

## On the Polynomial Realization of the Reduced $W$ -Algebra $U_\chi(\mathfrak{sl}_2, e)$

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**Abstract.** We study the structure of the reduced  $W$ -algebra  $U_\chi(\mathfrak{sl}_2, e)$  with  $e$  being regular nilpotent, over an algebraically closed field  $k$  of characteristic  $p > 2$ . As a consequence, we give a description of the center of the reduced enveloping algebra  $U_\chi(\mathfrak{sl}_2)$ .

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*Key Words:* Center, regular nilpotent, reduced  $W$ -algebra, reduced enveloping algebra.

### 1. Introduction

**1.1.** A finite  $W$ -algebra  $U(\mathfrak{g}_\mathbb{C}, \hat{e})$  is a certain associative algebra associated to a complex semisimple Lie algebra  $\mathfrak{g}_\mathbb{C}$  and a nilpotent element  $\hat{e} \in \mathfrak{g}_\mathbb{C}$ . The study of finite  $W$ -algebras can be traced back to Kostant's work, in which he considered the case when  $\hat{e}$  is a principal nilpotent element [9]. Kostant's results were generalized to arbitrary even nilpotent elements by Lynch [13]. Premet developed the finite  $W$ -algebras in full generality in [16]. On the basis of his proof of the celebrated Kac-Weisfeiler conjecture for Lie algebras of reductive groups in [15], Premet first constructed a modular version of finite  $W$ -algebras  $U_\chi(\mathfrak{g}, e)$  in [16] (they will be called reduced  $W$ -algebras in this paper). By means of a complicated but natural "admissible" procedure, the finite  $W$ -algebras over the field of complex numbers were introduced, arising from the modular version, which showed that they are filtrated deformations of the coordinate rings of Slodowy slices. The most important ingredient there is the construction of the PBW basis of finite  $W$ -algebras (cf. [16, §4]).

For finite  $W$ -algebras over the field of complex numbers, Gan and Ginzburg [4] reproved some of Premet's results, and recast them in the light of noncommutative algebraic geometry. Since then, finite  $W$ -algebras theory becomes a very active area, and the survey papers [1, 10, 19] have appeared. In particular, Brundan-Kleshchev showed that the finite  $W$ -algebras can be realized as shifted Yangians for the type  $A$  case in [3]. The results of [2, 12] are concerned with the finite-dimensional representations of finite  $W$ -algebras.

**1.2.** Premet proved in [17] that under the assumption  $p \gg 0$  for the positive characteristic field  $k = \overline{\mathbb{F}}_p$ , if the  $\mathbb{C}$ -algebra  $U(\mathfrak{g}_\mathbb{C}, \hat{e})$  has a one-dimensional repre-

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sentation (which has been proved by Losev [11], Goodwin-Röhrle-Ubly [5] and finally by Premet [18]), then the reduced enveloping algebra  $U_\chi(\mathfrak{g})$  of the modular counterpart  $\mathfrak{g}$  of  $\mathfrak{g}_\mathbb{C}$  possesses a simple module whose dimension is exactly the lower bound predicted by Kac-Weisfeiler conjecture mentioned above.

The PBW basis of reduced  $W$ -algebra  $U_\chi(\mathfrak{g}, e)$  is given in [16]. In the same paper, Premet showed that there exists mutually inverse equivalence between the representation category of the reduced enveloping algebras of Lie algebras and its reduced  $W$ -algebra counterparts over the positive characteristic field  $k$ .

In a recent work of Goodwin-Topley [6], the authors generalized Premet’s work to the case with  $p := \text{char}(k)$  and  $G$  satisfying Jantzen’s standard hypotheses on the corresponding reductive algebraic group  $G$  of  $\mathfrak{g}$  (see [7, §6.3]).

**1.3.** In this paper, we address the case where  $\mathfrak{g} = \mathfrak{sl}_2$  and  $\chi \in \mathfrak{g}^*$  being regular nilpotent. We first classify all the isomorphism class of simple  $U_\chi(\mathfrak{g}, e)$ -modules, which implies the following:

**Theorem 1.1.** *Let  $\mathfrak{g} = \mathfrak{sl}_2$  with  $\chi \in \mathfrak{g}^*$  being regular nilpotent, and let  $U_\chi(\mathfrak{g}, e)$  be the corresponding reduced  $W$ -algebra. Setting  $\theta := h^2 - 2h + 4e \in U_\chi(\mathfrak{g}, e)$  we have*

$$U_\chi(\mathfrak{g}, e) = k[\theta]/I(\theta),$$

where  $I(\theta)$  is the ideal of  $k[\theta]$  generated by  $\theta^p + a_1\theta^{p-1} + a_2\theta^{p-2} + \dots + a_{p-1}\theta + a_p$ . All the coefficients  $a_i \in \mathbb{F}_p$  for  $1 \leq i \leq p$  are determined by the equation

$$\theta^p + a_1\theta^{p-1} + a_2\theta^{p-2} + \dots + a_{p-1}\theta + a_p = \prod_{i=0}^{p-1}(\theta - (i^2 - 1)). \tag{1}$$

In conjunction with Theorem 1.1 we give the polynomial realization of the center  $Z_\chi(\mathfrak{g})$  of reduced enveloping algebra  $U_\chi(\mathfrak{g})$ , i.e.,

**Theorem 1.2.** *Let  $\mathfrak{g} = \mathfrak{sl}_2$  with  $\chi \in \mathfrak{g}^*$  being regular nilpotent, and let  $U_\chi(\mathfrak{g})$  be the corresponding reduced enveloping algebra of  $\mathfrak{g}$ . Write  $c := h^2 - 2h + 4ef$ , then we have*

$$Z_\chi(\mathfrak{g}) = k[c]/I(c),$$

where  $I(c)$  is the ideal of  $k[c]$  generated by  $c^p + a_1c^{p-1} + a_2c^{p-2} + \dots + a_{p-1}c + a_p$ . All the coefficients  $a_i \in \mathbb{F}_p$  for  $1 \leq i \leq p$  are determined by the equation

$$c^p + a_1c^{p-1} + a_2c^{p-2} + \dots + a_{p-1}c + a_p = \prod_{i=0}^{p-1}(c - (i^2 - 1)). \tag{2}$$

In the concluding section, we will discuss the coefficients  $a_i$  for  $1 \leq i \leq p$  in (1).

**Proposition 1.3.** *The coefficients in (1) are determined as follows:*

- (1)  $a_1 = a_2 = \dots = a_{\frac{p-3}{2}} = a_{p-1} = a_p = 0$ .
- (2) For  $\frac{p-1}{2} \leq i \leq p-2$ , we have  $a_i = 2(-1)^{i-\frac{p-3}{2}} \binom{i-1}{\frac{p-3}{2}}$ .

**1.4.** This article is organized as follows. In Section 2 we set the stage by recalling the notations and the definition of reduced  $W$ -algebras. The PBW basis of reduced  $W$ -algebras, and the mutually inverse category equivalences between the  $U_\chi(\mathfrak{g}, e)$ -modules and  $U_\chi(\mathfrak{g})$ -modules are also recalled. In Section 3, after the discussion of

various aspects on the irreducible representations of  $U_\chi(\mathfrak{g})$  and  $U_\chi(\mathfrak{g}, e)$ , the proof of Theorem 1.1 and Theorem 1.2 will be given. In Section 4, we will prove the Proposition above with the aid of related knowledge of basic number theory.

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## 2. Preliminaries

**2.1.** Let  $p$  be a prime number and  $k := \overline{\mathbb{F}}_p$  be an algebraically closed field of characteristic  $p$ . Let  $G$  be a linear algebraic group over  $k$ , and write  $\mathfrak{g} = \text{Lie}(G)$  for the Lie algebra of  $G$ . We write  $G^\circ$  for the identity component of  $G$ , and the derived subgroup of  $G$  is denoted by  $\mathcal{D}G$ .

Since  $\mathfrak{g}$  is the Lie algebra of an algebraic group, then it is a restricted Lie algebra in a natural way, and we write  $x \mapsto x^{[p]}$  for the  $p$ -map. The elements  $x^p - x^{[p]}$  for  $x \in \mathfrak{g}$  are central in the enveloping algebra  $U(\mathfrak{g})$  and the algebra  $Z_p(\mathfrak{g})$  that they generate is known as the  $p$ -center of  $U(\mathfrak{g})$ . It follows from the PBW theorem of  $U(\mathfrak{g})$  that  $Z_p(\mathfrak{g})$  is isomorphic to a polynomial algebra in  $\dim \mathfrak{g}$  variables.

Throughout the rest of the paper, we let  $G$  be a connected reductive algebraic group over  $k$ . Also we assume that  $G$  satisfies the standard hypotheses in [7, §6.3]:

- (H1) the derived group  $\mathcal{D}G$  of  $G$  is simply connected;
- (H2) the prime  $p$  is good for  $G$ ;
- (H3) there is a non-degenerate symmetric  $G$ -invariant bilinear form on  $\mathfrak{g}$ .

In the case of  $G$  being simple, those hypotheses hold for  $G$  under the following conditions: for type  $A_n$  we require  $p$  does not divide  $n + 1$ ; for types  $B_n, C_n$  or  $D_n$  we require  $p \neq 2$ ; for types  $G_2, F_4, E_6$  or  $E_7$  we require  $p \neq 2, 3$ ; for type  $E_8$  we require  $p \neq 2, 3, 5$  (see [7, §6.4]).

Write  $Z(\mathfrak{g})$  for the center of the universal enveloping algebra  $U(\mathfrak{g})$ . It is well-known that  $Z(\mathfrak{g})$  is generated by two distinguished subalgebras: the invariants  $U(\mathfrak{g})^G$  of the adjoint  $G$ -action and  $Z_p(\mathfrak{g})$ , the  $p$ -center of  $U(\mathfrak{g})$  (see e.g. [14, Theorem 1]).

For every maximal ideal  $H$  of  $Z_p(\mathfrak{g})$  there is a unique linear function  $\eta = \eta_H \in \mathfrak{g}^*$  such that

$$H = \langle x^p - x^{[p]} - \eta(x)^p \mid x \in \mathfrak{g} \rangle.$$

For any  $\xi \in \mathfrak{g}^*$  we denote by  $J_\xi$  the two-sided ideal of  $U(\mathfrak{g})$  generated by the central elements  $\{x^p - x^{[p]} - \xi(x)^p \mid x \in \mathfrak{g}\}$ . The quotient algebra  $U_\xi(\mathfrak{g}) := U(\mathfrak{g})/J_\xi$  is called the reduced enveloping algebra with  $p$ -character  $\xi$ .

Given  $\xi \in \mathfrak{g}^*$  we denote by  $Z_\xi(\mathfrak{g})$  the image of  $Z(\mathfrak{g})$  under the canonical homomorphism  $U(\mathfrak{g}) \rightarrow U_\xi(\mathfrak{g})$ . Obviously,  $Z_\xi(\mathfrak{g})$  is a subalgebra of the center  $Z(U_\xi(\mathfrak{g}))$  of  $U_\xi(\mathfrak{g})$ . In the case with  $\xi$  being regular nilpotent, it follows from [14, Theorem 12] that  $Z_\xi(\mathfrak{g})$  coincides with  $Z(U_\xi(\mathfrak{g}))$ .

**2.2.** Let  $g \in G$  and  $x \in \mathfrak{g}$ . We write  $g \cdot x$  for the image of  $x$  under  $g$  in the adjoint action,  $G^x$  for the centralizer of  $x$  in  $G$  and  $\mathfrak{g}^x$  for the centralizer of  $x$  in  $\mathfrak{g}$ .

We can fix a bilinear form as in (H3) and denote it by  $(\cdot, \cdot)$ . The existence of this form has two important consequences:

- (i) there exists an isomorphism of  $G$ -modules  $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ ;
- (ii)  $\mathfrak{g}^x = \text{Lie}(G^x)$  for all  $x \in \mathfrak{g}$ , see [8, §2.5].

Define  $G^{[x]} := \{g \in G \mid g \cdot x \in kx\}$ . We use similar notation when considering the coadjoint action of  $G$  on  $\mathfrak{g}^*$ .

Let  $T$  be a maximal torus in  $G$  and let  $\Phi$  be the root system of  $G$  with respect to  $T$ . Set  $\mathfrak{h} = \text{Lie}(T)$ . Given  $\alpha \in \Phi$ , we write  $\mathfrak{g}_\alpha$  for the root subspace of  $\alpha$  in  $\mathfrak{g}$ . We write  $X^*(T)$  for the group of characters of  $T$  and  $X_*(T)$  for the group of cocharacters of  $T$ . Let  $e$  be a nilpotent element. We can further assume that  $(T^{[e]})^\circ$  is a maximal torus of  $G^{[e]}$ . Write  $T_e := (T^e \cap \mathcal{D}G)^\circ$  and  $T_{[e]} := (T^{[e]} \cap \mathcal{D}G)^\circ$ .

Recall that a cocharacter  $\lambda : k^\times \rightarrow G$  is associated to  $e$  if  $\lambda(t) \cdot e = t^2e$  and  $\lambda(k^\times)$  is contained in the derived subgroup of a Levi subgroup in which  $e$  is distinguished (see [8, Definition 5.3]), and all associated cocharacters are conjugate under  $(G^e)^\circ$ , the identity component of the stabilizer of  $e$  in  $G$  (see [8, Lemma 5.3]).

Let  $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j; \lambda)$  be the Dynkin grading of  $\mathfrak{g}$ , where

$$\mathfrak{g}(j; \lambda) := \{x \in \mathfrak{g} \mid \lambda(t) \cdot x = t^j x \text{ for all } t \in k^\times\}.$$

As explained in [8, §5.5], the Dynkin grading is the analogue of the grading in characteristic 0 given by the  $\text{ad } h$ -eigenspace decomposition coming from an  $\mathfrak{sl}_2$ -triple  $(e, h, f)$ . We recall that  $\mathfrak{g}^e$  is contained in the non-negative part of the Dynkin grading, see [8, Proposition 5.8]. This yields

$$\mathfrak{g}^e = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathfrak{g}^e(j; \lambda).$$

We denote by  $\Phi^e \subseteq X^*(T_e)$  the restricted root system: the set of non-zero restrictions  $\alpha|_{T_e}$  where  $\alpha \in \Phi$ . Pick a set of positive roots  $\Phi_+^e \subseteq \Phi^e$ , and write  $\Phi_-^e = -\Phi_+^e$ . For  $\alpha \in X^*(T_e)$ , we write  $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid t \cdot x = \alpha(t)x \text{ for all } t \in T_e\}$  for the  $T_e$ -weight space corresponding to  $\alpha$ . So  $\mathfrak{g}_\alpha$  is zero unless  $\alpha \in \Phi^e \cup \{0\}$ , and we have  $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi^e} \mathfrak{g}_\alpha$ . Incorporating the Dynkin grading, we obtain the decomposition

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_0(j; \lambda) \oplus \bigoplus_{\substack{\alpha \in \Phi^e \\ j \in \mathbb{Z}}} \mathfrak{g}_\alpha(j; \lambda),$$

where  $\mathfrak{g}_\alpha(j; \lambda) := \mathfrak{g}_\alpha \cap \mathfrak{g}(j; \lambda)$ . For any  $x \in \mathfrak{g}$ , let  $\chi \in \mathfrak{g}^*$  be defined by  $\chi(x) := (e, x)$ . Note that the cocharacter  $\lambda \in X_*(T_{[e]}) \setminus X_*(T_e)$  for  $e$  satisfies the condition that the map  $\text{ad } e : \mathfrak{g}(j; \lambda) \rightarrow \mathfrak{g}(j+2; \lambda)$  is injective for all  $j \leq -1$ . We write  $\mathfrak{g}(j)$  instead of  $\mathfrak{g}(j; \lambda)$  to ease notation.

The skew-symmetric non-degenerate bilinear form  $\omega = \langle \cdot, \cdot \rangle$  is defined on  $\mathfrak{g}(-1)$  by

$$\langle x, y \rangle := (e, [x, y]) = \chi([x, y]).$$

Let  $\mathfrak{l} := \bigoplus_{\alpha \in \Phi_+^e} \mathfrak{g}_\alpha(-1)$ , which is a Lagrangian subspace of  $\mathfrak{g}(-1)$ . Then we write  $\mathfrak{l}' := \bigoplus_{\alpha \in \Phi_-^e} \mathfrak{g}_\alpha(-1)$ , so that  $\mathfrak{g}(-1) = \mathfrak{l} \oplus \mathfrak{l}'$ . Define the nilpotent subalgebra

$$\mathfrak{m} := \mathfrak{l} \oplus \bigoplus_{j \leq -2} \mathfrak{g}(j),$$

which is stable under the adjoint action of  $T$ . Set  $\bar{\mathfrak{p}} := \mathfrak{l}' \oplus \bigoplus_{j \geq 0} \mathfrak{g}(j)$ , which is a subspace of  $\mathfrak{g}$ . We fix a basis  $x_1, \dots, x_r, x_{r+1}, \dots, x_m$  for  $\bar{\mathfrak{p}}$  such that  $x_1, \dots, x_r$  span  $\mathfrak{g}^e$ . We choose this so that  $x_i \in \mathfrak{g}_{\alpha_i}(n_i)$ , where  $n_i \in \mathbb{Z}_{\geq -1}$  and  $\alpha_i \in \Phi^e \cup \{0\}$ . We define the generalized Gelfand-Graev  $\mathfrak{g}$ -module to be

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} k_\chi,$$

where  $k_\chi := k1_\chi$  is a one-dimensional  $\mathfrak{m}$ -module such that  $x.1_\chi = \chi(x)1_\chi$  for all  $x \in \mathfrak{m}$ . We denote by  $Q_\chi^x := Q_\chi/I_\chi Q_\chi$ , where  $I_\chi$  is the ideal of  $U(\mathfrak{g})$  generated by all  $x^p - x^{[p]} - \chi(x)^p 1$  with  $x \in \mathfrak{g}$ . Evidently,  $Q_\chi^x$  is a  $\mathfrak{g}$ -module with  $p$ -character  $\chi$ , and there exists a  $\mathfrak{g}$ -module isomorphism:

$$Q_\chi^x \cong U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{m})} k_\chi \cong U_\chi(\mathfrak{g})/K_\chi, \tag{3}$$

where  $K_\chi$  denotes the left ideal of  $U_\chi(\mathfrak{g})$  generated by all the elements  $x - \chi(x)$  with  $x \in \mathfrak{m}$ . The canonical projection is denoted by

$$\text{Pr} : U_\chi(\mathfrak{g}) \rightarrow Q_\chi^x. \tag{4}$$

**Definition 2.1.** Define a *reduced  $W$ -algebra*  $U_\chi(\mathfrak{g}, e)$  associated to  $\mathfrak{g}$  with  $p$ -character  $\chi$  by  $U_\chi(\mathfrak{g}, e) := (\text{End}_{\mathfrak{g}} Q_\chi^x)^{\text{op}}$ . ■

As any element of  $U_\chi(\mathfrak{g}, e)$  is uniquely determined by its effect on the generator  $1_\chi \in Q_\chi^x$ , then there exists a canonical isomorphism between  $U_\chi(\mathfrak{g}, e)$  and invariant algebra  $(Q_\chi^x)^{\text{ad}\mathfrak{m}}$  given by  $\theta \mapsto \theta(1_\chi)$  for all  $\theta \in U_\chi(\mathfrak{g}, e)$ . From now on, we shall identify  $U_\chi(\mathfrak{g}, e)$  with  $(Q_\chi^x)^{\text{ad}\mathfrak{m}}$ .

**Proposition 2.2.** ([16]) *Suppose that  $e \in \mathfrak{g}$  is regular nilpotent. Then the projection mapping in (4) induces a  $k$ -algebras isomorphism:*

$$\text{Pr} : Z_\chi(\mathfrak{g}) \xrightarrow{\sim} U_\chi(\mathfrak{g}, e). \tag{5}$$

The Kazhdan filtration of  $U(\mathfrak{g})$  is defined by setting  $F_j U(\mathfrak{g})$ , for  $j \in \mathbb{Z}$  to be spanned by monomials  $y_1 \cdots y_l$  with  $y_i \in \mathfrak{g}(n_i)$  and  $\sum_{i=1}^l (n_i + 2) \leq j$ . This induces non-negative filtrations  $(F_j Q_\chi^x)_{j \in \mathbb{Z}_{\geq 0}}$  and  $(F_j U_\chi(\mathfrak{g}, e))_{j \in \mathbb{Z}_{\geq 0}}$  on  $Q_\chi^x$  and  $U_\chi(\mathfrak{g}, e)$  respectively.

For  $m \in \mathbb{Z}_{\geq 1}$ , define  $\Lambda_m := \{(l_1, \dots, l_m) \mid l_j \in \mathbb{Z}, 0 \leq l_j \leq p - 1\}$ .

For  $\mathbf{a} := (a_1, \dots, a_m) \in \Lambda_m$ , we write  $x^{\mathbf{a}} := x_1^{a_1} \cdots x_m^{a_m}$ , an element of  $U(\mathfrak{g})$ . Then  $\{x^{\mathbf{a}} \otimes 1_\chi \mid \mathbf{a} \in \Lambda_m\}$  is a basis of  $Q_\chi^x$ . Define  $|\mathbf{a}| := \sum_{i=1}^m n_i$  and  $|\mathbf{a}|_e := \sum_{i=1}^m (n_i + 2)$  to be the total degree and the Kazhdan degree of  $x^{\mathbf{a}}$ .

In [16, §3.4] Premet showed that there exist  $\theta_1, \dots, \theta_r \in U_\chi(\mathfrak{g}, e)$  such that

$$\theta_l = (x_l + \sum_{|\mathbf{a}|_e = n_l + 2, |\mathbf{a}| \geq 2} \lambda_{\mathbf{a}, l} x^{\mathbf{a}} + \sum_{|\mathbf{a}|_e < n_l + 2} \lambda_{\mathbf{a}, l} x^{\mathbf{a}}) \otimes 1_\chi \tag{6}$$

for  $1 \leq l \leq r$ , where  $\lambda_{\mathbf{a}, l} \in k$  and  $\lambda_{\mathbf{a}, l} = 0$  if  $\mathbf{a}$  is such that  $a_{r+1} = \dots = a_m = 0$ . Moreover, we have

**Theorem 2.3.** ([16, Theorem 3.4]) *The monomials  $\theta_1^{a_1} \cdots \theta_r^{a_r}$  with  $0 \leq a_l \leq p - 1$  ( $1 \leq l \leq r$ ) form a basis of  $U_\chi(\mathfrak{g}, e)$ .*

**2.3.** We recall the mutually inverse equivalences between the category of all finite dimensional representations of the reduced enveloping algebras of Lie algebras and its reduced  $W$ -algebra counterparts. Let  $I_{\mathfrak{m}}$  denote the Jacobson radical of  $U_{\chi}(\mathfrak{m})$ , which is the left ideal of codimension one in  $U_{\chi}(\mathfrak{m})$  generated by  $\{x - \chi(x) \mid x \in \mathfrak{m}\}$ . Given a left  $U_{\chi}(\mathfrak{g})$ -module  $M$ , we set

$$M^{\mathfrak{m}} := \{v \in M \mid I_{\mathfrak{m}} \cdot v = 0\}.$$

In [16, Theorem 2.3(iv)], Premet proved that the reduced  $W$ -algebra  $U_{\chi}(\mathfrak{g}, e)$  is isomorphic to the  $k$ -algebra  $U_{\chi}(\mathfrak{g})^{\text{adm}} / (U_{\chi}(\mathfrak{g})^{\text{adm}} \cap U_{\chi}(\mathfrak{g})I_{\mathfrak{m}})$ , where  $U_{\chi}(\mathfrak{g})^{\text{adm}}$  denotes the centralizer of  $\mathfrak{m}$  in  $U_{\chi}(\mathfrak{g})$ . Therefore, any left  $U_{\chi}(\mathfrak{g}, e)$ -module can be considered as a  $U_{\chi}(\mathfrak{g})^{\text{adm}}$ -module with the trivial action of the ideal  $U_{\chi}(\mathfrak{g})^{\text{adm}} \cap U_{\chi}(\mathfrak{g})I_{\mathfrak{m}}$ .

**Theorem 2.4.** ([16]) *The functors*

$$Wh : U_{\chi}(\mathfrak{g})\text{-mod} \longrightarrow U_{\chi}(\mathfrak{g}, e)\text{-mod}, \quad M \mapsto M^{\mathfrak{m}}, \quad \text{and}$$

$$U_{\chi}(\mathfrak{g}) \otimes_{U_{\chi}(\mathfrak{g})^{\text{adm}}} - : U_{\chi}(\mathfrak{g}, e)\text{-mod} \longrightarrow U_{\chi}(\mathfrak{g})\text{-mod}, \quad V \mapsto U_{\chi}(\mathfrak{g}) \otimes_{U_{\chi}(\mathfrak{g})^{\text{adm}}} V,$$

are mutually inverse category equivalences.

Moreover, the proof of [16, Theorem 2.4] implies that

$$\dim M = p^{\dim \mathfrak{m}} \cdot \dim M^{\mathfrak{m}}. \tag{7}$$

### 3. Polynomial realization of reduced $W$ -algebra $U_{\chi}(\mathfrak{sl}_2, e)$

In this section we consider the case that  $\mathfrak{g} = \mathfrak{sl}_2$  with the usual basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with  $e$  being a regular nilpotent element in  $\mathfrak{g}$ . These elements satisfy  $e^{[p]} = 0$ ,  $f^{[p]} = 0$ , and  $h^{[p]} = h$ . It is clear that the Dynkin grading on  $\mathfrak{g}$  can be defined as

$$\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(2),$$

where  $\mathfrak{g}(-2) = kf$ ,  $\mathfrak{g}(0) = kh$ ,  $\mathfrak{g}(2) = ke$ . Moreover, we have  $\mathfrak{h} = kh$ ,  $\mathfrak{m} = kf$  and  $\bar{\mathfrak{p}} = kh \oplus ke$ . Fix the bilinear form on  $\mathfrak{g}$  to be the trace form  $\text{tr}(\cdot, \cdot)$ , and  $\chi(x) = (e, x)$  for any  $x \in \mathfrak{g}$ . It follows that  $\chi(e) = \chi(h) = 0$ ,  $\chi(f) = 1$ .

Let  $c := h^2 - 2h + 4ef$  be a multiple of the Casimir element of  $U(\mathfrak{g})$ . It is well-known that  $Z(\mathfrak{g})$  is generated by  $e^p, h^p - h, f^p$  and  $c$ . Therefore,  $Z_{\chi}(\mathfrak{g})$  can be realized as the quotient of a polynomial algebra in one variable  $c$ . In the following we will translate this problem into the one of the corresponding reduced  $W$ -algebra. Finally we will give the polynomial realization of  $Z_{\chi}(\mathfrak{g})$ .

From now on we will always assume that  $k = \overline{\mathbb{F}}_p$  is an algebraically closed field of characteristic  $p > 2$ .

**3.1.** Let  $G := SL(2)$  and  $R = \{\pm\alpha\}$  be the root system. We have the Weyl group  $W = \{1, s_{\alpha}\}$  with  $s_{\alpha}^2 = 1$ . The action of  $s_{\alpha}$  on  $\lambda' \in \mathfrak{h}^*$  is given by  $s_{\alpha}(\lambda') = \lambda' - \lambda'(h)\alpha$ . Set  $\rho = \frac{\alpha}{2} \in \mathfrak{h}^*$ . Then the dot action on  $\mathfrak{h}^*$  of any  $w \in W$  is defined by  $w \cdot \lambda' = w(\lambda' + \rho) - \rho$ .

Notice that  $\alpha(h) = 2$  and  $\rho(h) = \frac{\alpha}{2}(h) = 1$ . As  $\mathfrak{h} = kh$  is 1-dimensional, for any  $\lambda' \in \mathfrak{h}^*$  we may identify it with  $\lambda'(h) \in k$ , which will cause no confusion. Let  $M$  be an irreducible  $U_\chi(\mathfrak{g})$ -module. It follows from [7, §5.4] that there exists  $m_0 \in M$ ,  $m_0 \neq 0$ , such that  $e.m_0 = 0$  and  $h.m_0 = \lambda m_0$  for some  $\lambda \in k$ . The set  $\{v_i \equiv f^i \otimes m_0 \mid 0 \leq i \leq p-1\}$  is a basis of the baby Verma module  $Z_\chi(\lambda) := U_\chi(\mathfrak{sl}_2) \otimes_{U_\chi(kh+ke)} km_0$  and we have the relations

$$\begin{aligned} h.v_i &= (\lambda - 2i)v_i, \\ e.v_i &= \begin{cases} 0, & \text{if } i = 0, \\ i(\lambda - i + 1)v_{i-1}, & \text{if } i > 0, \end{cases} \\ f.v_i &= \begin{cases} v_{i+1}, & \text{if } i < p-1, \\ \chi(f)^p v_0, & \text{if } i = p-1. \end{cases} \end{aligned}$$

Since  $\chi(h) = 0$ , the possible  $\lambda$  are precisely the elements of  $\mathbb{F}_p$ , which we identify with the integers  $\{0, 1, \dots, p-1\}$ .

**Proposition 3.1.** (cf. [7]) *Each simple  $U_\chi(\mathfrak{g})$ -module has the form  $Z_\chi(\lambda)$  with  $\lambda \in \mathbb{F}_p$ , and  $Z_\chi(p - \lambda - 2) \cong Z_\chi(\lambda)$  as  $U_\chi(\mathfrak{g})$ -modules. Moreover, there cannot exist further isomorphisms.*

Owing to Proposition 3.1,  $U_\chi(\mathfrak{g})$  has  $\frac{p+1}{2}$  non-isomorphic simple modules, all of dimension  $p$ . The set  $\{Z_\chi(\lambda) \mid \lambda = 0, 1, \dots, \frac{p-3}{2}, p-1\}$  exhausts the class of all simple  $U_\chi(\mathfrak{g})$ -modules. We denote by  $Q_\chi(\lambda)$  the projective cover of  $Z_\chi(\lambda)$  as a  $U_\chi(\mathfrak{g})$ -module.

**Lemma 3.2.**  *$Q_\chi(\lambda)$  has length 2 for  $0 \leq \lambda \leq \frac{p-3}{2}$  and  $Q_\chi(p-1)$  has length 1. All composition factors of  $Q_\chi(\lambda)$  are isomorphic to  $Z_\chi(\lambda)$ .*

**Proof.** This is a direct consequence of Propostion 3.1 in conjunction with [7, Proposition 10.10]. ■

**3.2.** Now we consider the reduced  $W$ -algebra  $U_\chi(\mathfrak{g}, e)$ . In view of Proposition 2.2, there exists an isomorphism  $\text{Pr} : Z_\chi(\mathfrak{g}) \xrightarrow{\sim} U_\chi(\mathfrak{g}, e)$ , whence

$$\text{Pr}(c) = \text{Pr}(h^2 - 2h + 4ef) = h^2 - 2h + 4e.$$

From now on we put  $\theta := h^2 - 2h + 4e$ .

**Lemma 3.3.** *Suppose that  $\chi \in \mathfrak{g}^*$  is regular nilpotent. Then there exist  $\frac{p+1}{2}$  non-isomorphic simple  $U_\chi(\mathfrak{g}, e)$ -modules, all of dimension one. Moreover, the eigenvalues of these modules are  $\lambda^2 + 2\lambda$  with  $\lambda \in \mathbb{F}_p$ .*

**Proof.** According to Theorem 2.4 there exists a functor

$$Wh : U_\chi(\mathfrak{g})\text{-mod} \longrightarrow U_\chi(\mathfrak{g}, e)\text{-mod}, \quad M \mapsto M^{\mathfrak{m}},$$

sending  $U_\chi(\mathfrak{g})$ -modules to  $U_\chi(\mathfrak{g}, e)$ -modules, and all the finite-dimensional  $U_\chi(\mathfrak{g}, e)$ -modules can be obtained in this way. When combined with Proposition 3.1 this implies that all irreducible  $U_\chi(\mathfrak{g}, e)$ -modules have the form  $(Z_\chi(\lambda))^{\mathfrak{m}}$ .

Note that  $\mathfrak{m} = kf$  and  $I_{\mathfrak{m}} = \langle f - \chi(f) \rangle = \langle f - 1 \rangle$ , it follows that

$$(Z_\chi(\lambda))^{\mathfrak{m}} = \{v \in Z_\chi(\lambda) \mid (f - 1).v = 0\}.$$

Let  $m_0 \neq 0$  be in  $Z_\chi(\lambda)$  such that  $e.m_0 = 0$  and  $h.m_0 = \lambda m_0$  with  $\lambda \in \mathbb{F}_p$ . Since  $Z_\chi(\lambda)$  is spanned by the elements  $v_i \equiv f^i \otimes m_0$  with  $0 \leq i \leq p-1$ , we may thus assume that

$$a_0 m_0 + a_1 f \otimes m_0 + a_2 f^2 \otimes m_0 + \cdots + a_{p-1} f^{p-1} \otimes m_0 \in (Z_\chi(\lambda))^m$$

with  $a_i \in k$ , so that

$$\begin{aligned} & (f-1)(a_0 m_0 + a_1 f \otimes m_0 + a_2 f^2 \otimes m_0 + \cdots + a_{p-1} f^{p-1} \otimes m_0) \\ &= -a_0 m_0 + (a_0 - a_1) f \otimes m_0 + \cdots + (a_{p-2} - a_{p-1}) f^{p-1} \otimes m_0 + a_{p-1} \chi(f)^p m_0 \\ &= (a_{p-1} - a_0) m_0 + (a_0 - a_1) f \otimes m_0 + \cdots + (a_{p-2} - a_{p-1}) f^{p-1} \otimes m_0 = 0. \end{aligned}$$

As the elements  $f^i \otimes m_0$  are linearly independent, one can show by direct computation that  $a_0 = a_1 = a_2 = \cdots = a_{p-1}$ . Therefore, we can choose

$$v := m_0 + f \otimes m_0 + f^2 \otimes m_0 + \cdots + f^{p-1} \otimes m_0 \tag{8}$$

as a basis of  $(Z_\chi(\lambda))^m$ , whence

$$\begin{aligned} \theta.v &= (h^2 - 2h + 4e).(m_0 + f \otimes m_0 + f^2 \otimes m_0 + \cdots + f^{p-1} \otimes m_0) \\ &= (\lambda^2 - 2\lambda)m_0 + ((\lambda - 2)^2 - 2(\lambda - 2))f \otimes m_0 + 4\lambda m_0 + \cdots \\ &\quad + ((\lambda - 2(p-1))^2 - 2(\lambda - 2(p-1)))f^{p-1} \otimes m_0 + 4(p-1) \\ &\quad (\lambda - (p-1) + 1)f^{p-2} \otimes m_0 \\ &= (\lambda^2 + 2\lambda)(m_0 + f \otimes m_0 + f^2 \otimes m_0 + \cdots + f^{p-1} \otimes m_0) = (\lambda^2 + 2\lambda)v. \end{aligned} \tag{9}$$

As  $\lambda \in \mathbb{F}_p$ , it follows that  $\lambda^2 + 2\lambda \in \{0 \cdot 2, 1 \cdot 3, \dots, \frac{p-3}{2} \cdot \frac{p+1}{2}, p-1\}$ .

There are  $\frac{p+1}{2}$  pairwise different elements in the set  $\{0 \cdot 2, 1 \cdot 3, \dots, \frac{p-3}{2} \cdot \frac{p+1}{2}, p-1\}$ , as desired. ■

### 3.3. Proof of Theorem 1.1

We set  $S := \{0 \cdot 2, 1 \cdot 3, \dots, \frac{p-3}{2} \cdot \frac{p+1}{2}, p-1\}$ .

Note that all the elements of  $S$  are contained in  $\mathbb{F}_p$ .

**Proof.** (1) By virtue of our observation in Section 3.2, we know that the reduced  $W$ -algebra  $U_\chi(\mathfrak{g}, e)$  can be realized as the quotient algebra of  $k[\theta]$ . As  $k[\theta]$  is a principal ideal domain, the defining relation of  $U_\chi(\mathfrak{g}, e)$  is a principal ideal of  $k[\theta]$  generated by a polynomial. Thanks to Theorem 2.3, we may therefore assume that

$$I(\theta) := (\theta^p + a_1 \theta^{p-1} + a_2 \theta^{p-2} + \cdots + a_{p-1} \theta + a_p)$$

is the corresponding ideal, where  $a_i \in k$  for  $1 \leq i \leq p$ . We write

$$\theta^p + a_1 \theta^{p-1} + a_2 \theta^{p-2} + \cdots + a_{p-1} \theta + a_p = (\theta - x_0)(\theta - x_1) \cdots (\theta - x_{p-2})(\theta - x_{p-1}) \tag{10}$$

with  $x_i \in k$  for  $0 \leq i \leq p-1$ .

(2) We first show that all the  $x_i$ 's defined in (10) are contained in  $S$ . General theory ensures that all the irreducible  $k[\theta]/I(\theta)$ -modules are of dimension one. Let  $kv'$  be such a  $k[\theta]/I(\theta)$ -module, and set  $\theta.v' = \lambda'v'$  for some  $\lambda' \in k$ .

As  $kv'$  is an irreducible  $U_\chi(\mathfrak{g}, e)$ -module, Lemma 3.3 provides  $\lambda_0 \in \mathbb{F}_p$  such that  $\theta.v' = (\lambda_0^2 + \lambda_0)v'$ . As a result, all the  $x_i$ 's ( $0 \leq i \leq p-1$ ) are contained in  $S$ .

(3) Finally, we will calculate the multiplicity of  $x_0, \dots, x_{p-1}$  in  $S$ .

(Step 1) Consecutive application of Proposition 3.1 and Lemma 3.2 implies that  $Q_\chi(\lambda)$  are all non-isomorphic projective  $U_\chi(\mathfrak{g})$ -modules for  $\lambda \in \{0, 1, \dots, \frac{p-3}{2}, p-1\}$ ,  $\dim Q_\chi(\lambda) = 2p$  for  $\lambda \in \{0, 1, \dots, \frac{p-3}{2}\}$  and  $\dim Q_\chi(p-1) = p$ . Then Theorem 2.4 ensures that  $(Q_\chi(\lambda))^m$  are all non-isomorphic projective  $U_\chi(\mathfrak{g}, e)$ -modules. By virtue of (7), we obtain  $\dim Q_\chi(\lambda) = p \cdot \dim(Q_\chi(\lambda))^m$  for any  $\lambda \in \mathbb{F}_p$ . Therefore,  $\dim(Q_\chi(\lambda))^m = 2$  for  $\lambda \in \{0, 1, \dots, \frac{p-3}{2}\}$ , and  $\dim(Q_\chi(p-1))^m = 1$ .

Define 
$$Q := (Q_\chi(0))^m \oplus (Q_\chi(1))^m \oplus \dots \oplus (Q_\chi(\frac{p-3}{2}))^m \oplus (Q_\chi(p-1))^m, \tag{11}$$

so that  $Q$  is a projective  $U_\chi(\mathfrak{g}, e)$ -module, and  $\dim Q = 2 \cdot \frac{p-1}{2} + 1 = p$ . On the other hand, Theorem 2.3 yields  $\dim U_\chi(\mathfrak{g}, e) = p$ . By virtue of our observation above, we have  $U_\chi(\mathfrak{g}, e) \cong Q$  as  $U_\chi(\mathfrak{g}, e)$ -modules.

(Step 2) In (1), we have showed that  $U_\chi(\mathfrak{g}, e) = k[\theta]/I(\theta)$ . The principal indecomposable  $U_\chi(\mathfrak{g}, e)$ -modules  $(Q_\chi(\lambda))^m$  with  $\lambda \in \{0, 1, \dots, \frac{p-3}{2}, p-1\}$  are  $k[\theta]/I(\theta)$ -modules.

Since  $(Q_\chi(\lambda))^m$  is indecomposable, implication of (1) provides  $\lambda' \in S$  and  $n_{\lambda'} \in \mathbb{Z}_{\geq 1}$  such that  $(Q_\chi(\lambda))^m \cong k[\theta]/(\theta - \lambda')^{n_{\lambda'}}$ . In view of Lemma 3.2 and Theorem 2.4, all composition factors of  $(Q_\chi(\lambda))^m$  are isomorphic to  $(Z_\chi(\lambda))^m$ . We thus conclude that  $(Z_\chi(\lambda))^m = kv$  is one-dimensional and  $\theta.v = (\lambda^2 + 2\lambda)v$  (see (8) and (9)), so that  $\lambda' = \lambda^2 + 2\lambda$ . The dimension of  $(Q_\chi(\lambda))^m$  (see (Step 1)) ensures that  $n_{\lambda'} = 2$  if  $\lambda \in \{0, 1, \dots, \frac{p-3}{2}\}$  and  $n_{\lambda'} = 1$  if  $\lambda = p-1$ .

(4) Set  $S' := \{0 \cdot 2, 1 \cdot 3, \dots, \frac{p-3}{2} \cdot \frac{p+1}{2}\}$ , then  $S' \cup \{p-1\} = S$ . By (3), we know

$$U_\chi(\mathfrak{g}, e) \cong \bigoplus_{\lambda' \in S'} k[\theta]/(\theta - \lambda')^2 \oplus k[\theta]/(\theta - (p-1) \cdot 1). \tag{12}$$

For any  $\lambda' \in S'$ , set  $I_{\lambda'}$  to be the ideal of  $k[\theta]$  generated by  $(\theta - \lambda')^2$ , and  $I_{(p-1) \cdot 1}$  the ideal of  $k[\theta]$  generated by  $\theta - (p-1) \cdot 1$ . Obviously, the ideals  $I_{0 \cdot 2}, I_{1 \cdot 3}, \dots, I_{\frac{p-3}{2} \cdot \frac{p+1}{2}}, I_{(p-1) \cdot 1}$  are pairwise coprime, so that the Chinese Remainder Theorem yields that

$$k[\theta]/(I_{0 \cdot 2} \cap I_{1 \cdot 3} \cap \dots \cap I_{\frac{p-3}{2} \cdot \frac{p+1}{2}} \cap I_{(p-1) \cdot 1}) \cong \bigoplus_{\lambda' \in S'} k[\theta]/(\theta - \lambda')^2 \oplus k[\theta]/(\theta - (p-1) \cdot 1). \tag{13}$$

On the other hand, we have

$$I_{0 \cdot 2} \cap I_{1 \cdot 3} \cap \dots \cap I_{\frac{p-3}{2} \cdot \frac{p+1}{2}} \cap I_{(p-1) \cdot 1} = ((\theta - 0 \cdot 2)^2 \dots (\theta - \frac{p-3}{2} \cdot \frac{p+1}{2})^2 (\theta - (p-1) \cdot 1)), \tag{14}$$

and

$$\begin{aligned} & (\theta - 0 \cdot 2)^2 (\theta - 1 \cdot 3)^2 \dots (\theta - \frac{p-3}{2} \cdot \frac{p+1}{2})^2 (\theta - (p-1) \cdot 1) \\ &= (\theta - 0 \cdot 2) \dots (\theta - \frac{p-3}{2} \cdot \frac{p+1}{2}) (\theta - \frac{p-1}{2} \cdot \frac{p+3}{2}) \dots (\theta - (p-2) \cdot p) (\theta - (p-1) \cdot 1) \\ &= \prod_{\lambda=0}^{p-1} (\theta - \lambda(\lambda + 2)). \end{aligned} \tag{15}$$

Set  $i := \lambda + 1$  for any  $\lambda \in \mathbb{F}_p$ , then  $\lambda(\lambda + 2) = (i - 1)(i + 1) = i^2 - 1$ .

Hence 
$$\prod_{\lambda=0}^{p-1}(\theta - \lambda(\lambda + 2)) = \prod_{i=0}^{p-1}(\theta - (i^2 - 1)). \tag{16}$$

Recall in (1) we assumed that  $U_\chi(\mathfrak{g}, e) = k[\theta]/I(\theta)$ . Taking this into consideration, we combine (12) with (13), (14), (15) and (16) to conclude that

$$\theta^p + a_1\theta^{p-1} + a_2\theta^{p-2} + \dots + a_{p-1}\theta + a_p = \prod_{i=0}^{p-1}(\theta - (i^2 - 1)),$$

i.e., the numbers  $x_i$  for  $0 \leq i \leq p - 1$  in (10) can be given by

$$x_i = i^2 - 1, \tag{17}$$

which completes the proof of the theorem. ■

**3.4. Proof of Theorem 1.2.**

We return to the discussion on the center of the reduced enveloping algebra  $U_\chi(\mathfrak{g})$ .

**Proof.** Recall that  $Z(\mathfrak{g})$  is generated by  $e^p, h^p - h, f^p$ , and  $c$ , a multiple of the Casimir element of  $U(\mathfrak{g})$ . Then by definition  $\text{Pr}(c) = \text{Pr}(h^2 - 2h + 4ef) = h^2 - 2h + 4e$ . Thus, the theorem is a direct consequence of Theorem 1.1 and Proposition 2.2. ■

**Remark 3.4.** If one just considers the polynomial realization of the center of  $U_\chi(\mathfrak{g})$ , then the proof of Theorem 1.2 can be simplified. Direct computation shows that the Casimir element of  $U(\mathfrak{g})$  acts on the baby Verma module  $Z_\chi(\lambda)$  by scalar  $\lambda^2 + 2\lambda$  with  $\lambda \in \mathbb{F}_p$ . Hence, by a similar argument as in Section 3.3 one can also obtain Theorem 1.2.

**4. Further discussion of the coefficients in Theorem 1.1**

This section is devoted to the calculation of the coefficients  $a_i$ 's for  $1 \leq i \leq p$  in (1).

**4.1.** Retain the notation  $x_i = i^2 - 1$  for  $0 \leq i \leq p - 1$  as in (17). By the relationship between the coefficients and the roots, we have

$$\begin{aligned} -\tilde{a}_1 &= x_0 + x_1 + \dots + x_{p-1}, \\ \tilde{a}_2 &= x_0x_1 + x_0x_2 + \dots + x_0x_{p-1} + x_1x_2 + \dots + x_1x_{p-1} + \dots + x_{p-2}x_{p-1}, \\ &\dots\dots\dots \\ (-1)^j \tilde{a}_j &= \sum_{0 \leq i_1 < \dots < i_j \leq p-1} x_{i_1} \dots x_{i_j}, \\ &\dots\dots\dots \\ (-1)^p \tilde{a}_p &= x_0 \dots x_{p-1}, \end{aligned} \tag{18}$$

where the right side of (18) is the elementary symmetric polynomial in  $x_0, \dots, x_{p-1}$ , and  $\tilde{a}_j \equiv a_j \pmod{p}$  for  $1 \leq j \leq p$ .

**Lemma 4.1.** *We have  $a_{p-1} = a_p = 0$  in (1).*

**Proof.** Since

$$x_1 = 1^2 - 1 = 0 \quad \text{and} \quad x_{p-1} \equiv (-1)^2 - 1 = 0 \pmod{p}, \tag{19}$$

then  $(-1)^p \tilde{a}_p = x_0 \dots x_{p-1} = 0$ , i.e.,  $a_p = 0$ .

On the other hand, we have

$$(-1)^{p-1} \tilde{a}_{p-1} = \sum_{0 \leq i_1 < \dots < i_{p-1} \leq p-1} x_{i_1} \cdots x_{i_{p-1}}. \tag{20}$$

For each monomial in (20), there are  $p - 1$  different terms in  $\{x_0, \dots, x_{p-1}\}$  occur. Therefore, (19) yields  $a_{p-1} = 0$ . ■

**4.2.** For ease of reference we record the following well-known result:

- Lemma 4.2.** (1) For any  $(p-1) \nmid i$ , we have  $1^i + 2^i + \dots + (p-1)^i + p^i \equiv 0 \pmod{p}$ .  
 (2) For  $(p-1) \mid i$  with  $i \neq 0$ , we have  $1^i + 2^i + \dots + (p-1)^i + p^i \equiv -1 \pmod{p}$ .  
 (3) For  $i = 0$ , we have  $1^0 + 2^0 + \dots + (p-1)^0 + p^0 \equiv 0 \pmod{p}$ .

Set  $s_l := x_0^l + x_1^l + \dots + x_{p-1}^l$ . Then we have

**Lemma 4.3.** The following statements hold:

- (1) For any  $1 \leq l \leq \frac{p-3}{2}$ , we have  $s_l \equiv 0 \pmod{p}$ .  
 (2) For any  $\frac{p-1}{2} \leq l \leq p-2$ , we have  $s_l \equiv (-1)^{l-\frac{p-3}{2}} \binom{l}{l-\frac{p-1}{2}} \pmod{p}$ .

**Proof.** Note that

$$\begin{aligned} s_l &= \sum_{i=0}^{p-1} (i^2 - 1)^l = \sum_{i=1}^p (i^2 - 1)^l = \sum_{i=1}^p \left( \sum_{j=0}^l \binom{l}{j} (i^2)^{l-j} (-1)^j \right) \\ &= \sum_{j=0}^l \left( \binom{l}{j} (-1)^j \left( \sum_{i=1}^p i^{2(l-j)} \right) \right). \end{aligned} \tag{21}$$

(1) First we consider the case where  $1 \leq l \leq \frac{p-3}{2}$ . For any  $0 \leq j \leq l$ , we have  $0 \leq 2(l-j) \leq p-3$ . So we have  $(p-1) \nmid 2(l-j)$  or  $2(l-j) = 0$ . It follows from Lemma 4.2 that  $\sum_{i=1}^p i^{2(l-j)} \equiv 0 \pmod{p}$ . Therefore, the last term of (21) equals 0, i.e.,  $s_l \equiv 0 \pmod{p}$ .

(2) Now let  $\frac{p-1}{2} \leq l \leq p-2$ . As  $0 \leq j \leq l$ , then  $0 \leq 2(l-j) \leq 2(p-2)$ . Therefore, Lemma 4.2 implies that  $\sum_{i=1}^p i^{2(l-j)} \equiv -1 \pmod{p}$  for  $2(l-j) = p-1$ , and  $\sum_{i=1}^p i^{2(l-j)} \equiv 0 \pmod{p}$  otherwise. Then we have

$$\begin{aligned} \sum_{j=0}^l \left( \binom{l}{j} (-1)^j \left( \sum_{i=1}^p i^{2(l-j)} \right) \right) &\equiv \binom{l}{l-\frac{p-1}{2}} (-1)^{l-\frac{p-1}{2}} (-1) \\ &= (-1)^{l-\frac{p-3}{2}} \binom{l}{l-\frac{p-1}{2}} \pmod{p}, \end{aligned} \tag{22}$$

as required. ■

Recall that the Newton formula says

$$s_l + \tilde{a}_1 s_{l-1} + \dots + \tilde{a}_{l-1} s_1 + l \tilde{a}_l = 0 \tag{23}$$

for all  $1 \leq l \leq p$ . It follows that, for  $2 \leq l \leq p$ ,

$$\tilde{a}_l = -\frac{s_l + \tilde{a}_1 s_{l-1} + \dots + \tilde{a}_{l-1} s_1}{l} = -\frac{s_l + \sum_{i=1}^{l-1} \tilde{a}_i s_{l-i}}{l}. \tag{24}$$

**Lemma 4.4.** For any  $1 \leq l \leq \frac{p-3}{2}$ , we have  $\tilde{a}_l \equiv 0 \pmod{p}$ .

**Proof.** By virtue of Lemma 4.3 and (23), we have  $\tilde{a}_1 = -s_1 \equiv 0 \pmod{p}$ . For any  $1 \leq i \leq \frac{p-3}{2}$ , since  $s_i \equiv 0 \pmod{p}$  by Lemma 4.3, it follows from (24) that  $\tilde{a}_l \equiv 0 \pmod{p}$  for  $2 \leq l \leq \frac{p-3}{2}$ . ■

Now we consider the case where  $\frac{p-1}{2} \leq l \leq p-2$ . As  $s_1 \equiv \cdots \equiv s_{\frac{p-3}{2}} \equiv 0 \pmod{p}$  by Lemma 4.3, and  $\tilde{a}_1 \equiv \cdots \equiv \tilde{a}_{\frac{p-3}{2}} \equiv 0 \pmod{p}$  by Lemma 4.4, it follows that  $\sum_{i=1}^{l-1} \tilde{a}_i s_{l-i} \equiv 0 \pmod{p}$ . When combined with Lemma 4.3 and (24), we obtain

$$\tilde{a}_l = -\frac{s_l}{l} \equiv -\frac{(-1)^{l-\frac{p-3}{2}}}{l} \binom{l}{l-\frac{p-1}{2}} = \frac{(-1)^{l-\frac{p-1}{2}}}{l} \binom{l}{l-\frac{p-1}{2}} \pmod{p}. \quad (25)$$

**Lemma 4.5.** For  $\frac{p-1}{2} \leq l \leq p-2$ , we have  $\tilde{a}_l \equiv 2(-1)^{l-\frac{p-3}{2}} \binom{l-1}{\frac{p-3}{2}} \pmod{p}$ .

**Proof.** Since  $\binom{n}{m} = \frac{n}{m} \binom{n-1}{m-1}$  for any  $m, n \in \mathbb{Z}_{>0}$ , by (25) we have

$$\begin{aligned} \tilde{a}_l &\equiv \frac{(-1)^{l-\frac{p-1}{2}}}{l} \binom{l}{l-\frac{p-1}{2}} = \frac{(-1)^{l-\frac{p-1}{2}}}{l} \binom{l}{\frac{p-1}{2}} = \frac{(-1)^{l-\frac{p-1}{2}}}{\frac{p-1}{2}} \binom{l-1}{\frac{p-3}{2}} \\ &= 2(-1)^{l-\frac{p-3}{2}} \binom{l-1}{\frac{p-3}{2}} \pmod{p}. \end{aligned} \quad \blacksquare$$

### Proof of Proposition 1.3.

This proposition follows directly from Lemmas 4.1, 4.4 and 4.5.

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