(V,V)\right) \quad (b\in B,(U,V)\in\mathcal{D}(\Omega,Q)).$$

Then Lemma 2.4 and Theorem 7.7 show that

$$\Delta_{\xi,\xi'}\left(\tfrac{U(k)-\overline{U(k)}}{i} - 2Q(V(k),V(k))\right) = \overline{M_0M_0'^{-1}(k,(U,V))}\Delta_{\xi,\xi'}\left(\tfrac{U-\overline{U}}{i} - 2Q(V,V)\right)$$
$$= \Delta_{\xi,\xi'}\left(\tfrac{U-\overline{U}}{i} - 2Q(V,V)\right) \quad (k\in K,(U,V)\in\mathcal{D}(\Omega,Q)).$$

By the analytic continuation, we have for $k\in K,\ (U,V),(U',V')\in\mathcal{D}(\Omega,Q)$

$$\Delta_{\xi,\xi'}\left(\tfrac{U(k)-\overline{U'(k)}}{i} - 2Q(V(k),V'(k))\right) = \Delta_{\xi,\xi'}\left(\tfrac{U-\overline{U'}}{i} - 2Q(V,V')\right).$$

Putting $(U',V')=(iE,0)$ in the above equation, we obtain

$$\Delta_{\xi,\xi'}\left(\frac{U(k)-\overline{iE}}{i}\right) = \Delta_{\xi,\xi'}\left(\frac{U-\overline{iE}}{i}\right) \quad (k\in K,(U,V)\in\mathcal{D}(\Omega,Q)).$$

Thus we get equation (38), and hence the unitarizations of T_M and $T_{M'}$ are equivalent as unitary representations of G by Proposition 9.3. The proof is complete. ■

Proposition 9.4 together with the fact that $\mathcal{C}\colon\mathcal{D}\to\mathcal{D}(\Omega,Q)$ is a G-equivariant biholomorphism yields the following theorem.

Theorem 9.5. *Let $m,m'\colon G\times\mathcal{D}\to\mathbb{C}^\times$ be holomorphic multipliers. Suppose that there exist Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ of holomorphic functions on $\mathcal{D}$ which give the unitarizations of T_m and $T_{m'}$, respectively. Then the following two conditions are equivalent:*

(i) *$(T_m,\mathcal{H})$ and $(T_{m'},\mathcal{H}')$ are equivalent as unitary representations of G.*

(ii) *$(T_m,\mathcal{H})$ and $(T_{m'},\mathcal{H}')$ are equivalent as unitary representations of B, and $m(k,p)=m'(k,p)$ for all $k\in K$.*

10. Application to a certain non-symmetric domain

In this section, we see an application of Theorems 2.5 and 9.5. For a complex domain and a Lie group let us describe the set of unitary equivalence classes of the unitarizations of all holomorphic multiplier representations.

First we associate a holomorphic multiplier $m\colon G \times \mathcal{D} \to \mathbb{C}^\times$ to a one-dimensional complex representation $\theta_m\colon \mathfrak{g}_- \to \mathbb{C}$ by putting

$$\theta_m(Z) = \frac{d}{dt}\bigg|_{t=0} m(e^{tX}, p) + i\,\frac{d}{dt}\bigg|_{t=0} m(e^{tY}, p) \quad (Z = X + iY \in \mathfrak{g}_-).$$

Theorem 10.1 ([26, Theorem 3.6]). *The above correspondence induces a bijection from the set of* (G-)*equivalence classes of holomorphic multipliers to the set of one-dimensional complex representations of* $\mathfrak{g}_-$ *whose restrictions to* $\mathfrak{k}$ *lift to representations of* K.

By Lemma 7.3 and the above theorem, we obtain an algebraic description of G-equivariant holomorphic line bundles over $\mathcal{D}$, and we shall use this description for a classification of unitarizations.

Next let us deal with a particular complex domain. Put

$$\mathcal{U} = \left\{ x = \begin{bmatrix} x^1 & 0 & x^4 \\ 0 & x^2 & x^5 \\ x^4 & x^5 & x^3 \end{bmatrix}; x^1, \cdots, x^5 \in \mathbb{R} \right\},$$

and let $\Omega_5 = \mathcal{U} \cap \mathcal{P}(3,\mathbb{R})$, where $\mathcal{P}(3,\mathbb{R})$ denotes the homogeneous convex cone consists of all 3-by-3 real positive-definite symmetric matrices. We consider the following homogeneous domain $\mathcal{D}_5$ in $\mathcal{U}_\mathbb{C}$:

$$\mathcal{D}_5 = \mathcal{U} \oplus i\Omega_5.$$

Let $G = \mathrm{Aut}_{hol}(\mathcal{D}_5)^0$. To simplify the notation, we write $z = (z^1, z^2, z^3, z^4, z^5) \in \mathcal{U}_\mathbb{C}$ instead of

$$z = \begin{bmatrix} z^1 & 0 & z^4 \\ 0 & z^2 & z^5 \\ z^4 & z^5 & z^3 \end{bmatrix} \in \mathcal{U}_\mathbb{C}.$$

We define linear endomorphisms $a_1, a_2, a_3, a_{3,1}, a_{3,2}, e_1, e_2, e_3, e_{3,1}, e_{3,2}, w_1, w_2$ of $\mathcal{U}_\mathbb{C}$ by

$$\begin{aligned} a_1(z) &= (z^1, 0, 0, z^4/2, 0), \quad a_2(z) = (0, z^2, 0, 0, z^5/2), \\ a_3(z) &= (0, 0, z^3, z^4/2, z^5/2), \quad a_{3,1}(z) = (0, 0, 2z^4, z^1, 0), \\ a_{3,2}(z) &= (0, 0, 2z^5, 0, z^2), \quad e_1(z) = (1, 0, 0, 0, 0), \\ e_2(z) &= (0, 1, 0, 0, 0), \quad e_3(z) = (0, 0, 1, 0, 0), \\ e_{3,1}(z) &= (0, 0, 0, 1, 0), \quad e_{3,2}(z) = (0, 0, 0, 0, 1), \\ w_1(z) &= (-(z^1)^2 - 1, 0, -(z^4)^2, -z^1 z^4, 0), \\ w_2(z) &= (0, -(z^2)^2 - 1, -(z^5)^2, 0, -z^2 z^5), \end{aligned} \tag{40}$$

and we regard these maps as holomorphic vector fields on $\mathcal{D}_5$. Then by Geatti [6], holomorphic vector fields in (40) span the space $\mathfrak{X}$ of complete holomorphic vector fields on $\mathcal{D}_5$.

Let us consider the one-one map $\mathfrak{g} \ni X \mapsto X^{\#} \in \mathfrak{X}$, and let A_1, A_2, A_3, $A_{3,1}$, $A_{3,2}$, E_1, E_2, E_3, $E_{3,1}$, $E_{3,2}$, W_1, and W_2 be the elements of $\mathfrak{g}$ whose images under the map are a_1, a_2, a_3, $a_{3,1}$, $a_{3,2}$, e_1, e_2, e_3, $e_{3,1}$, $e_{3,2}$, w_1, and w_2, respectively. If we denote the Lie subgroup of G corresponding to the Lie algebra $\mathfrak{b} = \langle A_1, A_2, A_3, A_{3,1}, A_{3,2}, E_1, E_2, E_3, E_{3,1}, E_{3,2} \rangle$ by B, then B is an Iwasawa subgroup of G. Let $p = (i, i, i, 0, 0) \in \mathcal{D}_5$, let K be the isotropy subgroup of G at p, and let $\mathfrak{g}_- \subset \mathfrak{g}_{\mathbb{C}}$ be the complex subalgebra given by (29). For a basis $\{X_\lambda\}$ of $\mathfrak{g}$, we shall denote the dual basis by $\{X_\lambda^*\}$. Then every $\xi \in \mathfrak{g}^*$ satisfying $\xi([\mathfrak{g}_-, \mathfrak{g}_-]) = 0$ can be written as

$$\xi = \xi(\xi_3, \eta_3, n, n') = \xi_3 E_3^* + \eta_3 A_3^* + \frac{n}{2}(2W_1^* - E_1^*) + \frac{n'}{2}(2W_2^* - E_2^*),$$

where $\xi_3, \eta_3, n, n' \in \mathbb{R}$. If the representation $i\xi|_{\mathfrak{k}} \colon \mathfrak{k} \to \mathbb{C}$ lifts to a representation of K, then $n, n' \in \mathbb{Z}$. Put

$$\Theta^G = \left\{ \xi \in \mathfrak{g}^*;\ \begin{array}{c} \xi = \xi(\xi_3, \eta_3, n, n') \text{ with } \xi_3, \eta_3 \in \mathbb{R} \text{ and } n, n' \in \mathbb{Z}, \text{ and for a} \\ \text{holomorphic multiplier } m \colon G \times \mathcal{D}_5 \to \mathbb{C}^\times \text{ corresponding} \\ \text{to } i\xi|_{\mathfrak{g}_-}, \text{ the representation } T_m \text{ of } G \text{ is unitarizable} \end{array} \right\}.$$

Let us consider the isomorphism classes of G-equivariant holomorphic line bundles L over $\mathcal{D}_5$, and we denote the isomorphism class of L by $[L]$. Then the set Θ^G parametrizes the following set:

$$\left\{ [L];\ \begin{array}{c} L \text{ is a } G\text{-equivariant holomorphic line bundle over } \mathcal{D}_5 \text{ such} \\ \text{that the representation } \tau_L \text{ of } G \text{ is unitarizable} \end{array} \right\}.$$

By Theorem 2.5, the set Θ^G is also characterized by the unitarizabilities of $T_m|_B$. On the other hand, Ishi [13, Theorem 13] gives a classification of unitarizations of holomorphic multiplier representations of B, leading to the following: Put

$$\begin{aligned} \Theta_1^G(\eta_3) &= \{\xi(0, \eta_3, 0, 0)\} \quad (\eta_3 \in \mathbb{R}), \\ \Theta_2^G(\eta_3) &= \{\xi(0, \eta_3, n, 0); n \in \mathbb{Z}_{>0}\} \quad (\eta_3 \in \mathbb{R}), \\ \Theta_3^G(\eta_3) &= \{\xi(0, \eta_3, 0, n'); n' \in \mathbb{Z}_{>0}\} \quad (\eta_3 \in \mathbb{R}), \\ \Theta_4^G(\eta_3) &= \{\xi(0, \eta_3, n, n'); n, n' \in \mathbb{Z}_{>0}\} \quad (\eta_3 \in \mathbb{R}), \\ \Theta_5^G &= \{\xi(\xi_3, \eta_3, n, n'); \xi_3 < 0, \eta_3 \in \mathbb{R}, n, n' \in \mathbb{Z}_{>0}\}. \end{aligned} \tag{41}$$

Then the set Θ^G is the disjoint union of all the sets in (41). For elements ξ and ξ' of Θ^G, let m and m' be holomorphic multipliers which correspond to $i\xi|_{\mathfrak{g}_-}$ and $i\xi'|_{\mathfrak{g}_-}$, respectively. Then the unitarizations of T_m and $T_{m'}$ are equivalent as unitary representations of B if and only if ξ and ξ' belong to the same set in (41).

Finally we shall describe the set of unitary equivalence classes of the unitarizations as representations of G. By Theorem 9.5, we obtain the following: Put

$$\begin{aligned} \Theta_6^G(n, n') &= \{\xi(\xi_3, \eta_3, n, n'); \xi_3 < 0, \eta_3 \in \mathbb{R}\} \quad (n, n' \in \mathbb{Z}_{>0}), \\ \Theta_7^G(\eta_3, n, n') &= \{\xi(0, \eta_3, n, n')\} \quad (\eta_3 \in \mathbb{R}, n, n' \in \mathbb{Z}_{\geq 0}). \end{aligned} \tag{42}$$

Then the unitarizations of T_m and $T_{m'}$ are equivalent as unitary representations of G if and only if ξ and ξ' belong to the same set in (42).

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